The standard model in NCG

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Real structure

Definition

A spectral triple $(\mathcal{A},\mathcal{H},D)$, with $\mathcal{A}$ not necessarily commutative, satisfies the noncommutative order-one condition if there exists an anti-unitary operator J on $\mathcal{H}$ such that $[a,JbJ^{-1}]=0$ and $[[D,a],JbJ^{-1}]=0$ for all $a,b\in\mathcal{A}$.

J defines a real structure of KO-dimension $k\in\mathbb{Z}/8$ if

$$J^2=\epsilon\,,\qquad J\mathcal{D}=\epsilon' DJ\,,\qquad J\gamma=\epsilon''\gamma J \quad \text{for } p \text{ even}$$

where the signs $\epsilon,\epsilon',\epsilon''=\pm1$ are functions of $k \mod 8$:

k	0	1	2	3	4	5	6	7
ε	1	1	-1	-1	-1	-1	1	1
ε'	1	-1	1	1	1	-1	1	1
ε''	1		-1		1		-1	

Matrix-valued spectral triples in KO-dimension 6

We look for matrix solutions of the axioms for $(\mathcal{A}, \mathcal{H}, D, J, \gamma)$ with

$$J^2 = 1\,, \quad JD = DJ\,, \quad J\gamma = -\gamma J\,, \quad [a, JbJ] = 0\,, \quad [[D, a], JbJ] = 0\,,$$
$$\gamma = \gamma^*\,, \quad \gamma^2 = 1\,, \quad D\gamma = -\gamma D\,,$$

The conditions are implemented step by step and restrict an initial matrix algebra $\mathcal{A}$ eventually to $\mathcal{A}_F$.

Requirements

1. $\mathcal{H}$ has a separating vector, i.e. there is a $\xi \in \mathcal{H}$ such that $\mathcal{A} \ni a \mapsto a\xi \in \mathcal{H}$ is injective.
2. $(\mathcal{A}, \mathcal{H}, J)$ is irreducible, i.e. there is no non-trivial $(e \neq 0, 1)$ projector $e \in \mathcal{B}(\mathcal{H})$ which commutes with $\mathcal{A}$ and J.

Lemma

1. *Let $e \neq 1$ be a projector in the centre $Z(\mathcal{A})$. Then $eJeJ^{-1} = 0$.*
2. *Let $e_1, e_2 \in Z(\mathcal{A})$ be projectors with $e_1 e_2 = 0$. Then $e_1 J e_2 J^{-1} + e_2 J e_1 J^{-1} \in \{0, 1\}$*

Proposition

Let $\mathcal{A}_{\mathbb{C}}$ be the complexification of $\mathcal{A}$. Then one of the following cases is realised:

1. $Z(\mathcal{A}_{\mathbb{C}}) = \mathbb{C}1$.
2. $Z(\mathcal{A}_{\mathbb{C}}) = \mathbb{C}e_1 \oplus \mathbb{C}e_2$ and $e_1 = Je_2J^{-1}$ for the minimal projectors e_1, e_2.

Proposition

Let $Z(\mathcal{A}_{\mathbb{C}}) = \mathbb{C}1$ and γ a $\mathbb{Z}_2$-grading with $\gamma\mathcal{A}\gamma^{-1} = \mathcal{A}$. Then $\gamma J = J\gamma$.

In particular, $Z(\mathcal{A}_{\mathbb{C}}) = \mathbb{C}e_1 \oplus \mathbb{C}e_2$ in KO-dimension 6.

First steps in the proof:

- $A_{\mathbb{C}} = M_k(\mathbb{C})$ for some $k \in \mathbb{N}^{\times}$
- Existence of separating vector implies that $\mathcal{H}$ contains subspace isomorphic to $A_{\mathbb{C}} \otimes JA_{\mathbb{C}}J^{-1} = M_{k^2}(\mathbb{C})$
- Irreducibility implies $\mathcal{H} = M_{k^2}(\mathbb{C}) \ni x$ with $Jx = x^*$

$Z(\mathcal{A}_\mathbb{C}) = \mathbb{C}e_1 \oplus \mathbb{C}e_2$

- $\mathcal{A}_\mathbb{C} = M_{k_1}(\mathbb{C}) \oplus M_{k_2}(\mathbb{C})$ for $k_1, k_2 \in \mathbb{N}^\times$
- $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ with $\mathcal{H}_i = e_i\mathcal{H}$
- $e_2 = Je_1J^{-1} \quad \Rightarrow \quad J\mathcal{H}_1 = \mathcal{H}_2$ and $J\mathcal{H}_2 = \mathcal{H}_1$
- separating vector and irreducibility:
  $\mathcal{H}_1 \simeq M(k_1 \times k_2, \mathbb{C}) \ni x$ and $\mathcal{H}_2 \simeq M(k_2 \times k_1, \mathbb{C}) \ni y$
  with $J(x,y) = (y^*, x^*)$
- $\dim(\mathcal{A}_\mathbb{C}) = k_1^2 + k_2^2$ and $\dim(\mathcal{H}) = 2k_1k_2$
  separating vector $\quad \Rightarrow \quad k_1 = k_2 = k$.

Conclusion: $\mathcal{A}_\mathbb{C} = M_k(\mathbb{C}) \oplus M_k(\mathbb{C})$ and $\mathcal{H} = M_{k^2}(\mathbb{C}) \oplus M_{k^2}(\mathbb{C})$

Requirements from particle physics

1. One block of the real algebra $\mathcal{A}$ is quaternionic
   $\mathcal{A} = M_k(\mathbb{H}) \oplus M_{2k}(\mathbb{C})$
2. The quaternionic block has a non-trivial $\mathbb{Z}_2$-grading γ_1

The minimal solution of these requirements is
$\mathcal{A} = M_2(\mathbb{H}) \oplus M_4(\mathbb{C})$.

Up to isomorphisms, we have

$$J\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y^* \\ x^* \end{pmatrix},$$

$$\gamma = (\gamma_1, 0) - J(\gamma_1, 0)J^{-1} \in \mathcal{A}\mathcal{A}^{op}, \quad \gamma\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \gamma_1 x \\ -y\gamma_1 \end{pmatrix}$$

with $\gamma_1 = \mathrm{diag}(1_2, -1_2) \in M_2(\mathbb{H})$.

The even part $\mathcal{A}^{ev}$ of $\mathcal{A}$ is

$$\mathcal{A}^{ev} = (\mathbb{H} \oplus \mathbb{H}) \oplus M_4(\mathbb{C})$$

Order-one condition

We look for a D which

- satisfies the order-one condition
- connects the two blocks of the algebra, $e_1 D e_2 \neq 0$

We show that this is only possible if $\mathcal{A}^{ev}$ is further restricted to $\mathcal{A}_F$.

- Let $\pi_i(a) = e_i a$. The representations π_i are disjoint if there are no common subrepresentations.
- Let $T \in \mathrm{End}(\mathcal{H})$ with $[T, a] = 0$. Then $e_1 T e_2 \pi_2(a) = \pi_1(a) e_1 T e_2$.
- If π_i are disjoint, then $e_1 T e_2 = 0 = e_2 T e_1$.
- Same arguments for $[T, JbJ^{-1}] = 0$. Disjoint π_i imply $e_1 T e_2 = 0 = e_2 T e_1$.

Conclusion: The π_i are not disjoint.

Proposition

Up to isomorphisms of $\mathcal{A}^{ev}$ there is a unique involutive subalgebra $\mathcal{A}_F \subset \mathcal{A}^{ev}$ of maximal dimension which under the order-one condition permits a connecting D. This solution is the matrix algebra of the standard model

$$\mathcal{A}_F = \mathbb{H} \oplus \mathbb{C} \oplus M_3(\mathbb{C}) \subset M_2(\mathbb{H}) \oplus M_4(\mathbb{C})$$

$$\ni \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \bar{\lambda} & 0 & 0 \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & -\bar{\beta} & \bar{\alpha} \end{pmatrix} \oplus \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & m_{11} & m_{12} & m_{13} \\ 0 & m_{21} & m_{22} & m_{23} \\ 0 & m_{31} & m_{32} & m_{33} \end{pmatrix}$$

The Hilbert space is $M_4(\mathbb{C}) \oplus M_4(\mathbb{C}) \in \binom{x}{y}$. We have

$$J\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y^* \\ x^* \end{pmatrix}, \qquad \gamma\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \gamma_1 x \\ -y\gamma_1 \end{pmatrix}$$

The Hilbert space is $M_4(\mathbb{C}) \oplus M_4(\mathbb{C})$. Its elements are parametrised by elementary fermions:

$$\mathcal{H} \ni \left(\begin{array}{c|ccc}
\nu_R & u_{Rr} & u_{Rb} & u_{Rg} \\
e_R & d_{Rr} & d_{Rb} & d_{Rg} \\
\hline
\nu_L & u_{Lr} & u_{Lb} & u_{Lg} \\
e_L & d_{Lr} & d_{Lb} & d_{Lg} \\
\hline\hline
\multicolumn{4}{c}{\begin{array}{cc|cc}
\nu^c_R & e^c_R & \nu^c_L & e^c_L \\
\hline
u^c_{Rr} & d^c_{Rr} & u^c_{Lr} & d^c_{Lr} \\
u^c_{Rb} & d^c_{Rb} & u^c_{Lb} & d^c_{Lb} \\
u^c_{Rg} & d^c_{Rg} & u^c_{Lg} & d^c_{Lg}
\end{array}}
\end{array}\right)$$

We now look for the complete D (not only $e_1 D e_2$) which is even and satisfies order-one.

Lepton/quark decomposition

We want to represent D as a matrix acting on $\mathcal{H}_F$:

$$\mathcal{H}_F = \mathcal{H}_1 \oplus \mathcal{H}_3 \,, \quad \mathcal{H}_1 \simeq \mathbb{C}^8 \ni \begin{pmatrix} \ell_R \\ \ell_L \\ \ell_R^c \\ \ell_L^c \end{pmatrix} \,, \quad \mathcal{H}_3 \simeq \mathbb{C}^{24} \ni \begin{pmatrix} \boldsymbol{q}_R \\ \boldsymbol{q}_L \\ \boldsymbol{q}_R^c \\ \boldsymbol{q}_L^c \end{pmatrix}$$

with leptons $\nu, e \in \mathbb{C}$

$$\ell_R = \begin{pmatrix} \nu_R \\ e_R \end{pmatrix} , \ \ell_L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} , \ \ell_R^c = \begin{pmatrix} \nu_R^c \\ e_R^c \end{pmatrix} , \ \ell_L^c = \begin{pmatrix} \nu_L^c \\ e_L^c \end{pmatrix}$$

and quarks $\boldsymbol{u}, \boldsymbol{d} \in \mathbb{C}^3$

$$\boldsymbol{q}_R = \begin{pmatrix} \boldsymbol{u}_R \\ \boldsymbol{d}_R \end{pmatrix} , \ \boldsymbol{q}_L = \begin{pmatrix} \boldsymbol{u}_L \\ \boldsymbol{d}_L \end{pmatrix} , \ \boldsymbol{q}_R^c = \begin{pmatrix} \boldsymbol{u}_R^c \\ \boldsymbol{d}_R^c \end{pmatrix} , \ \boldsymbol{q}_L^c = \begin{pmatrix} \boldsymbol{u}_L^c \\ \boldsymbol{d}_L^c \end{pmatrix} .$$

Write operators on $\mathcal{H}_F$ as $T_F = \begin{pmatrix} T_{11} & T_{13} \\ T_{31} & T_{33} \end{pmatrix}$ with $T_{11} \in M_8(\mathbb{C})$ and $T_{33} \in M_{24}(\mathbb{C})$.

- $\mathcal{A}_F, J_F, \gamma_F$ diagonal (no 13,31-blocks)
- $D_{13} = 0$ and $D_{31} = 0$ from order-one and $D_F\gamma_F = -\gamma_F D_F$

diagonal parts of $\mathcal{A}_F$:

$$(q,\lambda,m)_{11} = \begin{pmatrix} q_\lambda & 0 & 0 & 0 \\ 0 & q & 0 & 0 \\ 0 & 0 & \lambda 1_2 & 0 \\ 0 & 0 & 0 & \lambda 1_2 \end{pmatrix}, \qquad q_\lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{pmatrix}$$

$$(q,\lambda,m)_{33} = \begin{pmatrix} q_\lambda \otimes 1_3 & 0 & 0 & 0 \\ 0 & q \otimes 1_3 & 0 & 0 \\ 0 & 0 & 1_2 \otimes m & 0 \\ 0 & 0 & 0 & 1_2 \otimes m \end{pmatrix},$$

with $q \in \mathbb{H}$, $\lambda \in \mathbb{C}$, $m \in M_3(\mathbb{C})$

diagonal part of J_F:

$$J_{11} = \begin{pmatrix} 0 & 0 & 1_2 & 0 \\ 0 & 0 & 0 & 1_2 \\ 1_2 & 0 & 0 & 0 \\ 0 & 1_2 & 0 & 0 \end{pmatrix} \circ CC$$

$$J_{33} = \begin{pmatrix} 0 & 0 & 1_2 \otimes 1_3 & 0 \\ 0 & 0 & 0 & 1_2 \otimes 1_3 \\ 1_2 \otimes 1_3 & 0 & 0 & 0 \\ 0 & 1_2 \otimes 1_3 & 0 & 0 \end{pmatrix} \circ CC$$

where CC means complex conjugation

diagonal part of γ_F:

$$\gamma_{11} = \begin{pmatrix} 1_2 & 0 & 0 & 0 \\ 0 & -1_2 & 0 & 0 \\ 0 & 0 & -1_2 & 0 \\ 0 & 0 & 0 & 1_2 \end{pmatrix},$$

$$\gamma_{33} = \begin{pmatrix} 1_2 \otimes 1_3 & 0 & 0 & 0 \\ 0 & -1_2 \otimes 1_3 & 0 & 0 \\ 0 & 0 & -1_2 \otimes 1_3 & 0 \\ 0 & 0 & 0 & 1_2 \otimes 1_3 \end{pmatrix}.$$

$\gamma_F^2 = 1$, $\gamma_F = \gamma_F^*$, $J_F^2 = 1$ and $\gamma_F J_F = -J_F \gamma_F$ satisfied

Diagonal part of D_F; from $\{D_F, \gamma_F\} = 0$, $[D_F, J_F] = 0$ and connecting part:

$$D_{11} = \left(\begin{array}{cc|cc} 0 & Y_1 & T & 0 \\ Y_1^* & 0 & 0 & 0 \\ \hline T^* & 0 & 0 & \overline{Y_1} \\ 0 & 0 & Y_1^t & 0 \end{array}\right), \quad D_{33} = \left(\begin{array}{cc|cc} 0 & \boldsymbol{Y}_3 & 0 & 0 \\ \boldsymbol{Y}_3^* & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \overline{\boldsymbol{Y}_3} \\ 0 & 0 & \boldsymbol{Y}_3^t & 0 \end{array}\right),$$

with $Y_1 \in M_2(\mathbb{C})$, $\boldsymbol{Y}_3 \in M_2(\mathbb{C}) \otimes M_3(\mathbb{C})$ and

$T = T^t = \begin{pmatrix} Y_R & 0 \\ 0 & 0 \end{pmatrix}$ with $Y_R \in \mathbb{C}$

- order-one: $\boldsymbol{Y}_3 = Y_3 \otimes 1_3$ with $Y_3 \in M_2(\mathbb{C})$
- Physical condition: D_F commutes with representation of $\mathbb{C}$ in $\mathcal{A}_F$, i.e. with $(q_\lambda, \lambda, 0)$
- leads to $Y_1 = \begin{pmatrix} Y_\nu & 0 \\ 0 & Y_e \end{pmatrix}, \qquad Y_3 = \begin{pmatrix} Y_u & 0 \\ 0 & Y_d \end{pmatrix}$

Cabibbo-Kobayashi-Maskawa matrices

- Standard model needs 3 copies of $\mathcal{H}_F$.
- leptons $\nu, e \in \mathbb{C}^3$, quarks $\boldsymbol{u}, \boldsymbol{d} \in \mathbb{C}^3 \otimes \mathbb{C}^3$
- $Y_{\nu,e,u,d,R} \in M_3(\mathbb{C})$ in D

We say that D_F, D'_F are equivalent if $D'_F = UD_FU^*$ for a unitary matrix U which commutes with $\mathcal{A}_F, J_F, \gamma_F$.

$$U_{11} = \begin{pmatrix} \mathrm{diag}(V_1, V_2) & 0 & 0 & 0 \\ 0 & \mathrm{diag}(V_3, V_3) & 0 & 0 \\ 0 & 0 & \mathrm{diag}(\overline{V_1}, \overline{V_2}) & 0 \\ 0 & 0 & 0 & \mathrm{diag}(\overline{V_3}, \overline{V_3}) \end{pmatrix}$$

$$U_{33} = \begin{pmatrix} \mathrm{diag}(W_1, W_2) \otimes 1_3 & 0 & 0 & 0 \\ 0 & \mathrm{diag}(W_3, W_3) \otimes 1_3 & 0 & 0 \\ 0 & 0 & \mathrm{diag}(\overline{W_1}, \overline{W_2}) \otimes 1_3 & 0 \\ 0 & 0 & 0 & \mathrm{diag}(\overline{W_3}, \overline{W_3}) \otimes 1_3 \end{pmatrix}$$

with $V_i, W_i \in U(3)$

Product of spectral triples

$$\mathcal{A} = \mathcal{A}_M \otimes \mathcal{A}_F\ , \qquad \mathcal{H} = \mathcal{H}_M \otimes \mathcal{H}_F\ , \qquad D = D_M \otimes 1_F + \gamma_M \otimes D_F\ ,$$

$$J = J_M \otimes J_F\ , \qquad \gamma = \gamma_M \otimes \gamma_F$$

with

- $\mathcal{A}_M = C^\infty(M)$, $\mathcal{H}_M = L^2(M, \mathcal{S})$, $\gamma_M = \gamma^5$, $J_M = \gamma^2 \circ CC$ and

$$D_M = \mathrm{i} e_a^\mu \gamma^a \nabla_\mu^S\ , \quad \nabla_\mu^S = \partial_\mu + \frac{1}{8}\omega_\mu^{ab}[\gamma_a, \gamma_b]$$

- e_μ^a vierbein, ω_μ^{ab} spin connection form:

$$\delta_{ab} e_\mu^a e_\nu^b = g_{\mu\nu}\ , \qquad \partial_\mu e_a^\nu - \Gamma_{\mu\rho}^\nu e_a^\rho + \omega_\mu^{ab} e_b^\nu = 0$$

- Lichnerowicz formula

$$D_M^2 = \Delta^{LC} + \frac{1}{4}R, \quad \Delta^{LC} = -g^{\mu\nu}(\partial_\mu\partial_\nu - \Gamma_{\mu\nu}^\rho \partial_\rho), \quad R = g^{\mu\nu}R_{\mu\nu}$$

Fluctuations

- replace D by fluctuated operator $D_A = D + A + JAJ^{-1}$, where $A = A^* = \sum_\alpha a_\alpha [D, b_\alpha]$
- D_A satisfies the same axioms as D
- $D_M \otimes 1$ generates 1-forms $G = e^\mu_a \gamma^a G_\mu$ with $G_\mu = \sum_\alpha a_\alpha \partial_\mu (b_\alpha) \in \mathcal{A}$
- $\gamma_M \otimes D_F$ generates 1-forms $\gamma^5 \Phi = \gamma^5 \sum_\alpha a_\alpha [D_F, b_\alpha]$

$$\Phi_{11} = \left(\begin{array}{cc|cc} 0 & \Phi_\ell & 0 & 0 \\ \Phi_\ell^* & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right), \qquad \Phi_\ell = \begin{pmatrix} M_\nu C_\ell \phi_1 & M_\nu C_\ell \phi_2 \\ -M_e \overline{\phi_2} & M_e \overline{\phi_1} \end{pmatrix}$$

$$\Phi_{33} = \left(\begin{array}{cc|cc} 0 & \Phi_q \otimes 1_3 & 0 & 0 \\ \Phi_q \otimes 1_3 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right), \quad \Phi_q = \begin{pmatrix} M_u C_q \phi_1 & M_u C_q \phi_2 \\ -M_d \overline{\phi_2} & M_d \overline{\phi_1} \end{pmatrix}$$

lepton part of D_A is:

$$D_{A,11} = \left(\begin{array}{cc|cc} \mathrm{i}\gamma^a e_a^\mu \nabla_\mu^{S,B} & \gamma_5 \Phi_\ell & T & 0 \\ \gamma_5 \Phi_\ell^* & \mathrm{i}\gamma^a e_a^\mu \nabla_\mu^{S,B,W} & 0 & 0 \\ \hline T^* & 0 & \mathrm{i}\gamma^a e_a^\mu \overline{\nabla_\mu^{S,B}} & \gamma_5 \overline{\Phi_\ell} \\ 0 & 0 & \gamma_5 \Phi_\ell^t & \mathrm{i}\gamma^a e_a^\mu \overline{\nabla_\mu^{S,B,W}} \end{array}\right)$$

with

$$\nabla_\mu^{S,B} = \begin{pmatrix} \nabla_\mu^S & 0 \\ 0 & \nabla_\mu^S - 2\mathrm{i}B_\mu \end{pmatrix} 1_Y \,,$$

$$\nabla_\mu^{S,B,W} = \begin{pmatrix} \nabla_\mu^S + \mathrm{i}(-B_\mu - W_\mu^3) & -\mathrm{i}(W_\mu^1 - \mathrm{i}W_\mu^2) \\ -\mathrm{i}(W_\mu^1 + \mathrm{i}W_\mu^2) & \nabla_\mu^S + \mathrm{i}(-B_\mu + W_\mu^3) \end{pmatrix} 1_3$$

$B_\mu, W_\mu^a \in C^\infty(X)$ real-valued quark part of D_A is

$$D_{A,33} = \left(\begin{array}{cc|cc} \mathrm{i}\gamma^a e_a^\mu \nabla_\mu^{S,B,G} & \gamma_5 \Phi_q & 0 & 0 \\ \gamma_5 \Phi_q^* & \mathrm{i}\gamma^a e_a^\mu \nabla_\mu^{S,W,G} & 0 & 0 \\ \hline 0 & 0 & \mathrm{i}\gamma^a e_a^\mu \overline{\nabla_\mu^{S,B,G}} & \gamma_5 \overline{\Phi_q} \\ 0 & 0 & \gamma_5 \Phi_q^t & \mathrm{i}\gamma^a e_a^\mu \overline{\nabla_\mu^{S,W,G}} \end{array}\right)$$

with

$$\nabla_\mu^{S,B,G} = \begin{pmatrix} \nabla_\mu^S 1_3 + \mathrm{i}((-G_\mu^0 + B_\mu)1_3 - G_\mu) & 0 \\ 0 & \nabla_\mu^S 1_3 + \mathrm{i}(-(G_\mu^0 + B_\mu)1_3 - G_\mu) \end{pmatrix} 1_3$$

$$\nabla_\mu^{S,W,G} = \begin{pmatrix} \nabla_\mu^S 1_3 + \mathrm{i}((-G_\mu^0 - W_\mu^3)1_3 + G_\mu) & -\mathrm{i}(W_\mu^1 - \mathrm{i}W_\mu^2)1_3 \\ -\mathrm{i}(W_\mu^1 + \mathrm{i}W_\mu^2)1_3 & \nabla_\mu^S 1_3 + \mathrm{i}((-G_\mu^0 + W_\mu^3) + G_\mu) \end{pmatrix} 1_3$$

$G_\mu^0 \in C^\infty(X)$ real $G_\mu \in M_3(C^\infty(X))$ hermitian and traceless

Unimodularity condition $\mathrm{tr}(A) = 0 \quad \Rightarrow \quad G_\mu^0 = -\frac{1}{3} B_\mu$

The action functional

Fermionic action functional

$$S_F = \langle J\psi, D_A\psi\rangle$$

where $\psi = \gamma\psi \in \mathcal{H}^+$ are Grassmann-valued

Spectral action principle [Chanseddine-Connes]

The bosonic action is a functional only of the spectrum of D_A^2.

functional calculus and Laplace transformation

$$S_A = \mathrm{Tr}(\chi(D_A^2)) = \int_0^\infty dt\ \hat{\chi}(t)\,\mathrm{Tr}(e^{-tD_A^2})\ ,$$

with $\chi(s) = \int_0^\infty dt\ e^{-st}\hat{\chi}(t)$

Proposition (heat kernel expansion)

Let F be a vector bundle over (M, g). A second-order differential operator $P = -(g^{\mu\nu}\partial_\mu\partial_\nu + A^\rho\partial_\rho + B)$ (locally), where $A^\mu, B \in End(\mathcal{F})$, has an asymptotic expansion

$$\mathrm{Tr}(e^{-tP}) \sim \sum_{k=0}^{\infty} t^{\frac{k-p}{2}} \int_M dx\ a_k(x, P) ,$$

where $a_k(x, P)$ are the Seeley-de Witt coefficients.

inversion of Laplace transformation

$$S_A \sim \sum_{k=0}^{\infty} \chi_{\frac{k-p}{2}} \int_M dx\ a_k(x, D_A^2) ,$$

$$\chi_z = \int_0^\infty dt\ t^z \hat{\chi}(t) = \begin{cases} \frac{1}{\Gamma(-z)} \int_0^\infty ds\ s^{-z-1}\chi(s) & \text{für } z \notin \mathbb{N} \\ (-1)^k \chi^{(k)}(0) & \text{für } z = k \in \mathbb{N} \end{cases}$$

The Seeley-de Witt coefficients are given in the book of Gilkey, but expressed in terms of $P = \Delta^F - \mathcal{E}$, where Δ^F is the connection Laplacian for $\nabla f = dx^\mu \otimes (\partial_\mu f + \omega_\mu f)$. One finds

$$P = \Delta^F - \mathcal{E} \quad \Leftrightarrow \quad \begin{aligned} \omega_\mu &= \frac{1}{2} g_{\mu\nu}(A^\nu + g^{\rho\sigma}\Gamma^\nu_{\rho\sigma}) \\ \mathcal{E} &= B - g^{\mu\nu}(\omega_\mu\omega_\nu + \partial_\mu\omega_\nu - \Gamma^\rho_{\mu\nu}\omega_\rho) \end{aligned}$$

The first coefficients are

$$a_0(x,P) = (4\pi)^{-\frac{p}{2}}\mathrm{tr}(\mathrm{id}) ,$$

$$a_2(x,P) = \frac{1}{6}(4\pi)^{-\frac{p}{2}}\mathrm{tr}(-R\mathrm{id} + 6\mathcal{E}) ,$$

$$a_4(x,P) = \frac{(4\pi)^{-\frac{p}{2}}}{360}\mathrm{tr}\Big((5R^2 - 2R_{\mu\nu}R^{\mu\nu} + 2R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 12\Delta^{LC}(R))\mathrm{id} \\ + 60\Delta^F(\mathcal{E}) - 60R\mathcal{E} + 180\mathcal{E}^2 + 30\Omega_{\mu\nu}\Omega^{\mu\nu}\Big)$$

where $\Omega_{\mu\nu} = \partial_\mu\omega_\nu - \partial_\nu\omega_\mu + \omega_\mu\omega_\nu - \omega_\nu\omega_\mu$

Spectral action

$$
\begin{aligned}
S_A = {} & \frac{1}{\pi^2}(48\chi_{-2} - c\chi_{-1} + d\chi_0) \int d^4x \sqrt{\det g} \\
& + \frac{1}{24\pi^2}(96\chi_{-1} - c\chi_0) \int d^4x \sqrt{\det g}\, R \\
& + \frac{\chi_0}{10\pi^2} \int d^4x \sqrt{\det g} \Big(\frac{11}{6} R^* R^* - 3 C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma}\Big) \\
& + \frac{1}{\pi^2}(-2a\chi_{-1} + e\chi_0) \int d^4x \sqrt{\det g}\, |\phi|^2 \\
& + \frac{\chi_0}{2\pi^2} \int d^4x \sqrt{\det g}\, a\Big(|D_\mu\phi|^2 - \frac{1}{6} R|\phi|^2\Big) \\
& + \frac{2\chi_0}{\pi^2} \int d^4x \sqrt{\det g} \Big(\frac{1}{2}\mathrm{tr}_3(G_{\mu\nu}G^{\mu\nu}) + \frac{1}{2}\mathrm{tr}_2(W_{\mu\nu}W^{\mu\nu}) + \frac{5}{3} B_{\mu\nu}B^{\mu\nu}\Big) \\
& + \frac{\chi_0}{2\pi^2} \int d^4x \sqrt{\det g}\, |\phi|^4
\end{aligned}
$$

with

$$
\begin{aligned}
G_{\mu\nu} &= \partial_\mu G_\nu - \partial_\nu G_\mu - \mathrm{i}(G_\mu G_\nu - G_\nu G_\mu) \in M_3(C^\infty(M)) \;, \\
W_{\mu\nu} &= \partial_\mu W_\nu - \partial_\nu W_\mu - \mathrm{i}(W_\mu W_\nu - W_\nu W_\mu) \in M_2(C^\infty(M)) \;, \\
B_{\mu\nu} &= \partial_\mu B_\nu - \partial_\nu B_\mu \in C^\infty(M) \\
|\phi|^2 &:= |\phi_1|^2 + |\phi_2|^2 \;,
\end{aligned}
$$

$$
\begin{pmatrix} D_\mu \phi_1 \\ D_\mu \phi_2 \end{pmatrix} = \begin{pmatrix} \partial_\mu \phi_1 \\ \partial_\mu \phi_2 \end{pmatrix} + \mathrm{i} \begin{pmatrix} W_\mu^3 - B_\mu & W_\mu^1 - \mathrm{i}W_\mu^2 \\ W_\mu^1 - \mathrm{i}W_\mu^2 & -W_\mu^3 - B_\mu \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}
$$

and

$$
\begin{aligned}
a &= \mathrm{tr}(Y_\nu^* Y_\nu + Y_e^* Y_e + 3Y_u^* Y_u + Y_d^* Y_d) \;, \\
b &= \mathrm{tr}((Y_\nu^* Y_\nu)^2 + (Y_e^* Y_e)^2 + 3(Y_u^* Y_u)^2 + (Y_d^* Y_d)^2) \;, \\
c &= \mathrm{tr}(Y_R^* Y_R) \;, \qquad d = 4\mathrm{tr}((Y_R^* Y_R)^2) \;, \qquad e = \mathrm{tr}((Y_R^* Y_R)(Y_\nu^* Y_\nu)) \;.
\end{aligned}
$$