# Introduction to classical and quantum Markov semigroups

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Hölder's inequality shows that $x \mapsto [x, y]$ is an element of $(\ell^p)^*$ for any $y \in \ell^q$; proving that every functional arises this way is an exercise. Furthermore, the same pairing gives an isomorphism between $(\ell^1)^*$ and $\ell^\infty$. [The dual of $\ell^\infty$ is much larger than $\ell^1$; it is isomorphic to the space $M(\beta\mathbb{N})$ of regular complex Borel measures on the Stone–Čech compactification of the natural numbers.]

Similarly, for conjugate indices $p$, $q \in (1, \infty)$, the dual of $L^p(\Omega, \mathcal{A}, \mu)$ is identified with $L^q(\Omega, \mathcal{A}, \mu)$, and the dual of $L^1(\Omega, \mathcal{A}, \mu)$ with $L^\infty(\Omega, \mathcal{A}, \mu)$, via the pairing

$$[f, g] := \int_\Omega f(x) g(x) \, \mu(\mathrm{d}x).$$

In particular, $\ell^2$ and $L^2(\Omega, \mathcal{A}, \mu)$ are conjugate-linearly isomorphic to their dual spaces. This is a general fact about Hilbert spaces, known as the *Riesz–Fréchet theorem*: if $H$ is a Hilbert space then

$$H^* = \big\{ \langle x| : x \in H \big\}, \qquad \text{where } \langle x|y := \langle x, y \rangle \quad \text{for all } y \in H.$$

If $K$ is a compact Hausdorff space then the dual of $C(K)$ is naturally isomorphic to the space $M(K)$ of regular complex Borel measures on $K$, with dual pairing

$$[f, \mu] := \int_K f(x) \, \mu(x) \qquad \text{for all } f \in C(K) \text{ and } \mu \in M(K).$$

The Hahn–Banach theorem implies that the dual space separates points: if $x \in X$ then there exists $\phi \in X^*$ such that $\|\phi\| = 1$ and $\phi(x) = \|x\|$.

**Definition 1.15.** A family of operators $T = (T_t)_{t \in \mathbb{R}_+} \subseteq B(X)$ is a *one-parameter semigroup* if

(i) $T_0 = I$, the identity operator

and (ii) $T_s T_t = T_{s+t}$ for all $s$, $t \in \mathbb{R}_+$.

The semigroup $T$ is *strongly continuous* if

$$\lim_{t \to 0+} \|T_t x - x\| = 0 \qquad \text{for all } x \in X.$$

The semigroup $T$ is *uniformly continuous* if

$$\lim_{t \to 0+} \|T_t - I\| = 0.$$

**Exercise 1.16.** Prove that a uniformly continuous semigroup is strongly continuous. Give an example to show that the converse is false.

**Theorem 1.17.** Let $T$ be a strongly continuous one-parameter semigroup on the Banach space $X$. There exist constants $M \geqslant 1$ and $a \in \mathbb{R}$ such that $\|T_t\| \leqslant M e^{at}$ for all $t \in \mathbb{R}_+$.

*Proof.* See [3, Theorem 6.2.1]. □

**Remark 1.18.** The semigroup $T$ of Theorem 1.17 is said to be of *type* $(M, a)$. A semigroup of type $(1, 0)$ is also called a *contraction semigroup*.

By replacing $T_t$ with $e^{-at}T_t$, one can often reduce to the case of semigroups with uniformly bounded norm. However, it is not always possible to go further and reduce to contraction semigroups; see [3, Example 6.2.3 and Theorem 6.3.8].

**Exercise 1.19.** Prove that a strongly continuous semigroup is continuous everwhere: if $t > 0$ then $\lim_{h\to 0} \|T_{t+h}x - T_t x\| = 0$. [The same is true, *mutatis mutandis*, for uniformly continuous semigroups.]

**Exercise 1.20.** Given any $A \in B(X)$, let $\exp(A) := \sum_{n=0}^{\infty} \frac{1}{n!} A^n$.

(i) Prove that this series is convergent, so that $\exp(A) \in B(X)$. Prove further that $\|\exp(A)\| \leqslant \exp\|A\|$.

(ii) Prove that if $A$, $B \in B(X)$ commute, in the sense that $AB = BA$, then so do $\exp(A)$ and $\exp(B)$, with $\exp(A)\exp(B) = \exp(A + B)$. [Hint: consider the derivatives of

$$t \mapsto \exp(tA)\exp(-tA) \qquad \text{and} \qquad t \mapsto \exp(tA)\exp(tB)\exp\big(-t(A + B)\big).]$$

(iii) Prove that setting $\exp(tA)$ for all $t \in \mathbb{R}_+$ defines a uniformly continuous one-parameter semigroup.

The converse of Exercise 1.20(iii) is true, and we state it as a theorem.

**Theorem 1.21.** If $T$ is a uniformly continuous one-parameter semigroup then there exists an operator $A \in B(X)$ such that $T_t = \exp(tA)$ for all $t \in \mathbb{R}_+$.

*Proof.* By continuity at the origin, there exists $t_0 > 0$ such that

$$\|T_s - I\| < 1/2 \qquad \text{for all } s \in [0, t_0].$$

Then

$$\Big\| t_0^{-1} \int_0^{t_0} T_s \,\mathrm{d}s - I \Big\| = t_0^{-1} \Big\| \int_0^{t_0} T_s - I \,\mathrm{d}s \Big\| \leqslant 1/2 < 1.$$

Hence $X := t_0^{-1} \int_0^{t_0} T_s \,\mathrm{d}s \in B(X)$ is invertible, because the Neumann series

$$\sum_{n=0}^{\infty} (I - X)^n = I + (I - X) + (I - X)^2 + \ldots$$

is convergent. Furthermore,

$$\begin{aligned}
h^{-1}(T_h - I)\int_0^{t_0} T_s \,\mathrm{d}s = h^{-1}\int_0^{t_0} T_{s+h} - T_s \,\mathrm{d}s &= h^{-1}\int_h^{t_0+h} T_s \,\mathrm{d}s - h^{-1}\int_0^{t_0} T_s \,\mathrm{d}s \\
&= h^{-1}\int_{t_0}^{t_0+h} T_s \,\mathrm{d}s - h^{-1}\int_0^{h} T_s \,\mathrm{d}s \\
&\to T_{t_0} - I
\end{aligned}$$

as $h \to 0+$. Hence

$$A := \lim_{h\to 0+} h^{-1}(T_h - I) = (T_{t_0} - I)(t_0 X)^{-1}.$$

Moreover, for any $t \in \mathbb{R}_+$,

$$\begin{aligned}
T_{t_0} = I + A\int_0^{t_0} T_s \,\mathrm{d}s &= I + A\Big(t_0 I + \int_0^{t_0}\int_0^{s} T_r \,\mathrm{d}r \,\mathrm{d}s\Big) \\
&= \dots \\
&= I + t_0 A + \frac{t_0^2}{2}A^2 + \dots + A^n \int_0^{t_0}\cdots\int_0^{t_n} T_{t_{n+1}} \,\mathrm{d}t_{n+1}\dots \mathrm{d}t_1 \\
&\to \sum_{n\geqslant 0}\frac{1}{n!}(t_0 A)^n = \exp(t_0 A) \qquad \text{as } n\to\infty,
\end{aligned}$$

since

$$\Big\| A^n \int_0^{t_0}\cdots\int_0^{t_n} T_{t_{n+1}} \,\mathrm{d}t_{n+1}\dots \mathrm{d}t_1 \Big\| \leqslant \frac{3t^{n+1}\|A\|^n}{2(n+1)!}.$$

This working shows that $T_t = \exp(tA)$ for any $t \in [0, t_0]$, so for all $t \in \mathbb{R}_+$, by the semigroup property: there exists $n \in \mathbb{Z}_+$ and $s \in [0, t_0)$ such that $t = nt_0 + s$, and

$$T_t = T_{t_0}^n T_s = \exp(nt_0 A + sA) = \exp(tA). \qquad \square$$

**Remark 1.22.** The integrals in the previous proof are *Bochner integrals*; they are an extension of the Lebesgue integral to functions which take values in a Banach space. We will only be concerned with continuous functions, so do not need to concern ourselves with notions of measurability. All the theorems that one would expect carry over to from the Lebesgue to the Bochner setting, together with the fact that if $T$ is a bounded operator then $T\int f = \int Tf$.

**Definition 1.23.** If $T$ is a uniformly continuous one-parameter semigroup then the operator $A \in B(X)$ such that $T_t = \exp(tA)$ for all $t \in \mathbb{R}_+$ is the *generator* of the semigroup.

**Exercise 1.24.** Prove that the generator of a uniformly continuous one-parameter semigroup $T$ is unique. [Hint: consider the limit of $t^{-1}(T_t - I)$ as $t \to 0+$.]

**Example 1.25.** Given $t \in \mathbb{R}_+$ and $f \in X := L^p(\mathbb{R}_+)$, where $p \in [1, \infty)$, let

$$(T_t f)(x) := f(x + t) \qquad \text{for all } x \in \mathbb{R}_+.$$

Then $T_t \in B(X)$, with $\|T_t\| = 1$, and $T = (T_t)_{t \in \mathbb{R}_+}$ is a one-parameter semigroup. If $f$ is continuous and has compact support then an application of the Dominated Convergence Theorem gives that $T_t f \to f$ as $t \to 0+$ ; since such functions are dense in $X$, it follows that $T$ is strongly continuous.

**Exercise 1.26.** Prove the assertions in Example 1.25. Prove further that if $f \in X$ is absolutely continuous, so that there exists $f' \in X$ such that

$$f(x) = f(0) + \int_0^x f'(y) \,\mathrm{d}y \qquad \text{for all } x \in \mathbb{R}_+,$$

then

$$\lim_{t \to 0+} t^{-1}(T_t f - f) = f',$$

where the limit exists in $X = L^p(\mathbb{R}_+)$. [Hint: show that

$$\|t^{-1}(T_t f - f) - f'\|_p^p = t^{-1} \int_0^t \|T_y f' - f'\|_p^p \,\mathrm{d}y$$

and then use the strong continuity of $T$ at the origin.]

## 1.2 Beyond uniform continuity

Throughout this section, $X$ denotes a Banach space.

**Definition 1.27.** An *unbounded operator in* $X$ is a linear transformation $A$ defined on a linear subspace $X_0 \subseteq X$, its *domain*; we write $\operatorname{dom} A = X_0$.

An *extension* of $A$ is an unbounded operator $B$ in $X$ such that $\operatorname{dom} A \subseteq \operatorname{dom} B$ and the restriction $B|_{\operatorname{dom} A} = A$.

An unbounded operator $A$ in $X$ is *densely defined* if $\operatorname{dom} A$ is dense in $X$ for the norm topology.

**Definition 1.28.** Given operators $A$ and $B$, let $A + B$ and $AB$ be defined by setting

$$\operatorname{dom}(A + B) := \operatorname{dom} A \cap \operatorname{dom} B, \quad (A + B)x := Ax + Bx$$

and

$$\operatorname{dom} AB := \{x \in \operatorname{dom} A : Ax \in \operatorname{dom} B\}, \quad (AB)x := A(Bx).$$

Note that neither $A + B$ nor $AB$ need be densely defined, even if $A$ and $B$ were.

**Definition 1.29.** Let $T$ be a strongly continuous one-parameter semigroup on $X$. Its *generator* $A$ has domain

$$\operatorname{dom} A := \big\{x \in X : \lim_{t \to 0+} t^{-1}(T_t x - x) \text{ exists in } X\big\}$$

and action

$$Ax := \frac{\mathrm{d}}{\mathrm{d}t} T_t x \Big|_{t=0} := \lim_{t \to 0+} t^{-1}(T_t x - x) \qquad \text{for all } x \in \operatorname{dom} A.$$

It is readily verified that $A$ is an unbounded operator.

**Exercise 1.30.** Prove that if $x \in X$ and $t \in \mathbb{R}_+$ then

$$\int_0^t T_s x \,\mathrm{d}s \in \operatorname{dom} A$$

and

$$(T_t - I)x = A \int_0^t T_s x \,\mathrm{d}s.$$

Deduce that $\operatorname{dom} A$ is dense in $X$. [Hint: begin by imitating the proof of Theorem 1.21.]

**Lemma 1.31.** Let the strongly continuous semigroup $T$ have generator $A$. If $x \in \operatorname{dom} A$ and $t \in \mathbb{R}_+$ then $T_t x \in \operatorname{dom} A$ and $T_t A x = A T_t x$; thus $T_t(\operatorname{dom} A) \subseteq \operatorname{dom} A$. Furthermore,

$$(T_t - I)x = \int_0^t T_s A x \,\mathrm{d}s = \int_0^t A T_s x \,\mathrm{d}s.$$

*Proof.* First, note that

$$h^{-1}(T_h - I)T_t x = T_t h^{-1}(T_h - I)x \to T_t A x \quad \text{as } h \to 0+,$$

by the boundedness of $T_t$, so $T_t x \in \operatorname{dom} A$ and $A T_t x = T_t A x$, as claimed. For the second part, let

$$F : \mathbb{R}_+ \to X;\ t \mapsto (T_t - I)x - \int_0^t T_s A x \,\mathrm{d}s.$$

Note that $F$ is continuous and $F(0) = 0$; furthermore, if $t > 0$ then

$$h^{-1}(F(t+h) - F(t)) = T_t h^{-1}(T_h - I)x - h^{-1} \int_0^h T_{s+t} A x \,\mathrm{d}s \to T_t A x - T_t A x = 0$$

as $h \to 0+$, whence $F \equiv 0$. ∎

**Definition 1.32.** An unbounded operator $A$ in $X$ is *closed* if, whenever $(x_n)_{n \in \mathbb{N}} \subseteq \operatorname{dom} A$ is such that $x_n \to x \in X$ and $A x_n \to y \in X$, it follows that $x \in \operatorname{dom} A$ and $Ax = y$. Note that a bounded operator is automatically closed.

The operator $A$ is *closable* if it has a closed extension, in which case its *closure* $\overline{A}$ is the smallest such.

**Exercise 1.33.** Prove that the *graph*

$$\mathcal{G}(A) := \{(x, Ax) : x \in \operatorname{dom} A\}$$

of an unbounded operator $A$ in $X$ is a normed vector space for the product norm

$$\| \cdot \| : (x, Ax)\| \mapsto \|x\| + \|Ax\|.$$

Prove further that $A$ is closed if and only if $\mathcal{G}(A)$ is a Banach space, and that $A$ is closable if and only if the closure of its graph in $X \oplus X$ is the graph of some operator.

**Exercise 1.34.** Let $A$ be the generator of the strongly continuous one-parameter semigroup $T$. Use Lemma 1.31 and Theorem 1.17 to show that $A$ is closed.

*Proof.* Suppose $(x_n)_{n\in\mathbb{N}} \subseteq \operatorname{dom} A$ is such that $x_n \to x$ and $Ax_n \to y$. Let $t > 0$ and note that

$$T_t x_n - x_n = \int_0^t T_s A x_n \,\mathrm{d}s \qquad \text{for all } n \geqslant 1.$$

Furthermore,

$$\left\| \int_0^t T_s A x_n \,\mathrm{d}s - \int_0^t T_s y \,\mathrm{d}s \right\| \leqslant \int_0^t M e^{as} \|Ax_n - y\| \,\mathrm{d}s \leqslant M t e^{\max\{a,0\}t} \|Ax_n - y\| \to 0$$

as $n \to \infty$, so

$$T_t x - x = \int_0^t T_s y \,\mathrm{d}s.$$

Dividing both sides by $t$ and letting $t \to 0+$ gives that $x \in \operatorname{dom} A$ and $Ax = y$, as required. □

**Definition 1.35.** Let $A$ be an unbounded operator in $X$. Its *spectrum* is the set

$$\sigma(A) := \{z \in \mathbb{C} : zI - A \text{ is not invertible in } B(X)\}$$

and its *resolvent* is the map

$$\mathbb{C} \setminus \sigma(A) \to B(X);\ z \mapsto (zI - A)^{-1}.$$

In other words, $z \in \mathbb{C}$ is not in the spectrum of $A$ if and only if there exists a bounded operator $B \in B(X)$ such that $B(zI - A) = I_{\operatorname{dom} A}$ and $(zI - A)B = I_X$. In particular, the operator $zI - A$ is a bijection from $\operatorname{dom} A$ onto $X$.

**Exercise 1.36.** Let $A$ be an unbounded operator in $X$ and suppose $z \in \mathbb{C}$ is such that $zI - A$ is a bijection from $\operatorname{dom} A$ onto $X$. Prove that $(zI - A)^{-1}$ is bounded if and only if $A$ is closed. [Thus algebraic invertibility of $zI - A$ is equivalent to its topological invertibility if and only if $A$ is closed.]

The following theorem shows that the resolvent of a semigroup generator may be thought of as the Laplace transform of the semigroup.

**Theorem 1.37.** Let $A$ be the generator of a one-parameter semigroup $T$ of type $(M, a)$ on $X$. Then $\sigma(A) \subseteq \{z \in \mathbb{C} : \operatorname{Re} z \leqslant a\}$. Furthermore, if $\operatorname{Re} z > a$ then

$$(zI - A)^{-1} x = \int_0^\infty e^{-zt} T_t x \,\mathrm{d}t \qquad \text{for all } x \in X \tag{1.2}$$