Real blow ups
Introduction to Analysis on Singular Spaces
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Abstract. These are notes based on (and at least in part for) three lectures on real blow up that were given as part of the Introductory workshop for the program at the Mathematical Sciences Research Institue on Analysis on (or is it of?) Singular Spaces, Fall 2008.

Contents

Introduction

In these three lectures I want to discuss real blow up as it relates to resolution of singular spaces and other analytic objects, especially Lie algebras of vector fields. Since this is quite a large subject, and other people will talk on certain aspects of it, I will concentrate on the geometric part – the definition and properties of blow up. Otherwise, as far as things I will use and also applications, I will simply summarize.

LECTURE 1

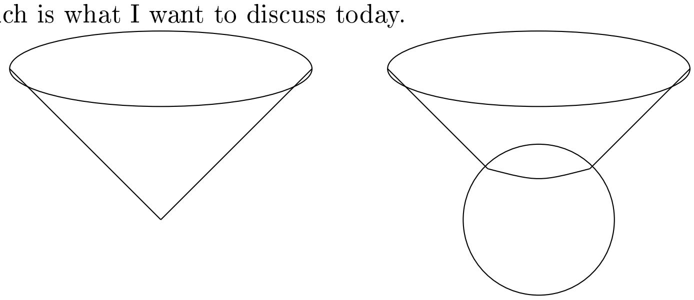
Why blow things up? – And the definition

Today I want to define the basic process of 'blowing up' a manifold around a submanifold. What I will describe is the real version of a procedure that is well known to algebraic geometers in the complex setting. In fact there are several variants, the main one is *radial blow up* which is what I will talk about almost exclusively. There is also the closely related *projective blow up* which is very similar, except one trades off the non-introduction of boundaries for a loss of orientability. I will indicate at some point why there are some reasons to prefer the radial procedure but in essence they are equivalent. There is also the notion of *parabolic blow up* which is similar but different – I will indicate what this is about but will probably not have time to go through it in any detail.

So, the basic question is:- Why blow up at all? If one is working in a genuinely smooth and uniform analytic setting there is not much reason to blow anything up. However, there are three closely related circumstances in which blow up can be very helpful. These correspond to trying to 'resolve'

(1) A singular function, e.g. $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$.
(2) A singular space, e.g. $C = \{t^2 = x^2 + y^2,\ t \geq 0\}$
(3) Degenerate vector fields, e.g. the span of $z_j \partial_{z_j}$ $j = 1, 2, 3$ on $\mathbb{R}^3$.

In all three cases these can be resolved by the introduction of polar coordinates – which is what I want to discuss today.



If one is constrained to work on a singular space – for instance the (one-sided) cone in Euclidean space C pictured above then one has a problem doing anything much right at the singular point. One can choose to work in small neighbourhoods away from, in some appropriate sense uniformly up to, the singular point but it is difficult to work directly around the singular point. A basic question for instance is: What is the space of smooth functions on C? In fact it is fair to say that there is no single answer to this but that the most obvious one is not very good. Namely one could say that a function on C is smooth if it is the restriction to C of a smooth function on $\mathbb{R}^3$. Then however, the usual properties of coordinate systems and Taylor series and so on fail, or get much more complicated.

In real blow up, the idea is simply to work in polar coordinates around the singular point. That is, we lift everything up to a manifold with boundary by using the polar map

$$\beta : [0,\infty) \times C_1 \ni (r,\theta) \longrightarrow r\theta \in C. \tag{1.1}$$

Here C_1 is the circle in $\mathbb{S}^2$ given by the intersection of C with the sphere of radius 1 in $\mathbb{R}^2$:

$$C_1 = C \cap \{t^2+x^2+y^2=1\} = \{(t,x,y) = (\frac{1}{\sqrt{2}}, \frac{\theta}{\sqrt{2}}),\ \theta \in \mathbb{R}^2,\ |\theta| = 1\}. \tag{1.2}$$

It is a (normalized) cross-section of the cone.

Now, $r > 0$ on the left in (1.1) is mapped diffeomorphically onto the smooth part of the cone; this is clear enough since it is immediate for the restriction to $r = 1$ and scaling in r on the left corresponds to radial scaling on the right. Thus the cone is 'blown up' to a manifold with boundary, where the whole boundary is mapped to the conic point under the 'blow down map' β. In fact we are really blowing up the ambient space, $\mathbb{R}^3$, and seeing what happens to the singular subset C. Notice most of all that the blow-down map β is itself smooth, it is the inverse of β which is singular – to the extent that it is undefined near $r = 0$.

So, what we are doing here is blowing up the origin in $\mathbb{R}^3$ and 'lifting' the previously singular subset to a smooth manifold with boundary. This is a procedure that works with great generality, when applied with sufficient diligence and care – as in Hironaka's remarkable result which asserts that by *appropriate* iteration of the complex version of this construction one can render any projective algebraic variety smooth.

1. Polar coordinates

The basic example of blow up then is to introduce polar coordinates around the origin in $\mathbb{R}^n$. Thus the model blow-down map in codimension n is

$$\beta : [0,\infty) \times \mathbb{S}^{n-1} \ni (r,\omega) \longmapsto r\omega \in \mathbb{R}^n. \tag{1.3}$$

Here $\mathbb{S}^{n-1}$ is the unit sphere in $\mathbb{R}^n$. This map is smooth! It is a diffeomorphism from $(0,\infty)\times\mathbb{S}^{n-1}$ onto the complement of the centre 0, i.e. onto $\mathbb{R}^n \setminus \{0\}$. On the other hand, the whole of the 'front face' $r = 0$ is mapped into the centre $\{0\}$. What other interesting features does this map have? Of course, it is not 1-1 so does not have an inverse but it is surjective.

For smooth maps there is a general notion of 'related vector fields'. Namely if $f : X \longrightarrow Y$ is a smooth map between manifolds (or open sets in Euclidean space if you prefer) then the differential $F_* : T_xX \longrightarrow T_{f(x)}Y$ is well-defined at each point. A vector field V on X and a vector field W on Y are f-related if $f_*(V_x) = W_{f(x)}$ for all $x \in X$.

LEMMA 1. *For every smooth vector field on $\mathbb{R}^n$ which vanishes at the origin, there is a unique smooth vector field on $[0,\infty)\times\mathbb{S}^{n-1}$ with is β-related to it.*

PROOF. Computing on the sphere is a bit tricky so I will not try to do it here! In fact this result is really a consequence of the homogeneity of β so let me give a full proof which involves little work. First, what is a smooth vector field on $\mathbb{R}^n$?

It is a smooth section of the tangent bundle, and hence a combination of the basic vector fields $\partial/\partial z_j$, $j = 1, \dots, n$ with smooth coefficients

$$W = \sum_j a_j(z) \frac{\partial}{\partial z_j}. \tag{1.4}$$

So, what does it mean for W to vanish at the origin? It means that each of the coefficients $a_j(z)$ must vanish at $z = 0$. By Taylor's theorem this means exactly that there are smooth functions a_{ij}, $i, j = 1, \dots, n$ such that $a_j(z) = \sum\limits_i a_{ij}(z) z_i$. Thus if W vanishes at the origin it can be written as a linear combination with smooth coefficients of the n^2 vector fields $z_i \partial_{z_j}$:

$$W = \sum_{ij} a_{ij}(z) z_i \frac{\partial}{\partial z_j}. \tag{1.5}$$

Now, it is a general fact that if a is a smooth function on the image space of $f : X \longrightarrow Y$ and V and W are f-related then $(f^*a)V$ and aW are f-related. This just comes from the fact that f_* is the transpose of f^* which is the pull-back on differential 1-forms.

Thus we only need to show that each of the vector fields $W_{ij} = z_i \partial_{z_j}$ is β-related to some smooth vector field V_{ij} on $[0, \infty) \times \mathbb{S}^{n-1}$. Such a vector field is of the form

$$V_{ij} = b_{ij}(r, \theta)\partial_r + T_{ij}(r) \tag{1.6}$$

where $b_{ij} \in \mathcal{C}^\infty([0, \infty) \times \mathbb{S}^{n-1})$ and the $T_{ij}(r)$ are smooth vector fields on the sphere depending smoothly on r as a parameter. Now β is a diffeomorphism in $r > 0$ so V_{ij} exists and is unique in $r > 0$. This is where the homogeneity completes the proof. Under the scaling diffeomorphism $r \longmapsto \tau r$, $\tau > 0$ the vector field V_{ij} changes to

$$V_{ij} = b_{ij}(\tau r, \theta)\tau^{-1}\partial_r + T_{ij}(\tau r) \tag{1.7}$$

but on the image this is the scaling $z \longmapsto \tau z$ under which W_{ij} is invariant. Thus from the uniqueness of the V_{ij} in $r > 0$ we see that

$$b_{ij}(\tau r, \theta) = \tau b_{ij}(r, \theta),\ T_{ij}(\tau r) = T_{ij}(r)\ \forall\ \tau, r > 0. \tag{1.8}$$

Thus the $T_{ij}(r) = T_{ij}(1)$ are independent of r and $b_{ij}(r, \theta) = r b_{ij}(\theta)$ is linear in r and hence

$$V_{ij} = b_{ij}(\theta) r \partial_r + T_{ij},\ b_{ij} \in \mathcal{C}^\infty(\mathbb{S}^{n-1}) \tag{1.9}$$

with the T_{ij} smooth vector fields on the sphere. Since $r\partial_r$ is certainly a smooth vector field, the V_{ij} are smooth down to $r = 0$ as claimed. □

In fact we conclude a little more from this proof than just the lifting. Namely we can say that the smooth vector fields on $\mathbb{R}^n$ which vanish at the origin lift to unique smooth vector fields on $[0, \infty) \times \mathbb{S}^{n-1}$ and that the lifted vector fields span, over $\mathcal{C}^\infty([0, \infty) \times \mathbb{S}^{n-1})$ *all* the smooth vector fields on $[0, \infty) \times \mathbb{S}^{n-1}$ which are tangent to the boundary. Why is this so? Well a smooth vector field on $[0, \infty) \times \mathbb{S}^{n-1}$ is of the form (1.6) as already noted. To be tangent to $r = 0$ the coefficient of ∂_r must vanish at $r = 0$ and hence it must be of the form

$$b_{ij}(r, \theta) r \partial_r + T_{ij}(r). \tag{1.10}$$

As is well-known, or can be proved directly, $z\cdot\partial_z = \sum_i z_i\partial_{z_i}$ lifts to $r\partial_r$ – since these are the generators of the respective radial actions. Thus the first term in (1.10) is in the span of the lift over $\mathcal{C}^\infty([0,\infty)\times\mathbb{S}^{n-1})$. It also follows from this that all the constant (in r) vector fields on the sphere, T_{ij} are in the span of the lift. Now, these must span the smooth vector fields on the sphere, since β is a diffeomorphism for $r>0$ and this finishes the proof.

2. Change of coordinates

So, this blow-up and smooth blow-down map (1.3) have nice properties which can be stated invariantly – the lifting of vector fields vanishing at the centre and for instance that the inverse image of the centre is the boundary with a smooth defining function r. What about coordinate invariance? Really it is coordinate-invariance which makes blow up important and separates it from 'just introducing polar coordinates' (although that is precisely what we are doing).

Lemma 2. *If U_1 and U_2 are open neighourhoods of $0\in\mathbb{R}^n$ and $F: U_1 \longrightarrow U_2$ is a diffeomorphism such that $F(0)=0$ then there is a diffeomorphism*

$$\tilde F : \tilde U_1 = \{(r,\theta)\in[0,\infty)\times\mathbb{S}^{n-1}; r\theta\in U_1\} \longrightarrow \tilde U_2 = \{(r,\theta)\in[0,\infty)\times\mathbb{S}^{n-1}; r\theta\in U_2\} \tag{1.11}$$

giving a commutative diagram

$$\begin{tikzcd} \tilde U_1 \arrow[r, "\tilde F"] \arrow[d, "\beta"'] & \tilde U_2 \arrow[d, "\beta"] \\ U_1 \arrow[r, "F"] & U_2. \end{tikzcd} \tag{1.12}$$

Clearly $\tilde F$ is unique if it exists, since it is determined by continuity from $r>0$.

Proof. If F is an orthogonal transformation then $\tilde O$ is just the restriction of O to $\mathbb{S}^{n-1}$ acting trivially on r. In particular this means that we can replace F by OF if necessary to arrange that $L = F_*(0)\in \mathrm{GL}(n,\mathbb{R})$ is orientation-preserving and so is connected to the identity by a smooth curve L_t, $t\in[0,1]$ so $L_0=\mathrm{Id}$, $L_1 = F_*(0)$. The vector field, W_t, defined by differentiating this family,

$$\frac{d}{dt}L_t^* g = L_t^*(W_t g) \tag{1.13}$$

is a smooth curve of linear vector fields – i.e. is a combination of the $z_i\partial_{z_j}$ with coefficients depending smoothly on t. Thus we can lift L_t to a family of diffeomorhisms, $\tilde L_t$, 'upstairs' generated in the same way by the lifts V_t of the W_t.

Thus we are reduced to the case that $F_*(0)=\mathrm{Id}$ as well as $F(0)=0$. Then, in a possibly smaller neighbourhood U of 0, F itself is connected to the identity by a curve of diffeomorphisms (onto their images) fixing 0 and with differential Id their. Namely,

$$F(z)_i = z_i + \sum_{jk} a_{ijk}(z) z_j z_k,\ F_t(z) = z_i + t\sum_{jk} a_{ijk}(z)z_jz_k,\ t\in[0,1]. \tag{1.14}$$

Now the same argument applies, showing that $\frac{d}{dt}F_t^*(g) = F_t^*(V_t g)$ where V_t vanishes at 0 (and in fact vanishes to second order at 0) so lifts to a curve of diffeomorphisms with end point $\tilde{F}$. Of course away from $r = 0$ $\tilde{F}$ is unique and known to exist anyway. □

So, one reason to say 'blow up the origin' instead of 'introduce polar coordinates around the origin' is that it draws attention to this coordinate invariance. In fact another way of saying this is that the blow-up of a point p in a manifold M is well defined – it is a new manifold with a blow down map which is smooth

$$\beta : [M, \{p\}] \longrightarrow M. \tag{1.15}$$

Invariantly one can take $[M; \{p\}]$ – which is M with p blown up – to be $(M \setminus \{p\}) \cup (T_pM \setminus \{0\})/\mathbb{R}_+$. The claim is that this has a unique $\mathcal{C}^\infty$ structure as a manifold with boundary, where the first part is the interior and the second part, which is just a sphere, (written out invariantly as the quotient of the complement of the origin in a vector space by the radial action – if you like it is the space of half-lines through the origin) is the boundary, such that in local coordinates near p this reduces to exactly the local picture we had above. To see this, just think how the differential F_* acts on the sphere.

3. Projective coordinates

There are other ways of looking at the blow up of a point which are helpful, especially in computations. I did not do this in the lectures but here is a brief description. First of all, what are coordinates on the sphere – clearly this is involved here. Well, if we introduce the homogeneous functions on the sphere $(\mathbb{R}^n \setminus \{0\})/\mathbb{R}_+$ which are the

$$\omega_j = \frac{z_j}{r},\ r = \left(z_1^2 + \cdots + z_n^2\right)^{\frac{1}{2}} \tag{1.16}$$

then $\sum\limits_j \omega_j^2 = 1$ and

$$\sum_j \omega_j d\omega_j = 0 \text{ on } \mathbb{S}^{n-1}. \tag{1.17}$$

So, we can get local coordinates at any point on the sphere by choosing $n-1$ of these provided we abide by two rules. First, don't choose one with $\omega_j = \pm 1$ at the point, since its differential is zero and it cannot be a coordinate. Secondly, choose all of the ω_j which vanish at the point, since their differentials are not dependent on any of the others! Apart from this you are free to choose as you can easily check.

So r and appropriate choice of the ω_i's give coordinates on the blown up space near each point. However, such 'polar coordinates' are not so easy to compute with. Instead one can use the corresponding projective coordinates at the point. At least one of the ω_j's is non-zero (not limiting yourself to the ones you chose as coordinates). Choose one of the corresponding z_j's, (if one $\omega_j = \pm 1$ of course it has to be that one,) as a 'radial variable' – it might be negative nearby, but no matter. Then as projective coordinates we can use z_j and the $t_k = z_k/z_j$ for $k \neq j$. As a little exercise you can check that

LEMMA 3. *Near any (boundary is the only interesting case) point of* $[0, \infty) \times \mathbb{S}^{n-1}$ *the* t_k *and one* z_j *described above give local coordinates in terms of which the* *lifts of the linear vector fields are*

$$z_l\partial_{z_s} \longmapsto \begin{cases} t_l\partial_{t_s} & l,s \neq j \\ \partial_{t_s} & l=j,\ s\neq j \\ t_k(z_l\partial_{z_l} - \sum\limits_r t^r\partial_{t_r}) & l\neq j, s=l \\ z_l\partial_{z_l} - \sum\limits_r t^r\partial_{t_r} & l=s=j. \end{cases} \tag{1.18}$$

So one can certainly cover the blow-up by patches in which such projective coordinates are valid.

4. Vector bundles

The blow up of a point in a manifold, as described above, is coordinate invariant. For a real vector bundle $E \longrightarrow M$ over a manifold M the zero section is a submanifold of E which is diffeomorphic to M but is just given by a point in each fibre. It follows that we can blow up each point 'of M' (thought of as the zero section) in the corresponding fibre and more significantly that the fibres will fit together smoothly as the point varies.

PROPOSITION 1. *For a real vector bundle $E \longrightarrow M$ the set $[M;0_E] = (E \setminus 0_E) \cup (\mathbb{S}E)$ where $\mathbb{S}E \longrightarrow M$ is the bundle of spheres $(E\setminus 0_E)/\mathbb{R}_+$, has a natural structure as a manifold with boundary and smooth blow-down map*

$$\beta : [E;0_E] \longrightarrow E \tag{1.19}$$

which restricts to the blow-down map for $[E_p;\{p\}]$ for each $p \in M$, and is consistent with local trivializations of E over open sets of M.

PROOF. I will not dwell too much on this although it is important. Taking a trivialization of E over an open set U identifies everything with a product $U \times [\mathbb{R}^n;\{0\}]$ and everything is seen to make sense as stated. A change of trivialization is, on the overlap in the bases, a smooth family of linear maps on the fibres. The discussion above shows that this lifts to a smooth family of maps on the fibres of the blown up spaces proving the result, but one should do it more carefully than I am. $\square$

The preimage of 0_E under the blow-down map is the 'front fact' of the blown up space – in this case it is diffeomorphic to the sphere bundle of E.

Note that bundle isomorphisms $E \longrightarrow F$ lift to diffeomorphisms of the blown up spaces $[E,0_E] \longrightarrow [F;0_F]$ by the same arguments as above (although general smooth bundle maps do not – they do not 'know where to go'). What is more important in the sequel is that general smooth diffeomorphisms preserving the zero section also lift smoothly.

LEMMA 4. *If $E \longrightarrow M$ is a vector bundle and $U_1, U_2 \supset 0_E$ are open neighbourhoods of the zero section with $F : U_1 \longrightarrow U_2$ a diffeomorphism such that $F(0_E) = 0_E$, then F lifts to a diffeomorphism between neighbourhoods of the front face of $[M;0_E]$.*

PROOF. Time is short so I will not go through this in detail. It can be proved in a way that is quite close in spirit to the proof of Lemma 2 above proceeding in steps. First, a diffeormorphism of M lifts to a diffeomorphism of E which is the identity on the fibres. These diffeomorphisms lift to the blown up space and hence we can assume that F is actually the identity on 0_E. The differential of F at the zero section is then the identity on tangent vectors to 0_E and hence projects to a bundle isomorphism of E. Again this lifts, so this projection can also be arranged to be the identity. It then follows by a partition of unity argument that F can be connected to the identity through a smooth family of diffeomorphisms which all have these two properties. Again these are given by integration of a one-parameter family of vector fields which vanishes at 0_E. In local trivializations it is easy to see that such a vector field lifts to be smooth – using the arguments above – and then the integration can be done on the blown-up space to construct the lifted diffeomorphism.

Alternatively you can sit down and compute the lift in local coordinates. It is not all that hard. □

5. Embedded submanifolds

Now the final step, for the moment, is to show that if $Y \subset M$ is an embedded submanifold of another manifold then there is a well defined blown-up manifold with boundary $[M;Y]$ which is such that in local coordinates in which Y is given by the vanishing of the first k coordinates then $[M;Y]$ is just the product of the blow up of the origin in these variables with the coordinate space in the other variables. One way to see this without doing too much work is to use Lemma 4 and the collar neighbourhood theorem. The latter shows that for an embedded submanifold there is always a diffeomorphism of a neighbourhood of Y in M to the total space E of a vector bundle over M such that Y is mapped to the zero section. This in fact characterizes the condition that the submanifold be embedded. The vector bundle in question is the normal bundle to Y in M, the quotient T_YM/TY. From this the existence of the blown up manifold with boundary, as $[M;Y] = (M \setminus Y) \cup (TY \setminus 0_Y)/\mathbb{R}_+$ with a natural $\mathcal{C}^\infty$ structure and blow-down map

$$\beta : [M;Y] \longrightarrow M \tag{1.20}$$

follows. It has the 'obvious properties' being a diffeomorphism of the interior onto $M \setminus Y$ and restricting to the boundary, which is the front face $\mathbb{S}Y = (TY \setminus 0_{TY})/\mathbb{R}_+$ as the projection to Y.

The generalization of the discussion above of vector fields is

PROPOSITION 2. *Under the blow up of an embedded submanifold Y of a manifold M the smooth vector fields on M which are tangent to Y lift under the blow down map (are related under it) to unique smooth vector fields on $[M;Y]$. The lifted vector fields span, over $\mathcal{C}^\infty([M;Y])$ all smooth vector fields on $[M;Y]$ which are tangent to the boundary – the front face produced by the blow up of Y.*

6. Projective blow up

In projective blow up, we simply use 'two-sided' polar coordinates. In other words instead of the polar coordinate map (1.3) we use the closely related map

$$\beta_{\mathbb{P}} : \mathbb{R} \times \mathbb{P}^2 \ni (\rho, \eta) \longmapsto \rho\eta \in \mathbb{R}^2,\ \mathbb{P}^2 = \mathbb{S}^2/\pm\,. \tag{1.21}$$

This is still smooth and surjective and is locally near each point 'the same map'. The advantage is that there is no boundary on the left. The disadvantage (not very serious generally) is that $\mathbb{P}^2$ is not orientable. One can go ahead and check that projective blow up is indeed globally well-defined as for the radial case. The relationship between them is pictured here:

$$
\begin{tikzcd}
{[M;Y] \equiv [[M;Y]_{\mathbb{P}};H]} \arrow[r, "\beta(H)"] \arrow[dr, "\beta(Y)"'] & {[M;Y]_{\mathbb{P}}} \arrow[d, "\beta_{\mathbb{P}}(Y)"] \\
 & M.
\end{tikzcd}
\tag{1.22}
$$

Here H is the hypersurface $\{\rho = 0\} \subset [M;Y]_{\mathbb{P}}$ which is the inverse image of Y under the projective blow up. Thus, radial blow up factors through projective blow up and in that sense the latter is more 'fundamental'.

Why not use projective blow up? There are at least two reasons. One is that the functions we deal with often do not lift to be smooth across H under projective blow up, but 'smooth up to it from both sides' so the simiplication is only apparent. The other is that we are often dealing with boundaries in the first place and then the projective blow up does not really make sense anyway, or rather reduces to the same thing.

7. Parabolic blow up

I did not talk about this in the end. It is discussed extensively in the book on the C.L. Epstein, [**?**].

8. What does this buy us?

So what can we do with this blow up? We can resolve orbifolds and other manifolds which look like bundles of cones over a smooth manifold. We can also 'resolve' Morse functions. Suppose that M is a compact manifold, then it always carries a Morse function, a smooth function $u \in \mathcal{C}^\infty(M)$ with the property that at every point of M either the function is non-stationary, $du(p) \neq 0$, or else if $du(p) = 0$ then the Hessian is invertible, where the Hessian is the map

$$T_pM \ni v \longmapsto H_u v(p) \in T_p^*M \tag{1.23}$$

which is induced by taking a smooth vector field V on M with $V(p) = v$ and considering $d(Vu)(p) \in T_p^*M$ – which can be seen not do depend on the particular choice of V. A Morse function only has finitely many critical points $\{p_1, \dots, p_N\}$, and if these are blown up then near each of the new front faces it takes the form

$$u = u(p_i) + r_i^2 U_i \tag{1.24}$$

were U_i is smooth and has $dU_i \neq 0$ on $U_i = 0$. In particular this means that the level sets of u are all unions of smooth manifolds which meet transversally – they are resolved to normal crossings. The level set for a critical value has been 'resolved' to $r_i = 0$, the new front face, plus $U_i = 0$ which is a smooth hypersurface which is transversal to $r_i = 0$.

[Picture please!]

Let me try at this stage to anticipate some of what I will show later about such a 'resolution'. Why should such a blow up help? One thing to look at is the Lie algebra of smooth vector fields which annihilate the function u. Where $du \neq 0$ on a level set, this is just the Lie algebra of vector fields tangent to the fibres. At the singular point, on the singular stratum, it becomes much more complicated. However, after the single blow up of the critical point, as described above, the smooth vector fields annihilating u, i.e. pairing to zero with du, are locally the ones tangent to $r_i = 0$ or $U_i = 0$ away from the intersection but at the intersection we can take $U_i = s$ to be one of the coordinates, y_k the others and then the vector fields are locally spanned by

$$r\partial_r - 2s\partial_s,\ \partial_{y_k}. \tag{1.25}$$

So, this is rather degenerate, but what I want to show later is that we can 'resolve' such vector fields and as a result discuss the properties of differential operators which are in the enveloping algebra.

If we want to do more than this – resolve more complicated singular objects or objects more complicated than spaces – for instance Lie algebras of vector fields – then we need to do two things. We need to iterate blow ups, and we need to blow up submanifolds of manifolds with corners. The latter obviously will arise on iteration of blow ups since each time we blow up a new boundary hypersurface emerges and the simplest case is when these meet transversally. I will talk about both these things tomorrow, but just suppose it works out well! Then we can resolve arbitrary projective algebraic varieties (courtesy of Hironaka), we can resolve smooth actions of compact Lie groups on compact manifolds (or proper actions of compact groups). I cannot cover all these things but I will try to describe some of them and also try to give an idea of what I really mean by resolution.

9. A list of theorems!

Still to come!

LECTURE 2

Iterated blow ups and manifolds with corners

Last time I went through the definition of the manifold $[M;Y]$ obtained from M by blowing up along a closed embedded submanifold Y with it natural blow-down map

$$\beta : [M;Y] \longrightarrow M. \tag{2.1}$$

This is a smooth map, so pull-back gives $\beta^* : \mathcal{C}^\infty(M) \longrightarrow \mathcal{C}^\infty([M;Y])$. This is injective but cannot be surjective, namely there are more functions which are smooth in polar coordinates. This in fact is is one of the reasons to blow up.

What about vector fields? The vector fields which lift to be smooth under β are precisely those which are tangent to Y. There are always local coordiantes z, y in M near any point of Y in which Y is locally defined by the $z_1 = \cdots = z_k = 0$ and the y_i's become coordinates on Y. The vector fields tangent to Y are then the $\mathcal{C}^\infty$ combinations of the ∂_{y_i} and the $z_j\partial_{z_i}$. The approach I took last time shows that these lift to be smooth, to be tangent to the new boundary $r = 0$ and to span, over $\mathcal{C}^\infty$ coefficients on $[M;Y]$ *all* the vector fields tangent to $\partial[M;Y]$. For any manifold with boundary this latter space consists of all the sections of a vector bundle

$$\{V \in \mathcal{C}^\infty(X;TX); V \text{ is tangent to } \partial X\} = \mathcal{C}^\infty(X; {}^{\mathrm{b}}TX). \tag{2.2}$$

This is already an important fact, since a Lie algebra of vector fields consisting of all the smooth sections of a vector bundle is getting close to the standard case of *all* the smooth vector fields on a (compact) manifold without boundary.

1. Manifolds with corners

Each blow up introduces a boundary, so in order to do iterated blow up we have to work in the context of manifolds with corners. I will be brief about these, really there is not much to worry about in the basic theory. In summary the definition of a manifold in the usual sense is as a set X with a covering by local coordinates systems with $\mathcal{C}^\infty$ transition maps. For a manifold with corners we allow the coordinate 'model' to be the intersection of an open subset of $\mathbb{R}^n$ with one of the k-corners $[0,\infty)^k \times \mathbb{R}^{n-k}$. Smoothness of a map is the existence of a smooth extension to an open set in $\mathbb{R}^n$, by Whitney's (easy) extension theorem this is the same as local boundedness of all derivatives. So, locally a manifold with corners looks like $[0,\infty)^k \times \mathbb{R}^{n-k}$ at a boundary point of codimension k; I will write the local coordinates $x_1, \dots, x_k$, $y_1, \dots, y_{n-k}$. This means that there are boundary hypersurfaces, connected sets given locally by the vanishing of one of the x_j. To make sure that these are manifolds with corners in the same sense I insist that the boundary hypersurfaces be embedded. This means that each of them, $H \subset X$ is given by $\rho_H = 0$ where $\rho_H \in \mathcal{C}^\infty(X)$, $d\rho_H \neq 0$ on H and $\rho_H \geq 0$.

So, as a little exercise you can go back to what I did last time and see that we can blow up closed 'embedded' submanifolds $Y \subset X$ for a manifold with corners, provided that embedded implies that an appropriate version of the collar neighbourhood theorem holds. This is the condition that H be a p-submanifold. More precisely this means that near each point of X there are local coordinates of the adapted sort that I described above, x and y such that locally

$$Y = \{x_1 = \cdots = x_j = 0,\ y_i = \cdots = y_l = 0\} \tag{2.3}$$

where either $j = 0$ (no x equations) of $l = 0$ (no y equations) is permitted. Note that $l = 0$ makes Y into a boundary face – a component of the intersection of boundary hypersurfaces. The other extreme $j = 0$ corresponds to an 'interior p-submanifold' which is most like the usual case.

Proposition 3. *It is always possible to blow up a closed embedded p-submanifold Y in a manifold with corners X giving a new manifold with corners $[X; Y]$ with maximal boundary codimension either the same or increased by one and with a smooth blow down map*

$$\beta : [X; Y] \longrightarrow X. \tag{2.4}$$

Although there is nothing much to manifolds with corners at the level I have described here, there is something more significant in the maps between them which I want to emphasize. Smoothness itself is straighforward, but smooth maps between manifolds with corners $f : X \longrightarrow X'$, can, and should, be required to 'preserve some of the boundary structure'. The natural condition is that inverse images of boundary faces should be boundary faces, in a wide sense that they be unions of boundary faces. In terms of boundary defining functions this means

$$f^*\rho'_i = a_i \prod_j \rho_j^{\alpha_{ij}},\ 0 < a_i \in \mathcal{C}^\infty(X) \tag{2.5}$$

where the $\rho'_i \in \mathcal{C}^\infty(X')$ and $\rho_j \in \mathcal{C}^\infty(X)$ are listings of the boundary defining functions for X' and X respectively; the α_{ij} are necessarily non-negative integers but 0 is allowed.

Definition 1. A smooth map $f : X \longrightarrow X'$ between manifolds with cornes which satisfies (2.5) is called an *interior b-map*.

Here the 'b-' just stands for boundary. Note that the composite of two b-maps is again a b-map. A general, not necessarily 'interior' b-map is one which is an interior b-map into one of the boundary faces of X'. This just corresponds to either (2.5) or $f^*\rho'_i \equiv 0$ holding for each boundary defining function of X'.

2. Examples again

Now, I can describe one of the original applications of blow up – to define the b-calculus (although this had already been done, maybe it is better to say it gives a clear characterization).

Above, I emphasized p-submanifolds. However, one of the most interesting examples of embedded submanifolds in the usual setting of a compact manifold without boundary is the diagonal

$$\mathrm{Diag} = \{(m, m) \in M^2; m \in M\}. \tag{2.6}$$

This is certainly embedded. However in the case of a manifold with boundary it is not a p-submanifold as we see even in the one-dimensional case

$$\mathrm{Diag} = \{(x_1, x_2) \in [0,1]^2; x_1 = x_2\}. \tag{2.7}$$

[Sketch]

As we have been hearing from Michael Taylor, pseudodifferential operators correspond to kernels with rather simple 'conormal' singularities at the diagonal and smooth elsewhere. In this case there are several possibilities about what to do. We can ignore the boundary, defining pseudodifferential operators by restriction from $\mathbb{R} \times \mathbb{R}$ for instance. However, ignoring boundaries that are really there is not wise. We can follow Boutet de Monvel and consider *transmission conditions* – perhaps Gerd Grubb will talk more about this. However, we can also think of defining pseudodifferential operators instead as generalizations of the *tangent* vector fields

$$x\partial_x,\ \partial_{y_j}. \tag{2.8}$$

So, this brings us to two questions simultaneously.

(1) If we think of non-p-submanifolds as singular, how can we resolve the diagonal in the case of a manifold with boundary (or for that matter with corners).
(2) What does it mean to 'resolve' the algebra of vector fields tangent to the boundary on a manifold with boundary (of with corners).

In the first case we can say a blow up of some Y resolves the diagonal if the lift of Diag to $[M^2; Y]$ is a p-submanifold. In the second case we say the blow up resolves the Lie algebra if its elements lifts, from one of the factors, to be smooth and also to be collectively transversal to the diagonal.

We already know that for the vector fields to lift to be smooth, they must be tangent to Y. Clearly $Y \supset \partial\,\mathrm{Diag}$ is also necessary, since otherwise there are points at which nothing is changed and Diag cannot have been resolved. In the case of the tangent vector fields (2.8), these two conditions force

$$Y = \partial X \times \partial X. \tag{2.9}$$

This is a boundary face, and hence a p-submanifold. I am assuming here that ∂X is connected.

LEMMA 5. *The diagonal* Diag $\subset X^2$ *for a compact manifold with boundary lifts to* $X^2_{\mathrm{b}} = [X^2; (\partial X)^2]$ *to a p-submanifold and* $\mathcal{V}_{\mathrm{b}}(X)$, *the Lie algebra of smooth vector fields lifts to be transversal to the (lifted) diagonal.*

PROOF. Computation. In fact if you think about it this really reduces to the 1-dimensional case. I have not yet defined the lift of a submanifold under blow up, so you should continue reading to find out what this means. □

3. Commutation

Now, if Y_1 and Y_2 are both subsets of M we can ask what happens to Y_2 after we blow up Y_1 – which better be a p-submanifold for this to be possible. We have to distinguish between the two cases where $Y_2 \subset Y_1$ and $Y_2 \setminus Y_1 \neq \emptyset$. In the first case we define the lift $\widetilde{Y_2} = \beta^!(Y_2) \subset [X; Y_1]$ to be $\beta^{-1}(Y_2)$. In the second case we define it to be the closure of $\beta^{-1}(Y_2 \setminus Y_1)$ – although this doesn't make much sense unless Y_2 meets Y_1 reasonably sensibly.

[Sketch]

It is easy to think of a 'joint p-submanifold' condition on Y_1, $Y_2 \subset X$ – namely that they are each p-submanifolds and near any point of their intersections there is *one* adapted coordinate system in M in terms of which the *both* take the form (2.3), with different 'index sets' of course so we should generalize this by saying

$$Y_p = \{x_i = 0,\ i \in I_p \subset \{1, \dots, k\},\ y_j = 0,\ j \in J_p \subset \{1, \dots, n-k\}\}. \tag{2.10}$$

PROPOSITION 4. *If Y_1 and Y_2 are joint p-submanifolds in the sense of* (2.10) *then the lift of Y_2 to $[X; Y_1]$ is a p-submanifold. In the special case that in addition* either $Y_1 \subset Y_2$ *or* $Y_2 \subset Y_1$ *or* $Y_1 \pitchfork Y_2$ *there is a canonical isomorphism*

$$[[X; Y_1]; \beta_1^!(Y_2)] = [[X; Y_2]; \beta_2^!(Y_1)] \tag{2.11}$$

but not otherwise.

Generally we denote the iterated blow up, $[[X; Y_1]; \beta_1^!(Y_2)]$ as $[X; Y_1; Y_2]$ and then the commutation result becomes

$$[X; Y_1; Y_2] = [X; Y_2; Y_1]. \tag{2.12}$$

PROOF. I doubt that I will have time to do this in the lecture but it is not so hard. Note that the transversal case, $Y_1 \pitchfork Y_2$ is the easy one. In terms of (2.10) it means that $I_1 \cap I_2 = \emptyset = J_1 \cap J_2$. What this amounts to is that one can locally decompose $M = M_1 \times M_2$ as a product, so that $Y_1 = Y_1' \times M_2$ and $Y_2 = M_1 \times Y_2'$ where $Y_1' \subset M_1$ and $Y_2' \subset M_2$ are p-submanifolds. Then it follows easily.

The case of inclusion one way or the other can be done by computation. One way to think about it is to consider the radial vector fields around the submanifolds. For Y in (2.3) this would be

$$x_1 \partial_{x_1} + \dots + x_j \partial_{x_j} + y_1 \partial_{y_1} + \dots + y_l \partial_{y_l}. \tag{2.13}$$

The condition of inclusion means that one of these vector fields (the one for the smaller submanifold) is obtained from the other by adding terms. Since this radial vector field lifts to $r\partial_r$ it follows that the radial actions commute and this leads to the commutation of the blow ups.

The fact that this *doesn't* work otherwise can also be seen by lifting the radial vector fields. □

4. Tangent vector fields again

Let me point out that this commutation result allows us to resolve the Lie algebra of vector fields, $\mathcal{V}_{\mathrm{b}}(X)$, tangent to all the boundaries of a manifold with corners, and hence as I will indicate below, to define the b-calculus in this context too. Namely, for a manifold with corners X consider all the products $H_i \times H_i$ of a boundary hypersurface with itself. These are all transversal one to another. So we need a little result to proceed.

LEMMA 6. *Under blow up of a boundary face all other boundary faces lift to boundary faces and transversal boundary faces remain transversal.*

So, combining this with the commutation result for transversal p-submanifolds discussed above, we can define unambiguously

$$X_{\mathrm{b}}^2 = [X^2; H_1^2; H_2^2; \dots; H_N^2] \tag{2.14}$$

giving a manifold with corners, independent of the order, to which Diag lifts to be a p-submanifold. The the tangent vector fields, forming $\mathcal{V}_{\mathrm{b}}(X)$, lift to be collectively transversal to this lifted diagonal.

5. Another commutation result

In Proposition 4 it is noted that for joint p-submanifolds Y_1 and Y_2 which are neither comparable (meaning one is contained in the other) nor transversal, the two manifolds $[X;Y_1;Y_2]$ and $[X;Y_2;Y_1]$ are different. How then can one 'blow up' such a subset. It is possible to show that one can 'correct' the blow up in two ways.

The first, and most frequent 'solution' is to simply blow up the intersection first and then check that

$$[X;Y_1\cap Y_2;Y_1;Y_2]\equiv[X;Y_1\cap Y_2;Y_2;Y_1]. \tag{2.15}$$

In fact, after under the blow up of $Y_1\cap Y_2$ the two bigger manifolds Y_1 and Y_2 lift to p-submanifolds which are disjoint, and hence transversal – giving (2.15).

There is a second alternative, which is rarely used (and may not even be in the literature). That is, one can blow up $Y_1\cap Y_2$ *last*:

PROPOSITION 5. *For any pair of embedded joint p-submanifolds there is a natural diffeomorphism*

$$[X;Y_1;Y_2;Y_1\cap Y_2]\equiv[X;Y_2;Y_1;Y_1\cap Y_2]. \tag{2.16}$$

However this manifold is different to the one in (2.15). Note that on the left in (2.16) $Y_1\cap Y_2$ first lifts as a submanifold of Y_1 but *is not* a submanifold of the lift of Y_2 – so the notation is a bit dangerous.

6. Fibrations and b-fibrations

Perhaps the most important smooth maps between manifolds are diffeomorphisms. However, in geometric and other settings *fibrations* are also of vital importance. A smooth map $f:X\longrightarrow X'$ between manifolds without boundary is a *submersion* if its differential is everywhere surjective, $f_*:T_xX\longrightarrow T_{f(x)}X'$, for all $x\in X$. If X and X' are compact the Implicit Function Theorem shows that f is actually a fibration, meaning that it is surjective and each point $x'\in X'$ has an open neighbourhood U for which there is a diffeomorphism F giving a commutative diagram

$$\begin{tikzcd} f^{-1}(U) \arrow[r, "F"] \arrow[d, "f"'] & U\times Z \arrow[dl, "\pi_1"] \\ U; & \end{tikzcd} \tag{2.17}$$

here π_1 is projection onto the first factor. The manifold Z is then determined up to diffeomorphism (provided X' is connected) and such a triple $f:X\longrightarrow X'$ may be written

$$\begin{tikzcd} Z \arrow[r, dash] & X \arrow[d, "f"] \\ & X' \end{tikzcd} \tag{2.18}$$

(there is now actual map from the model fibre Z, rather each fibre is diffeomorphic to it) and thought of as a fibre bundle, with fibre Z and structure group $\mathrm{Diff}(Z)$. One reason such maps are particularly well-behaved is that Fubini's theorem shows that fibre-integration preserves smoothness:

$$f_* : \mathcal{C}^\infty_c(X;\Omega) \longrightarrow \mathcal{C}^\infty_c(X';\Omega). \tag{2.19}$$

Here Ω is the (trivial) bundle of densities, those things which can be invariantly integrated.

We can easily set up fibrations in the category of compact manifolds with corners. However, the submersion condition is not enough – for instance just take the identity map $[0,1] \longrightarrow \mathbb{R}$ which has surjective differential but is not surjective. Insisting that a smooth map between manifolds with corners be surjective as well as have surjective differential at every point does lead to a fibration; it also ensures that the map be an interior b-map.

However, in the category of manifolds with corner there is a class of maps that is larger than this direct generalization of a fibration but which has enough of the properties to be very useful. It consists of the *b-fibrations*. To see where the defining conditions come from, recall that the differential f_* of a smooth map may be defined by duality from the pull-back. Namely the cotangent space of a manifold X at a point x is the quotient $T^*_x X = \mathcal{J}(x)/\mathcal{J}(x)^2$ of the ideal $\mathcal{J}(x) \subset \mathcal{C}^\infty(X)$ of smooth functions which vanish as x by the smaller ideal of functions which vanish to second order at x – which is spanned by the products of elements of $\mathcal{J}(x)$. Then $f^*\mathcal{J}(f(x)) \subset \mathcal{J}(x)$ and hence $f^* : T^*_{f(x)}X' \longrightarrow T^*_x X$ has dual which by definition is the differential $f_* : T_x X \longrightarrow T_{f(x)}X'$.

I have written out all this elementary stuff since on a manifold with corners there is a not-quite-obvious, but natural, generalization of it. First if ρ_1 and ρ_2 are defining functions for the same boundary hypersurface then $\rho_1 = a\rho_2$ where $0 < a \in \mathcal{C}^\infty(X)$. Thus $\log\rho_1 = \log\rho_2 + \log a$ where $\log a \in \mathcal{C}^\infty(X)$. It follows that the larger space of functions

$$\mathcal{C}^\infty_{\log}(X) = \{f : X \setminus \partial X \longrightarrow \mathbb{C}; f = \sum_j c_j \log\rho_j + f',\ f' \in \mathcal{C}^\infty(X),\ c_j \in \mathbb{C}\} \tag{2.20}$$

is independent of the boundary defining functions, ρ_j, used to define it and is therefore intrinsic. Moreover, interior b-maps define pull-back operations on it since under an such a map, see (2.5),

$$f^*\log(\rho'_i) = \sum_i \alpha_{ij}\log\rho_j + \log(a_i). \tag{2.21}$$

In local admissible coordinates $x_i = \rho_i$, the differentials of these functions are locally of the form

$$\sum_i (c_i + x_i u_i)\frac{dx_i}{x_i} + \sum_j v_j dy_j \tag{2.22}$$

for smooth functions u_i, v_j. Evaluating the coefficients at a point, i.e. taking the quotient, gives vector spaces ${}^{\mathrm{b}}T^*_x X$ which are therefore naturally defined and combine to give a smooth vector bundle ${}^{\mathrm{b}}T^*X$. The dual bundle, ${}^{\mathrm{b}}TX$, is the one that the tangent vector fields, spanned locally by $x_i\partial_{x_i}$ and ∂_{y_j} form *all* the smooth sections of

$$\mathcal{V}_{\mathrm{b}}(X) = \mathcal{C}^\infty(X; {}^{\mathrm{b}}TX). \tag{2.23}$$

With this alternative tangent bundle in mind the b-differential is well-defined for any interior b-map by duality. What it does is tell us how the tangent vector fields behave under f; at a boundary point it has a little more information in it than the usual differential. I will still denote it f_* since you can tell the difference since this $f_* : {}^{\mathrm{b}}T_x X \longrightarrow {}^{\mathrm{b}}T_{f(x)}X'$.

Now, with this preamble it is not surprising that we define a *b-submersion* to be an interior b-map which has everywhere surjective b-differential. It is not quite clear that this condition is satisfied by fibrations in the category of manifolds with corners; it is but it is satisfies by other maps too. In particular

PROPOSITION 6. *The blow-down map $\beta : [M; F] \longrightarrow M$ corresponding to blow up of any boundary face of a manifold with corners is a b-submesion.*

Blow maps for interior p-submanifolds, or any non-boundary face, are not b-submersions.

This is quite a useful concept but is not very close to that of a fibration. To get what we want, we need to impose another condition as well. This can be seen in various ways but the simplest is a global condition. Namely an interior b-map is said to be *b-normal* if no boundary hypersurface is mapped under it into a boundary face of codimension two (or higher of course). In terms of (2.5) this means that for each j there is at most one i such that $\alpha_{ij} \neq 0$. Indeed, $\{\rho_j = 0\}$ is mapped into $\{\rho'_i = 0\}$ under f if $\alpha_{ij} \neq 0$. Again a fibration is automatically b-normal, but a blow-down map (at least a non-trivial one, for a boundary face of codimension 2 or higher) is not b-normal.

DEFINITION 2. An interior b-map is a b-fibration if it is both a b-submersion and is b-normal.

It might be an iteresting result if this condition implied that f was a fibration, but the truth is more interesting, namely it does not. To see a non-trivial example of a b-fibration consider the composite map of a blow-down and projection

$$f : [[0,1]^2; \{0\}] \xrightarrow{\beta} [0,1]^2 \xrightarrow{\pi_1} [0,1]. \tag{2.24}$$

Both maps a are b-submersions, hence so is the composite which is clearly an interior b-map. Since the target manifold is a manifold with boundary, and hence has no boundary faces of codimension 2 or higher, the b-normality condtion is automatically satisfied.

So, the claim I want to emphasize here is that b-fibrations are the replacements for fibrations in the category of manifolds with corners. I will try to justify this in various ways in the sequel. For the moment let me incorporate it into a definition. As I will explain below this definition needs to be expanded – here we are only considering the way smooth vector fields can degenerate at the boundary.

DEFINITION 3. Let $\mathcal{V} \subset \mathcal{V}_{\mathrm{b}}(X)$ be a Lie algebra of smooth vector fields on a compact manifold with corners which and suppose that $\mathcal{V}$ contains all the smooth vector fields of compact support in the interior. A resolution of $\mathcal{V}$ consists of a manifold with corners $X^2_{\mathcal{V}}$ which is obtained from X^2 by iterated blow up of p-submanifolds of the boundary (meaning at each stage – they do not have to be lifts of manifolds from X^2), so there is an overall blow-down map

$$\beta^2_{\mathcal{V}} : X^2_{\mathcal{V}} \longrightarrow X^2 \tag{2.25}$$

which is an identification of the interiors. It is further required that

- The factor exchange map lifts (extends by continuity from the interior) to be a smooth involution on $X^2_{\mathcal{V}}$.
- The diagonal lifts (to the closure of its intersection with the interior) to be a p-submanifold $\mathrm{Diag}_{\mathcal{V}} \subset X^2_{\mathcal{V}}$.
- The elements of $\mathcal{V}$ acting on the left factor lift (extend by continuity from the interior) to be smooth on $X^2_{\mathcal{V}}$ and to be collectively transversal to $\mathrm{Diag}_{\mathcal{V}}$ at each point (i.e. they span the normal bundle to $\mathrm{Diag}_{\mathcal{V}}$.
- The composite map $\pi_{L,\mathcal{V}} = \pi_L \circ \beta : X^2_{\mathcal{V}} \longrightarrow X$ is a b-fibration which is transversal to the lifted diagonal.

The first condition means that the last conditon holds for the corresponding right stretched projection and similarly the third condition holds for the lift of the vector fields from the right factor.

These properties are enough to allow one to 'microlocalize' the Lie algebra to a 'small' space of pseudodifferential operators and to a 'large' space of pseudodifferential operators. Further properties (discussed below) ensure that the first is an alegbra and in the second composition is possible under natural growth constraints. There are plenty of Lie algebras which cannot be resolved in this way (and also as we shall see there are more general notions of resolution if the conditions at the beginning that the vector fields be smooth and be arbitrary in the interior is dropped). Still there are lots of known examples.

Problem 1 (Resolution problem). Is it possible to give a direct characterization of which Lie algebras are resolvable in this way?

Problem 2 (Uniqueness problem). It is easy to see that $X^2_{\mathcal{V}}$ with the properties listed need not be unique. However, there should be some sort of uniqueness condition, meaning different resolutions should be closely related.

As noted above such a resolution is enough to define a space of operators. To prove composition results it is very convenient to go one step further and define a corresponding triple space.

Definition 4. A *triple resolution* of X^3 associated to a resolution of a Lie algebra $\mathcal{V}$ as in Definition 3 is a manifold with corners $X^3_{\mathcal{V}}$ obtained by iterated blow up of boundary p-submanifolds from X^3, with overall blow-down map $\beta^3 : X^3_{\mathcal{V}} \longleftrightarrow X^3$, in such a way that

- The three factor exchange maps lift to diffeomorphisms
- The projection $\pi_F : X^3 \longrightarrow X^2$ onto the right two factors lifts to a b-fibration $\pi_{F,\mathcal{V}} : X^3_{\mathcal{V}} \longrightarrow X^2_{\mathcal{V}}$ giving a commutative diagam

$$\begin{tikzcd} X^3_{\mathcal{V}} \arrow[r, "\pi_{F,\mathcal{V}}"] \arrow[d, "\beta^3"'] & X^2_{\mathcal{V}} \arrow[d, "\beta"] \\ X^3 \arrow[r, "\pi_F"'] & X^2. \end{tikzcd} \tag{2.26}$$

  Hence from the first condition the same is true of the other two projections π_S and π_C.
- The diagonal $\mathrm{Diag}_{\mathcal{V}}$ in $X^2_{\mathcal{V}}$ lifts (to the closure of the inverse images of its interior) under each of the projections to three joint p-submanifolds which intersect precisely at the lift of the triple diagonal (which is therefore also a p-submanifold).

- The map $\pi_{F,\mathcal{V}}$ is transversal to the lifts of the diagonal under the other two projections.

The existence of such a triple resolution for $\mathcal{V}$ guarantees the composition properties for operators mentioned above – these are made more precise later.

CONJECTURE 1. *There is always a triple resolution for any Lie algebra which has a resolution in the sense of Definition 3.*

In Definition 3 it was assumed that the initial object was a Lie algebra of smooth vector fields including all vector fields with compact support in the interior. This is rather an unreasonable restriction! I will include some examples below without giving a general *a priori* definition of resolution. The point is that both the single space, replacing X, and a replacement for the diagonal need to be chosen or constructed.

7. Examples of resolution of a vector fields

I did not cover these examples in the lectures at all, but I include here a substantial list (but by no means exhaustive) of Lie algebras which are known to have resolutions of this type introduced in Definition 3. Before doing so, let me give a result which reduces the workload a bit.

PROPOSITION 7. *If $\mathcal{V} \subset \mathcal{V}_{\mathrm{b}}(X)$ is a Lie algebra of vector fields on a compact manifold with corners which has a resolution in the sense of Definition 3 then so does $\rho^\alpha\mathcal{V}$, with elements $\rho^\alpha V$, $V \in \mathcal{V}$, for any product of boundary defining functions ρ^α.*

Note that the Lie algebras obtained this way are by no means uninteresting and some are included in the list below.

CONJECTURE 2. *Let $\mathcal{V}$ be a Lie algebra with a resolution as in Definition 3 and suppose $F \subset X$ is a boundary face of codimension 2 or greater. Then the Lie algebra $\mathcal{J}(F)\mathcal{V}$, formed by the span of the products of elements of $\mathcal{V}$ and smooth function vanishing on F, has a resolution when lifted to $[X; F]$.*

(A)=b So, we start with a compact manifold with boundary X. The basic Lie algebra is $\mathcal{V}_{\mathrm{b}}(X)$ itself. In boundary-adapted coordinates (which we always use) x and y_j it is spanned locally by

$$x\partial_x \text{ and } \partial_{y_j},\ j = 1, \dots, n-1,\ n = \dim X. \tag{2.27}$$

It is resolved, as mentioned above, by blowing up the corner

$$X^2_{\mathrm{b}} = [X^2; (\partial X)^2] \text{ resolves } \mathcal{V}_{\mathrm{b}}(X) \tag{2.28}$$

if ∂X is connected. If there are several components, $\partial X = \cup_j H_j$, of the boundary then there are different possible resolutions. The usual choice is just to take the products of the components of the boundary and consider

$$X^2_{\mathrm{b}} = [X^2; H_1 \times H_1; H_2 \times H_2; \dots H_N \times H_N] \text{ resolves } \mathcal{V}_{\mathrm{b}}(X). \tag{2.29}$$

One can also consider *all* the products between different boundary components. These products are disjoint in X^2 so the blow-up is independent of order

$$X^2_{\mathrm{ob}} = [X^2; \mathcal{M}_1(X) \times \mathcal{M}_1(X)] \text{ resolves } \mathcal{V}_{\mathrm{b}}(X). \tag{2.30}$$

Here $\mathcal{M}_1(X)$ is the collection of boundary components; this is sometimes called the 'overblown' resolution.

The triple resolution associated to (2.29) is

$$X^3_{\mathrm{b}} = [X^3; (\partial X)^3; X \times (\partial X)^2; \partial X \times X \times \partial X; (\partial X)^2 \times X]. \tag{2.31}$$

References:

(B)=0 The next simplest case is the 'zero Lie algebra', $\mathcal{V}_0(X)$. (Other names have been used, especially in relation to conformal compactification because 'zero' seems to be interpretated as perjorative!) This consists of the smooth vector fields on X (in the usual sense) which vanish (hence the 'zero') at ∂X. It is spanned by

$$x\partial_x,\ x\partial_{y_j}. \tag{2.32}$$

Then

$$X^2_0 = [X^2; \partial \operatorname{Diag}(X)] \text{ resolves } \mathcal{V}_0(X). \tag{2.33}$$

The associated triple resolution is analogous to (2.31)

$$X^3_0 = [X^3; \partial \operatorname{Diag}_3; X \times \partial \operatorname{Diag}; ..] \tag{2.34}$$

where Diag_3 is the triple diagonal and the dots are the other boundaries of the other two partial diagonals – the images of the first one under the factor exchange maps.

References:

(C)=ϕ-b More generally, and this is a construction we will apply several times below, we can consider a fibration of the boundary $\phi : \partial X \longrightarrow B$. Then the fibred-boundary, also called 'edge' Lie algebra is

$$\mathcal{V}_{\phi\text{-b}}(X) = \{V \in \mathcal{V}_{\mathrm{b}}(X); V \text{ is tangent to the fibres of } \phi\}. \tag{2.35}$$

We can now choose 'boundary coordinates' which are divided into two groups, z_l which are lifted from the base and y_j which induce coordinates on the fibres of ϕ. In terms of these the Lie algebra is spanned by

$$x\partial_x,\ x\partial_{y_k},\ \partial_{z_l} \tag{2.36}$$

near the boundary. Within $\partial X \times \partial X$, the corner of X^2, consider the fibre diagonal Diag_ϕ of ϕ. This is the set of pairs projecting to the same point in B, i.e. lying in the same fibre of ϕ. Then

$$X^2_\phi = [X; \operatorname{Diag}_\phi] \text{ resolves } \mathcal{V}_{\phi\text{-b}}(X). \tag{2.37}$$

In fact this includes the previous two cases as the special fibrations with one fibre (giving $\mathcal{V}_{\mathrm{b}}$) and with point fibres (giving $\mathcal{V}_0(X)$). The triple resolution is given by the natural generalization

$$X^3_\phi = [X^3; \operatorname{Diag}^3_\phi; \pi_F^{-1}(\operatorname{Diag}_\phi); \pi_S^{-1}(\operatorname{Diag}_\phi); \pi_C^{-1}(\operatorname{Diag}_\phi)]. \tag{2.38}$$

References:-

(D)=cu The next basic case is the cusp algebra $\mathcal{V}_{\mathrm{cu}}(X)$. This is actually not well-defined but depends on the choice of some additional data. Namely one should fix a defining function for the boundary $x \in \mathcal{C}^\infty(X)$ up to a (positive) constant multiple and additional term $O(x^2)$. Geometrically this corresponds to an isomorphism of $N^*\partial X$ to $\partial X \times L$ for some real 1-dimensional vector space L. Different choices give different algebras but they are identified by appropriate diffeomorphisms. Given the choice of defining function the cusp algebra

$$\mathcal{V}_{\mathrm{cu}}(X) = \{V \in \mathcal{V}_{\mathrm{b}}(X); Vx \in x^2\mathcal{C}^\infty(X)\} \tag{2.39}$$

is locally spanned by

$$x^2\partial_x,\ \partial_{y_j}. \tag{2.40}$$

Then

$$X^2_{\mathrm{cu}} = [X^2; \partial X \times \partial X; S] \text{ resolves } \mathcal{V}_{\mathrm{cu}}(X) \tag{2.41}$$

where the first blow up gives X^2_{b} and $S \subset \mathrm{ff}(X^2_{\mathrm{b}})$ is a p-submanifold which can be defined as the flow-out of the lifted diagonal under the lift of elements of the cusp algebra. More usefully it can be written down as $s = 0$ where $s = (x - x')/(x + x')$ is a smooth function on X^2_{b} obtained from the given defining function x on the left factor of X and x' on the right.

The triple resolution is now getting a little harder! We can start from X^3_{b} in (2.31). Then we need to consider the three lifts of S from the three copies of X^2_{b} and the corresponding triple submanifold T. The complexity comes from the fact that the inverse image of S under each of the stretched projections consists of *two* p-submanifolds, one in the 'front face' of X^3_{b} (formed by the blow up of $(\partial X)^3$) and the other in the face coming from the corresponding corner of codemsion two. Thus we have seven p-submanifolds to blow up T, three S^{ff}_i's and three S_i's and this is the order we need to use or we do not get a triple resolution

$$X^3_{\mathrm{cu}} = [X^3_{\mathrm{b}}; T; S^{\mathrm{ff}}_i; S_i]. \tag{2.42}$$

Even a sketch of this is rather hard.

(E)=ϕ-cu The fibred-cusp algebras arising from a choice of cusp structure (actually less is neeeded, namely it is only needed 'along the fibres') and a fibration of the boundary ϕ as above:

$$\mathcal{V}_{\phi\text{-cu}}(X) = \{V \in \mathcal{V}_{\phi\text{-b}}(X); Vx \in x^2\mathcal{C}^\infty(X)\}. \tag{2.43}$$

It is locally spanned by

$$x^2\partial_x,\ x\partial_{z_l},\ \partial_{y_j} \tag{2.44}$$

where the coordinates z are lifted from the base of the fibration on the boundary; it is resolved by a similar blow up to the cusp case:

$$X^2_{\phi\text{-cu}} = [X^2_{\mathrm{b}}; S_{\phi\text{-cu}}] \text{ resolves } \mathcal{V}_{\phi\text{-cu}}(X) \tag{2.45}$$

and a similar triple resolution, which I will not write down.

(F)=sc This is an extreme case of the previous example, where the fibration has points as fibres. It is spanned by

$$x^2\partial_x,\ x\partial_{y_j}. \tag{2.46}$$

I only mention it because it is important in applications. Note that it also follows from case (A) above and Proposition 7 that it has a resolution.

(G)=I-ϕ One can also iterate fibrations. That is, if one has a tower of fibrations

$$\partial X \xrightarrow{\phi_1} Y_1 \xrightarrow{\phi_2} Y_2 \xrightarrow{\phi_3} Y_3 \qquad \cdots \xrightarrow{\phi_N} Y_N \tag{2.47}$$

then one can define a Lie algebra of vector fields with some higher jet information. Namly one can take a product decomposition of the manifold and extend the fibrations a little way out into the manifold so that the final base becomes $[0,\epsilon)_x \times Y_N$; denote these extended fibrations $\tilde{\phi}_j$. Then set

$$\begin{aligned}\mathcal{V}_{\text{I-}\phi} = \big\{V \in \mathcal{V}_{\text{b}}(X); V = V_1 + x^1 V_2 + x^2 V_3 + \cdots + x^{N-1} V_N + x^N V',\\ \text{where } V_j \text{ is tangent to the fibres of } \tilde{\phi}_j \text{ and } V' \in \mathcal{V}_{\text{b}}(X)\big\}.\end{aligned} \tag{2.48}$$

Of course there are many ways to do the extension and the Lie algebra will depend on some of this information. There is a resolution using iterated blow ups and indeed a triple resolution.

(I)=b-H In fact it is not necessary to have a fibration of the boundary to produce an interesting Lie algebra. Suppose we simply have a subbundle $H \subset T\partial X$. Let $\alpha_i \in \mathcal{C}^\infty(X;\Lambda^1)$ be smooth 1-forms which define a lift of H from the boundary, in the sense that their joint null spaces at the boundary form a subbundle $\tilde{H} \subset T_{\partial X}X$ which is of rank one greater than H and for which $\tilde{H} \cap T\partial X = H$. Then we can set

$$\mathcal{V}_{\text{b-H}}(X) = \{V \in \mathcal{V}_0(X); \alpha_i(V) \in x^2\mathcal{C}^\infty(X)\}. \tag{2.49}$$

This is a Lie algebra since

$$\alpha_i([V,W]) = V\alpha_i(W) - W\alpha_i(V) - d\alpha_i(V,W). \tag{2.50}$$

It is locally spanned by

$$x\partial_x,\ xV_l, x^2W_j \tag{2.51}$$

where the V_l resrict to the boundary to span H. Despite the notation the Lie algebra depends on more than H, rather on $\tilde{H}$. It has a resolution an a triple resolution.

(J)=ad The next example, the adiabatic algebra, is the first which does not satisfy the assumptions of Definition 3. It is fixed by a fibration, say of a compact manifold without boundary, $\phi : X \longrightarrow Y$ with typical fibre Z. The vector fields we are interested in are on X but depend on a parameter, ϵ. For $\epsilon > 0$ they are just arbitrary vector fields depending smoothly on ϵ but at $\epsilon = 0$ we demand that they become tangent to the fibres of ϕ. Now, we can regard the parameter dependent vector fields as smoooth vector fields on $\tilde{X} = X \times [0,1]_\epsilon$ which satisfy

$$\mathcal{V}_{\text{ad}}(X) = \{V \in \mathcal{V}_{\text{b}}(\tilde{X}); V\epsilon = 0,\ V \text{ tangent to the fibres of } \phi \text{ at } \epsilon = 0\}. \tag{2.52}$$

In coordinates adapted to the fibration $\mathcal{V}_{\text{ad}}$ is spanned by

$$\partial_{z_i},\ \epsilon\partial_{y_j}. \tag{2.53}$$

Notice that the full space here is X_{ad}, not X, so $\mathcal{V}_{\text{ad}}(X_{\text{ad}})$ does not restrict to arbitrary vector fields in the interior – since there is no ϵ-derivative. Thus Definition 3 does not apply directly. Nevertheless there is a resolution in an essentially similar sense. The point however is that we do not need more than one 'copy' of the ϵ parameter, since it is a parameter. The rôle of the diagonal is played by the fibre diagonal in ϵ. Thus the resolved space is

$$X^2_{\text{ad}} = [X^2 \times [0,1]; \text{Diag}(\phi) \times \{0\}] \tag{2.54}$$

where $\text{Diag}(\phi)$ is the fibre diagonal of ϕ. There are two maps back to the single space X_{ad} and all are b-fibrations with $\mathcal{V}_{\text{ad}}(X)$ lifting under each of them to be smooth and collectively transversal to the lifted fibre diagonal $\text{Diag}(X) \times [0,1]$. The triple space follows the pattern that can be seen from the examples above.

References:-

(K)=b-cu Next consider an example of a 'transition algebra.' One such is the transition from $\mathcal{V}_{\text{b}}(X)$ to $\mathcal{V}_{\text{cu}}(X)$ for some compact manifold with boundary as a parameter ϵ approaches 0. Given the local bases (2.27) and (2.44) of these two algebras, the 'obvious' transtion basis is

$$(x^2+\epsilon^2)^{\frac12} x\partial_x,\ \partial_{y_j}. \tag{2.55}$$

The first vector field is not smooth. It can be replaced by $(x+\epsilon)x\partial_x$ but this does not really mitigate the 'lack of smoothness'. So the single space itself needs to be resolved

$$X_{\text{b-cu}} = [X \times [0,1]; \partial \times \{0\}] \tag{2.56}$$

to which the vector fields in $\mathcal{V}_{\text{b-cu}}$ lift to be smooth. So in fact this is a setting rather similar to the preceeding one and with a similar resolution:

$$X^2_{\text{b-cu}} = [X^2\times[0,1]; (\partial X)^2\times\{0\}; (\partial X)^2; S_{\text{b-cu}}; \partial X\times X\times\{0\}; X\times\partial X\times\{0\}]. \tag{2.57}$$

Here $S_{\text{b-cu}}$ is a submanifold of the fact produced by the first blow up, of $(\partial X)^2 \times \{0\}$ which corresponds closely to S in the cusp case discussed above. There is a similar triple resolution.

References:-

(L)=cu-ϕ-cu Similar to the preceeding case again but now a transition from cusp to fibred cusp, so the local spanning vector fields are

$$x^2\partial_x,\ (x^2+\epsilon^2)^{\frac12}\partial_{y_j},\ \partial_{z_l} \tag{2.58}$$

corresponding to a fibration of the boundary of a compact manifold with boundary as in case (E) above.

References:-

There are other such 'transition algebras'.

(M)=b-f Now passing to rather more general cases, suppose X is a compact manifold with corners and $f : X \longrightarrow X'$ is a b-fibration. Consider the space of smooth vector fields tangent to the fibres of f :

$$\mathcal{V}_{\text{b-fi}}(X) = \{V \in \mathcal{V}_{\text{b}}(X); Vf^*u = 0\ \forall\ u \in \mathcal{C}^\infty(X')\}. \tag{2.59}$$

This has a resolution $X^2_{\text{b-f}}$ which is obtained from the fibre diagonal for f in X^2 by blow up in X^2; as usual there is a triple space.

Reference: None at the moment.

(N)=b-St The boundary stratification algebras. Consider an iterated conic space, a special type of stratified space. This is too hard to describe in a few sentences, but think of it as an iterted cone bundle. The space itself has a resolution to a compact manifold with corners X where the boundary hypersurfaces H_i are strictly order, corresponding to the 'depth' of the stratum. Thus H_1 corresponds to the smallest singular stratum and H_N to the largest. Each of these H_j's carries a fibration which 'remembers' the original stratum – thus its base is a resolution of the corresponding stratum. There are compatibility conditions for the strata at the corners – the leaves decrease as i increases. Here we are interested in the 'finite length' vector fields on the original manifold – which do not form a Lie algebra. To make them all smooth they are multiplied by a defining function for each boundary hypersurface of X. With arbitrary smooth coefficients the resulting vector fields on X form a Lie algebra which is an iterated version of the fibred boundary Lie algebra in (C). Near a point in the interior of H_i the Lie algebra reduces to the ϕ-b case. Near a corner, say of codimension 3, where the first three boundary hypersurfaces, H_1, H_2 and H_3 meet, the algebra is spanned by

$$x_1x_2x_3\partial_{x_1},\ x_2x_3\partial_{x_3},\ x_3\partial_{x_3},\ x_1x_2x_3\partial_{y'_{l'}}, x_2x_3\partial_{y''_{l''}},\ x_3\partial_{y'''_{l'''}},\ pa_{z_l} \tag{2.60}$$

where the tangential vector fields correspond to tangency to the different fibrations. Again this has a resolution, which I will not disucss.

(O)=m-sc There are verious Lie algebra which correspond to the cases discussed above for a compact manifold with boundary but on a manifold with corners with no ordering of the boundary faces. For instance the basic Lie algebra $\mathcal{V}_{\mathrm{b}}(X)$ on a manifold with corners, which is a special case of (M) where the b-fibration is the map to a point can be scaled as described in (7) by multiplying by a boundary defining function for each boundary hypersurface. This gives the multi-scattering algebra which is locally spanned in adapted coordinates by

$$\rho_{\mathrm{tot}}x_1\partial x_1,\ \rho_{\mathrm{tot}}x_2\partial_{x_2},\ \rho_{\mathrm{tot}}\partial_{y_l}\rho_{\mathrm{tot}} = x_1\cdots x_n \tag{2.61}$$

near a corner of codimension two. The appropriate single space is X_{tot} obtained from X by blowing up all the boundary faces in order increasing with the dimension. There is a double and a triple resolution with which I will not bother you!

There are lots of other examples too. Some worked out in detail some not (yet).

8. Morse case again

9. b-calculus

10. Duality and distributions

11. Pull-back and push-forward

12. Smoothness under blow-up

13. Conormal distributions

14. Examples

15. More theorems!

LECTURE 3

Resolutions and compactifications

Bibliography

[1] C.L. Epstein and R. B. Melrose, *The Heisenberg algebra, index theory and homology*, In preparation.