Wave trace invariants

Lecture II: Global aspects of Fourier integral operators

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Yesterday's result

- X – $\mathscr{C}^\infty$ closed manifold, $\dim X = d$.
- $P \in \Psi^1(X;\Omega^{1/2})$ elliptic, $P = P^* > 0$.
- $\lambda_1 \le \lambda_2 \le \lambda_3 \le \dots$ – eigenvalues of P, with associated eigenfunctions $\phi_j \in \mathscr{C}^\infty(X;\Omega^{1/2})$ which (are chosen to) form an ONB in $L^2(X;\Omega^{1/2})$.
- Wave kernel is $U(t,x,y) = \sum_{j=1}^\infty e^{it\lambda_j}\phi_j(x)\overline{\phi_j(y)}\,|dt|^{1/2}$.
- Wave trace is $w(t) = \sum_{j=1}^\infty e^{it\lambda_j}\,|dt|^{1/2}$.

Lecture I: $w \in \mathscr{S}'(\mathbb{R};\Omega^{1/2})$ and

$$\mathrm{WF}(w) \subseteq \{(t,\tau) \mid \exists (x,\xi) : \chi_t(x,\xi) = (x,\xi),\ \xi \neq 0,\ \tau > 0\}.$$

Upon writing $w = \pi_* \Delta^* U$ (Δ^* – restriction to the diagonal, π_* – fiber integration), this follows from the propagation result for the first-order hyperbolic operator $-i\partial_t - P$:

$$\mathrm{WF}(U) = \{(t,x,\tau,\xi;y,-\eta) \mid \tau = p(x,\xi),\ (x,\xi) = \chi_t(y,\eta),\ \xi \neq 0\}.$$

Historical comment

Analysis of the “big” singularity of $w(t)$ at $t = 0$ yields Weyl’s law (Hörmander ’68):

For $P \in \Psi^m(X;\Omega^{1/2})$, $m > 0$, $P = P^* > 0$ work with $P^{1/m} \in \Psi^1(X;\Omega^{1/2})$ to get

$$N_P(\lambda) = \sharp\{j \mid \lambda_j \leq \lambda\} = C\lambda^{d/m} + O(\lambda^{(d-1)/m}) \quad \text{as } \lambda \to \infty,$$

where

$$C = (2\pi)^{-d} \int_{p(x,\xi)\leq 1} dx d\xi.$$

Today's lecture

To determine the nature of the singularity of $w \in \mathscr{S}'(\mathbb{R})$ at $t = T$, $T \neq 0$ the period of a periodic bicharacteristic, we need to know more about the analytic structure of the kernel $U = U(t, x, y)$.

- $U \in \mathscr{D}'(\mathbb{R} \times X \times X; \Omega^{1/2})$ is a Lagrangian distribution with respect to $\Lambda = \mathrm{WF}(U)$.
- $\pi_* \Delta^* : \mathscr{C}^\infty(\mathbb{R} \times X \times X; \Omega^{1/2}) \to \mathscr{C}^\infty(\mathbb{R}; \Omega^{1/2})$ is a Fourier integral operator with underlying canonical relation $\{(t, \tau; t, x, x, \tau, \xi, \xi) \mid \tau \neq 0\}$.
- Under suitable assumptions, $\pi_* \Delta^*$ and U compose cleanly. Near $t = T$, this implies that $\pi_* \Delta^* U \in \mathscr{D}'(\mathbb{R})$ is a Lagrangian distribution with respect to $\Lambda_T^+ = \{(t, \tau) \mid t = T, \tau > 0\}$.

Lagrangian submanifolds

- $(T^*X \setminus 0, \sigma)$, where $\sigma = d\alpha$ and $\alpha = \xi\, dx$, is a homogeneous symplectic manifold.
- Homogeneous means that $M_\lambda^* \sigma = \lambda \sigma$ for all $\lambda > 0$, where M_λ is multiplication by λ in the fibers.
- $\Lambda \subset T^*X \setminus 0$ is a homogeneous Lagrangian submanifold if $\dim \Lambda = d$ and $\alpha\big|_\Lambda = 0$ ($\Longleftrightarrow \sigma|_\Lambda = 0$ and $M_\lambda \Lambda = \Lambda$ for all $\lambda > 0$).
- $\chi_t \colon T^*X \setminus 0 \to T^*X \setminus 0$ for $t \in \mathbb{R}$ is a canonical transformation (i.e., a homogeneous symplectomorphism), where $\{\chi_t\}_t$ is the flow of H_p.

Phase functions

Let $U \subseteq \mathbb{R}^d$ be open, $N \in \mathbb{N}$.

- $\varphi \in \mathscr{C}^\infty(U \times (\mathbb{R}^N \setminus 0); \mathbb{R})$ is said to be a phase function if
  - $\varphi(x, \lambda\theta) = \lambda\varphi(x,\theta)$ for $\lambda > 0$,
  - $d\varphi \neq 0$ everywhere on $U \times (\mathbb{R}^N \setminus 0)$.
- A phase function φ is said to be clean, of excess e, if, in addition, the critical set

$$C_\varphi = \{(x,\theta) \mid \varphi'_\theta(x,\theta) = 0\}$$

  is a $(d+e)$-dimensional submanifold of $U \times (\mathbb{R}^N \setminus 0)$, with $T_{(x,\theta)}C_\varphi$ being given by the vanishing of the differentials $d\left(\frac{\partial\varphi}{\partial\theta_1}\right), \ldots, d\left(\frac{\partial\varphi}{\partial\theta_N}\right)$.
- A phase function φ is said to be non-degenerate if it is clean of excess 0.

Parametrization of Lagrangian submanifolds

Let φ be a clean phase function, of excess e.

Proposition

$\Lambda_\varphi = \{(x, \varphi'_x(x,\theta)) \mid \varphi'_\theta(x,\theta) = 0\}$ is an immersed homogeneous Lagrangian submanifold of $T^*U \setminus 0$, and the map

$$C_\varphi \to \Lambda_\varphi, \quad (x,\theta) \mapsto (x, \varphi'_x(x,\theta))$$

is a fibration, with fibers of dimension e.

For $(x,\xi) \in \Lambda_\varphi$, denote by $C_{\varphi,(x,\xi)}$ the fiber over (x,ξ).

Lagrangian distributions

Definition

$u \in \mathscr{D}'(X;\Omega^{1/2})$ is said to be a Lagrangian distribution of order m with respect to Λ if

- $\mathrm{WF}(u) \subseteq \Lambda$,
- near any $\lambda^* \in \Lambda$, u is (microlocally) of the form $u(x)\,|dx|^{1/2}$, where

$$\boxed{u(x) = (2\pi)^{-(d+2N-2e)/4} \int e^{i\varphi(x,\theta)}\, a(x,\theta)\, d\theta}$$

and

- $\varphi \in \mathscr{C}^\infty(\mathbb{R}^d \times (\mathbb{R}^N \setminus 0), \mathbb{R})$ is a clean phase function, of excess e, parametrizing Λ $(= \Lambda_\varphi)$ near λ^*,
- $a \in S^{m+(d-2N-2e)/4}(\mathbb{R}^d \times \mathbb{R}^N)$.

We write $u \in I^m(X,\Lambda;\Omega^{1/2})$.

An example: Conormal distributions

For $Y \subset X$ a $\mathscr{C}^\infty$ submanifold, the conormal bundle

$$\Lambda = N^* Y \setminus 0 = \left\{ (x,\xi) \,\middle|\, x \in Y, \xi\big|_{T_x Y} = 0 \right\} \subset (T^* X \setminus 0)\big|_Y$$

is a Lagrangian submanifold of $T^* X \setminus 0$.

In this case, we write $I^m(X, Y; \Omega^{1/2}) = I^m(X, N^* Y \setminus 0; \Omega^{1/2})$, and distributional 1/2-densities belonging to this space are said to be conormal with respect to Y.

Lemma

a) *Let* $Q \in \Psi^1(X; \Omega^{1/2})$. *Then*

$$Q: I^m(X, \Lambda; \Omega^{1/2}) \to I^{m+1}(X, \Lambda; \Omega^{1/2}).$$

b) *If, in addition,* $\sigma^0(Q)\big|_\Lambda = 0$, *then*

$$Q: I^m(X, \Lambda; \Omega^{1/2}) \to I^m(X, \Lambda; \Omega^{1/2}).$$

The Keller-Maslov bundle

Introduce the Keller-Maslov bundle as a (locally constant) complex line bundle $M_\Lambda \to \Lambda$ as follows:

- Let φ, ψ be clean phase functions such that $\Lambda_\varphi \subseteq \Lambda$, $\Lambda_\psi \subseteq \Lambda$, and $\Lambda_\varphi \cap \Lambda_\psi \neq \varnothing$.
- Transition from Λ_φ to Λ_ψ is by multiplying by $e^{\frac{i\pi}{4}\left(\mathrm{sgn}\,\varphi''_{\theta\theta} - \mathrm{sgn}\,\psi''_{\tilde\theta\tilde\theta}\right)}$ which takes values in $\mathbb{Z}_4 = \{\pm 1, \pm i\}$.

The principal symbol map

For $u(x) = (2\pi)^{-(d+2N-2e)/4} \int e^{i\varphi(x,\theta)} a(x,\theta)\, d\theta$ as above, define the principal symbol $\sigma^{m+d/4}(u(x)|dx|^{1/2})$ of $u(x)|dx|^{1/2}$ as a section of $M_\Lambda \otimes \Omega^{1/2}$ by

$$\Lambda_\varphi \ni (x,\xi) \mapsto \int_{C_{\varphi,(x,\xi)}} e^{\frac{i\pi}{4}\operatorname{sgn}\Phi} a_{\overline{m}}(x,\theta',\theta'')|\det\Phi|^{-1/2}\, d\theta'',$$

where $\overline{m} = m + (d-2N-2e)/4$. Further, split $\theta = (\theta',\theta'')$ in such a way that the map $C_{\varphi,(x,\xi)} \ni (x,\theta) \mapsto \theta''$ has bijective differential, and let $\Phi = \begin{pmatrix} \varphi''_{xx} & \varphi''_{x\theta'} \\ \varphi''_{\theta' x} & \varphi''_{\theta'\theta'} \end{pmatrix}$.

Theorem

The principal symbol map $\sigma^{m+d/4}$ fits into a short exact sequence

$$0 \to I^{m-1}(X,\Lambda;\Omega^{1/2}) \to I^m(X,\Lambda;\Omega^{1/2}) \xrightarrow{\sigma^{m+d/4}} S^{(m+d/4)}(\Lambda; M_\Lambda \otimes \Omega^{1/2}) \to 0$$

which splits.

Definition of FIOs

- $C \subset (T^*X \setminus 0) \times (T^*Y \setminus 0)$ is said to be a canonical relation if

$$\Lambda = C' = \{(x,y,\xi,-\eta) \mid (x,\xi;y,\eta) \in C\} \subset T^*(X \times Y) \setminus 0$$

  is a Lagrangian submanifold.

- An operator $A : \mathscr{C}_c^\infty(Y;\Omega^{1/2}) \to \mathscr{C}^\infty(X;\Omega^{1/2})$ is said to be an FIO of order m with underlying canonical relation C, written $A \in I^m(X,Y,C;\Omega^{1/2})$, if its kernel K belongs to $I^m(X \times Y, \Lambda;\Omega^{1/2})$.
- Then $A : \mathscr{E}'(Y;\Omega^{1/2}) \to \mathscr{D}'(X;\Omega^{1/2})$ and

$$\mathrm{WF}(Au) \subseteq C \circ \mathrm{WF}(u), \quad \forall u \in \mathscr{E}'(Y;\Omega^{1/2}).$$

Examples of FIO

For these lectures, the most important examples are:

- $U(t) = e^{itP} \in I^0(X, X, \operatorname{graph} \chi_t; \Omega^{1/2})$ for $t \in \mathbb{R}$.
- $U \in I^{-1/4}(\mathbb{R} \times X, X, C; \Omega^{1/2})$, where

$$C = \{(t, x, \tau, \xi; y, \eta) \mid \tau = p(x, \xi), (x, \xi) = \chi_t(y, \eta), \xi \neq 0\}.$$

- $\pi_* \Delta^* \in I^0(\mathbb{R}, \mathbb{R} \times X \times X, C_0; \Omega^{1/2})$, where

$$C_0 = \{(t, \tau; t, x, x, \tau, \xi, -\xi) \mid \tau \neq 0\}.$$

Composition

Let $A \in I^m(X, Y, C_0; \Omega^{1/2})$, $B \in I^p(Y, Z, C_1; \Omega^{1/2})$. Suppose that

- $C_0 \times C_1$ and $(T^*X \setminus 0) \times \Delta_{T^*Y \setminus 0} \times (T^*Z \setminus 0)$ intersect cleanly in C, with excess e.
- The fibers of the canonical map $C \to C_0 \circ C_1$ are connected and compact.
- The map $(\operatorname{supp} K_A \times \operatorname{supp} K_B) \cap (X \times \Delta_Y \times Z) \to X \times Z$, $(x, y, y, z) \mapsto (x, z)$ is proper.

Then $A \circ B \in I^{m+p+e/2}(X, Z, C_0 \circ C_1; \Omega^{1/2})$ and the principal symbol of $A \circ B$ is computable in terms of the principal symbol of A and the principal symbol of B.

Elliptic FIO and parametrices

Let $A \in I^m(X,X,C;\Omega^{1/2})$, where $C = \operatorname{graph}\chi$ for some canonical transformation $\chi\colon T^*X \setminus 0 \to T^*X \setminus 0$.

- A is said to be elliptic if $\sigma^{m+d/4}(A)$ nowhere vanishes.
- In this case, there exists a parametrix $B \in I^{-m}(X,X,C;\Omega^{1/2})$, i.e., both

  $$A \circ B - I,\ B \circ A - I$$

  have $\mathscr{C}^\infty$ kernels.
- (Egorov's theorem) If $P \in \Psi^p(X;\Omega^{1/2})$, then $B \circ P \circ A \in \Psi^p(X;\Omega^{1/2})$, with principal symbol $\chi^*\sigma^p(P)$.

Remark The parametrix construction can be microlocalized.

The main result

Assume that the periodic bicharacteristics, of period $T \neq 0$, as a point set are $\sqcup_{l=1}^{r} Y_l \subset S^*X$, where each Y_l is a clean fixed point set of χ_T; $\dim Y_l = d_l$, $1 \leq l \leq r$.

Let $F: X \to X$ be a $\mathscr{C}^\infty$ diffeomorphism and $Y \subset X$ be a submanifold consisting of fixed points of F. Then Y is said to be a clean fixed point set of F if

$$T_y Y = \{t \in T_y X \mid dF_y(t) = t\}, \quad \forall y \in Y.$$

Theorem

Under this assumption the period T is isolated and

$$w(t) \sim \sum_{s=1}^{\overline{d}+1} b_s (t - T + i0)^{-s/2} |dt|^{1/2} + O(\log(t - T)) \quad \text{as } t \to T,$$

where $\overline{d} = \max_{1 \leq l \leq r} d_l$.

Furthermore, $b_s = 0$ *for* $s \equiv \overline{d} \pmod 2$ *if* $d_l \equiv d_{l'} \pmod 2$ *for all* $1 \leq l, l' \leq r$.

Proof of theorem

Working microlocally, the contribution coming from Y_l in the composition $\pi_* \Delta^* U$ is clean and of excess d_l, hence the result belongs to $I^{d_l/2-1/4}(\mathbb{R}, \Omega^{1/2}; \Lambda_T^+)$.
By its very definition, $I^{d_l/2-1/4}(\mathbb{R}, \Omega^{1/2}; \Lambda_T^+)$ consists of 1/2-distributional densities having asymptotic expansions as $t \to T$ in scalar multiples of

$$\left(\int_0^\infty e^{it\tau} \tau^{\frac{d_l-1}{2}-k} \, d\tau \right) |dt|^{1/2}, \quad k \in \mathbb{N}_0,$$

from which the result follows. $\square$

The case $d_1 = \ldots = d_r = 1$

In this case, the Y_l are closed orbits and the cleanness assumption is equivalent with the periodic bicharacteristics, γ, of period T, be non-degenerate.

The linearized Poincaré map is the map

$$\Pi_\gamma \colon T_{\gamma(0)}(S^*X)/\operatorname{span} H_p \to T_{\gamma(0)}(S^*X)/\operatorname{span} H_p$$

induced by $d\chi_T(\gamma(0))$ (note that $\gamma(T) = \gamma(0)$) which is symplectic.

- Eigenvalues come in quadruples: $\lambda, \overline{\lambda}, \lambda^{-1}, \overline{\lambda}^{-1}$
- γ is non-degenerate if 1 is not an eigenvalue.
- A non-degenerate γ is elliptic if $|\lambda| = 1$ for all eigenvalues λ, i.e., $\lambda = \overline{\lambda}^{-1}$.
- A non-degenerate γ is hyperbolic if $\lambda \in \mathbb{R}$ for all eigenvalues λ, i.e., $\lambda = \overline{\lambda}$.
- A non-degenerate γ is loxodromic if all four $\lambda, \overline{\lambda}, \lambda^{-1}, \overline{\lambda}^{-1}$ are different.
- Mixed cases are possible.

The case $d_1 = \ldots = d_r = 1$, cont.

The result in this case reads

$$w(t) \sim c_{-1\,T}(t - T + i0)^{-1}|dt|^{1/2} + \sum_{r \geq 0} c_{rT}(t - T + i0)^{r}\log(t - T + i0)|dt|^{1/2} \text{ as } t \to T$$

which is implied by $w \in I^{1/4}(\mathbb{R}, \Omega^{1/2}; \Lambda_T^{+})$ where the latter holds near $t = T$.