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Aufgabenblatt 2

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Aufgabe 1 (2 Punkte)

Consider functions $f_k : \mathbb{R} \to \mathbb{K}$ with

$$f_k(x) = 0 \quad \text{for } |x| > \tfrac{1}{k} \quad \text{and} \int_{|x|\leq\frac{1}{k}} f_k(x)\,dx = 1$$

Show that for all $k \in \mathbb{N}$, the map $u_{f_k} : \mathcal{S}(\mathbb{R}) \to \mathbb{K}$, $g \mapsto \int f_k(x)g(x)\,dx$ is a tempered distribution. Prove that u_{f_k} converges to δ_0 (as $k \to \infty$) in $\mathcal{S}'(\mathbb{R})$, i.e. in the wk-*-topology ("$f_k \to \delta_0$").

How would you approximate δ_a for $a \neq 0$?

Lösung

For each k we have, $u_{f_k}(g) \leq \|f_k\|_{L^1} \sup_x |g(x)| = \sup_x |g(x)|$ so $u_{f_k} \in \mathcal{S}'$. Moreover, $|g(0) - u_{f_k}(g)| = |\int (g(0) - g(x)) f_k(x) dx| \leq \sup_{|x|<1/k} |g(0) - g(x)|$, so for $k \to \infty$, $u_{f_k}(g) \to g(0)$. Since $\delta_0 \in \mathcal{S}'(\mathbb{R}^n)$, we are done.
More generally, δ_a can be approximated using functions f_k with $\int_{|x-a|\leq\frac{1}{k}} f_k(x) = 1$.

Aufgabe 2 (4 Punkte)

Prove directly, without using the theorems on induced topologies discussed in the lecture, the equivalence of the two assertions (C1) and (C2) regarding the continuity of a linear functional $u : \mathcal{D}(\Omega) \to \mathbb{K}$.

(C1) For all compact sets $K \subset \Omega$, there are $L \in \mathbb{N}_0$ and $C > 0$, such that

$$|u(g)| \leq C \sum_{|\alpha|\leq L} \sup_x |\partial^\alpha g(x)| \qquad \text{for all } g \in C_0^\infty(K) \; .$$

(C2) For any sequence $\{g_j\}_{j\geq 1}$ in $\mathcal{C}_0^\infty(\Omega)$ that converges to 0 in the sense that $\sup_x |\partial^\alpha g(x)| \to 0$ for each multiindex α and that for all j, the supports $\operatorname{supp} g_j$ are contained in some fixed compactum $K \subset \Omega$, we have that $u(g_j) \to 0$.

Suggestion: The direction "$\Rightarrow$" is easy. For the other direction, assume that (C1) is not valid, i.e. there is a compact set $K \subset \Omega$ such that there are no $C > 0$ and no $L \in \mathbb{N}_0$ such that the above estimate holds for all $g \in C_0^\infty(K)$. Use this to construct a sequence with $g_j \to 0$ (as $j \to \infty$) in the sense of (C2) but with $u(g_j) = 1$.

Remark: In the two characterizations (C1) and (C2) of continuity you can see the inductive limit topology "in action" and you should try to understand (C1) and (C2) in terms of the general construction discussed in the lecture. Alternatively, you can ignore this background and take e.g. (C1) as the definition of continuity on $\mathcal{D}(\Omega)$.

Lösungshinweis

Proof of equivalence: see Hörmander theorem 2.1.4