and $\|(zI - A)^{-1}\| \leqslant M(\operatorname{Re} z - a)^{-1}$.

Proof. Fix $z \in \mathbb{C}$ with $\operatorname{Re} z > a$ and note first that

$$R : X \mapsto X;\ x \mapsto \int_0^\infty e^{-zt} T_t x \,\mathrm{d}t$$

is a bounded linear operator, with $\|R\| \leqslant M(\operatorname{Re} z - a)^{-1}$.
If $x \in X$ and $y = Rx$ then

$$T_t y = \int_0^\infty e^{-zs} T_{s+t} x \,\mathrm{d}s = \int_t^\infty e^{-z(u-t)} T_u x \,\mathrm{d}u = e^{zt} \int_t^\infty e^{-zu} T_u x \,\mathrm{d}u,$$

and therefore, if $t > 0$,

$$\begin{aligned} t^{-1}(T_t - I)y &= t^{-1} e^{zt} \int_t^\infty e^{-zs} T_s x \,\mathrm{d}s - t^{-1} \int_0^\infty e^{-zs} T_s x \,\mathrm{d}s \\ &= -t^{-1} e^{zt} \int_0^t e^{-zs} T_s x \,\mathrm{d}s + t^{-1}(e^{zt} - 1) \int_0^\infty e^{-zs} T_s x \,\mathrm{d}s \\ &\to -x + zy \qquad \text{as } t \to 0+. \end{aligned}$$

Thus $y \in \operatorname{dom} A$ and $(zI - A)y = x$. It follows that $\operatorname{ran} R \subseteq \operatorname{dom} A$ and $(zI - A)R = I_X$. However, since $(T_t - I)R = R(T_t - I)$ and R is bounded, the same working shows that

$$RAx = -x + zRx \iff R(zI - A)x = x \qquad \text{for all } x \in \operatorname{dom} A.$$

Thus $R(zI - A) = I_{\operatorname{dom} A}$ and $R = (zI - A)^{-1}$, as claimed. □

Theorem 1.38. (Feller–Miyadera–Phillips) A closed, densely defined operator A in the Banach space X is the generator of a strongly continuous semigroup of type (M, a) if and only if

$$\sigma(A) \subseteq \{z \in \mathbb{C} : \operatorname{Re} z \leqslant a\}$$

and

$$\|(zI - A)^{-m}\| \leqslant M(z - a)^{-m} \qquad \text{for all } z > a \text{ and } m \in \mathbb{N}.$$

Proof. Suppose A is the generator of a strongly continuous semigroup T of type (M, a). The spectral condition is a consequence of Theorem 1.37, and applying (1.2) repeatedly shows that

$$(zI - A)^{-m} x = \int_0^\infty \cdots \int_0^\infty e^{-z(t_1 + \cdots + t_m)} T_{t_1 + \cdots + t_m} x \,\mathrm{d}t_n \ldots \mathrm{d}t_1$$

for all $z > a$, $m \geqslant 1$ and $x \in X$. Thus

$$\|(zI - A)^{-m}\| \leqslant M\Big(\int_0^\infty e^{-(z-a)t} \,\mathrm{d}t\Big)^m = M(z - a)^{-m}.$$

For the converse, see [3, Theorem 8.3.1]. The idea is to approximate the generator A by bounded operators $A_z = zA(zI - A)^{-1}$, where $z \in (a, \infty)$, then show that the uniformly continuous semigroup with generator A_z converges to a strongly continuous semigroup with generator A as $z \to \infty$. In particular, it suffices to assume that $\sigma(A) \cap (a, \infty)$ is empty. ☐

Theorem 1.39. (Hille–Yosida) Let A be a closed, densely defined operator in the Banach space X. The following are equivalent.

(i) A is the generator of a strongly continuous contraction semigroup.

(ii) $\sigma(A) \subseteq \{z \in \mathbb{C} : \operatorname{Re} z \leqslant 0\}$ and

$$\|(zI - A)^{-1}\| \leqslant (\operatorname{Re} z)^{-1} \qquad \text{whenever } \operatorname{Re} z > 0.$$

(iii) $\sigma(A) \cap (0, \infty)$ is empty and

$$\|(zI - A)^{-1}\| \leqslant z^{-1} \qquad \text{whenever } z > 0.$$

Proof. Note that (i) implies (ii), by Theorem 1.37, and (ii) implies (iii) trivially. That (iii) implies (i) follows from the extension of Theorem 1.38 noted in its proof. ☐

1.3 The Lumer–Phillips theorem

Throughout this section, X denotes a Banach space and X^* its topological dual.

Definition 1.40. For all $x \in X$, let

$$T(x) := \{\phi \in X^* : \phi(x) = \|x\|^2 = \|\phi\|^2\}$$

is the set of *normalised tangent functionals* to x. The Hahn–Banach theorem implies that $T(x)$ is non-empty for all $x \in X$.

Exercise 1.41. Prove that if H is a Hilbert space then $T(x) = \{\langle x|\}$ for all $x \in H$, where the Dirac functional $\langle x|$ is such that $\langle x|y := \langle x, y\rangle$ for all $y \in H$. [Recall the Riesz–Fréchet theorem, that $H^* = \{\langle x| : x \in H\}$.]

Exercise 1.42. Prove that if $f \in X = C(K)$ and $x_0 \in K$ is such that $|f(x_0)| = \|f\|$ then setting $\phi(g) := \overline{f(x_0)}g(x_0)$ for all $g \in X$ defines a normalised tangent functional for f. Deduce that $T(f)$ may contain more than one element.

Definition 1.43. An unbounded operator A on X is *dissipative* if and only if, for all $x \in \operatorname{dom} A$, there exists $\phi \in T(x)$ such that $\operatorname{Re} \phi(Ax) \leqslant 0$.

Exercise 1.44. Prove that an operator A on a Hilbert space H is dissipative if and only if $\operatorname{Re}\langle x, Ax\rangle \leqslant 0$ for all $x \in \operatorname{dom} A$.

Exercise 1.45. Prove that an operator A on a Hilbert space H is dissipative if and only if $\|(I+A)x\| \leqslant \|(I-A)x\|$ for all $x \in \operatorname{dom} A$.

Exercise 1.46. Suppose T is a contraction semigroup with generator A. Prove that A is dissipative.

Proof. If $x \in \operatorname{dom} A$ and $\phi \in T(x)$ then

$$\operatorname{Re}\phi(Ax) = \lim_{t\to 0+} t^{-1}\operatorname{Re}\phi(T_t x - x) \leqslant \lim_{t\to 0+} t^{-1}\|\phi\|\,\|x\| - \|x\|^2 = 0,$$

so A is dissipative. □

Lemma 1.47. If A is dissipative then

$$\|(zI-A)x\| \geqslant z\|x\| \qquad \text{for all } z>0 \text{ and } x \in \operatorname{dom} A. \tag{1.3}$$

If, further, $zI - A$ is surjective for some $z > 0$ then $z \notin \sigma(A)$ and $\|(zI-A)^{-1}\| \leqslant z^{-1}$.

Proof. If $z > 0$, $x \in \operatorname{dom} A$ and $\phi \in T(x)$ is such that $\operatorname{Re}\phi(Ax) \leqslant 0$ then

$$\|x\|\,\|(zI-A)x\| \geqslant |\phi((zI-A)x)| = |z\|x\|^2 - \phi(Ax)| \geqslant z\|x\|^2.$$

In particular, $zI - A$ is injective.

If $zI - A$ is also surjective then the inequality (1.3) implies that $\|y\| \geqslant z\|(zI-A)^{-1}y\|$ for all $y \in X$, whence the second claim. □

Remark 1.48. In fact, being dissipative is equivalent to the condition (1.3), but this is harder to prove. See [2, Proposition 3.1.14].

Exercise 1.49. Let A be dissipative. Prove that $zI - A$ is surjective for some $z > 0$ if and only if $zI - A$ is surjective for all $z > 0$. [Hint: for a suitable choice of z and z_0, consider the series $R_z := \sum_{n=0}^{\infty}(z-z_0)^n(z_0I-A)^{-(n+1)}$.]

Proof. Suppose that $z_0 > 0$ is such that $z_0I - A$ is surjective. It follows from Lemma 1.47 that $\|(z_0I-A)^{-1}\| \leqslant z_0^{-1}$. The series

$$R_z = \sum_{n=0}^{\infty}(z_0 - z)^n(z_0I-A)^{-(n+1)}$$

is norm convergent for all $z \in (0, 2z_0)$; if we can show that $R_z = (zI-A)^{-1}$ then the result follows.

If $B \in B(X)$ is such that $\|B\| < 1$ then $I - B$ is invertible, with $(I-B)^{-1} = \sum_{n=0}^{\infty} B^n$. Hence if $B = (z_0 - z)(z_0I - A)^{-1}$ then

$$R_z = (z_0I-A)^{-1}(I-B)^{-1} = (I-B)^{-1}(z_0I-A)^{-1},$$

so $\operatorname{ran} R_z \subseteq \operatorname{dom}(z_0 I - A) = \operatorname{dom}(zI - A)$,

$$(zI - A)R_z = ((z - z_0)I + (z_0 I - A))R_z = ((z - z_0)(z_0 I - A)^{-1} + I)(I - B)^{-1} = I_X$$

and

$$R_z(zI - A) = R_z((z - z_0)I + (z_0 I - A)) = (I - B)^{-1}((z - z_0)(z_0 I - A)^{-1} + I) = I_{\operatorname{dom} A}. \quad \square$$

Theorem 1.50. (Lumer–Phillips) A closed, densely defined operator A generates a strongly continuous contraction semigroup if and only if A is dissipative and $zI - A$ is surjective for some $z > 0$.

Proof. One implication follows from Exercise 1.49, Lemma 1.47 and Theorem 1.39. The other implication follows from Theorem 1.39 and Exercise 1.46. $\square$

Example 1.51. Let $X = L^2[0, 1]$, and let $Af := g$, where

$$\operatorname{dom} A := \left\{ f \in X : \text{there exists } g \in X \text{ such that } f(t) = \int_0^t g(x) \, \mathrm{d}x \text{ for all } t \in [0, 1] \right\}.$$

Thus $f \in \operatorname{dom} A$ if and only if $f(0) = 0$ and f is absolutely continuous on $[0, 1]$, with $Af = f'$ almost everywhere. For such f, note that

$$\operatorname{Re}\langle f, Af \rangle = \operatorname{Re} \int_0^1 \overline{f(t)} f'(t) \, \mathrm{d}t = \frac{1}{2} \int_0^t (\overline{f} f)'(t) \, \mathrm{d}t = \frac{1}{2} |f(1)|^2 \geqslant 0,$$

so $-A$ is a dissipative operator (but A is not).

Let $g \in X$ and $z > 0$; we wish to find $f \in \operatorname{dom} A$ such that

$$(zI + A)f = g \iff zf + f' = g \iff f = \textstyle\int(g - zf).$$

We proceed by iterating this relation: given $h \in \{f, g\}$, let $h_0 := h$ and, for all $n \in \mathbb{Z}_+$, let $h_{n+1} \in X$ be such $h_{n+1}(t) = \int_0^t h_n(s) \, \mathrm{d}s$ for all $t \in [0, 1]$. Then

$$f = g_1 - z \textstyle\int f = g_1 - z \int \int (g - zf) = \cdots = \displaystyle\sum_{j=0}^{n-1} (-z)^j g_{j+1} + (-z)^n f_n$$

for all $n \in \mathbb{N}$. The series $\sum_{j=0}^\infty (-z)^j g_{j+1}$ is uniformly convergent on $[0, 1]$, so defines a function $F \in \operatorname{dom} A$, whereas $(-z)^n f_n \to 0$ as $n \to \infty$. Thus

$$(zI + A)F = -\sum_{j=0}^\infty (-z)^{j+1} g_{j+1} + \sum_{j=0}^\infty (-z)^j g_j = g_0 = g,$$

so $zI + A$ is surjective. By the Lumer–Phillips theorem, $-A$ generates a contraction semigroup.

Exercise 1.52. Fill in the details at the end of Example 1.51. [Hint: with the notation of the example, show that if $h \in \{f, g\}$ then $|h_n(t)|^2 \leqslant t^n \|h\|_2^2/n!$ for all $n \in \mathbb{N}$.]

Remark 1.53. We can explain informally why the operator A defined in Example 1.51 does not generate a semigroup, and why $-A$ does. Recall that each element of a semigroup leaves the domain of the generator invariant, by Lemma 1.31, and A would generate a left-translation semigroup, which does not preserve the boundary condition $f(0) = 0$. Moreover, $-A$ generates the right-translation semigroup, and this does preserve the boundary condition.

If we let A_0 be the restriction of A to the domain

$$\operatorname{dom} A_0 := \{f \in \operatorname{dom} A : f(1) = 0\},$$

so adding a further boundary condition, then both A_0 and $-A_0$ are dissipative, but neither generates a semigroup. We cannot solve the equation $(zI \pm A_0)f = g$ for all g when subject to the constraint that $f \in \operatorname{dom} A_0$. [Take $g \in L^2[0,1]$ such that $g(t) = t$ for all $t \in [0,1]$, construct F as in Example 1.51 and note that $F(1) \neq 0$.]

Example 1.54. Recall the definition of the Sobolev spaces $H^k(\mathbb{R}^d)$ given in Example 1.8. The *Laplacian*

$$\Delta := \sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2} = \sum_{j=1}^{d} D^{2e_j},$$

where $2e_j \in \mathbb{Z}_+^d$ is the multi-index with 2 in the jth coordinate and 0 elsewhere. It may be shown that

$$\langle \Delta f, g \rangle_{L^2(\mathbb{R}^d)} = -\langle \nabla f, \nabla g \rangle_{L^2(\mathbb{R}^d)} \qquad \text{for all } f, g \in H^2(\mathbb{R}^d), \tag{1.4}$$

where

$$\nabla := (D^{e_1}, \ldots, D^{e_d}) : f \mapsto \Big(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_d} \Big).$$

It follows that Δ is dissipative when regarded as an operator in $L^2(\mathbb{R}^d)$ with domain $\operatorname{dom} \Delta := H^2(\mathbb{R}^d)$. Note that $C_c^\infty(\mathbb{R}^d) \subseteq H^2(\mathbb{R}^d)$, so Δ is densely defined.

One way to show that (1.4) holds is to use the Fourier transform. Fourier-theoretic results can also be used to prove that $zI - \Delta$ is surjective for all $z > 0$, essentially because the map $x \mapsto 1/(z + |x|^2)$ is bounded on $\mathbb{R}^n$. Thus the Laplacian generates a contraction semigroup.