BERICHT ÜBER GEGENSTÄNDE, ZIELE UND ERGEBNISSE DER BISHERIGEN FORSCHUNGSTÄTIGKEIT - RESEARCH STATEMENT

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1. Description of research area

My primary field of research is algebraic geometry with connections to commutative and homological algebra, representation theory and invariant theory.
Algebraic geometry like many other mathematical disciplines has gone through at least three different phases over the last 100-150 years: a naive period connected with the names of Clebsch, Noether, Enriques, Severi, Castelnuovo starting from the late 19th century until about 1930 where the fine-details of the proofs were often short-cut because of the lack of proper foundations and simultaneous rapid growth of the subject, a critical and foundational period around 1930-1960 under van der Waerden, Weil, Zariski, Grothendieck striving for the right conceptual framework and language to express the ideas of the subject and building a formidable abstract machinery, and finally the phase from 1960 on to the present where people tend to revert to the initial geometric problems to test the techniques and achievements of the second phase against them (often extending them in the process). Also much of the charm of algebraic geometry derives from the fact that on the one hand it deals with very down-to-earth objects which are useful in the construction of a windshield wiper, robotic design or phylogeny, whilst on the other hand it offers a wealth of elaborate abstract tools which can be used for the study of and tested against concrete examples. To quote from Mumford's Preface to his 1974 Lectures on Curves and Their Jacobians:

> *And certainly Grothendieck's work contributed to the field some very abstract and very powerful ideas which are quite hard to digest. But this subject, like all subjects, has a dual aspect in that all these abstract ideas would collapse of their own weight were it not for the underpinning supplied by concrete classical geometry.*

A unifying theme of my research up to now has been the *explicit* or *classical geometry of varieties*, in particular *the explicit birational geometry of moduli spaces*, where the term "explicit" means that one wants to go beyond mere existence results and study varieties in terms of defining equations or concrete parametrizations that bring out their geometric or arithmetic features as clearly as possible. Thus the aim of [B05] and [B08-1] was to find an explicit characterization of Gorenstein A-algebras R in codimension 2 in terms of the symmetry properties of the matrices occurring in a minimal free resolution of R over A, and in particular to investigate how the ring structure of the A-module R is encoded in these matrices. The result was used in [B07] to analyze the birational properties of the moduli space of canonical surfaces with $p_g = 5$, $q = 0$, $K^2 = 11$ in $\mathbb{P}^4$ (with some genericity assumption on the singularities of the canonical image) by exhibiting an explicit parametrization of these surfaces in terms of Gorenstein symmetric matrices which occur in the minimal free resolution of their canonical rings as modules over the homogeneous coordinate ring of $\mathbb{P}^4$. In a different direction, [B06] dealt with the explicit study of the derived categories of coherent sheaves on rational homogeneous varieties G/P, in particular finding complete exceptional sequences or small generating sets of homogeneous vector bundles in these categories. This relates to the theory of moduli in the sense that exceptional sheaves are certain rigid sheaves with 0-dimensional moduli. Finally the series of papers [B08-2],[BvB08-1], [BvB08-2], [BvB09-1], [B09] relates to the so-called *rationality problem in invariant theory* which has been one of my main research interests recently. Therefore I will discuss this in a little more detail here.

The main problem can be phrased as follows: let G be a connected linear algebraic group (over $\mathbb{C}$), V a finite-dimensional linear representation of G. Is the quotient V/G (or, what is stronger, $\mathbb{P}(V)/G$) rational? This is an old problem going back to Emmy Noether. No examples of non-rational such quotients are known if G is assumed to be connected. But if G is not connected, e.g. finite, then Saltman [Sa] has given examples of nonrational V/G. If we work over non-closed fields, there are examples by Merkurjev [Mer] of spaces V/G with G connected and semisimple which are non-rational (in fact, both in Saltman's and Merkurjev's examples, the quotients are not even stably rational). One reason why the rationality problem is important is that many moduli spaces in algebraic geometry are of the type $\mathbb{P}(V)/G$. For example, the quotient of the action of $\mathrm{PGL}_n(\mathbb{C})$ on pairs of $n\times n$-matrices by simultaneous conjugation is birational to the moduli space of stable rank n vector bundles on $\mathbb{P}^2$ with Chern numbers $c_1 = 0$, $c_2 = n$, but it is also birational to the relative Jacobian of degree $g-1$ line bundles over the parameter space of smooth plane curves of degree n and genus g. In addition, the function field of this quotient coincides with the field of fractions of the centre of the generic division ring on two $n\times n$ generic matrices. The determination whether this quotient is rational or stably rational remains a tantalizing open problem in general.
Many notions have been introduced which capture certain properties of rational varieties (unirationality, stable rationality, retract rationality, coprime rationality..., the technically most versatile being the notion of rational connectedness of Kollár [Koll]). It has turned out that rational varieties and varieties close to the rational ones are also the type of varieties that occur most frequently in applications.
Of particular interest is the case where $V_d = \mathrm{Sym}^d(\mathbb{C}^{n+1})^\vee$, $G = \mathrm{SL}_{n+1}(\mathbb{C})$, i.e. $H_d^n = \mathbb{P}(V_d)/G$ is the moduli space of hypersurfaces of degree d in $\mathbb{P}^n$. The main results of other authors on the rationality of H_d^n can be summarized as follows: H_d^1 is rational for all d, [Bo-Ka], [Kat83], [Kat84]; H_d^2 is rational for $d \equiv 1 \pmod 4$ ([Shep]), $d \equiv 1 \pmod 9$, $d \geq 19$ ([Shep]), and $d \equiv 0 \pmod 3$, $d \geq 210$ ([Kat89]). H_d^2 is known to be rational for some smaller values of d, too, but these -though very diffcult to handle sometimes- are somewhat sporadic cases and we do not see a general pattern emerging; see [Kat92/2], [Kat96], [B08-2] for the case $d = 4$.

2. Summary of results

The article [B05] deals with the question how the ring structure of a Gorenstein A-algebra R is encoded in its minimal free resolution; here A is a Cohen-Macaulay local ring with 2 invertible in A, and R is a finite perfect A-algebra, $\dim A - \dim_A R = 2$, with $R \simeq \mathrm{Ext}^2(R, A)$ as R-modules. It is shown that an A-module R with Gorenstein symmetric length 2 minimal free resolution is automatically an A-algebra once a certain depth condition on an ideal of minors of a presentation matrix of R over A is satisfied. Gorenstein symmetric means of the form

$$0 \longrightarrow A^n \xrightarrow{\begin{pmatrix} -\beta^t \\ \alpha^t \end{pmatrix}} A^{2n} \xrightarrow{(\alpha\, \beta)} A^n \longrightarrow R \longrightarrow 0\,.$$

This is in accordance with a conjecture put forward by L. Szpiro. The question was considered previously by Eisenbud and Ulrich, but their result could not be used for applications to canonical surfaces since they imposed too restrictive conditions. This result was strengthened in [B08-1] where it was shown that in the preceding setting one can obtain a resolution of R which is simultaneously Gorenstein symmetric and of Koszul module type (i.e. $\det(\alpha)$, $\det(\beta)$ is a regular sequence in A), which answered a question of M. Grassi. The theory was applied in [B07] to show that the moduli space of canonical surfaces in $\mathbb{P}^4$ with $p_g = 5$, $q = 0$, $K^2 = 11$ satisfying a genericity assumption on the singularities of the canonical image is irreducible, unirational of dimension 38. The main point here was the fact that Gorenstein symmetric 3×6-matrices $(\alpha\, \beta)$ (so $\alpha\beta^t = \beta\alpha^t$ and α and β are 3×3) with entries in the first row cubic forms on $\mathbb{P}^4$ and linear entries otherwise, can be reduced to a certain normal form preserving the symmetry.

The paper [B06] studied derived categories of coherent sheaves on flag manifolds G/P, especially the question of finding small generating sets or if possible complete exceptional sequences in them. Recall that an object $E \in D^b(\mathrm{Coh}(G/P))$ is called exceptional if $\mathrm{RHom}^\bullet(E, E) \simeq \mathbb{C}$ (in degree 0) and an n-tuple $(E_1, \dots, E_n)$ is called an exceptional sequence if all E_i are exceptional and $\mathrm{RHom}^\bullet(E_i, E_j) = 0$ whenever $i > j$. As usual, the set $\{E_1, \dots, E_n\}$ is said to generate $D^b(\mathrm{Coh}(G/P))$ if the smallest full triangulated subcategory of $D^b(\mathrm{Coh}(G/P))$ containing all E_i is equivalent to $D^b(\mathrm{Coh}(G/P))$. In [B06] small generating sets were found for symplectic isotropic Grassmannians, and a structure theorem for the derived categories of quadric bundles was proved. The latter was applied to derived categories of coherent sheaves on orthogonal isotropic Grassmannians. We also outlined a new approach to proving Beilinson-type theorems for flag manifolds G/P, based on a degeneration of the diagonal to a union of products of Schubert varieties and their duals due to M. Brion, and cellular resolutions of monomial ideals. We also confirmed a conjecture of F. Catanese on the structure of $D^b(\mathrm{Coh}(G/P))$ for type A-Grassmannians and quadrics. The conjecture is taken up in recent work of Kaneda and Ye.

Let us now summarize our work on the rationality problem: [B08-2] recasts Katsylo's important work [Kat92/2], [Kat96] in geometric terms, replacing some of his arguments by more geometric ones, e.g. by linking them to classical work of Clebsch and Salmon.
The article [BvB08-1] proves the rationality of all moduli spaces of plane curves of sufficiently large degree d which presents an important advance over the previous

work of Shepherd-Barron [Shep] and Katsylo [Kat89]. The hard part was to check that some data satisfied a genericity requirement, and this was done by showing that the data becomes periodic over a finite field $\mathbb{F}_p$ and using upper-semicontinuity over $\mathrm{Spec}(\mathbb{Z})$.

[BvB08-2] contains a method to calculate matrix representatives for equivariant bilinear maps of $\mathrm{SL}_3(\mathbb{C})$-representations in an algorithmically efficient way and a criterion for the stable rationality of quotients of some Grassmannians by an SL-action. This is subsequently used to prove in particular the rationality of the moduli space of plane curves of degree 34.

[BvB09-1] develops algorithmic tools that allow us to prove in this paper the rationality of all moduli spaces of plane curves of degree d, with the possible exception of 15 values of d for which rationality remains unknown.

[B09] is a survey paper on the rationality problem, but it also contains some new results on the moduli spaces of plane curves together with a theta-characteristic and a detailed exposition of the relation of the Hesselink stratification of the cone of Hilbert nullforms to the rationality problem. It also contains an account of the proof of the rationality of the moduli space of plane curves of degree 34 in a more conceptual setting.

3. Research projects and future directions

Here is a list of some projects that I am currently working on or which are interesting topics I would like to turn to in the future. I have tried to keep them relatively concrete in the form of a problem list.

The rationality problem

- Let $G = \mathrm{SL}_{n+1}(\mathbb{C})$ acting on $V_d = \mathrm{Sym}^d(\mathbb{C}^{n+1})^\vee$, so that $H_d^n = \mathbb{P}(V_d)/G$ is the moduli space of hypersurfaces of degree d in $\mathbb{P}^n$. The method of [BvB08-1] showing that H_d^2 is rational for d sufficiently large can be applied more generally. A realistic goal could be:

  > Prove that for each $n \in \mathbb{N}$ there exists an integer $N = N(n)$ such that for all d coprime to $n+1$ and $d \geq N$ the space H_d^n is rational.

  For small *fixed* n ($n = 3, 4$ say) there is no substantial difficulty to generalize the method of [BvB08-1]. The assumption that $n+1$ should be coprime to d can certainly be removed for those $n+1$ for which an almost free $\mathrm{PGL}_{n+1}(\mathbb{C})$-representation with stably rational quotient is known: i.e. for $n+1$ a divisor of 420. Otherwise the problem is much more difficult if $n+1$ and d are not coprime.
- If $G = \mathrm{PGL}_n(\mathbb{C})$ acts on pairs of matrices in $V_n = \mathfrak{gl}_n \oplus \mathfrak{gl}_n$ by simultaneous conjugation, it is a very important and hard problem to decide if $Q_n = V_n/G$ is stably rational or rational. The function field of Q_n is the function field of the relative Jacobian $\mathcal{J}ac_n^{g-1}$ of degree $g-1$ line bundles over the parameter space of (smooth) plane curves of degree n and genus g. In [vdBer] rationality of Q_3 is proven by a geometric method based on the study of linear series on plane cubics. It would be interesting to extend this method of proof to Q_4 and possibly other Q_i.
- Calculate unramified cohomology $H^i_{\mathrm{nr}}(\mathbb{C}(V)^G,\ \mu_n)$ of invariant function fields $\mathbb{C}(V)^G$for $i \geq 3$ in terms of the group cohomology of G in the spirit of [Bogo87] (the case $i = 2$). This would be particularly interesting for

$G = \mathrm{PGL}_8(\mathbb{C})$ where no almost free linear action with stably rational quotient is known. This relates to the general question of the previous item.
- Decide the rationality of the configuration space $(\mathbb{P}^2)^{(6)}/\mathrm{SL}_3(\mathbb{C})$ of 6 unordered points in $\mathbb{P}^2$ which by classical work of Coble has a concrete model as a hypersurface in a weighted projective space of dimension 5. Stable rationality is known in this case.
- As an application of invariant theoretic methods, I would like to investigate the relation of the Hesselink stratification of the Hilbert nullcone to the Poincaré center problem, jointly with Hans-Christian Graf von Bothmer.

Torsors over Del Pezzo surfaces

- Study universal torsors over Del Pezzo surfaces over a Dedekind scheme (or universal torsors over a family of Del Pezzo surfaces), and generalize the work of [Se-Sk] to the relative case.

Vector bundles on flag manifolds

- The general problem whether there exist complete exceptional sequences in the derived categories of coherent sheaves on flag manifolds G/P remains open. In particular, the relation of the non-vanishing of Hom-spaces for certain strong complete exceptional sequences to the Bruhat-Chevalley order on G/P should be further investigated. Moreover, there are recent ideas of Masaharu Kaneda and Jiachen Ye how one may construct exceptional sequences using Frobenius splitting techniques in positive characteristic, and I would be interested to develop this picture further.

Canonical surfaces

- Can the technique of [B07] be generalized to analyze the moduli spaces of canonical surfaces in $\mathbb{P}^4$ with $q = 0$, $p_g = 5$, $K^2 = 12$ and particularly $K^2 = 13$ with only improper double points as singularities of the canonical image? For $K^2 = 13$ is the moduli space irreducible/unirational?

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