Dynamics and Potentials

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(Based on a joint work [2] with Hajime MORIYA)

reference
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Traditional Approach: Potential $\Longrightarrow$ Dynamics

Our Basic Result: Dynamics $\Longleftrightarrow$Potential
(Existence and uniqueness)

The uniqueness from the following **standardness** requirement for $\{P(I)\}$

(i) Convergence condition (new)

(ii) Properties under a conditional expectation (new and useful).

Some Applications (Theorem 3 and 2):

(1) Energy estimates (consequence of the property(ii)).

(2) Useful for equivalence proof of equilibrium conditions

2 Dynamics and Potential

Dynamics:$t \in \mathbb{R} \rightarrow \alpha_t \in \mathrm{Aut}\mathcal{A}$ (group)

Assumption: If $I \Subset L$ and $A \in \mathcal{A}(I)$, then

$$(\dot{A} =)\delta_\alpha(A) = \frac{d}{dt}\alpha_t(A)\Big|_{t=0} \quad \text{exists.}$$

δ_α: $*$-derivation on $\mathcal{A}_0 \equiv \bigcup\{\mathcal{A}(I); I \Subset L\}$

Potential: $I \Subset L \rightarrow P(I) = P(I)^* \in \mathcal{A}(I)_+$
(Interaction energy among local objects in I)

Normal approach:

- construction of δ_α out of {P(I)}
  $\delta_\alpha(A) = i\sum_I [P(I), A]$
  $(I \cap \mathrm{supp}A = \emptyset \Rightarrow [P(I), A] = 0)$
  (Formally, $\dot{A} = i[H, A]$ with Hamiltonian
  $H = \sum_I P(I)$)

- construction of α_t out of δ_α

New result:

Theorem 1: Existence of $P(I)$ for any $*$-derivation δ on $\mathcal{A}_0$, satisfying

(1) Natural convergence condition(new):
$H_I = \sum_J \{P(J); J \cap I \neq \emptyset\}$ convergent.
$\Longrightarrow (\dot{A}) = i[H_I, A]$ for $A \in \mathcal{A}(I)$

(2) Standardness (new):

$$E_J^{\varphi}(P_\varphi(I)) = \begin{cases} 0 & \text{if}\, J \not\supset I \\ P_\varphi(I) & \text{if}\, J \supset I. \end{cases}$$

3 Supplementary Explanation

(1)**Equivalent Potential**

Def: Two potentials $\{P_1(I)\}$ and $\{P_2(I)\}$ are equivalent if δ_α derived from them are the same.

Example of equivalent potential

Let I_2 and I_3 be finite subsets with $|I_2| = 2$, and $|I_3| = 3$. For a given potential $\{P_1(I)\}$ with $P_1(I_2) \neq 0$, define another potential P_2 by

$$P_2(I) = P_1(I) \qquad \text{if} I \neq I_2, I_3$$

$$P_2(I_2) = 0, \;\; P_2(I_3) = P_1(I_3) + P_1(I_2).$$

Then P_1 and P_2 satisfy the minimal conditions for potential $(P(I) = P(I)^* \in \mathcal{A}(\mathcal{I})_+)$, $P_1 \neq P_2$ and they give the same time derivative δ_α. So they are equivalent potentials.

The standardness condition(2) in Theorem 1 is for the purpose of avoiding the situation described in the above example.

(2)**(non-commutartive) conditional expectation** E_J^φ

A state φ is a Θ-even product state, i.e.
$$\varphi(\Theta(A)) = \varphi(A), \;\; \varphi(AB) = \varphi(A)\varphi(B)$$
for any $A \in \mathcal{A}(I)$, $B \in \mathcal{A}(J)$, $I \cap J = \emptyset$.

(Such product states for the fermion and spin algebra exist in abundance. A tracial state τ, the fermion vacuum state are examples.)

E_I^{φ} is a linear, positive, unital projection from $\mathcal{A}$ to $\mathcal{A}(I)$, characterized by $E_J^{\varphi}(A) \in \mathcal{A}(J)$ and the following set of equations;

$$\varphi(CAB) = \varphi(CE_J^{\varphi}(A)B) \quad (B, C \in \mathcal{A}(J)).$$

It has the following property;
$E_I^{\varphi}E_J^{\varphi} = E_{I\cap J}^{\varphi}, \qquad E_I^{\varphi}E_{I^c}^{\varphi} = E_{I^c}^{\varphi}E_I^{\varphi} = \varphi$
$\lim_{\nu} E_{K_\nu}^{\varphi}(A) = E_K^{\varphi}(A) \quad \text{if } K_\nu \longrightarrow K$
$\lim_{\nu \nearrow \mathbb{L}} E_{K_\nu}^{\varphi}(A) = A$
Here $K_\nu \longrightarrow K$ for a net K_ν means

$$K = \bigcap_{\mu}\left(\bigcup_{\nu \geq \mu} K_\nu\right) = \bigcup_{\mu}\left(\bigcap_{\nu \geq \mu} K_\nu\right)$$

4 Translation Covariant System

For $\mathbb{L} = \mathbb{Z}^d$, it is an additive group, which we call **translation group**. There is a representation of this group: $n \in \mathbb{L} \to \tau_n \in \mathrm{Aut}\mathcal{A}$ such that

$$\tau_n(\mathcal{A}(I)) = \mathcal{A}(I+n), \;\; \tau_n\Theta = \Theta\tau_n \text{ for all } n \in \mathbb{L}$$

A potential P is translation covariant if

$$\tau_n(P(I)) = P(I+n) \text{ forall } I \Subset \mathbb{L}.$$

Theorem 2: The real linear space of all translation covariant standard potentials form a separable Banach space with the norm

$$||P|| = ||H_P(n)|| \qquad \text{(independent of } n\text{)}.$$

This theorem provides a tool for the Application(2) above.

We have introduced earlier

$H_P(I) =$ the total interaction enregy of the partial system in I as an open system

where open system refers to the fact that the interactions between I and outside (I^c) are all included. We also define

$$U_P(I) = \sum_{J \subset I} P(J)$$

for the total interaction energy of the system I as a closed system. We have the following Energy Estimate:

Theorem 3: $||U_P(I)|| \leq ||H_P(I)|| \leq ||P|| \cdot |I|$.
The first inequality is immediate from
$||E_I^\varphi|| = 1$ and $U_P(I) = E_I^\varphi H_P(I)$

Proof of $\|H_P(I)\| < \|P\| \cdot |I|$

$$H_P(I)=\lim_{J \nearrow L} \sum_K \{P(K) \mid K \cap I \neq \phi,\ K \subset J\} \qquad \text{(Def.)}$$

Interaction Energy of I as an open system

i.e. Interaction among I as well as

between I and the outside of I.

Let $I=\{i_1, i_2, \quad , i_m\}$, $m=|I|$.

$F_k=\{K \mid i_k \in K \subset \{i_1,\dots,i_{k-1}\}^c, K \subset J\}$ $(k=1, \cdots, m)$

i.e. $F_1=\{K \mid i_1 \in K\}$,

$F_2=\{K \mid i_2 \in K,\ i_1 \notin K\}=\{K \mid i_2 \in K \subset \{i_1\}^c\}$

$F_3=\{K \mid i_3 \in K,\ i_1, i_2 \notin K\}=\{K \mid i_3 \in K \subset \{i_1, i_2\}^c\}$

....

$$H_P(I)=\sum_{k=1}^{m} h_k,$$

$$h_k=\lim_{J \nearrow L} \sum_K \{P(K) \mid K \in F_k\}$$

Note that E_S^{φ} selects those terms with $K \subset S$.
and annihilate all other terms.
due to the standardness of the potential.

Hence, for $S=\{i_1,i_2,\ldots,i_{k-1}\}^c$,

$$\sum_{K\in F_k} P(K)=E_S^{\varphi} H_p(i_k)$$

This implies

$$h_k=E_S^{\varphi} H_p(i_k)$$

By $\| E_S^{\varphi} \|=1$, we obtain $| h_k | \leqq \| H_p(i_k) \| = \| P \|$

Therefore

$$\| H_p(I) \| \leqq \sum_{k=1}^{m} \| h_k \| \leqq m \| P \|.$$