Mikrolokale Analysis
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Aufgabenblatt 2

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Aufgabe 1 (2 Punkte)

Consider functions $f_k : \mathbb{R} \to \mathbb{K}$ with

$$f_k(x) = 0 \quad \text{for } |x| > \tfrac{1}{k} \quad \text{and} \int_{|x|\leq\frac{1}{k}} f_k(x)\,dx = 1$$

Show that for all $k \in \mathbb{N}$, the map $u_{f_k} : \mathcal{S}(\mathbb{R}) \to \mathbb{K}$, $g \mapsto \int f_k(x)g(x)\,dx$ is a tempered distribution. Prove that u_{f_k} converges to δ_0 (as $k \to \infty$) in $\mathcal{S}'(\mathbb{R})$, i.e. in the wk-*-topology ("$f_k \to \delta_0$").

How would you approximate δ_a for $a \neq 0$?

Aufgabe 2 (4 Punkte)

Prove directly, without using the theorems on induced topologies discussed in the lecture, the equivalence of the two assertions (C1) and (C2) regarding the continuity of a linear functional $u : \mathcal{D}(\Omega) \to \mathbb{K}$.

(C1) For all compact sets $K \subset \Omega$, there are $L \in \mathbb{N}_0$ and $C > 0$, such that

$$|u(g)| \leq C \sum_{|\alpha|\leq L} \sup_x |\partial^\alpha g(x)| \qquad \text{for all } g \in C_0^\infty(K)\ .$$

(C2) For any sequence $\{g_j\}_{j\geq 1}$ in $\mathcal{C}_0^\infty(\Omega)$ that converges to 0 in the sense that $\sup_x |\partial^\alpha g(x)| \to 0$ for each multiindex α and that for all j, the supports $\operatorname{supp} g_j$ are contained in some fixed compactum $K \subset \Omega$, we have that $u(g_j) \to 0$.

Suggestion: The direction "$\Rightarrow$" is easy. For the other direction, assume that (C1) is not valid, i.e. there is a compact set $K \subset \Omega$ such that there are no $C > 0$ and no $L \in \mathbb{N}_0$ such that the above estimate holds for all $g \in C_0^\infty(K)$. Use this to construct a sequence with $g_j \to 0$ (as $j \to \infty$) in the sense of (C2) but with $u(g_j) = 1$.