Wave trace invariants

Lecture I: Introduction to microlocal analysis

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Summer School on Spectral Geometry

Göttingen, September 9-12, 2014

Format of this course

- Tuesday, 14:30–15:30 – Lecture I: Introduction to microlocal analysis
- Wednesday, 17:00–18:00 – Lecture II: Global aspects of Fourier integral operators
- Thursday, 16:00–17:00 – Lecture III: The wave trace and periodic bicharacteristics
- Friday, 10:30–11:30 – Lecture IV: Computation of higher wave trace invariants

The setup

- X – $\mathscr{C}^\infty$ closed manifold, $\dim X = d$.
- $P \in \Psi^1(X; \Omega^{1/2})$ elliptic, $P = P^* > 0$.
- P has purely discrete spectrum (as an unbounded operator in $L^2(X; \Omega^{1/2})$).
- $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots$ – eigenvalues of P, with associated eigenfunctions $\phi_j \in \mathscr{C}^\infty(X; \Omega^{1/2})$ which are chosen to form an ONB in $L^2(X; \Omega^{1/2})$, $\lambda_j \sim c j^{1/d}$ as $j \to \infty$ by Weyl's law.

The wave trace is

$$\boxed{w(t) = \sum_{j=1}^{\infty} e^{i\lambda_j t}, \quad t \in \mathbb{R},}$$

which is (formally) the trace of the wave group $\{e^{itP}\}_{t\in\mathbb{R}}$. The latter yields the solving operator of the first-order hyperbolic equation $-i\partial_t u = Pu$.

The problem

Problem. Study the singularities of $e \in \mathscr{S}'(\mathbb{R})$.

Key observation. e^{itP}, $t \in \mathbb{R}$, is zeroth-order elliptic FIO.

Proposition

$0 \neq T \in \operatorname{singsupp} w$ *implies that there is a periodic trajectory of the Hamilton vector field* H_p*, of period* T*, where* $p = \sigma^1(P) \in S^{(1)}(T^*X \setminus 0; \mathbb{R})$.

Question. How does a non-degenerate periodic trajectory of H_p contribute to the singularities of the wave trace, e?

Let γ be such a periodic trajectory. Its contribution is

$$e_\gamma(t) \sim c_{-1\gamma}(t-T+i0)^{-1} + \sum_{r\geq 0} c_{r\gamma}(t-T+i0)^r \log(t-T+i0) \text{ as } t \to T.$$

Wave trace invariants

- The $c_{rT} = \sum_\gamma c_{r\gamma}$, $r \geq -1$, where summation is over all periodic trajectories, γ, of period T, are the wave trace invariants of the title.
- $$\boxed{c_{-1\gamma} = \frac{T_\gamma}{2\pi} \frac{i^{\mu_\gamma}}{\sqrt{\det(I - \Pi_\gamma)}} e^{i\int_\gamma \sigma^0_{\text{sub}}(P)},}$$

  where
  - T_γ – primitive period of γ,
  - μ_γ – Maslov index,
  - Π_γ – linearized Poincaré map,
  - $\sigma^0_{\text{sub}}(P)$ – subprincipal symbol of P.
- The $c_{r\gamma}$, $r \geq -1$, determine the Birkhoff normal form of γ completely.

α-densities

Linear algebra part. Let V be a real vector space, $\dim_{\mathbb{R}} V = d$. For $\alpha \in \mathbb{C}$, an α-density is a map $\ell : \mathscr{F}(V) \to \mathbb{C}$, where $\mathscr{F}(V)$ is the set of all linear bases of V, such that

$$\ell(Ae_1, \dots, Ae_d) = |\det A|^{\alpha} \ell(e_1, \dots, e_d)$$

for all $(e_1, \dots, e_d) \in \mathscr{F}(V)$, $A \in \mathrm{GL}(V)$. The space $\Omega^{\alpha}(V)$ of such α-densities is a one-dimensional complex vector space.

For X a $\mathscr{C}^{\infty}$ manifold, the α-density bundle $\Omega^{\alpha}(X) = \bigsqcup_{p \in X} \Omega^{\alpha}(T_p^* X)$ is a complex line bundle over X.

- Transition functions are $\left|\det\left(\partial y / \partial x\right)\right|^{-\alpha}$ for a coordinate change $y = y(x)$.
- Sections of $\Omega^{\alpha}(X)$ are called α-densities and are locally written as $u(x)\,|dx|^{\alpha}$.

We shall encounter the cases $\alpha = 0$, $\alpha = 1/2$, and $\alpha = 1$.

Integration and distributions on manifolds

- There is an invariant integration on X,

$$\mathscr{C}_c^\infty(X;\Omega^1) \to \mathbb{C}, \quad \omega \mapsto \int_X \omega.$$

- $L^2(X;\Omega^{1/2})$ is a Hilbert space with scalar product

$$(\omega,\eta) = \int_X \omega\overline{\eta}.$$

- $\mathscr{D}'(X;\Omega^{1/2})$ is the dual space of $\mathscr{C}_c^\infty(X;\Omega^{1/2})$.
    - Then $L^1_{\mathrm{loc}}(X;\Omega^{1/2}) \hookrightarrow \mathscr{D}'(X;\Omega^{1/2})$ via

$$f \mapsto \Big(\phi \mapsto \int_X \underbrace{f(x)\phi(x)}_{\in L^1_{\mathrm{comp}}(X;\Omega^{1/2})}\Big).$$

Schwartz kernel theorem

Theorem

There is a 1-to-1 correspondence between linear (sequentially continuous) operators $A: \mathscr{C}_c^\infty(Y;\Omega^{1/2}) \to \mathscr{D}'(X;\Omega^{1/2})$ *and distributions* $K \in \mathscr{D}'(X \times Y;\Omega^{1/2})$ *such that*

$$\langle Au, \phi\rangle = \langle K, \phi \otimes u\rangle$$

for $u \in \mathscr{C}_c^\infty(Y;\Omega^{1/2})$, $\phi \in \mathscr{C}_c^\infty(X;\Omega^{1/2})$.

This is often written as $Au(x) = \int_Y K(x,y)u(y)$. Instead of K one also writes $A(x,y)$.

Now. Describe linear operators A through their kernels K.

Pseudodifferential operators

Example

$A \in \Psi^m(X;\Omega^{1/2})$ if and only if $K \in I^m(X \times X, \Delta_X; \Omega^{1/2})$ is conormal with respect to the diagonal $\Delta_X = \{(x,x) \mid x \in X\}$.

- In local coordinates, full symbol of $A \in \Psi^m(X;\Omega^{1/2})$ is

$$a(x,\xi) \sim a_m(x,\xi) + a_{m-1}(x,\xi) + \dots$$

- Principal symbol is $\sigma^m(A)(x,\xi) = a_m(x,\xi)$.
- Subprincipal symbol is

$$\sigma^{m-1}_{\text{sub}}(A)(x,\xi) = a_{m-1}(x,\xi) - \frac{1}{2i}\sum_j \frac{\partial^2 a_m}{\partial x_j \partial \xi_j}(x,\xi).$$

The "calculus" of wave front sets

Let $A\colon \mathscr{C}_c^\infty(Y;\Omega^{1/2}) \to \mathscr{D}'(X;\Omega^{1/2})$ be linear and K be its kernel. Utilize the identification $T^*(X\times Y) \cong T^*X \times T^*Y$.

- $\mathrm{WF}'(A) = \{(x,\xi;y,\eta) \in (T^*X \setminus 0)\times(T^*Y\setminus 0) \mid (x,y,\xi,-\eta) \in \mathrm{WF}(K)\}$ is the wave front relation of A.
- $\mathrm{WF}_X(A) = \{(x,\xi) \in T^*X \setminus 0 \mid \exists y\colon (x,y,\xi,0) \in \mathrm{WF}(K)\}$.
- $\mathrm{WF}'_Y(A) = \{(y,\eta) \in T^*Y \setminus 0 \mid \exists x\colon (x,y,0,-\eta) \in \mathrm{WF}(K)\}$.

Proposition

Let $u \in \mathscr{D}'(Y;\Omega^{1/2})$. Suppose that $\mathrm{WF}(u) \cap \mathrm{WF}'_Y(A) = \varnothing$ and the projection $(\operatorname{supp} K \times \operatorname{supp} u) \cap (X \times \Delta_Y) \to X$, $(x,y,y) \mapsto x$ is proper. Then $Au \in \mathscr{D}'(X;E)$ is defined. Moreover,

$$\mathrm{WF}(Au) \subseteq \mathrm{WF}'(A) \circ \mathrm{WF}(u) \cup \mathrm{WF}_X(A).$$

An example: pull-backs

Let $f : X \to Y$ be $\mathscr{C}^\infty$. Then $f^* : \mathscr{C}^\infty(Y) \to \mathscr{C}^\infty(X)$, $u \mapsto u \circ f$, has kernel

$$K(x, y) = \delta(y - f(x))$$

and, therefore,

$$\mathrm{WF}(K) = \{(x, y, \xi, \eta) \mid y = f(x), \xi + {}^t df(x)\eta = 0, \eta \neq 0\}.$$

Here, ${}^t df(x) : T^*_{f(x)} Y \to T^*_x X$ is the dual map of $df(x) : T_x X \to T_{f(x)} Y$.

We conclude that

$$\mathrm{WF}(f^*) = \{(x, {}^t df(x)\eta; f(x), \eta) \mid {}^t df(x)\eta \neq 0\},$$
$$\mathrm{WF}'_Y(f^*) = \{(f(x), \eta) \mid {}^t df(x)\eta = 0, \eta \neq 0\},$$

and $\mathrm{WF}_X(f^*) = \emptyset$.

Basic symplectic structure

- $\alpha = \xi\, dx = \sum_{j=1}^{d} \xi_j\, dx^j$ – canonical 1-form on $T^*X \setminus 0 = \{(x,\xi) \mid \xi \neq 0\}$.
- $\sigma = d\alpha = d\xi \wedge dx = \sum_{j=1}^{d} d\xi_j \wedge dx^j$ – symplectic form on $T^*X \setminus 0$, σ is non-degenerate.
- $H_f = \dfrac{\partial f}{\partial \xi}\dfrac{\partial}{\partial x} - \dfrac{\partial f}{\partial x}\dfrac{\partial}{\partial \xi}$ – Hamilton vector field associated with $f \in \mathscr{C}^\infty(T^*X \setminus 0; \mathbb{R})$, $df = -H_f \lrcorner\, \sigma = \sigma(\cdot, H_f)$.
- $\{f,g\} = H_f g = \dfrac{\partial f}{\partial \xi}\dfrac{\partial g}{\partial x} - \dfrac{\partial f}{\partial x}\dfrac{\partial g}{\partial \xi}$ – Poisson bracket.
    - g is constant along integral curves of H_f iff $\{f,g\} = 0$.
    - In particular, f is constant along integral curves of H_f.
- $\left(\mathscr{C}^\infty(T^*X \setminus 0; \mathbb{R}), \{\,,\,\}\right)$ is Lie algebra.
- $H_{\{f,g\}} = [H_f, H_g]$, so $f \mapsto H_f$ is Lie algebra homomorphism.

Energy inequalities

Consider the Cauchy problem for the operator $-i\partial_t - P$:

(CP) $$\boxed{-i\partial_t u = Pu + f \quad \text{on } (0,T)\times X, \quad u(0,\cdot) = \phi}$$

for the unknown $u = u(t,x)$.

Proposition

Given $\phi \in H^\sigma(X;\Omega^{1/2})$ *and* $f \in L^1((0,T);H^\sigma(X;\Omega^{1/2}))$ *for some* $\sigma \in \mathbb{R}$*, Eq. (CP) possesses a unique solution* $u \in \mathscr{C}([0,T];H^\sigma(X;\Omega^{1/2}))$*.*

Indeed,

$$u(t) = U(t)\phi + \int_0^t U(t-t')f(t')\,dt',$$

where $u(t) = u(t,\cdot)$ and $U(t) = e^{itP}$ (defined via the spectral theorem).

Propagation of singularities

Statement on the propagation of singularities is microlocalized versions of the energy inequalities: Let $p = \sigma^1(P)$ and $\{\chi_t\}_{t\in\mathbb{R}}$ be the flow of H_p (= the bicharacteristic flow of P).

Proposition

$\mathrm{WF}(U(t)\phi) = \chi_t \mathrm{WF}(\phi)$ *for all* $\phi \in \mathscr{D}'(X;\Omega^{1/2})$.

Consistent with this result is:

Proposition

Let $U\colon \mathscr{C}^\infty(X;\Omega^{1/2}) \to \mathscr{C}^\infty(\mathbb{R}\times X;\Omega^0 \boxtimes \Omega^{1/2})$ *be the solution operator of the Cauchy problem for* $-i\partial_t - P$. *Then*

$$\mathrm{WF}'(U) = \{(t,x,\tau,\xi;y,\eta) \mid \tau = p(x,\xi),\ (x,\xi) = \chi_t(y,\eta)\}.$$

Wave front set of w

The kernel of U is

$$U(t,x,y) = \sum_j e^{it\lambda_j}\phi_j(x)\overline{\phi_j(y)}.$$

Therefore, $w = \pi_*\Delta^* U$, where

- $\Delta : \mathbb{R}\times X \to \mathbb{R}\times X\times X$ is diagonal map and $\Delta^* : \mathscr{C}^\infty(\mathbb{R}\times X\times X; \Omega^0 \boxtimes \Omega^{1/2} \boxtimes \Omega^{1/2}) \to \mathscr{C}^\infty(\mathbb{R}\times X; \Omega^0 \boxtimes \Omega^1)$,
- $\pi_* : \mathscr{C}^\infty(\mathbb{R}\times X; \Omega^0 \boxtimes \Omega^1) \to \mathscr{C}^\infty(\mathbb{R})$ is integration along fibers.

Lemma

$$\mathrm{WF}(w) \subseteq \{(t,\tau) \mid \exists (x,\xi) \in T^*X \setminus 0 : \chi_t(x,\xi) = (x,\xi),\ \tau > 0\}.$$

Proof of lemma

We already know that

$$\begin{aligned}
\mathrm{WF}(U) &= \{(t,x,y,\tau,\xi,-\eta) \mid \tau = p(x,\xi),\ (x,\xi) = \chi_t(y,\eta),\ \xi \neq 0\},\\
\mathrm{WF}'(\Delta^*) &= \{(t,x,\tau,\xi+\eta; t,x,x,\tau,\xi,\eta) \mid (\tau,\xi+\eta) \neq 0\},
\end{aligned}$$

and then

$$\mathrm{WF}(\Delta^* U) \subseteq \{(t,x,\tau,\xi-\eta) \mid \tau = p(x,\xi),\ (x,\xi) = \chi_t(x,\eta),\ \xi \neq 0\}.$$

One further computes

$$\begin{aligned}
\mathrm{WF}'(\pi_*) &= \{(t,\tau; t,x,\tau,0) \mid \tau \neq 0\},\\
\mathrm{WF}(\pi_*\Delta^* U) &\subseteq \{(t,\tau) \mid \exists (x,\xi) : \chi_t(x,\xi) = (x,\xi),\ \xi \neq 0,\ \tau > 0\}
\end{aligned}$$

which concludes the proof. $\square$