POLYHOMOGENEOUS FUNCTIONS

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Contents

Roughly speaking, a function u of a single variable $x > 0$ is polyhomogeneous if it has an asymptotic expansion as $x \to 0$ of the form

$$u(x) \sim \sum_{z,k} a_{z,k}\, x^z \log^k x, \quad a_{z,k} \in \mathbb{C} \tag{1}$$

Here $z \in \mathbb{C}$, $k \in \mathbb{N}_0$, and for each z only finitely many $a_{z,k}$ are non-zero. We write $\log^k x = (\log x)^k$. Functions of this sort arise in various ways:

- As solutions of differential equations.
- As results of integrating smooth functions (see the push-forward theorem).
- etc.

We will do the following:

(1) Make precise the meaning of the asymptotic expansion; this includes fixing the sets of (z, k) which may occur in the expansion (index sets). We will also want to be able to 'differentiate the asymptotics'[1], so we make this requirement part of the definition of (1).
(2) Allow dependence of u and the $a_{z,k}$ on additional variables (parameters); geometrically this means considering functions on a half space rather than a half line
(3) Characterize polyhomogeneity in terms of differential equations

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[1] since we want to use asymptotics in solving differential equations

(4) Extend this to asymptotics in terms of several variables approaching a limit. That is, consider functions on quadrants/octants etc. instead of a half line; or more generally quadrants/octants... times Euclidean spaces if parameters are present.
(5) Think about coordinate invariance. This leads to generalization to manifolds with corners.

0.1. **Preliminaries, polyhomogeneous functions on the half line.** First note that (all asymptotics are meant as $x \to 0$)

$$x^z \log^k x = o(x^{z'} \log^{k'} x) \text{ iff } \begin{cases} \mathrm{Re} z > \mathrm{Re} z' & \text{or} \\ \mathrm{Re} z = \mathrm{Re} z', k < k' \end{cases}$$

For example, the sequence of functions

$$\log^2 x,\ \log x,\ x^i \log x,\ 1,\ x \log x,\ x,\ x^2,\ x^\pi \log^{10} x$$

is decreasing with respect to the order $g > f :\iff f = o(g)$.

Therefore, the asymptotic series in (1) makes sense if the sum runs over $(z,k) \in E$ where E satisfies condition (a) in the following definition.

Definition 1. *An **index set** is a subset $E \subset \mathbb{C} \times \mathbb{N}_0$ satisfying*

(a) *For each $s \in \mathbb{R}$ the set*

$$E_{\leq s} := \{(z,k) \in E : \mathrm{Re} z \leq s\}$$

is finite.

(b) $(z,k) \in E$, $0 \leq l \leq k \Rightarrow (z,l) \in E$.

*E is a C^∞**-index set** if in addition*

(c) $(z,k) \in E \Rightarrow (z+1,k)$.

We also denote

$$\inf E := \min\{\mathrm{Re} z : (z,k) \in E \text{ for some } k\}$$

Condition (b) means that with any $\log x$ power also all the lesser powers may appear (with the same x^z). We will see presently why this is useful. Condition (c) will be important when considering invariance under coordinate changes.

In the sequel the differential expression (operator) $x\partial_x := x\frac{\partial}{\partial x}$ will occur frequently. One reason for this is that it behaves very nicely (much better than ∂_x) with functions of the form $x^z \log^k x$:

$$x\partial_x(x^z) = z\, x^z, \quad x\partial_x(\log^k x) = k \log^{k-1} x$$

and therefore

$$x\partial_x(x^z \log^k x) = z\, x^z \log^k x + k\, x^z \log^{k-1} x \tag{2}$$

so the space spanned by $x^z \log^j x$, $j = 0, \dots, k$, is invariant under the operator $x\partial_x$ for any k. This would be false for the operator ∂_x.

In the following definitions we write

$$\mathbb{R}_+ = [0, \infty), \quad \mathrm{int}(\mathbb{R}_+) = (0, \infty)$$

Definition 2. *Let E be an index set. A* ***polyhomogeneous function on $\mathbb{R}_+$ with index set*** *E is a smooth function $u : \mathrm{int}(\mathbb{R}_+) \to \mathbb{C}$ for which there are $a_{z,k} \in \mathbb{C},\ (z,k) \in E$, so that for all $j \in \mathbb{N}_0$ and $s \in \mathbb{R}$ we have*

$$(x\partial_x)^j \left(u(x) - \sum_{(z,k)\in E_{\leq s}} a_{z,k}\, x^z \log^k x \right) = O(x^s) \tag{3}$$

In this case we write

$$u(x) \sim \sum_{z,k} a_{z,k}\, x^z (\log x)^k$$

The space of polyhomogeneous functions on $\mathbb{R}_+$ with index set E is denoted

$$\mathcal{A}^E(\mathbb{R}_+)$$

Here we use the

Convention: All O estimates are to be understood as locally uniform on the spaces in question.

That is, $f(x) = O(x^s)$ means that for any compact set $K \subset \mathbb{R}_+$ there is a constant C so that $|f(x)| \leq Cx^s$ for all $x \in K$ at which f is defined. The main point is that we have a statement about behavior as $x \to 0$ (since K may contain zero), but none about behavior as $x \to \infty$.

Remarks 3.

1. *$u(x)$ is only defined for $x > 0$, but we say that u is polyhomogeneous on $[0, \infty)$ since there is a condition on the behavior of u in arbitrarily small pointed neighborhoods of 0.*
2. *If we required (3) only for $j = 0$ then we would get the standard notion of asymptotic series (no derivatives).*
3. *We would obtain the same space of functions if we required*

$$(x\partial_x)^j \left(u(x) - \sum_{\substack{(z,k)\in E \\ \mathrm{Re}\, z<s}} a_{z,k}\, x^z \log^k x \right) = O(x^{s-\varepsilon}) \tag{4}$$

*for all j, s and $\varepsilon > 0$. This is the definition used in [**?**].*

Proposition 4. *Let E be an index set. Then $\mathcal{A}^E(\mathbb{R}_+)$ is a vector space, and is mapped by $x\partial_x$ to itself.*

This is obvious from (2). Note that for the last statement one needs condition (b) in Definition 1 and all j in Definition 2.

Definition 2 may be made more digestible by introducing some notation. We will do this after considering parameters.

Examples 5.

(1) *A function is smooth on $\mathbb{R}_+$ if and only if it is polyhomogeneous with index set $\mathbb{N}_0 \times \{0\}$. One implication in this equivalence follows directly from Taylor's theorem, applied to the function and its derivatives. The other directions is an exercise .*
(2) *Clearly, x^{-2}, x^e, $\log x$ are polyhomogeneous on $\mathbb{R}_+$ with suitable index sets.*
(3) *The function $\sin \frac{1}{x}$ is not polyhomogeneous on $\mathbb{R}_+$ for any index set: Its fast oscillation as $x \to 0$ cannot be modelled using functions of the form $x^z \log^k x$. If we used the Taylor series $\sin \frac{1}{x} = \frac{1}{x} - \frac{1}{6}\frac{1}{x^3} \pm \dots$ then arbitrarily large negative powers of x would appear. This is not allowed for an index set.*

(refer to manifolds with corners chapter)

0.2. **Polyhomogeneous functions on the half space.** We now consider functions on the half space

$$H^n := \mathbb{R}_+ \times \mathbb{R}^{n-1}$$

It is standard to denote the variables

$$x \in \mathbb{R}_+, \quad y = (y_1, \dots, y_{n-1}) \in \mathbb{R}^{n-1}$$

The definition of polyhomogeneity extends in a straightforward way, where we want to assume smooth dependence on y and also the possibility to differentiate the asymptotic series in y.

Definition 6. *Let E be an index set. A* ***polyhomogeneous function on H^n with index set*** *E is a smooth function $u : \operatorname{int}(H^n) \to \mathbb{C}$ for which there are $a_{z,k} \in C^\infty(\mathbb{R}^{n-1})$, $(z,k) \in E$, so that for all $j \in \mathbb{N}_0$, $\alpha \in \mathbb{N}_0^{n-1}$ and $s \in \mathbb{R}$ we have*

$$(x\partial_x)^j \partial_y^\alpha \left(u(x,y) - \sum_{(z,k)\in E_{\leq s}} a_{z,k}(y)\, x^z \log^k x \right) = O(x^s) \tag{5}$$

In this case we write

$$u(x,y) \sim \sum_{(z,k)\in E} a_{z,k}(y)\, x^z (\log x)^k$$

The space of polyhomogeneous functions on H^n with index set E is denoted

$$\mathcal{A}^E(H^n)$$

Recall the convention that O estimates are meant to be uniform on compact subsets. Here this means compact subsets of H^n.

The following definitions are designed to focus attention on various aspects of this definition. First, it is useful to give a name to combinations of derivatives as they occur in (5).

Definition 7. *A* ***b-differential operator*** *on H^n is an operator of the form*

$$\sum_{j,\alpha} b_{j,\alpha}(x,y)(x\partial_x)^j \partial_y^\alpha$$

where $b_{j,\alpha} \in C^\infty(H^n)$ for all $j \in \mathbb{N}_0, \alpha \in \mathbb{N}_0^{n-1}$ and only finitely many terms of the sum are non-zero. The space of b-differential operators on H^n is denoted $\operatorname{Diff}_b^*(H^n)$.

As usual the **order** of $P \in \mathrm{Diff}_b^*(H^n)$ is defined as the largest $j + |\alpha|$ for which $b_{j,\alpha}$ is not identially zero, and by $\mathrm{Diff}_b^m(H^n)$ we denote the set of b-operators of order at most m. Clearly this is a vector space, and the composition of two b-operators is a b-operator.

We now introduce spaces in which the remainders – the expressions in parantheses in (5) – lie.

Definition 8. *For $s \in \mathbb{R}$ let*

$$\mathcal{A}^s(H^n) = \{u \in C^\infty(\mathrm{int}(H^n)) : \ Pu = O(x^s) \ \textit{for all}\ P \in \mathrm{Diff}_b^*(H^n)\}$$

Functions in $\mathcal{A}^s(H^n)$ are sometimes called **conormal** with respect to the boundary ∂H^n. The definition of polyhomogeneity translates directly as:

Lemma 9. *A function $u \in C^\infty(\mathrm{int}(H^n))$ is polyhomogeneous, $u(x,y) \sim \sum_{(z,k)\in E} a_{z,k}(y)\, x^z (\log x)^k$ if and only if for each $s \in \mathbb{R}$ we can write*

$$u = \sum_{(z,k)\in E_{\leq s}} a_{z,k}\, x^z \log^k x \ + \ r_s, \quad r_s \in \mathcal{A}^s(H^n) \tag{6}$$

Remark 10. *The condition $r_s \in \mathcal{A}^s(H^n)\ \forall s$ is equivalent to the seemingly weaker condition that $r_s \in \mathcal{A}^{s'}(H^n)\ \forall s$, for some s' which tends to infinity as $s \to \infty$. (For example $s' = s-1$ or $s' = s/2$.)*

(Proof as exercise.)

As before, we have

Proposition 11. *Let E be an index set. Then $\mathcal{A}^E(H^n)$ and $\mathcal{A}^s(H^n)$ are vector spaces, and are mapped by $\mathrm{Diff}_b^*(H^n)$ to themselves.*

0.3. **Characterization by differential operators.** In order to prove some basic properties of polyhomogeneous functions, it is useful to characterize them in a different way.

The starting point is the observation that $(x\partial_x - z)x^z = 0$, and more generally

$$(x\partial_x - z)x^z \log^k x = kx^z \log^{k-1} x \tag{7}$$

which implies $(x\partial_x - z)^{k+1}x^z \log^k x = 0$. A neat way to understand this is by noticing that $x\partial_x - z = x^z x\partial_x x^{-z}$ (conjugation of $x\partial_x$ by the operator of multiplication by x^z), which reduces the claims to the case $z = 0$.

More precisely and more generally, we have for any finite subset $S \subset \mathbb{C}$ and numbers $p_z \in \mathbb{N}_0$

$$\ker \prod_{z\in S} (x\partial_x - z)^{p_z+1} = \{\sum_{z\in S} \sum_{k=0}^{p_z} a_{z,k}\, x^z \log^k x,\ a_{z,k} \in \mathbb{C}\} \tag{8}$$

as functions on $\mathrm{int}(\mathbb{R}_+)$: Clearly the functions on the right are in the kernel, and then the equality follows from a dimensional argument.

The right side of (8) is simply a 'piece' of the polyhomogeneous expansion! This makes the following theorem plausible.

Theorem 12. *For each $s \in \mathbb{R}$ define the differential operator*

$$B_{E,s} = \prod_{(z,k)\in E_{\leq s}} (x\partial_x - z)$$

Then

$$\mathcal{A}^E(H^n) = \{u \in C^\infty(\mathrm{int}(H^n)) : B_{E,s}u \in \mathcal{A}^s(H^n) \text{ for all } s \in \mathbb{R}\} \tag{9}$$

In this characterization of polyhomogeneity the coefficients $a_{z,k}$ do not appear explicitly!

Note that the factor $x\partial_x - z$ appears $p+1$ times in $B_{E,s}$ if $p = \max\{k : (z,k) \in E\}$. This implies

$$\{v \in C^\infty(\mathrm{int}(H^n) : B_{E,s}v = 0\} = \{ \sum_{(z,k)\in E_{\leq s}} a_{z,k}(y) x^z \log^k x : a_{z,k} \in C^\infty(\mathbb{R}^{n-1})\} \tag{10}$$

by the remarks before the theorem (applied for any fixed y; the $a_{z,k}$ must be smooth in y by smoothness of u).

Proof. First, let $u \in \mathcal{A}^E(H^n)$. For any $s \in \mathbb{R}$, write u as in (6). Then (10) implies $B_{E,s}u = B_{E,s}r_s$, and by Proposition 11 this lies in $\mathcal{A}^s(H^n)$ since $B_{E,s} \in \mathrm{Diff}_b^*(H^n)$.

We have proved the inclusion '$\subset$' of (9). To prove the converse, we use the following lemma.

Lemma 13. *Let $s \in \mathbb{R}$, $z \in \mathbb{C}$.*

(1) $u \in \mathcal{A}^s \Rightarrow x^z u \in \mathcal{A}^{s+\mathrm{Re}z}$
(2) *If $u \in \mathcal{A}^s$ then there is $w \in \mathcal{A}^s$ such that $(x\partial_x - z)w = u$.*

Proof. Note that $x\partial_x x^z = x^z(x\partial_x + z)$. Applying this repeatedly, we see that for any $P \in \mathrm{Diff}_b^*(H^n)$ there is $P' \in \mathrm{Diff}_b^*(H^n)$ satisfying $Px^z = x^z P'$. This implies (1) since $|x^z| = x^{\mathrm{Re}z}$. Using conjugation by x^z and (1) we may assume $z = 0$ in (2). Set

$$w(x,y) = \begin{cases} \int_0^x u(t,y)\frac{dt}{t} & \text{if } s > 0 \\ \int_1^x u(t,y)\frac{dt}{t} & \text{if } s \leq 0 \end{cases}$$

Then $x\partial_x w = u$, and $u = O(x^s)$ implies $w = O(x^s)$ if $s > 0$, and $w = O(1 + x^s) = O(x^s)$ if $s \leq 0$, and similarly for the $\partial_y^\alpha w$ estimates. The estimates of $(x\partial_x)^j \partial_y^\alpha w$ for $j \geq 1$ follow from $x\partial_x w = u$ and the estimates for u. □

To finish the proof of Theorem 12 assume $u \in C^\infty(\mathrm{int}(H^n))$ satisfies $B_{E,s}u \in \mathcal{A}^s(H^n)$ for all $s \in \mathbb{R}$. Fix s and let $\tilde{u} = B_{E,s}u$. Applying the lemma iteratively find $w \in \mathcal{A}^s$ with $\tilde{u} = B_{E,s}w$. Then $B_{E,s}(u - w) = 0$, hence by (10) there are $a_{z,k} \in C^\infty(\mathbb{R}^{n-1})$ for $\mathrm{Re}z \leq s$ so that $u - w = \sum_{(z,k)\in E_{\leq s}} a_{z,k}\, x^z \log^k x$. It is easy to check that when the same procedure is done for $s' > s$, producing coefficients $a'_{z,k}$, then one must have $a'_{z,k} = a_{z,k}$ for $\mathrm{Re}z \leq s$. It follows that $u \sim \sum_{(z,k)\in E} a_{z,k}(y)\, x^z (\log x)^k$, so $u \in \mathcal{A}^E(H^n)$. □

0.4. **Polyhomogeneous functions on a quadrant.** We will first discuss polyhomogeneous functions on the simplest manifold with corners, the quadrant $\mathbb{R}^2_+$. This will guide us how to proceed for general manifolds with corners.

What's the idea? Polyhomogeneity of a smooth function u on $\mathrm{int}(\mathbb{R}^2_+)$ should involve three things:

(1) $u(x,y)$ should be polyhomogeneous as $x \to 0$, smoothly in $y > 0$.
(2) $u(x,y)$ should be polyhomogeneous as $y \to 0$, smoothly in $x > 0$.
(3) These expansions should be uniform, in a suitable sense, at the corner, i.e. for $x \to 0$ and $y \to 0$.

The smooth dependence in 1. and 2. should be as in the case of the half space, but it is less clear how to make 3. precise. There are different ways to do this. First, we should fix index sets for both side faces.

Definition 14. *Let M be a manifold with corners. An **index family** $\mathcal{E}$ for M is an assignment of an index set $\mathcal{E}(H)$ to each boundary hypersurface H of M.*

For $M = \mathbb{R}^2_+$ we denote an index family simply by (E,F), where E is considered as index set for $\{x=0\}$ and F is an index set for $\{y=0\}$.[4]

Next, we extend the definitions of b-differential operators and conormal spaces to this case.

Definition 15. *A **b-differential operator** on $\mathbb{R}^2_+$ is an operator of the form*

$$\sum_{j,l} b_{j,l}(x,y)(x\partial_x)^j(y\partial_y)^l$$

where $b_{j,l} \in C^\infty(\mathbb{R}^2_+)$ for all $j,l \in \mathbb{N}_0$ and only finitely many terms of the sum are non-zero. The space of b-differential operators on $\mathbb{R}^2_+$ is denoted $\mathrm{Diff}^*_b(\mathbb{R}^2_+)$.

Definition 16. *For $s,t \in \mathbb{R}$ let*

$$\mathcal{A}^{(s,t)}(\mathbb{R}^2_+) = \{u \in C^\infty(\mathrm{int}(\mathbb{R}^2_+)) : Pu = O(x^s y^t) \text{ for all } P \in \mathrm{Diff}^*_b(\mathbb{R}^2_+)\}$$

Here the local uniformity implicit in the O is for compact subsets of $\mathbb{R}^2_+$.

Definition 17. *Let (E,F) be an index family for $\mathbb{R}^2_+$. A **polyhomogeneous function on** $\mathbb{R}^2_+$ **with index family** (E,F) is a smooth function $u : \mathrm{int}(\mathbb{R}^2_+) \to \mathbb{C}$ for which there are*

$$a_{z,k} \in \mathcal{A}^F(\mathbb{R}_+),\ (z,k) \in E \text{ and } b_{w,l} \in \mathcal{A}^E(\mathbb{R}_+),\ (w,l) \in F$$

and $N \in \mathbb{R}$ so that for all $s \in \mathbb{R}$ we have

$$u = \sum_{(z,k)\in E_{\le s}} a_{z,k}(y) x^z \log^k x + r_s, \quad r_s \in \mathcal{A}^{(s,-N)}(\mathbb{R}^2_+) \tag{11}$$

$$u = \sum_{(w,l)\in F_{\le s}} b_{w,l}(x) y^z \log^k y + r'_s, \quad r'_s \in \mathcal{A}^{(-N,s)}(\mathbb{R}^2_+) \tag{12}$$

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[4] This is opposite to the notation used in [**?**].

The $-N$ should be thought of as any number smaller than $\inf E$ and $\inf F$. It is needed since both u and the sum on the right in (11) will behave like $y^{\inf E}$ times logarithms, and similarly for (12).

Examples 18.

(1) *u is smooth on $\mathbb{R}^2_+$ if and only if it is polyhomogeneous with index sets $E = F = \mathbb{N}_0 \times \{0\}$.*

(2) *The function $u(x,y) = \sqrt{x^2+y^2}$ is smooth on $\mathbb{R}^2 \setminus \{(0,0)\}$, so it has polyhomogeneous expansions in the interior of each boundary hypersurface. However, u is not polyhomogeneous on $\mathbb{R}^2_+$. To see this, we find the expansion at the face $x = 0$ by writing, for $y > 0$,*

$$\sqrt{x^2+y^2} = y\sqrt{1+(x/y)^2} = y\sum_0^\infty c_i(\frac{x}{y})^{2i} \tag{13}$$

$$= y + \frac{1}{2}\frac{x^2}{y} - \frac{1}{8}\frac{x^4}{y^3} + \dots \tag{14}$$

with the Taylor series $\sqrt{1+t} = \sum_0^\infty c_i t^i = 1 + t/2 - t^2/8 + \dots$ (for $|t| < 1$). Thus, in the expansion $u(x,y) \sim \sum_{i=0}^\infty a_{2i}(y)x^{2i}$ the coefficients are $a_{2i}(y) = c_i y^{1-2i}$. Although each a_{2i} is polyhomogeneous as $y \to 0$, there is no index set F so that each a_{2i} has the same index set F. Therefore, polyhomogeneity at the corner fails.

Remark 19. *Equations* (11) *and* (12) *imply that the coefficient functions $a_{z,k}$, $b_{w,l}$ must satisfy compatibility conditions at the corner: When we write*

$$a_{z,k}(y) \sim \sum_{(w,l)\in F} c_{z,k,w,l}\, y^w \log^l y, \quad b_{2,l}(x) \sim \sum_{(z,k)\in E} c'_{z,k,w,l}\, x^z \log^k x$$

then necessarily $c_{z,k,w,l} = c'_{z,k,w,l}$ for all z,k,w,l.

Again we have a characterization of polyhomogeneity by differential operators, which avoids explicit appearance of the coefficient functions.

Theorem 20. *For each $s \in \mathbb{R}$ define the differential operators*

$$B^x_{E,s} = \prod_{(z,k)\in E_{\le s}} (x\partial_x - z)$$

$$B^y_{F,s} = \prod_{(w,l)\in F_{\le s}} (y\partial_y - w)$$

Then $u \in \mathcal{A}^{(E,F)}(\mathbb{R}^2_+)$ iff u is smooth in the interior and there is $N \in \mathbb{R}$ so that for all $s \in \mathbb{R}$

$$B^x_{E,s}u \in \mathcal{A}^{(s,-N)}(\mathbb{R}^2_+), \quad B^y_{F,s}u \in \mathcal{A}^{(-N,s)}(\mathbb{R}^2_+) \tag{15}$$

Proof. Analogous to the proof of Theorem 12. ∎

(15) implies that

$$B^x_{E,s} B^y_{F,s} u \in \mathcal{A}^{(t,t)}(\mathbb{R}^2_+), \quad t = \frac{s-N}{2}$$

since the spaces $\mathcal{A}^{(s,t)}(\mathbb{R}^2_+)$ are invariant under b-differential operators and $\mathcal{A}^{(s,-N)}(\mathbb{R}^2_+) \cap \mathcal{A}^{-N.s}(\mathbb{R}^2_+) \subset \mathcal{A}^{(t,t)}(\mathbb{R}^2_+)$. The proof of this inclusion is left as an exercise.

0.5. **Polyhomogeneous functions on general model spaces.** Next we consider the spaces

$$\mathbb{R}^n_k := \mathbb{R}^k_+ \times \mathbb{R}^{n-k}$$

which are the local models for general manifolds with corners. The extension of the previous discussion to this case is straightforward: The $\mathbb{R}^{n-k}$ variables are treated like (smooth) parameters as in the case of a halfspace, and having a codimension k corner is analogous to a codimension 2 corner. This should really be worked out by the reader as an exercise, but we provide the main steps.

By convention, we denote the coordinates as $x_1, \dots, x_k \in \mathbb{R}_+$, $y_1, \dots, y_{n-k} \in \mathbb{R}$ and also $x = (x_1, \dots, x_k)$, $y = (y_1, \dots, y_{n-k})$. Then **b-differential operators** are, by definition, operators of the form

$$\sum_{\alpha \in \mathbb{N}_0^k, \beta \in \mathbb{N}_0^{n-k}} b_{\alpha,\beta}(x,y)(x_1\partial_{x_1})^{\alpha_1} \dots (x_k \partial_{x_k})^{\alpha_k} \partial_y^\beta \tag{16}$$

with all $b_{\alpha,\beta}$ smooth on $\mathbb{R}^n_k$. For $s_1, \dots, s_k \in \mathbb{R}$ we define the **conormal spaces**

$$\mathcal{A}^{(s_1,\dots,s_k)}(\mathbb{R}^n_k) = \{u \in C^\infty(\mathrm{int}(\mathbb{R}^n_k)) : \, Pu = O(x_1^{s_1} \dots x_k^{s_k}) \text{ for all } P \in \mathrm{Diff}^*_b(\mathbb{R}^n_k)\}$$

As always, the O is to be understood as being locally uniform on $\mathbb{R}^n_k$.

An index family for $\mathbb{R}^n_k$ is given by k index sets $E_1, \dots, E_k$, with E_j associated to the boundary hypersurface $\{x_j = 0\}$ for each j. We want to define the space

$$\mathcal{A}^{(E_1,\dots,E_k)}(\mathbb{R}^n_k)$$

of polyhomogeneous functions on $\mathbb{R}^n_k$ with index family $(E_1, \dots, E_k)$. The definition of polyhomogeneity works by induction over k. Suppose we have defined polyhomogeneous functions on spaces $\mathbb{R}^n_{k-1}$ for any n, then a **polyhomogeneous function on $\mathbb{R}^n_k$ with index family** $(E_1, \dots, E_k)$ is a smooth function $u : \mathrm{int}(\mathbb{R}^n_k) \to \mathbb{C}$ so that there are $N \in \mathbb{R}$ and functions for each $j = 1, \dots, k$

$$a^{(j)}_{z,k} \in \mathcal{A}^{(E_1,\dots,\widehat{E_j},\dots,E_k)}(\mathbb{R}^{n-1}_{k-1}), \quad (z,k) \in E_j$$

with the hat denoting omission, such that for all $s \in \mathbb{R}$ and for each j we have

$$u = \sum_{(z,k)\in(E_j)_{\le s}} a^{(j)}_{z,k}(x_{\neq j}, y)\, x_j^z \log^k x_j + r^{(j)}_s, \quad r^{(j)}_s \in \mathcal{A}^{(-N,\dots,s,\dots,-N)}(\mathbb{R}^n_k) \tag{17}$$

where $x_{\neq j} = (x_1, \dots, \widehat{x_j}, \dots, x_k)$ and the s is at the jth spot.

As for $\mathbb{R}^2_+$ this implies (multiple) compatibility relations for the expansion coefficients for different j, and also we have a characterization by differential operators analogous to Theorem 20.

Note that if we allow the functions $a^{(j)}_{z,k}$ in (17) to depend on *all* variables (including x_j) then by expanding them in Taylor series around $x_j = 0$ we get an expansion of the type (17), but with additional terms involving $x_j^{z+m} \log^k x_j$ for $m \in \mathbb{N}_0$. This is where part (c) of Definition 1 matters, and we get:

Lemma 21. *Suppose each E_j is a C^∞ index set. Then $\mathcal{A}^{(E_1,\dots,E_k)}(\mathbb{R}^n_k)$ is equal to the space of functions having expansions as in* (17) *but with the $a^{(j)}_{z,k}$ depending on all variables x, y.*

Note that allowing these more general coefficients has the disadvantage that they are not uniquely determined by u (while those in (17) are). However, the lemma will be needed when discussing coordinate invariance.

0.6. **The invariant perspective: Manifolds with corners.** The only additional issue which arises when we consider manifolds with corners is invariance under coordinate changes. We first discuss this for $\mathbb{R}_+$. The standard coordinate x on $\mathbb{R}_+$ is a boundary defining function for the boundary hypersurface $\{0\}$. A general boundary defining function is a function on $\mathbb{R}_+$ which vanishes at 0, has non-vanishing derivative there, and is positive on $(0,\infty)$. By Taylor's theorem, it can be written as

$$x' = x\rho(x), \quad \rho \in C^\infty(\mathbb{R}_+),\ \rho > 0 \text{ on } \mathbb{R}_+$$

Then

$$(x')^z \log^k x' = x^z \rho(x)^z (\log x + \log \rho(x))^k$$

Now ρ is smooth and positive, hence $\log \rho$ and ρ^z are smooth. Expanding these functions in Taylor series around $x = 0$ and multiplying out, we see that

$$(x')^z \log^k x' \sim \sum_{m=0}^{\infty} \sum_{l=0}^{k} \gamma_{m,l}\, x^{z+m} \log^l x$$

for certain coefficients $\gamma_{m,l}$. The fact that also the powers x^{z+m} appear on the right is another reason, besides Lemma 21, for the condition (c) in Definition (1).

Now consider coordinate changes on $\mathbb{R}^n_k$. A general boundary defining function for the boundary hypersurface $\{x_j = 0\}$ is of the form

$$x'_j = x_j \rho(x, y), \quad \rho \in C^\infty(\mathbb{R}^n_k), \rho > 0$$

A simple inductive argument together with Lemma 21 shows that:

Proposition 22. *Let $\mathcal{E}$ be an index family for $\mathbb{R}^n_k$. If each index set in $\mathcal{E}$ is a C^∞-index set then the space $\mathcal{A}^{\mathcal{E}}(\mathbb{R}^n_k)$ is invariant under changes of coordinates.*

Also, it is clear that the definitions of these spaces are local in the sense that $u \in \mathcal{A}^{\mathcal{E}}(\mathbb{R}^n_k)$ if and only if $\rho u \in \mathcal{A}^{\mathcal{E}}(\mathbb{R}^n_k)$ for every $\rho \in C^\infty(\mathbb{R}^n_k)$. Therefore, if we define, for any open subset $U \subset \mathbb{R}^n_k$,

$$\mathcal{A}^{\mathcal{E}}(U) = \{u \in C^\infty(U \cap \operatorname{int}(\mathbb{R}^n_k)) :\ \rho u \in \mathcal{A}^{\mathcal{E}}(\mathbb{R}^n_k) \text{ for all } \rho \in C^\infty_0(U)\}$$

then this is compatible with the previous definition in case $U = \mathbb{R}^n_k$. This allows us to define:

Definition 23. *Let M be a manifold with corners and $\mathcal{E}$ a C^∞ index set for M. A* ***polyhomogeneous function on M with index family $\mathcal{E}$*** *is a smooth function $u : \mathrm{int}(M) \to \mathbb{C}$ which is polyhomogeneous with corresponding index family in any local chart.*

Explicitly, this means that for any chart $\varphi : \tilde{U} \to U$, $\tilde{U} \subset \mathbb{R}^n_k$ open, we have $\varphi^* u \in \mathcal{A}^{\mathcal{E}'}(\tilde{U})$, where $\mathcal{E}'(\phi^{-1}(H \cap U)) := \mathcal{E}(H)$ for every boundary hypersurface H of M which intersects U.

We can also generalize the remaining discussion to the manifold case. We start by reformulating the definition of b-differential operators on $\mathbb{R}^n_k$, (16), in an invariant way. First, consider first order operators annihilating constants, i.e. vector fields. Note that any smooth vector field on $\mathbb{R}^n_k$ can be written

$$\sum_{j=1}^{k} a_j \partial_{x_j} + \sum_{l=1}^{n-k} b_l \partial_{y_l}$$

with smooth functions a_j, b_l. Such a vector field is tangent to the boundary hypersurface $\{x_j = 0\}$ if and only if $a_j = 0$ at $x_j = 0$, which is equivalent to $a_j = x_j a'_j$ for some smooth function a'_j. This shows that

$$\begin{aligned}&\{\text{smooth vector fields on } \mathbb{R}^n_k \text{ which are tangent to all boundary hypersurfaces}\}\\ &\quad = \mathrm{span}_{C^\infty(\mathbb{R}^n_k)}\{x_1\partial_{x_1}, \dots, x_k\partial_{x_k},\ \partial_{y_1}, \dots, \partial_{y_{n-k}}\} :=\\ &\quad \{\sum_{j=1}^{k} a_j\, x_j \partial_{x_j} + \sum_{l=1}^{n-k} b_l \partial_{y_l},\ a_j, b_l \in C^\infty(\mathbb{R}^n_k)\ \forall j, l\}\end{aligned}$$

Then clearly, for $m \in \mathbb{N}_0$,

$$\mathrm{Diff}^m_b(\mathbb{R}^n_k) = \{a + \sum_{r=1}^{m} \sum_{V_1,\dots,V_r \in \mathcal{V}_b(\mathbb{R}^n_k)} V_1 \dots V_r :\ a \in C^\infty(\mathbb{R}^n_k)\}$$

This generalizes naturally to manifolds:

Definition 24. *Let M be a manifold with corners. Then define*

$$\mathcal{V}_b(M) = \{\textit{smooth vector fields on } M \textit{ which are tangent to all boundary hypersurfaces}\}$$

and for $m \in \mathbb{N}_0$

$$\mathrm{Diff}^m_b(M) = \{a + \sum_{r=1}^{m} \sum_{V_1,\dots,V_r \in \mathcal{V}_b(M)} V_1 \dots V_r :\ a \in C^\infty(M)\}$$

This leads directly to conormal spaces:

Definition 25. *Let M be a manifold with corners. A* ***weight family*** *for M is a map $\mathcal{M}_1(M) \to \mathbb{R}$, where $\mathcal{M}_1(M)$ is the set of boundary hypersurfaces of M.*

For a set of boundary defining functions ρ_H for each $H \in \mathcal{M}_1(M)$ and for a weight family $\mathfrak{s}$ define $\rho^{\mathfrak{s}} = \prod_{H\in\mathcal{M}_1(M)} \rho_H^{\mathfrak{s}(H)}$. Finally, define the conormal space

$$\mathcal{A}^{\mathfrak{s}}(M) = \{u \in C^\infty(\mathrm{int}(M)) : \ Pu = O(\rho^{\mathfrak{s}}) \text{ for all } P \in \mathrm{Diff}_b^*(M)\}$$

As always, the O is understood locally uniformly on M. This is a reasonable definition since clearly the set of functions which are $O(\rho^{\mathfrak{s}})$ is independent of the choice of the ρ_H. Also, we see that in the case of $M = \mathbb{R}^n_k$ we get back the previous definition.

Now we have generalizations of all previous results:

- $C^\infty(M) = \mathcal{A}^{\mathcal{E}_0}(M)$ where $\mathcal{E}_0(H) = \mathbb{N}_0 \times \{0\}$ for all H.
- $\mathcal{A}^{\mathfrak{s}}(M)$ and $\mathcal{A}^{\mathcal{E}}(M)$ are vector spaces and invariant under $\mathrm{Diff}_b^*(M)$.
- There is also a characterization of $\mathcal{A}^{\mathcal{E}}(M)$ using vector fields. This is a little subtle since we need to think carefully about an invariant generalization of the vector fields $x_i\partial_{x_i}$ in Theorem 20. The main observation is that, for $M = \mathbb{R}^n_k$, the vector field $x_i\partial_{x_i}$ $(i \in \{1,\dots,k\})$ turns under any coordinate change into a vector field of the form

$$x_i\partial_{x_i} + x_iV, \ V \in \mathcal{V}_b(\mathbb{R}^n_k) \tag{18}$$

  and that the space of these vector fields is invariant under coordinate changes. We call a b-vector field on M **radial** with respect to the boundary hypersurface H if it has the form (18) in any coordinate system, with x_i defining H.

  Also, it is easily checked that (15) remains true if $x\partial_x$, $y\partial_y$ are replaced by any radial vector fields for the respective boundary hypersurfaces. In light of this, the following theorem is natural, and we leave the details of the proof to the reader.

  Theorem 26. *Let M be a manifold with corners and $\mathcal{E}$ a C^∞ index set for M. For each $H \in \mathcal{M}_1(M)$ choose a radial vector field V_H and define*

$$B^H_{\mathcal{E},s} = \prod_{(z,k)\in\mathcal{E}(H)_{\leq s}} (V_H - z)$$

  Then a smooth function u on $\mathrm{int}(M)$ is in $\mathcal{A}^{\mathcal{E}}(M)$ if and only if there is $N \in \mathbb{R}$ so that for all $s \in \mathbb{R}$ and all $H \in \mathcal{M}_1(M)$

$$B^H_{\mathcal{E},s}u \in \mathcal{A}^{\mathfrak{s}_H}(M), \quad \mathfrak{s}_H(H') := \begin{cases} s & H' = H \\ -N & H' \neq H \end{cases}$$