HEAT TRACE EXPANSIONS AND WEYL'S LAW ON THE ASYMPTOTICS OF EIGENVALUES

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ABSTRACT. We construct the generalized 'heat operator' $\exp(-tP)$ and the complex powers P^s, $s \in \mathbb{C}$, associated with certain elliptic pseudodifferential operators P on a closed n-dimensional manifold X.

We show that P^s is a pseudodifferential operator of order $\mu \operatorname{Re} s$, where μ is the order of P. The trace $\operatorname{Tr}(P^s)$ exists whenever $\operatorname{Re} s$ is sufficiently negative. It defines a holomorphic function on a half plane in $\mathbb{C}$ which then extends meromorphically to all of $\mathbb{C}$ with at most simple poles. In a similar vein we prove that $\exp(-tP)$ is a trace class operator and that $\operatorname{Tr}(\exp(-tP))$ has an asymptotic expansion in powers of t and log-terms as $t \to 0^+$. From this we derive Weyl's law on the asymptotics of the eigenvalues of P.

CONTENTS

Introduction

In his famous article 'Can one hear the shape of a drum' [12] Mark Kac introduced the following problem.

Consider a two-dimensional membrane, represented by a bounded domain in the plane with sufficiently smooth boundary. If the membrane is fixed at the boundary and set in motion by a drumstick, then its displacement U in the direction perpendicular to the plane will satisfy the wave equation

$$\partial_t^2 U - c^2 \Delta U = 0, \quad U|_{\partial\Omega} = 0.$$

Here c is a constant depending on the material. Without loss of generality let $c = 1$. One is particularly interested in time harmonic solutions ('standing waves') of the form

$$U(t, x) = u(x)e^{i\omega t}$$

for some function u on Ω and some $\omega \in \mathbb{R}$. They represent the pure tones the membrane is capable of producing.

Substituting U in the wave equation, we find that u satisfies the equation

$$\omega^2 u + \Delta u = 0, \quad u|_{\partial\Omega} = 0.$$

In other words: $\lambda = -\omega^2$ is an eigenvalue of the Dirichlet problem and u an eigenfunction. We know (or otherwise will see a corresponding statement later in these talks) that the eigenvalues of the Dirichlet problem form a sequence $0 > \lambda_1 \geq \lambda_2 \geq \lambda_3 \ldots$ going to $-\infty$.

The question Mark Kac asked is whether it is possible to determine Ω from the sequence of eigenvalues (λ_k) including multiplicities, i.e. to 'hear the shape of the drum'.

We know today, that this is not possible, at least if we admit domains bounded by piecewise smooth curves, see the article by Gordon, Webb and Wolpert [4]. It is, however, possible, to extract a lot of information about Ω from the sequence (λ_k). One of the basic results is that the sequence of eigenvalues of the Dirichlet problem determines the volume of Ω. This is not limited to two dimensions but works for domains in $\mathbb{R}^n$.

Mark Kac traces the problem of 'hearing the shape of a drum' back to a question posed by the Dutch physicist H.A. Lorentz on the occasion of a lecture series which took place here in Göttingen. Let me quote Kac:

> Lorentz gave five lectures under the overall title "Alte und neue Fragen der Physik" – Old and new problems of physics – and at the end of the fourth lecture he spoke as follows (in free translation from the original German): "In conclusion, there is a mathematical problem which perhaps will arouse the interest of mathematicians who are present. It originates in the radiation theory of Jeans.
>
> "In an enclosure with a perfectly reflecting surface there can form standing electromagnetic waves analogous to the tones of an organ pipe; we shall confine our attention to very high overtones. Jeans asks for the energy in the frequency interval $d\nu$. To this end he calculates the number of overtones which lie between the frequencies ν and $\nu + d\nu$ and multiplies this number by the energy which belongs to the frequency ν and which, according to a theorem of statistical mechanics, is the same for all frequencies.
>
> "It is here that arises the mathematical problem to prove that the number of sufficiently high overtones which lies between ν and $\nu + d\nu$ is independent of the shape of the enclosure and is simply proportional to its volume. For many simple shapes for which

calculations can be carried out, this theorem has been verified in a Leiden dissertation. There is no doubt that it holds in general even for multiply connected regions. Similar theorems for other vibrating structures like membranes, air masses, etc. should also hold."

If one expresses this conjecture of Lorentz in terms of our membrane, it emerges in the form[1]

$$N(\lambda) = \sum_{\lambda_k < \lambda} 1 \sim \frac{\operatorname{vol}\Omega}{2\pi}\lambda.$$

Here $N(\lambda)$ is the number of eigenvalues less than λ, $\operatorname{vol}\Omega$ the area of Ω and '$\sim$' means that

$$\lim_{\lambda\to\infty} \frac{N(\lambda)}{\lambda} = \frac{\operatorname{vol}\Omega}{2\pi}.$$

The function $N(\lambda)$ is known as the 'counting function' for the eigenvalues. According to Kac, Hermann Weyl got interested in the question and solved the problem [23]. In these notes, we will consider more generally certain elliptic pseudodifferential operators P of order μ on closed manifolds and show that their counting functions satisfy a corresponding law.

Using pseudodifferential techniques, will first construct parameter-dependent resolvents of these operators. In a second step, we will study the complex powers and then analyze the behavior of $\operatorname{Tr} P^s$. A similar analysis can be applied to define the generalized 'heat' operator e^{-tP} and to study its trace. Under suitable assumptions on P we shall show for the complex powers:

Theorem 0.1. *P^s is a pseudodifferential operator of order $\mu \operatorname{Re} s$. It is of trace class if $\operatorname{Re} s < -n/\mu$. For these values of s, the zeta function $\zeta_P(s) = \operatorname{Tr} P^s$ defines a holomorphic function.*

Moreover ζ_P extends to a meromorphic function on $\mathbb{C}$ with at most simple poles in the points $s_j = (j-n)/\mu$. The residue in s_j is explicitly computable. The residue in $s = 0$ vanishes, hence 0 is a regular point. If P is a differential operator, then the residues vanish whenever s_j is an integer.

For the heat operator we find:

Theorem 0.2. *The operator e^{-tP}, $t > 0$, is a regularizing pseudodifferential operator and therefore of trace class. The trace $\operatorname{Tr} e^{-tP}$ has an asymptotic expansion as $t \to 0^+$ of the form*

$$\operatorname{Tr} e^{-tP} \sim \sum_{j\in\mathbb{N}_0, j-n\notin\mathbb{N}} c_j t^{(j-n)/\mu} + \sum_{j\in\mathbb{N}_0, j-n\in\mathbb{N}} c_j' t^{(j-n)/\mu} \ln t + \sum_{k\in\mathbb{N}} c_j'' t^k.$$

If P is additionally assumed to be self-adjoint and positive, Weyl's law then follows from a Tauberian theorem. We have

Theorem 0.3. *$N(t) \sim c_{p_\mu} t^{n/\mu}$ with a coefficient explicitly computable from the principal symbol p_μ of P.*

These results are not new; the crucial techniques were developed by Seeley [20] and refined in the 80's. In the text and in the notes at the end of the sections I will give references to more modern developments.

[1] Note that Kac works here with $\frac{1}{2}\Delta$

1. Pseudodifferential Operators

Pseudodifferential operators are an indispensable tool in modern analysis. Understanding the basics of this theory is worthwhile in many respects; I therefore include a short presentation. Pseudodifferential operators originated from the study of singular integral equations. Probably the first paper, where a complete calculus was developed, is Kohn and Nirenberg's [13]. Good sources to read more are the books by Hörmander [10], Kumano-go [14], Shubin [18], and Taylor [21].

1.1. **Symbols.** An important notion in connection with pseudodifferential operators is the function $\mathbb{R}^n \ni x \mapsto \langle x \rangle = (1+|x|^2)^{1/2} \in \mathbb{R}_+$.

Definition 1.1. (a) **Symbol classes.** We let $S^\mu = S^\mu(\mathbb{R}^n \times \mathbb{R}^n)$ denote the space of all smooth functions $p = p(x,\xi)$ on $\mathbb{R}^n \times \mathbb{R}^n$ satisfying the estimates

$$|D_\xi^\alpha D_x^\beta p(x,\xi)| \le C_{\alpha\beta} \langle \xi \rangle^{\mu - |\alpha|}.$$

We call these elements symbols of order μ. The estimates furnish a Fréchet topology for S^μ.

We write $S^{-\infty} = \bigcap_\mu S^\mu$. Elements in this space are often referred to as regularizing or smoothing.

(b) **Asymptotic expansion.** A symbol $p \in S^\mu$ has the asymptotic expansion $p \sim \sum_{j=0}^\infty p_{\mu-j}$ with symbols $p_{\mu-j} \in S^{\mu-j}$ if, for each N, we have

$$p - \sum_{j=0}^N p_{\mu-j} \in S^{\mu-N}$$

(c) **Classical symbols.** The symbol p is classical, if it has an expansion $p \sim \sum p_{\mu-j}$ with $p_{\mu-j} \in S^{\mu-j}$ being positively homogeneous of degree $\mu - j$ in ξ for all $|\xi| \ge 1$, i.e.

$$p_{\mu-j}(x,t\xi) = t^{\mu-j} p(x,\xi), \;\; x \in \mathbb{R}^n, \xi \in \mathbb{R}^n, |\xi| \ge 1, t \ge 1.$$

(d) **Ellipticity.** The symbol $p \in S^\mu$ is elliptic of order μ, if there exists an $R \ge 0$ such that $p(x,\xi)$ is invertible for all (x,ξ) with $|\xi| \ge R$, and

$$|p(x,\xi)^{-1}| \le c\langle \xi \rangle^{-\mu}.$$

For a classical symbol, this condition simplifies to the requirement that $p_\mu(x,\xi)$ is invertible for $x \in \mathbb{R}^n, |\xi| = 1$.

Theorem 1.2. *Given a sequence $(p_j)_{j=0}^\infty$ with $p_j \in S^{\mu-j}$, there exists a symbol $p \in S^\mu$ such that $p \sim \sum p_j$.*

Remark 1.3. The function p in Definition 1.1 need not be scalar, it could take values in matrices of arbitrary size; this is actually important in order to accommodate the case of systems of operators or operators acting on sections of vector bundles over a manifold. Of course, a symbol can only be elliptic, if it takes values in square matrices.

1.2. **Sobolev spaces.** We denote by $\mathscr{S} = \mathscr{S}(\mathbb{R}^n)$ the space of rapidly decreasing functions on $\mathbb{R}^n$ and by $\mathscr{S}' = \mathscr{S}'(\mathbb{R}^n)$ its dual space, the space of tempered distributions.

Definition 1.4. The Fourier transform of $u \in \mathscr{S}$ is the function $\mathscr{F}u$ or $\hat{u}$ on $\mathbb{R}^n$ defined by

$$\mathscr{F}u(\xi) = \hat{u}(\xi) = \int e^{ix\xi} u(x) \, đx$$

with $đx = (2\pi)^{-n/2} dx$.

Definition 1.5. By $H^s(\mathbb{R}^n)$ we denote the L^2-based Sobolev space on $\mathbb{R}^n$. It is the space of all tempered distribution u for which the Fourier transform $\mathscr{F}u$ is a regular function and $\langle\xi\rangle^s\mathscr{F}u \in L^2$. We endow it with the norm

$$\|u\|^2_{H^s} = \int \langle\xi\rangle^{2s}|\mathscr{F}u|^2(\xi)\, đ\xi.$$

Example 1.6. For $y \in \mathbb{R}^n$ the delta distribution $\delta_y : \mathscr{S} \ni \varphi \mapsto \varphi(y) \in \mathbb{C}$ is an element of $H^s(\mathbb{R}^n)$ whenever $s < -n/2$, since its Fourier transform is the constant function $(2\pi)^{-n/2}e^{-iy\xi}$.

Theorem 1.7. (Sobolev embedding theorem) *Let $s > n/2$. Then $H^s(\mathbb{R}^n)$ consists of continuous functions.*

Recall that we call an operator K acting on functions over a space M an integral operator with kernel $k = k(x, y)$, where k is a function on $M \times M$, if

$$Ku(x) = \int_M k(x, y)u(y)\, dy.$$

Proposition 1.8. *Let $s > n/2$ and A be a bounded linear operator $A : H^{-s} \to H^s$. Then A is an integral operator with the continuous kernel $k_A(x, y) = \langle A\delta_y, \delta_x\rangle$.*

If even $A : H^{-s-k} \to H^{s+k}$ is continuous, then the kernel is C^k.

The pairing $\langle A\delta_y, \delta_x\rangle$ makes sense in view of the mapping properties of A. It is not difficult to check that it furnishes the kernel. For the second statement note that $D^\alpha_x D^\beta_y k_A(x, y)$ is the kernel of $(-1)^{|\beta|}D^\alpha A D^\beta$.

It is sometimes useful to consider also weighted Sobolev spaces.

Definition 1.9. For $s_1, s_2 \in \mathbb{R}$ let $H^{s_1,s_2}(\mathbb{R}^n) = \langle x\rangle^{-s_2}H^{s_1}(\mathbb{R}^n)$.

Theorem 1.10. (a) *It is clear that $H^0 = H^{0,0} = L^2$ and that $H^{s_1,s_2} \subseteq H^{t_1,t_2}$ whenever $s_1 \geq t_1$ and $s_2 \geq t_2$.*

(b) *The imbedding $H^{s_1,s_2} \hookrightarrow H^{t_1,t_2}$ is compact whenever $s_1 > t_1$ and $s_2 > t_2$. This is a special case of Rellich's theorem.*

(c) *The imbedding $H^{s_1,s_2} \hookrightarrow H^{t_1,t_2}$ is trace class whenever $s_1 - t_1 > n$ and $s_2 - t_2 > n$.*

1.3. Operators.

Definition 1.11. To a symbol $p \in S^\mu$ we associate the pseudodifferential operator $\operatorname{op} p$ by

$$(\operatorname{op} p)u(x) = \int e^{ix\xi}p(x, \xi)\hat{u}(\xi)\, đ\xi,\ u \in \mathscr{S}(\mathbb{R}^n), x \in \mathbb{R}^n. \tag{1.1}$$

Here, $\mathscr{S}(\mathbb{R}^n)$ is the Schwartz space of rapidly decreasing functions on $\mathbb{R}^n$, $đ\xi = (2\pi)^{-n/2}d\xi$, and $\hat{u}$ is the Fourier transform of u:

$$\hat{u}(\xi) = \int e^{-ix\xi}u(x)\, đx.$$

Theorem 1.12. *Let $p \in S^\mu$.*

(a) $\operatorname{op} p : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$ *is continuous.*

(b) *For each $s \in \mathbb{R}$,*

$$\operatorname{op} p : H^s(\mathbb{R}^n) \to H^{s-\mu}(\mathbb{R}^n)$$

is bounded. The operator norm can be estimated by finitely many symbol semi-norms.

A simple proof for the continuity of zero order operators in L^p-spaces (which implies (b)) can be found in Hwang's paper [11]. Simple considerations show that continuity extends to many weighted Sobolev spaces [17]

Theorem 1.13. (a) *Let $p \in S^\mu$ and $q \in S^\nu$. Then there is an element $r \in S^{\mu+\nu}$ such that*

$$\operatorname{op} p \circ \operatorname{op} q = \operatorname{op} r.$$

The element r has the asymptotic expansion

$$r(x,\xi) \sim \sum_\alpha \frac{1}{\alpha!} \partial_\xi^\alpha p(x,\xi) D_x^\alpha q(x,\xi).$$

One often writes $r = p\#q$ and calls r the Leibniz product of p and q. The corresponding map $S^\mu \times S^\nu \to S^{\mu+\nu}, (p,q) \mapsto r$ is continuous.

(b) *Let $p \in S^\mu$. Then the formal adjoint $(\operatorname{op} p)^*$ of $\operatorname{op} p$ is of the form $\operatorname{op} q$ for some $q \in S^\mu$. It has the asymptotic expansion*

$$q(x,\xi) \sim \sum \frac{1}{\alpha!} \partial_\xi^\alpha D_x^\alpha \overline{p(x,\xi)}.$$

The corresponding map $S^\mu \to S^\mu$ is continuous.

Theorem 1.14. *If $p \in S^\mu$ is elliptic, then there is a symbol q such that*

$$p\#q - 1 = r_1 \text{ and } q\#p - 1 =: r_2$$

are elements of $S^{-\infty}$. The symbol q is called a parametrix to p.

Proof. Here is the idea: One starts with a symbol $q_0 \in S^{-\mu}$, coinciding with $p(x,\xi)^{-1}$ for $|\xi| \geq R+1$; it can be taken of the form $\chi(\xi)p(x,\xi)^{-1}$ for a zero excision function χ. Then $p\#q_0 = 1 - s_1$ for an element s_1 of order -1. Iteration ($q_1 = q_0\#s_1$, $q_2 = q_0\#s_1\#s_1$ then gives a sequence of elements $q_j \in S^{\mu-j}$ such that

$$p\# \sum_0^N q_j - 1 \in S^{\mu-N}.$$

Asymptotic summation, cf. Theorem 1.15, below, then gives an element q such that $p\#q - 1$ is in $S^{-\infty}$. Similarly one can construct a q' such that $q'\#p - 1 \in S^{-\infty}$. It is then easy to see that also $q - q' \in S^{-\infty}$. ☐

Theorem 1.15. *Let $\mu \in \mathbb{R}$ and $q_j \in S^{\mu-j}$, $j \in \mathbb{N}_0$. Then there exists a symbol $q \in S^\mu$ with $q \sim \sum q_j$.*

1.4. **Pseudodifferential operators on manifolds.** Let X be a closed (i.e. compact, without boundary) manifold. By $H^s(X)$ we denote the space of all distributions on X which, in local coordinates on a patch $U \subset \mathbb{R}^n$, belong to $H^s(\mathbb{R}^n)$ after multiplication by a function in $C_c^\infty(U)$.

From Theorem 1.10 we immediately obtain:

Theorem 1.16. (a) *It is clear that $H^0(X) = L^2(X)$, and that $H^s(X) \subseteq H^t(X)$ whenever $s \geq t$.*

(b) *The imbedding $H^s(X) \hookrightarrow H^t(X)$ is compact whenever $s > t$.*

(c) *The imbedding $H^s \hookrightarrow H^t$ is trace class whenever $s - t > n$.*

Definition 1.17. An operator $P : C^\infty(X) \to C^\infty(X)$ is called a pseudodifferential operator of order μ on X, provided that for all smooth functions ϕ, ψ with support in a single coordinate neighborhood with coordinate chart κ, the pullback of $\phi P \psi$ under κ is a pseudodifferential operator on $\mathbb{R}^n$ with symbol in S^μ.

We call P classical, if these local symbols are all classical. We write $\Psi^\mu(X)$ for the space of all pseudodifferential operators of order μ on X. By $\Psi^\mu_{cl}(X)$ we denote the subspace of classical operators.

Let ϕ, ψ be $\equiv 1$ in a neighborhood of a point $x \in X$. We call P elliptic near x, if any symbol of the pullback of $\phi P \psi$ under κ satisfies the ellipticity condition in

1.1(d) near the pre-image of x under κ. We call P elliptic, if it is elliptic near every $x \in X$.

Remark 1.18. (a) If P is classical, then one can associate to P a homogeneous principal symbol $\sigma_\psi(P)$. It is a function on $T^*X \setminus \{0\}$, the cotangent bundle with the zero section removed, homogeneous of degree μ. It can be obtained from the principal symbols of the localized symbols as described in the last part of Definition 1.17. They are first defined for $|\xi|$ large and can then be extended by homogeneity to $\xi \neq 0$.

(b) Pseudodifferential operators acting on sections of vector bundles can be defined in an analogous way. They are locally given by matrices of symbols. They are called classical, if all entries are classical symbols. The principal symbol of a classical pseudodifferential operator $P : C^\infty(X; E_1) \to C^\infty(X, E_2)$ of order μ then is an endomorphism $p_\mu : \pi^* E_1 \to \pi^* E_2$, where π^* denotes the pull-back of vector bundles (see e.g. [2]) under the projection $\pi : T^*X \setminus 0 \to X$.

Definition 1.19. A pseudodifferential operator P is called regularizing or smoothing, provided it can be written with local symbols in $S^{-\infty}$.

Lemma 1.20. *A pseudodifferential operator P is regularizing if and only if it can be written as an integral operator with a C^∞-kernel.*

Proof. It follows from Proposition 1.8 that P then has a smooth kernel. In fact, it follows form (1.1) that, in local coordinates, we can write

$$k_P(x, y) = (2\pi)^{-n} \int e^{i(x-y)\xi} p(x, \xi)\, d\xi. \tag{1.2}$$

Conversely, a corresponding formula shows that every integral operator with smooth kernel has a symbol in $S^{-\infty}$. □

Lemma 1.21. *Let P be a pseudodifferential operator and $\varphi, \psi \in C^\infty(X)$ have disjoint supports. Then $\varphi P \psi$ is regularizing.*

Proof. If in Equation 1.2 $p(x, \xi)$ is replaced by $\varphi(x)p(x, \xi)\psi(y)$, then the integral can also be given sense for arbitrary $p \in S^\mu$ via integration by parts ans then furnishes the desired kernel. □

This property of pseudodifferential operators is called pseudolocality.

Theorem 1.22. *Let P be a classical pseudodifferential operator of order μ on the closed manifold.*

(a) *For all $s \in \mathbb{R}$, the operator P extends to a bounded operator*

$$P : H^s(X) \to H^{s-\mu}(X). \tag{1.3}$$

(b) *If P in (1.3) is a Fredholm operator for one choice of s, then it is a Fredholm operator for every choice of s, and there exists a Fredholm inverse which is a pseudodifferential operator of order $-\mu$. This is the case precisely if the principal symbol of P is invertible for all $(x, \xi) \in T^*X \setminus \{0\}$, i.e. if P is elliptic.*

(c) *If P in (1.3) is invertible for one choice of s, then it is invertible for every choice of s. The inverse is a pseudodifferential operator of order $-\mu$.*

Remark 1.23. The key to proving Theorem 1.22(b) is a local parametrix construction. The local parametrices can be patched to define a pseudodifferential operator Q of order $-\mu$ such that

$$PQ - I =: R_1 \text{ and } QP - I =: R_2$$

are operators of order $-\infty$.

Statement (c) follows from (b) together with the observation that the kernel and the cokernel of an elliptic pseudodifferential operator consist of smooth functions and therefore do not depend on s.

Remark 1.24. The pseudodifferential operators of order zero actually form a Fréchet subalgebra of $\mathcal{L}(L^2(X))$ and more generally of $\mathcal{L}(H^s)$ for every $s \in \mathbb{R}$. The fact that this algebra contains its inverses whenever they exist is often called 'spectral invariance' and has many interesting consequences, see Gramsch [5]. It extends to many classes of weighted spaces on $\mathbb{R}^n$, see [17].

Theorem 1.25. *Let P be elliptic, $f \in H^s(X)$ and let $u \in H^{-N}(X)$ (for some $N \in \mathbb{R}$) be a solution of the equation $Pu = f$. Then $u \in H^{s+\mu}(X)$.*

Proof. Apply a parametrix Q to P to the equation $Pu = f$. Using the notation of Remark 1.23, $(I+R_2)u = Qu \in H^{s+\mu}$. Since $R_2 u \in C^\infty$, we obtain the assertion. □

Remark 1.26. The property in Theorem 1.25 is called elliptic regularity.

Theorem 1.27. *Let P be a pseudodifferential operator of order $< -n = -\dim X$. Then P is an integral operator with a continuous kernel function k_P given by Equation* (1.2)*. Moreover, P is a trace class operator and*

$$\operatorname{Tr} P = \int_X k_P(x,x)\, dx.$$

2. Complex Powers and 'Heat' Operators

Let P be an unbounded operator on a Hilbert space H.

Definition 2.1. We say that the ray $R_\theta = \{re^{i\theta} \in \mathbb{C} : r \geq 0\}$ is a *ray of minimal growth* for P, provided R_θ does not intersect the spectrum of P and there exists a constant $c \geq 0$ with

$$\|(P-\lambda)^{-1}\| \leq c\langle\lambda\rangle^{-1}.$$

2.1. **Complex powers.** Complex powers of pseudodifferential operators were first studied in a by now classical paper by Seeley [20]. Let us first recall the definition, which works in a more general context.

Let R_θ be a ray of minimal growth for the operator P. In particular, zero then is not in the spectrum of P and hence there is a $\delta_0 > 0$ such that $B(0, 2\delta_0)$ is contained in the resolvent set. For $\operatorname{Re} s < 0$ define P_s by

$$P_s = \frac{i}{2\pi}\int_{\mathscr{C}} \lambda^s (P-\lambda)^{-1}\, d\lambda. \tag{2.4}$$

Here $\mathscr{C}$ is the contour in $\mathbb{C}$ from ∞ to $\delta_0 e^{i\theta}$ along R_θ, clockwise about the circle $\{|z| = \delta_0\}$ to $\delta_0 e^{i\theta}$ and back to ∞ along R_θ. The integral converges, since $|\lambda^s| \leq c_s|\lambda|^s$.

A crucial point is that on the incoming ray the argument of λ is considered to be θ, while on the outgoing ray, it is $\theta - 2\pi$. Hence the pieces along the ray do *not* cancel unless s is a negative integer.

Remark 2.2. Expressions of the form

$$f(P) = \frac{i}{2\pi}\int_{\mathscr{C}} f(\lambda)(P-\lambda)^{-1}\, d\lambda \tag{2.5}$$

with a contour $\mathscr{C}$ which 'surrounds' the spectrum of P and a function f which is holomorphic on the spectrum of P are called *Dunford integrals* for $f(P)$.

The underlying idea is Cauchy's theorem in complex analysis: For a holomorphic function on a simply connected domain and a contour $\mathscr{C}$ which simply surrounds z

$$f(z) = \frac{1}{2\pi i}\int_{\mathscr{C}} \frac{f(w)}{w-z}\, dw.$$

Note that the shift in the sign is simply due to the fact that we consider $(P-\lambda)^{-1}$ instead of $(\lambda - P)^{-1}$.

The notation P^s is justified by the following theorem:

Theorem 2.3. *Let $s, t \in \mathbb{C}$ with negative real parts.*

(a) *$s \mapsto P_s$ is an analytic family of bounded operators*

(b) *$P_s P_t = P_{s+t}$*

(c) *$P_{-1} = P^{-1}$ is the inverse to P.*

Proof. (a) follows by differentiating under the integral sign. For (b) let $\mathscr{C}'$ be a contour which lies inside $\mathscr{C}$ and close to $\mathscr{C}$. By Cauchy's theorem we can then replace $\mathscr{C}$ by $\mathscr{C}'$. Then

$$\begin{aligned}
P_s P_t &= -\frac{1}{4\pi^2}\int_{\mathscr{C}'}\left(\int_{\mathscr{C}} (P-\lambda)^{-1}(P-\mu)^{-1}\lambda^s\mu^t\, d\mu\right) d\lambda \\
&= \frac{1}{4\pi^2}\int_{\mathscr{C}'}\int_{\mathscr{C}} \frac{\lambda^s\mu^t}{\lambda-\mu}\left((P-\lambda)^{-1} - (P-\mu)^{-1}\right) d\mu d\lambda \\
&= \frac{i}{2\pi}\int_{\mathscr{C}'} \lambda^{s+t}(P-\lambda)^{-1}\, d\lambda + \frac{1}{4\pi^2}\int_{\mathscr{C}} (P-\lambda)^{-1}\int_{\mathscr{C}'} \frac{\lambda^s\mu^t}{\lambda-\mu} d\lambda d\mu,
\end{aligned}$$

where Fubini's theorem has been applied. The last integral vanishes, since μ lies outside of $\mathscr{C}'$.

(c) For a negative integer, the integration contour reduces to the circle of radius δ_0 surrounded counterclockwise. Denote by C the opposite contour. We can then write the expression (2.4) for P_{-1} as

$$\begin{aligned}&\frac{1}{2\pi i}\int_C \lambda^{-1}(P-\lambda)^{-1}\,d\lambda\\ &\quad= \frac{1}{2\pi i}\int_C \lambda^{-1}\lambda^{-1}P^{-1}(P^{-1}-\lambda^{-1})^{-1}d\lambda\\ &\quad= -\frac{1}{2\pi i}\int_C (P^{-1}-\mu)^{-1}\,d\mu\; P^{-1}\end{aligned}$$

with the inverse P^{-1} to P. Now we observe that the spectrum of P^{-1} lies inside C. Holomorphic functional calculus for the bounded operator P^{-1} and $f \equiv 1$ then shows the assertion. $\square$

We can therefore define the powers P^s for all $s \in \mathbb{C}$: We let

$$P^s = \begin{cases} P_s, & \operatorname{Re} s < 0 \\ P^k P_{s-k}, & k \text{ integer}, -1 \le \operatorname{Re} s - k < 0. \end{cases}$$

2.2. **Heat operators.** Instead of only making the assumption of the existence of a ray of minimal growth, we assume that $P-\lambda$ is invertible for all λ in a sector

$$\Lambda = \Lambda_\theta = \{re^{i\varphi} \in \mathbb{C} : r \ge 0 \text{ and } |\varphi| \ge \theta\} \text{ for some } \theta < \pi/2 \tag{2.6}$$

and that

$$\|(P-\lambda)^{-1}\| \le c\langle\lambda\rangle^{-1}, \quad \lambda \in \Lambda, \tag{2.7}$$

for a suitable constant c.

For $t > 0$ we then define

$$e^{-tP} = \frac{i}{2\pi}\int_{\mathscr{C}} e^{-t\lambda}(P-\lambda)^{-1}\,d\lambda, \tag{2.8}$$

where $\mathscr{C}$ is the contour from ∞ to $\delta_0 e^{i\theta}$ along the ray R_θ, clockwise about the origin on the circle $|z| = \delta_0$ to $\delta_0 e^{-i\theta}$ and back to ∞ along $R_{-\theta}$.

The integral converges, since $e^{-\lambda t}$ decays exponentially along the rays. Note that for this to be the case it is important that the rays lie in the right half plane.

Remark 2.4. The name 'heat' operator stems from the fact that $e^{-tP}u_0$ solves the equation $\partial_t u + Pu = 0$, $u(0) = u_0$, which becomes the heat equation for $P = -\Delta$.

Theorem 2.5. (a) *$t \mapsto e^{-tP}$ is a smooth function on $R_{>0}$ with values in bounded operators.*

(b) *Let $s, t > 0$. Then $e^{-sP}e^{-tP} = e^{-(s+t)P}$.*

Proof. (a) follows by differentiating under the integral sign. For (b) use a similar argument as in the proof of Theorem 2.3. $\square$

2.3. **Domains.** In the above, we are making assumptions on the spectrum of the operator P as an unbounded operator on a Hilbert space. As a consequence we have to specify the domain of P. In general, there are many possible choices. We will focus, however, on the case where P is an elliptic pseudodifferential operator (in fact, we will make even stronger assumptions on P) considered as an unbounded operator on $L^2(X)$. In that case, there only is one closed extension.

Definition 2.6. Let $A : C^\infty(X) \to C^\infty(X)$ be an arbitrary operator. By $\mathcal{D}_{\min}$, the *minimal domain*, we denote the domain of the closure of A, while $\mathcal{D}_{\max}$ is the set of all $u \in L^2$ such that $Au \in L^2$.

Clearly, $\mathcal{D}_{\min}$ is the domain of the smallest closed extension and $\mathcal{D}_{\max}$ that of the largest.

Theorem 2.7. *Let P be an elliptic pseudodifferential operator of order $\mu > 0$. Then $\mathcal{D}_{\min} = \mathcal{D}_{\max} = H^\mu(X)$.*

Proof. Let $u \in H^\mu(X)$. Then there exists a sequence $u_m \in C^\infty(X)$ with $u_m \to u \in H^\mu(X)$. Hence $H^\mu \subseteq \mathcal{D}_{\min}$. Conversely, suppose that $u \in \mathcal{D}_{\max}$, i.e. $u \in L^2$ and $Pu \in L^2$. Elliptic regularity then implies that $u \in H^\mu(X)$. Hence $\mathcal{D}_{\max} \subseteq H^\mu$. This shows the assertion. □

Theorem 2.8. *Let P be an elliptic pseudodifferential operator of order $\mu > 0$. Then either the L^2-spectrum of P is all of $\mathbb{C}$, or it consists of a countable number of eigenvalues with no accumulation point.*

Proof. If $P - \lambda$ is invertible for some λ, then

$$(P-\lambda)^{-1} : L^2 \to \mathcal{D}(P) = H^\mu \hookrightarrow L^2$$

is compact. This implies that the spectrum of $(P - \lambda)^{-1}$ is discrete with only possible accumulation point in zero. This shows the assertion since the spectral values of $P - \lambda$ are just the inverses of the elements in the spectrum of $(P - \lambda)^{-1}$. □

2.4. **Strategy.** In a first step, we shall see that the resolvent can be replaced by a parameter-dependent parametrix with a classical symbol having special homogeneity properties. This is the decisive step for the construction of both P^s and e^{-tP}.

We shall see that P^s is a pseudodifferential operator of order $\mu \operatorname{Re} s$, where μ is the order of P and that e^{-tP} is a smoothing operator. According to Theorem 1.16 it makes sense to take the trace of e^{-tP} and that of A^s, provided $\operatorname{Re} s < -n/\mu$.

From the asymptotic expansion of the parametrix symbol we then derive (under suitable additional assumptions on P and its symbol), the meromorphic structure of the trace of P^s and the asymptotic expansion of the trace of e^{-tP}.

2.5. **Notes.** The fundamental paper here is Seeley's article [20] on complex powers, where also the strategy of the resolvent analysis was developed. Kumano-go and Tsutusmi [15] simplified the technique; it is worthwhile having a look at Kumano-go's book [14]. In principle, the same information can be extracted from the traces of the resolvent, the heat kernel, and the complex powers, see [9]. For pseudodifferential boundary value problems, corresponding results on asymptotic expansions are harder to obtain, see Grubb's book [6].

In connection with noncommutative residues, for example, it is important to consider not only the traces of P^s or e^{-tP}, but more generally traces of operators QP^s or Qe^{-t} for general pseudodifferential operators Q. In the closed manifold case, the analysis is mostly parallel. For boundary value problems see e.g. [7].

3. Construction of a Parameter-dependent Parametrix

3.1. **Notation and preliminary results.** Let X be a closed manifold. From now on, we shall fix a classical pseudodifferential operator $P : C^\infty(X, E) \to C^\infty(X, E)$ of order μ, acting on sections of a vector bundle E over X.

In a fixed generic coordinate neighborhood U we shall denote the full symbol of P by p with the asymptotic expansion $p \sim \sum p_{\mu-j}$ and $p_{\mu-j}(x, \xi)$ homogeneous of degree $\mu - j$ for $|\xi| \geq 1$. We shall sometimes need the fully homogeneous variant $p^h_{\mu-j}$, which is homogeneous for all $\xi \neq 0$:

$$p^h_\mu(x, \xi) = |\xi|^\mu p^h_\mu(x, \xi/|\xi|). \tag{3.9}$$

Note that the symbol and the components take values in quadratic matrices.

In the sequel we fix a ray $R_\theta = \{re^{i\theta} : r \geq 0\}$ as above. This will be sufficient for the construction of complex powers, where the existence of the resolvent is only required on one ray. For the analysis of the 'heat' operator, we will let λ vary in a sector. Technically, this does not make a difference.

We will make the following assumption:

Assumption 3.1. *Write $\lambda = \eta^\mu$ with $\eta = re^{i\theta/\mu} \in R_{\theta/\mu}$, $r \geq 0$. Then there exists a $C_R \geq 0$ such that $p(x, \xi) - \eta^\mu$ is invertible whenever $|\xi| + |\eta| \geq C_R$ and satisfies*

$$|(p(x, \xi) - \eta^\mu)^{-1}| \leq \langle \xi, \eta \rangle^{-\mu}. \tag{3.10}$$

This property is often referred to as parameter-ellipticity.

It is easy to see (cf. [6, Lemma 1.5.4] for details) the following

Lemma 3.2. *Equivalently to* (3.10) *we could ask that*

(i) $p^h_\mu(x, \xi)$ *has no eigenvalues on R_θ for $\xi \neq 0$ or that,*

(ii) $p_\mu(x, \xi)$ *has no eigenvalues on R_θ for $|\xi| = 1$.*

Remark 3.3. Due to the compactness of X we can actually redefine p_μ so that p_μ has no eigenvalues on R_θ for *all* (x, ξ). In fact, let $L = \max\{|p_\mu(x, \xi)| : x \in U, |\xi| \leq 1\}$ and choose a non-negative function ω with compact support and $\omega(\xi) = 1$ for $|\xi| \leq 1$. Then set

$$\tilde{p}_\mu(x, \xi) = p_\mu(x, \xi) + (L + 1)\omega(\xi)e^{i\theta} I \tag{3.11}$$

As $\omega(\xi)$ is regularizing, we can (and will) replace p_μ by $\tilde{p}_\mu$ in the sequel and assume that:

$$|(p_\mu(x, \xi) - \eta^\mu)^{-1}| \leq c\langle \xi, \eta \rangle^{-\mu}, \quad x \in U, \xi \in \mathbb{R}^n, \eta \in R_{\theta/\mu}. \tag{3.12}$$

3.2. **Parametrix.** We next define a sequence of symbols $q_{-\mu-j} = q_{-\mu-j}(x, \xi, \eta)$ for $x \in U, \xi \in \mathbb{R}^n, \eta \in R_{\theta/\mu}$, $j = 0, 1, \ldots$, by

$$q_{-\mu}(x, \xi, \eta) = (p_\mu(x, \xi) - \eta^\mu)^{-1} \tag{3.13}$$

$$\begin{aligned} &q_{-\mu-j}(x, \xi, \eta) \\ &\quad = \; -\sum \frac{1}{\alpha!} \partial^\alpha_\xi q_{-\mu-k}(x, \xi, \eta) D^\alpha_x p_{\mu-l}(x, \xi)(p_\mu(x, \xi) - \eta^\mu)^{-1}, \end{aligned} \tag{3.14}$$

$j = 1, 2, \ldots$, where the sum extends over all α, k and l such that $k + l + |\alpha| = j$ and $k < j$.

For an arbitrary derivative ∂ with respect to x or ξ we have

$$\partial(p_\mu - \eta^\mu)^{-1} = -(p_\mu - \eta^\mu)^{-1} \partial p_\mu (p_\mu - \eta^\mu)^{-1}.$$

We therefore obtain:

Lemma 3.4. *$\partial_\xi^\alpha \partial_x^\beta q_{-\mu-j}$ is a linear combination of terms of the form*

$$(p_\mu - \eta^\mu)^{-1} \partial_\xi^{\alpha_1} \partial_x^{\beta_1} p_{\mu-k_1} (p_\mu - \eta^\mu)^{-1} \ldots \partial_\xi^{\alpha_r} \partial_x^{\beta_r} p_{\mu-k_r} (p_\mu - \eta^\mu)^{-1}$$

with

$$k_1 + \ldots k_r + |\alpha_1| + \ldots |\alpha_r| = j + |\alpha|.$$

Moreover

$$\partial_\xi^\alpha \partial_x^\beta q_{-\mu-j}(x, t\xi, t\eta) = t^{-\mu-j} \partial_\xi^\alpha \partial_x^\beta q_{-\mu-|\alpha|-j}(x, \xi, \eta). \tag{3.15}$$

There are at least two factors $(p_\mu - \eta^\mu)^{-1}$ if either $j > 0$ or $|\alpha| + |\beta| > 0$.

This leads us to the following estimate:

Proposition 3.5.

$$\partial_\xi^\alpha \partial_x^\beta q_{-\mu-j}(x, \xi, \eta) = O(\langle \xi, \eta \rangle^{-\mu} \langle \xi \rangle^{-j-|\alpha|}) \text{ for all } x, \xi, \eta, \alpha, \beta \tag{3.16}$$

$$\partial_\xi^\alpha \partial_x^\beta q_{-\mu-j}(x, \xi, \eta) = O(\langle \xi, \eta \rangle^{-2\mu} \langle \xi \rangle^{\mu-j-|\alpha|}) \text{ if } |\alpha| + |\beta| + j > 0 \tag{3.17}$$

We now choose a symbol $q = q(x, \xi, \eta)$ with $q \sim \sum q_{-\mu-j}$. It satisfies

$$\partial_\xi^\alpha \partial_x^\beta q(x, \xi, \eta) = O(\langle \xi, \eta \rangle^{-\mu} \langle \xi \rangle^{-j-|\alpha|}) \tag{3.18}$$

for $x \in U, \xi \in \mathbb{R}^n, \eta \in R_{\theta/\mu}$ and arbitrary multi-indices α, β.

Moreover, by construction, we have

$$\partial_\xi^\alpha \partial_x^\beta (q(x, \xi, \eta) - \sum_{j<K} q_{-\mu-j}(x, \xi, \eta)) = O(\langle \xi, \eta \rangle^{-2\mu} \langle \xi \rangle^{\mu-j-|\alpha|-K}) \tag{3.19}$$

Theorem 3.6. *We have*

$$q(x, \xi, \eta) \# (p(x, \xi) - \eta^\mu) - I = r(x, \xi, \lambda),$$

where, for arbitrary N, the seminorms of $r(x, \xi, \eta)$ in S^{-N} are $O(\eta^{-\mu})$.

Note that $|\eta^\mu| = |\lambda|$.

Proof. Fix some $K \in \mathbb{N}$ and write $q = \sum_{j<K} q_{-\mu-j} + r_q^K$ and $p = \sum_{j<K} p_{\mu-j} + r_p^K$.

It then follows from the estimates (3.16) and the construction of q that the seminorms of $q_{-\mu-j}$ in S^{-j} and of r_q^K in S^{-K} are $O(\eta^{-\mu})$. Theorem 1.13(a) then implies that the seminorms of $t_1 := r_q^K \# (p - \mu)$ and $t_2 = \sum_{j<K} q_{-\mu-j} \# r_p^K$ in $S^{\mu-K}$ are $O(\eta^{-\mu})$.

It remains to consider the composition

$$\sum_{j<K} q_{-\mu-j} \# \sum_{l<K} p_{\mu-l}.$$

We divide it into two parts: Those compositions, where $j + l < K$, and the rest. Again, the seminorms of the compositions belonging to the rest are $O(\eta^{-\mu})$ in $S^{\mu-K}$.

As for the remaining compositions, we consider the terms

$$\sum_{j,l,\alpha} \frac{1}{\alpha!} \partial_\xi^\alpha q_{-\mu-j} D_x^\beta p_{\mu-l}$$

where $j + l + |\alpha| < K$. By construction, cf. (3.13), the sum furnishes the identity. By the same argument as above, the terms, where $j + l + |\alpha| \geq K$ are $O(\eta^{-\mu})$ in $S^{\mu-K}$. □

Theorem 3.7. *These local parametrices can be patched to a parameter-dependent pseudodifferential operator $Q(\eta)$ on X such that*

$$Q(\eta)(P-\eta^\mu)-I=R_1(\eta)$$

where the symbol seminorms of $R(\eta)$ in each S^{-N} are $O(\eta^{-\mu})$.

This tricky patching procedure can be found in Seeley's original paper, [20].

In a similar way, we find an operator $\tilde{Q}(\eta)$ such that

$$(P-\eta^\mu)\tilde{Q}(\eta)-I=R_2(\eta),$$

where the symbol seminorms of $R_2(\eta)$ are $O(\eta^{-\mu})$ in each S^{-N}.

Remark 3.8. In fact, we can then take $\tilde{Q}=Q$, since

$$\begin{aligned} Q(\eta) &= Q(\eta)(P-\eta^\mu)\tilde{Q}(\eta)-Q(\eta)R_2(\eta)\\ &= (I+R_1(\eta))\tilde{Q}(\eta)-Q(\eta)R_2(\eta)=\tilde{Q}(\eta)-QR_2(\eta)+R_1(\eta)\tilde{Q}(\eta). \end{aligned}$$

so that Q and $\tilde{Q}$ only differ by a regularizing operator with seminorms $O(\eta^{-2\mu})$.

Corollary 3.9. *For $|\eta|$ sufficiently large, $P-\eta^\mu$ is invertible on $L^2(X)$ and*

$$\|(P-\eta^\mu)^{-1}-Q(\eta)\|=O(\eta^{-2\mu}).$$

Note that the estimate is due to the fact that

$$(Q(\eta)(P-\eta^\mu))^{-1}=(I+R_1(\eta))^{-1}=\sum_{j=0}^{\infty}R_1(\eta)^j \tag{3.20}$$

by Neumann's series, so that

$$(P-\eta^\mu)^{-1}-Q(\eta)^{-1}=Q(\eta)((Q(\eta)P)^{-1}-I)=Q(\eta)\sum_{j=1}^{\infty}R_1(\eta)^j=O(\eta^{-2\mu}),$$

since the norm of $Q(\eta)$ is $O(\eta^{-\mu})$.

Remark 3.10. (a) Writing $\eta=\lambda^{1/\mu}$, the estimate in Corollary 3.9 says that

$$(\|(P-\lambda)^{-1}-Q(\lambda^{1/\mu})\|=O(\lambda^{-2}).$$

(b) We can actually write the resolvent as a parameter-dependent pseudodifferential operator: From

$$Q(\mu)(P-\eta^\mu)=I+R_1(\eta) \text{ and } (P-\eta^\mu)Q(\eta)=I+R_2(\eta) \tag{3.21}$$

we conclude that

$$(P-\eta^\mu)^{-1}=Q(\eta)-R_1(\eta)Q(\eta)+R_1(\eta)(P-\eta^\mu)R_2(\eta)$$

differs from $Q(\eta)$ only by an operator whose symbol seminorms in each S^{-N} are $O(\eta^{-2\mu})$. Hence we may as well assume that $Q(\eta)$ *is* the resolvent.

In fact, using the estimates in Proposition 3.5 and (3.19) we see:

Proposition 3.11. *If $Q_K(\eta)$ is the parameter-dependent parametrix taking only into account the first K terms of the parametrix, then $(P-\eta^\mu)^{-1}-Q_K(\eta)$ is an operator of order $\mu-K$, with symbol seminorms uniformly $O(\eta^{-2\mu})$.*

Corollary 3.12. *We see from the above that the inverse of $P-\lambda$ is an elliptic pseudodifferential operator of order $-\mu$. Hence it is a compact operator on $L^2(X)$. In particular, the spectrum of P is discrete with no accumulation point.*

Remark 3.13. If the spectrum of the principal symbol $p_\mu(x, \xi)$ will not intersect the ray R_θ for $|\xi| = 1$ it will neither intersect rays $R_{\theta'}$ for θ' close to θ, hence there is a whole sector of minimal growth.

According to Corollary 3.9, $P - \lambda$ will then be invertible for large λ in the sector (the corresponding bounds can be taken uniform in the angle). As the spectrum is discrete by Corollary 3.12, there will be at most finitely many spectral points of P in the sector. Assuming only that P is invertible, there will be a ray of minimal growth.

4. Complex Powers

4.1. **The symbol of P^s.** We shall next analyze the properties of the complex powers P_s defined by (2.4) for $\operatorname{Re} s < 0$ and then go over to the general powers P^s.

Theorem 4.1. *P_s is a pseudodifferential operator of order $\mu \operatorname{Re} s$ and the symbol $\sigma(P_s)$ of P_s has an asymptotic expansion $\sigma(P_s) \sim \sum_{j=0}^{\infty} c_j(x,\xi;s)$, where, for $\lambda = \eta^\mu$*

$$c_j(x,\xi;s) = \frac{1}{2\pi i}\int_{\mathscr{C}} \lambda^s q_{-\mu-j}(x,\xi,\lambda^{1/\mu})\, d\lambda \tag{4.22}$$
$$= \frac{i}{2\pi}\int_{\mathscr{C}'} \mu\eta^{\mu s+\mu-1} q_{-\mu-j}(x,\xi,\eta)\, d\eta$$

with c_j homogeneous of degree $\mu s - j$ for $|\xi| \geq 1$. Here we have again written $\eta = \lambda^{1/\mu}$ to express the relation $\eta^\mu = \lambda$, and $\mathscr{C}'$ is the path given by the change $\lambda \mapsto \eta$.

Proof. In local coordinates we have

$$\begin{aligned} P_s &= \frac{i}{2\pi}\int_{\mathscr{C}} \lambda^s \operatorname{op}(q(x,\xi,\lambda^{1/\mu}))\, d\lambda \\ &= \operatorname{op}\left(\frac{i}{2\pi}\int_{\mathscr{C}} \lambda^s q(x,\xi,\lambda^{1/\mu})\, d\lambda\right) \\ &\sim \sum_{j=0}^{\infty} \operatorname{op}\left(\frac{i}{2\pi}\int_{\mathscr{C}} \lambda^s q_{-\mu-j}(x,\xi,\lambda^{1/\mu})\, d\lambda\right). \end{aligned}$$

Substituting $\lambda = t^\mu\sigma$ and using the homogeneity relation (3.15) yields

$$\begin{aligned} c_j(x,t\xi;s) &= \frac{i}{2\pi}\int_{\mathscr{C}} \lambda^s q_{-\mu-j}(x,\xi,\lambda^{1/\mu})\, d\lambda \\ &= t^{\mu s+\mu}\frac{i}{2\pi}\int_{\mathscr{C}} \sigma^s q_{-\mu-j}(x,t\xi,(t^\mu\sigma)^{1/\mu})\, d\sigma \\ &= t^{\mu s+\mu-\mu-j}\frac{i}{2\pi}\int_{\mathscr{C}} \sigma^s q_{-\mu-j}(x,\xi,\sigma^{1/\mu})\, d\sigma \\ &= t^{\mu s-j} c_j(x,\xi;s). \end{aligned}$$

A similar argument implies that the integral

$$\int_{\mathscr{C}} \lambda^s \operatorname{op}\left(q(x,\xi,\lambda^{1/\mu}) - \sum_{j<K} q_{-\mu-j}(x,\xi,\lambda^{1/\mu})\right) d\lambda$$

furnishes an operator of order $\mu \operatorname{Re} s - K$, so that the asymptotic expansion is justified. $\square$

Remark 4.2. Note that $q_{-\mu}(x,\xi,\lambda^{1/\mu}) = (p_\mu(x,\xi) - \lambda)^{-1}$ and therefore $c_0 = p_\mu^s$.

Lemma 4.3. *There exist constants $c_1, c_2 > 0$ such that $p_\mu(x,\xi) - \eta^\mu$ lies in the annulus*

$$\Omega_{|\xi|} = \{z \in \mathbb{C} : c_1\langle\xi\rangle \leq |z| \leq c_2\langle\xi\rangle\}.$$

Proof. For $|\xi| \geq 1$ this follows from homogeneity and the fact that $p_\mu(x,\xi)$ is invertible (note that 0 lies on R_θ). For small ξ use Remark 3.3. $\square$

Corollary 4.4. *The symbols $q_{-\mu-j}(x,\xi,\eta)$, $j = 0, 1, \ldots$, are holomorphic functions of η outside $\Omega_{|\xi|}$.*

By Cauchy's theorem we can therefore replace the contour $\mathscr{C}'$ in Equation (4.22) by the boundary of $\Omega_{|\xi|} \setminus R_{\theta/\mu}$ oriented in the mathematically positive sense.

$$c_j(x,\xi,s) = \frac{i}{2\pi}\int_{\partial\Omega_{|\xi|}} \mu\eta^{\mu s+\mu+1} q_{-\mu-j}(x,\xi,\eta)\,d\eta. \tag{4.23}$$

For fixed (x,ξ), the function $s \mapsto c_j(x,\xi;s)$ then is a holomorphic function on $\mathbb{C}$.

More is true: For $\operatorname{Re} s < 0$, $\mapsto c_j(s)$ is a holomorphic family of symbols of order zero.

Theorem 4.5. *For all $s \in \mathbb{C}$, P^s is a pseudodifferential operator of order $m \operatorname{Re} s$. Its symbol has the asymptotic expansion $\sigma(P^s) \sim \sum c_j(s)$, where c_j is given by the formula* (4.23).

Proof. The first assertion is immediate from the fact that $P^s = P^k P_{s-k}$ for k so large that $\operatorname{Re} s - k < 0$ together with Theorem 4.1.

The symbol of P^s is the Leibniz product of the symbol of P^k and that of P_{s-k}. Hence the homogeneous components are holomorphic functions on $\operatorname{Re} s < k$, taking values in symbols of order $k\mu$. As (4.23) provides a holomorphic extension, it must be the one. □

Proposition 4.6. *For $\operatorname{Re} s < 0$, $s \mapsto P_s$ is a holomorphic family of bounded operators on $H^t(X)$ for arbitrary t. More generally, $s \mapsto P^s$ is holomorphic from $\mu \operatorname{Re} s < c$ to operators in $\mathcal{L}(H^t(X), H^{t-c}(X))$.*

4.2. **Integral kernels.** Since P^s is a pseudodifferential operator, it has a distributional kernel k_s on $X \times X$ by the Schwartz kernel theorem. This kernel is smoother than one might expect:

Theorem 4.7. *In local coordinates, the kernel of P^s has the following properties:*

(a) *For each $k \in \mathbb{N}_0$, $s \mapsto k_s$ is a holomorphic function from $\mu \operatorname{Re} s < -n - k$ into matrices of C^k-functions on $X \times X$.*

(b) *If C is any compact set in $X \times X$ disjoint from the diagonal, then the restriction $k_s|_C$ is an entire function with values in matrices of C^∞-functions.*

(c) *For each fixed $x \in X$ the map $s \mapsto k_s(x,x)$ extends from $\operatorname{Re} s < -n$ to a meromorphic function on the complex plane with at most simple poles in the points $s_j = (j-n)/\mu$. The residue in s_j is given by*

$$\frac{1}{(2\pi)^{n+1} i\mu}\int_{|\xi|=1}\int_{\mathscr{C}} \lambda^{(j-n)/\mu} q_{-\mu-j}(x,\xi,\lambda^{1/\mu}) d\lambda dS(\xi).$$

(d) *The point $s = 0$ always is a regular point. The value $k_0(x,x)$ is given by*

$$\frac{1}{(2\pi)^n \mu}\int_{|\xi|=1}\int_0^\infty q_{-\mu-n}(x,\xi,t^{1/\mu} e^{i\theta/\mu}) dt dS(\xi).$$

(e) *If P is a differential operator then the residues in the integers vanish.*

(a) follows from Proposition 1.8 and Proposition 4.6.

For (b) and (c) we first make the following observation: According to Proposition (3.11) the parametrix Q_K constructed from the first K terms in the asymptotic expansion of the symbol of Q furnishes an $O(\eta^{-2\mu})$ approximation of $(P - \eta^\mu)^{-1}$ up to operators of order $\mu - K$. Next recall from Equation 1.2 that the kernel of a pseudodifferential operator P with symbol p is given by

$$(2\pi)^{-n}\int e^{i(x-y)\xi} p(x,\xi) d\xi, \tag{4.24}$$

where the formula holds whenever the order of p is $< -n$. Since

$$\frac{i}{2\pi}\int_{\mathscr{C}} \lambda^s((P-\lambda)^{-1} - Q_K(\lambda^{1/\mu}))d\lambda$$

exists and is holomorphic for $\operatorname{Re} s < 1$, taking values in operators or order $\mu - K$, the singularities of the kernels of P^s and those of Q_K will agree, provided K is so large that $\mu - K < -n$.

Now Q_K is locally given by the finite sum of terms $\sum_{j<K} \operatorname{op}(q_{-\mu-j})$. Hence

$$\frac{i}{2\pi}\int \lambda^s Q_K(x,\xi,\lambda^{1/\mu})d\lambda = \sum \operatorname{op} c_j(x,\xi;s),$$

and the associated kernel is

$$k_{Q_K,s} = (2\pi)^{-n}\sum \int e^{i(x-y)\xi}c_j(x,\xi;s)d\xi$$

For (b) these terms can be analyzed easily by integrating by parts, since $x \neq y$: $k_{Q_K,s}$ is given by

$$k_{Q_K,s}(x,y) \quad = \quad (2\pi)^{-n}|x-y|^{-2K}\sum_{j<K}\int e^{i(x-y)\xi}\Delta_\xi^K c_j(x,\xi;s)\,d\xi.$$

Since $\Delta^K c_j(s)$ is a symbol of order $\mu \operatorname{Re} s - j - 2K$, we can here extend the range of s to the whole complex plane.

Still we have the restriction $\operatorname{Re} s < 1$ for the coincidence of the singularities of the kernels. In order to remove it, recall that for $k > \operatorname{Re} s \geq k-1$ we write

$$P^s = P^k P_{s-k} = \frac{i}{2\pi}\int_{\mathscr{C}} \lambda^{s-k}P^k(P-\lambda)^{-1}\,d\lambda.$$

Now we observe that $P^kQ_K - P^k(P-\lambda)^{-1}$ will be pseudodifferential of order $\mu(k+1) - K$ and $O(\lambda^{-2})$. Hence the singularities of the kernel of P^s coincide with those of

$$\frac{i}{2\pi}\int_{\mathscr{C}} \lambda^{s-k}P^kQ(\lambda^{1/\mu})\,d\lambda \tag{4.25}$$

for $\operatorname{Re} s < k+1$. Next write the asymptotic expansion of the symbol of P^kQ as $\sum q_j^{(k)}$. Then the definition of $c_j(s)$ and the analyticity imply that

$$\frac{i}{2\pi}\int_{\mathscr{C}} \lambda^{s-k}q_j^{(k)}(x,\xi,\lambda^{1/\mu})\,d\lambda = c_j(x,\xi;s).$$

Moreover, $P^kQ - \sum_{j<K} q_j^{(k)}$ is $O(\lambda^{-2})$ uniformly in symbols of order $\mu k - K$. For K large, the singularities of (4.25) therefore coincide with those computed from the $\sum_{j<K} \operatorname{op} c_j$ for $\operatorname{Re} s < k+1$.

(c) We again consider first the case $\operatorname{Re} s < 1$ and argue that it suffices to study the terms (note that $x = y$ in (4.24))

$$(2\pi)^{-n}\int c_j(x,\xi;s)d\xi.$$

We divide the integration into the part over $|\xi| \leq 1$ and that over $|\xi| \geq 1$. The first produces an entire function and thus does not contribute to the singularities. To determine the second, we introduce polar coordinates. Using that c_j is homogeneous of degree $\mu s - j$ we obtain an integral of the form

$$(2\pi)^{-n}\int_S c_j(x,\omega;s)dS(\omega)\int_1^\infty r^{\mu s-j+n-1}dr. \tag{4.26}$$

The last integral is $-\frac{1}{\mu s-j+n}$. This furnishes the pole in $s_j = (j-n)/\mu$ together with the residue in this point.

(d) The possible pole in zero arises from the term associated with c_{-n}. But as $P^0 = I$, this component of the symbol is zero. According to Equation (4.26) the residue in zero vanishes.

It is somewhat more involved to determine the value in zero and to show (e).

Corollary 4.8. *For* $\operatorname{Re} s < -n/\mu$ *the operator* P^s *is trace class, and we can define*

$$\zeta_P(s) = \operatorname{Tr}(P^s).$$

This is a holomorphic function on $\operatorname{Re} s < -n/\mu$. *It extends to a meromorphic function on* $\mathbb{C}$ *with at most simple poles in the points* $s_j = (j-n)/m$. *The origin is not a pole.*

Proof. For $\operatorname{Re} s < -n/\mu$ the operator P^s is of order $< -n$. By Theorem 1.27 it is trace class and

$$\operatorname{Tr} P^s = \int_X k_{P^s}(x,x)\,dx.$$

The assertion then follows from Theorem 4.7. $\square$

5. Heat Operators

The analysis of the heat operator

$$e^{-tP} = \frac{i}{2\pi}\int_{\mathscr{C}} e^{-tp}(P-\lambda)^{-1}\,d\lambda$$

proceeds in a similar way.

Fix a sector Λ as in (2.6). We assume that, in local coordinates the principal symbol p_μ of P satisfies the assumption (3.12). We then construct the parameter-dependent parametrix as before.

Theorem 5.1. (a) *For each $t > 0$, e^{-tP} is an integral operator with smooth kernel.*

(b) *For $t \geq 0$, $t \mapsto e^{-tP}$ is continuous as a family of pseudodifferential operators of order zero.*

(c) *In this sense, the symbol $v(x,\xi;t)$ of e^{-tP} has an asymptotic expansion*

$$v \sim \sum_{j=0}^{\infty} v_{-j}$$

with

$$v_0 = e^{-tp_\mu} \quad \textit{and} \tag{5.27}$$

$$v_{-j} = \frac{i}{2\pi}\int_{\mathscr{C}} e^{-t\lambda} q_{-\mu-j}(x,\xi,\lambda^{1/\mu})\,d\lambda. \tag{5.28}$$

The continuous extension of v_{-l} to $t=0$ is zero.

Moreover, the v_{-j} satisfy the homogeneity relation

$$v_{-j}(x,s\xi;s^{-\mu}t) = s^{-j}v_{-j}(x,\xi;t) \quad |\xi|\geq 1, s\geq 1. \tag{5.29}$$

(d) *There exists a $c > 0$ such that*

$$|\partial_\xi^\alpha D_x^\beta v_{-j}(x,\xi;t)| \leq C\langle\xi\rangle^{-j-|\alpha|}e^{-ct\langle\xi\rangle^\mu}(t^{1/\mu}\langle\xi\rangle)^a \quad \textit{for any } a \leq \min\{\mu, j+|\alpha|\}. \tag{5.30}$$

(e) *The symbol v can be chosen in such a way that*

$$\left|\partial_\xi^\alpha D_x^\beta\left(v - \sum_{j<K} v_{-l}\right)\right| \leq C\langle\xi\rangle^{-K-|\alpha|}e^{-ct\langle\xi\rangle^\mu}(t^{1/\mu}\langle\xi\rangle)^a \quad \textit{for any } a \leq \min\{\mu, M+|\alpha|\}. \tag{5.31}$$

Let us look at the proof of some of these facts.

(a) follows from the estimates in (d) and (e).

Now consider (c). By the consideration in 4.4 the contour can actually be closed for fixed (x,ξ) and then the expression can be considered for all $t\in\mathbb{R}$. It follows from the structure of the terms in the parametrix that the $q_{-\mu-j}$ for $j>0$ have at least two factors $(p_\mu - \eta^\mu)^{-1}$. Hence there are no simple poles, and integration over the closed contour produces a Taylor series starting with t^1, so that $v_{-j}(x,\xi;0) = 0$.

The homogeneity relations follow from a calculation as for the complex powers. The estimates for the v_{-j} and the difference in (e) are a result of the construction of the $q_{-\mu-j}$.

Remark 5.2. In order to obtain the trace expansion in Theorem 0.2 one can either mimic the strategy of the proof of Theorem 4.7 or else use the fact that the resolvent, the complex powers and the heat operator are related by transition formulae like

$$\begin{aligned} P^{-s} &= \tfrac{1}{(s-1)\cdots(s-k)}\tfrac{i}{2\pi}\int_{\mathscr{C}}\lambda^{k-s}\partial_\lambda^k(P-\lambda)^{-1}\,d\lambda = \tfrac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-tP}\,dt,\\ e^{-tP} &= t^{-k}\tfrac{i}{2\pi}\int_{\mathscr{C}'}e^{-t\lambda}\partial_\lambda^k(P-\lambda)^{-1}\,d\lambda = \tfrac{1}{2\pi i}\int_{\operatorname{Re}s=c}t^{-s}\Gamma(s)P^{-s}\,ds \end{aligned}$$

$k \in \mathbb{N}_0$.

The first formula states that P^s is the Mellin transform of e^{-tP}, multiplied by $1/\Gamma$. Now it is well-known that the asymptotic behavior $\sim t^{s_j}\log^k t$ as $t \to 0^+$ is translated by the Mellin transform into a pole of order $k+1$ in the point s_j and vice versa (under suitable conditions, see [9] for details).

6. Weyl Asymptotics

We now additionally assume that P is self-adjoint and positive. Then P has an orthonormal basis of eigenfunctions in L^2 with associated eigenvalues λ_j, $j = 1, 2 \ldots$.. Since we required P to be invertible, the eigenvalues are positive numbers with $\lambda_j \to \infty$, see Theorem 2.8. Recall that

$$N(\lambda) = \sum_{\lambda_j < \lambda} 1.$$

We will now compute the asymptotics of $N(\lambda)$ from our knowledge of $\operatorname{Tr} P^s$.

For this, we first observe that, by functional calculus, the spectrum of P^s consists of the values λ_j^s, and hence the trace of P^s is given by

$$\operatorname{Tr} P^s = \sum_j \lambda_j^s.$$

6.1. **A Tauberian theorem.** We will use a Wiener-Ikehara Tauberian theorem. You find a proof of Ikehara's theorem e.g. in Donoghue's book [3, Section 47] and Wiener's paper is [24]. I am relying here on the version given in Aramaki [1, Proposition 1] who also provides a useful extension to the case where $\operatorname{Tr} P^s$ has higher order poles.

Theorem 6.1. *Let P be a positive and self-adjoint operator on a Hilbert space. Assume that $\operatorname{Tr} P^s$ is holomorphic in a half-plane $\{\operatorname{Re} s < a < 0\}$ and that there exists a constant A such that*

$$TrP^s - \frac{A}{s-a}$$

has a continuous extension to $\{\operatorname{Re} s = a\}$. Then we have

$$N(\lambda) = \frac{A}{a}\lambda^{-a}(1 + o(1)) \quad \text{as } \lambda \to \infty.$$

6.2. **Application to the counting function of P.** For the operator P we can compute the residue from the formula in Theorem 4.7(c) for $j = 0$ from the principal symbol of P, or else from Equation (4.26), with $j = 0$, noting that

$$-\frac{1}{\mu s + n} = -\frac{1}{\mu}\frac{1}{(s - (-n/\mu))}.$$

Specifically: Since $\operatorname{Tr} P^s = \int_X k_s(xx)\,dx$ we have

$$N(\lambda) \sim \frac{\lambda}{n}\int_X \int_S c_0(x, \omega; -n/\mu) dS(\omega) dx.$$

6.3. **The case of the negative Laplacian.** Here, the symbol is $|\xi|^2$. We can take the ray to be the negative real axis. We then obtain $q_{-\mu}(x, \xi, \eta) = |\xi|^2 - \eta^2$ with $\eta^2 \in \mathbb{R}_-$ and $c_0(x, \xi; s) = |\xi|^{2s}$. The value on the unit sphere in T^*X is constant 1. Hence (4.26) computes the residue as $(2\pi)^{-n} \operatorname{vol} S^{n-1}$. The residue of $\operatorname{Tr} P^s$ is given by integration over X; it therefore has the value

$$A = -(2\pi)^{-n}\frac{1}{\mu}\operatorname{vol} S^{n-1}\operatorname{vol} X.$$

Finally the fact that $a = -n/\mu = -n/2$ yields Weyl's law:[2]

$$N(\lambda) \sim \frac{\operatorname{vol} S^{n-1}\operatorname{vol} X}{n(2\pi)^n}\lambda = \frac{\operatorname{vol} B^n \operatorname{vol} X}{(2\pi)^n}\lambda,$$

where B is the unit ball in $\mathbb{R}^n$.

[2]There is no contradiction to the statement of Kac. The additional factor 2 we have in the denominator originates from the fact that we use Δ while Kac works with $\frac{1}{2}\Delta$.

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