# TUTORIAL ON PSEUDODIFFERENTIAL OPERATORS SUMMERSCHOOL ON SPECTRAL THEORY, GÖTTINGEN SEPTEMBER 2014. PART I: MOTIVATION AND DEFINITIONS

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## Introduction

The aim of this talk is to give a down-to-earth introduction to pseudodifferential operators centered on its usage in the spectral theory of elliptic operators. The focus will be on general ideas and conceptual notions rather than precise proofs.

This being a tutorial, the time is quite loosely fitted. Questions are encouraged. The assumed prerequisites are that you know some basic functional analysis (topological vector spaces, distributions) and what a (vector) bundle on a smooth manifold is.

The plan for this talk is:

(1) Quick motivation
(2) Oscillatory integrals
(3) Pseudodifferential operators
(4) Operators on manifolds

The main reference for this tutorial is the notes of Joshi [4]. Everything can as always be found in the works of Hörmander, in this case [3], to which we refer some proofs.

## 1. Quick motivation

1.1. **Differential operators and spectral theory.** Let $A$ be a differential operator of order $m$ on a $d$-dimensional manifold $M$. This means that in local coordinates

$$A = \sum_{|\alpha|\leq m} a_\alpha(x) D^\alpha, \quad \text{where} \quad \alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d, \quad D_j = i\partial_{x_j} \quad \text{and} \quad D^\alpha = D_1^{\alpha_1} D_d^{\alpha_d}.$$

For instance, a Bochner-Laplacian $A = \nabla^*\nabla$, for a connection $\nabla$, or the Hodge-de Rahm Laplacian $A = \mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d}$.

Technical conditions aside (domains, closed extensions to $L^2$, self-adjointness), spectral theory asks for which complex numbers $\lambda$ there is a solution to $Au = \lambda u$. Rather, it asks for when the operator $A - \lambda$ is (non-) invertible.

*Exercise* 1.1. Check that if an object such as $(A - \lambda)^{-1}$ exists, it is not a differential operator unless $m = 0$.

More zealously we could ask for a "full" description of $(A-\lambda)^{-1}$. This is what spectral theory encompasses. Two important spectral properties that one often studies are

- Qualitative properties[1].
- Quantative properties[2]

1.2. **Laplacian on euclidean space.** Let us restrict to $M = \mathbb{R}^d$ and

$$A = -\Delta = -\sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2}.$$

We denote the Fourier transform by $\mathscr{F}$, which acts continuously on the Schwartz space $\mathscr{F} : \mathscr{S}(\mathbb{R}^d) \to \mathscr{S}(\mathbb{R}^d)$ and extends to a unitary on $L^2(\mathbb{R}^d)$.

For $f \in \mathscr{S}(\mathbb{R}^d)$ it holds that

$$\mathscr{F}(A-\lambda)\mathscr{F}^* f(\xi) = (|\xi|^2 - \lambda) f(\xi).$$

Hence, for $\lambda \in \mathbb{C} \setminus \mathbb{R}_{\geq 0}$, $A-\lambda$ is "invertible" with

$$\begin{aligned}(A-\lambda)^{-1} f(x) &= [\mathscr{F}^*(|\xi|^2-\lambda)^{-1}\mathscr{F} f](x) \\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathrm{e}^{i(x-y,\xi)}(|\xi|^2-\lambda)^{-1} f(y)\mathrm{d}y\mathrm{d}\xi \qquad (1)\\ &= \int_{\mathbb{R}^d} k_\lambda(x,y) f(y)\mathrm{d}y, \quad \text{where} \quad k_\lambda(x,y) = \int_{\mathbb{R}^d} \frac{\mathrm{e}^{i(x-y,\xi)}}{|\xi|^2-\lambda}\mathrm{d}\xi.\end{aligned}$$

For $d = 3$, one can compute that $k_\lambda(x,y) = \mathrm{e}^{i\sqrt{\lambda}|x-y|}|x-y|^{-1}$ where the branch of the square root satisfies $\mathrm{Im}(\sqrt{\lambda}) > 0$.

*Exercise* 1.2. Find all the missing $2\pi$:s.

1.3. **Ambition.** We wish for a general machinery where one can:

(1) place the formal computations above in a solid analytic framework;
(2) carry out the same computations for "general" operators on "manifolds".

The quotation marks in the last point on our wish list comes from: finding resolvents often depend on some ellipticity assumption and the procedure only works well on closed manifolds unless placing quite subtle controlling assumptions at the boundary or at infinity. We of course also wish to carry this out in such a way that both qualitative as well as quantative properties of the spectrum is understood.

## 2. Oscillatory integrals

We see from the computations of (1) that we need to understand distributions given by integrals of the form

$$u(z) \text{ “=” } \int_{\mathbb{R}^d} \mathrm{e}^{i\phi(z,\theta)} a(z,\theta)\mathrm{d}\theta.$$

[1] Given $\lambda$ such that $(A-\lambda)^{-1}$ doesn't exist, why doesn't it?

[2] How big is the set of $\lambda$ such that $(A-\lambda)^{-1}$ doesn't exist?

To do this, we restrict our attention to $\phi$ being a "phase function" and $a$ a "symbol" as in (1).

We say that a function $\phi = \phi(z,\theta)$ depending on $(z,\theta) \in \mathbb{R}^N \times \mathbb{R}^d \setminus \{0\}$ is homogeneous of degree $m$ in $\theta$ if

$$\phi(z,\lambda\theta) = \lambda^m \phi(z,\theta) \quad \forall \lambda > 0.$$

**Definition 2.1** (Phase function, Definition 3.1 of [4])**.** Let $U \subseteq \mathbb{R}^N$ be open. A function $\phi \in C^\infty(U \times \mathbb{R}^d \setminus \{0\})$ is called a phase function if

(1) $\phi$ is homogeneous of degree $1$ in $\theta$;
(2) $\mathrm{d}\phi$ is nowhere vanishing.

The example to keep in mind is $N = 2d$ with $z = (x,y) \in \mathbb{R}^{2d}$

$$\phi(z,\theta) = (x-y,\theta).$$

We use the notation $\langle\theta\rangle = (1+|\theta|^2)^{1/2}$.

**Definition 2.2** (Symbol, Definition 18.1.1 of [3] or Definition 3.2 of [4])**.** Let $U \subseteq \mathbb{R}^N$ be open. A function $a = a(z,\theta) \in C^\infty(U \times \mathbb{R}^d)$ is said to be a symbol of degree $m$ if for any compact $K \subseteq U$, $\alpha \in \mathbb{N}^N$ and $\beta \in \mathbb{N}^d$, there is a $C = C(\alpha,\beta,K)$ such that

$$\sup_{x\in K} |\partial_z^\alpha \partial_\theta^\beta a(z,\theta)| \leq C\langle\theta\rangle^{m-|\beta|}. \tag{2}$$

The linear space $S^m(U \times \mathbb{R}^d)$ of symbols of order $m$ is a Frechet space in the topology defined from the semi norms defined from (2).

*Exercise* 2.1. Check that if $a$ is a rational function in $\theta$, with coefficients in $C^\infty(U)$, then $a$ is a symbol of the same order as $a$ has as a rational function.

*Exercise* 2.2. Check that if $a$ is homogeneous of degree $m$, $a$ is a symbol of degree $m$.

*Exercise* 2.3. Check that multiplication $S^m \times S^{m'} \to S^{m+m'}$ as well as differentiation $\partial_z^\alpha \partial_\theta^\beta : S^m \to S^{m-|\beta|}$ are well defined and continuous operations.

*Remark* 2.3. The definition of symbols is made to allow for partial integration being used as an order-reducing operation.

*Remark* 2.4. One can also ask for globally estimated symbol, placing conditions on the behavior at infinity. This is needed for considering operators on $\mathbb{R}^d$ and goes in similar spirit for complete, non-compact manifolds. Such considerations are not needed on closed manifolds.

*Remark* 2.5. Sometimes it is needed to consider $(\rho,\delta)$-symbols. We avoid them.

**Theorem 2.6.** *The distribution $u \in \mathscr{D}'(U) = C_c^\infty(U)'$ can be made well defined and satisfies that*

$$\mathrm{singsupp}(u) \subseteq \pi_U\left(\{(z,\theta) : \mathrm{d}_\theta\phi(z,\theta) = 0\}\right).$$

In our favorite example $\phi(z,\theta) = (x-y,\theta)$, for $U = V \times V$ the singular support is always contained in the diagonal

$$\Delta_V = \{(x,x) : x \in V\}.$$

*Sketch of proof, for details see Section 3 of* [4]. We need to make sense of the expression

$$u(\psi) = \int_U \int_{\mathbb{R}^d} \mathrm{e}^{i\phi(z,\theta)} a(z,\theta)\psi(z)\mathrm{d}\theta\mathrm{d}z,$$

for $\psi \in C_c^\infty(U)$. After decomposing $a = a_0 + a_1$ where $a_1$ vanishes near $\theta = 0$ and $a_0$ is compactly supported in $\theta$, we can assume $a = a_1$.

We pull the following operator out of the hat:

$$L = \sum_{j=1}^N \frac{\partial \phi}{\partial z_j}\frac{\partial}{\partial z_j} + \sum_{l=1}^d |\theta|^2 \frac{\partial \phi}{\partial \theta_l}\frac{\partial}{\partial \theta_l}.$$

It holds that

$$L\mathrm{e}^{i\phi} = \mu \cdot \mathrm{e}^{i\phi},$$

where

$$\mu(z,\theta) = |\mathrm{d}_z \phi|^2 + |\theta|^2 |\mathrm{d}_\theta \phi|^2.$$

*Exercise* 2.4. Take a cutoff $\chi = \chi(\theta) \in C_c^\infty(\mathbb{R}^d)$ such that $(1-\chi)a = a$. Show that

$$\mu^{-1}L(1-\chi) : S^m \to S^{m-1}.$$

We conclude from the exercise that for any $r \in \mathbb{N}$

$$\begin{aligned} u(\psi) &= \int_U \int_{\mathbb{R}^d} (\mu^{-1}L(1-\chi))^r \mathrm{e}^{i\phi(z,\theta)} a(z,\theta)\psi(z)\mathrm{d}\theta\mathrm{d}z \\ &= \sum_{|\alpha|\leq r} \int_U \int_{\mathbb{R}^d} \mathrm{e}^{i\phi(z,\theta)} a_\alpha(z,\theta)\partial_z^\alpha \psi(z)\mathrm{d}\theta\mathrm{d}z, \end{aligned}$$

where $a_\alpha \in S^{m-r}$. For $r > m+d$, this expression converges. □

## 3. Pseudodifferential operators

For a symbol $a \in S^m(\mathbb{R}^{2d} \times \mathbb{R}^d)$ we define the operator

$$Op(a) : C_c^\infty(\mathbb{R}^d) \to C^\infty(\mathbb{R}^d), \quad Op(a)f(x) := \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} a(x,y,\xi)\mathrm{e}^{i(x-y,\xi)} f(y)\mathrm{d}y\mathrm{d}\xi.$$

We interpret this as an oscillatory integral.

*Exercise* 3.1. Check that $Op(a)$ is well defined.

*Exercise* 3.2. Check that if $a$ does not depend on $y$, the operator $Op(a)$ can be defined as in (1) without the usage of oscillatory integrals.

*Exercise* 3.3. Compute $Op(a)$ when $a(x,\xi) = \sum_{|\alpha|\leq m} a_\alpha \xi^\alpha$, where $a_\alpha$ are numbers.

**Proposition 3.1.** *The operator* $Op(a)$ *is continuous and extends by duality to a continuous operator*

$$Op(a) : \mathscr{E}'(\mathbb{R}^d) \to \mathscr{D}'(\mathbb{R}^d).$$

*Remark* 3.2. If $U, V \subseteq \mathbb{R}^d$ are open sets and $a \in S^m(U \times V \times \mathbb{R}^d)$, we obtain an operator $Op(a) : C_c^\infty(V) \to C^\infty(U)$.

3.1. **Schwartz kernels.** The Schwarz kernel theorem guarantees that associated with any continuous linear operator $A : C_c^\infty(U) \to \mathscr{D}'(U)$ there is a unique *Schwartz kernel* $K_A \in \mathscr{D}'(U \times U)$ satisfying that

$$A\psi(\varphi) = K_A(\varphi \otimes \psi), \quad \varphi, \psi \in C_c^\infty(U).$$

The operators we are interested in factors over the dense inclusion $C^\infty(U) \subseteq \mathscr{D}'(U)$ or in nice cases even over $C_c^\infty(U) \subseteq \mathscr{D}'(U)$.

**Proposition 3.3.** *The Schwartz kernel of $Op(a)$ is given by the oscillatory integral*

$$K_a(x,y) := \int_{\mathbb{R}^d} a(x,y,\xi) e^{i(x-y,\xi)} d\xi.$$

*Remark* 3.4. By Theorem 2.6, $K_a \in C^\infty(U \times U \setminus \Delta_U)$. In fact, the singularity of $K_a$ at $\Delta_U$ (being a *conormal* distribution at $\Delta_U$) characterizes the fact that $K_a$ defines a pseudo-differential operator.

*Remark* 3.5. It holds that $K_a \in C^\infty(U \times U)$ if and only if $a \in S^{-\infty}(U \times U \times \mathbb{R}^d) = \cap_{m\in\mathbb{R}} S^m(U \times U \times \mathbb{R}^d)$. In this case $a$, and $K_a$ as well as $Op(a)$, are said to be a smoothing symbol, kernel respectively operator.

**Definition 3.6.** Let $A : C_c^\infty(U) \to C^\infty(U)$ be a continuous operator.

- $A$ is said to be a pseudodifferential operator of order $m$ if $A = Op(a) + R$ where $a \in S^m(U \times U \times \mathbb{R}^d)$ and $K_R \in C^\infty(U \times U)$.
- A pseudodifferential operator $A$ with $a = 0$ is said to be smoothing.

The space of pseudo-differential operators on $U$ of order $m$ is denoted by $\Psi^m(U)$.

3.2. **Asymptotic expansions.** A very useful property of pseudodifferential operators comes from the following lemma.

**Lemma 3.7.** *Assume that $(m_j)_{j\in\mathbb{N}}$ is a decreasing sequence converging to $-\infty$ and $a_j \in S^{m_j}(U \times U \times \mathbb{R}^d)$ a collection of symbols. For $m = m_0$ there is an $a \in S^m(U \times U \times \mathbb{R}^d)$ with*

$$\mathrm{supp}(a) \subseteq \cup_j \mathrm{supp}(a_j) \quad \textit{and} \quad a - \sum_{j=0}^{N} a_j \in S^{m_{N+1}} \ \forall N.$$

*The symbol $a$ is uniquely determined in $S^m/S^{-\infty}$.*

*Proof.* By a partition of unity argument, we can assume that everything is supported inside a compact subset of $U$. Choose a bump function $\chi \in C_c^\infty(\mathbb{R}^d)$, i.e. $\chi = 1$ near $\xi = 0$. Some computations show that $1 - \chi(\varepsilon\cdot) \to 0$ in $S^1$. By continuity of multiplication, there exists a sequence $\varepsilon_j \searrow 0$ such that

$$\left|\partial_z^\alpha \partial_\xi^\beta \left((1-\chi(\varepsilon_j\xi))a_j(x,y,\xi)\right)\right| \leq 2^{-j}\langle\xi\rangle^{m_j+1-|\beta|}.$$

We now set

$$a(x,y,\xi) := \sum_{j=0}^{\infty} (1-\chi(\varepsilon_j\xi))a_j(x,y,\xi),$$

which is a well defined locally finite sum. Since $2^{-j}$ is summable, the Lemma follows from a quick and dirty computation. ∎

One writes $a \sim \sum_{j=0}^{\infty} a_j$ even though the right hand side is not well defined.

**Corollary 3.8.** *If $A_0, A_1, \dots$ are pseudo-differential operators of order $m_j \to -\infty$ there is a pseudodifferential operator $A$ of order $m$ with $A - \sum_{j=0}^{N} A_j$ being of order $\tilde{m}_{N+1}$.*

Similarly to the case of symbols, one writes $A \sim \sum_{j=1}^{\infty} A_j$. We mention another corollary, whose importance at this stage mainly is conceptual.

**Corollary 3.9.** *There exists $a_L, a_R \in S^m(\mathbb{R}^d \times \mathbb{R}^d)$ such that (mod $C^\infty$-kernels)*

$$Op(a)f(x) = \int_{\mathbb{R}^d} a_L(x,\xi)\mathscr{F}f(\xi)\mathrm{d}\xi = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} a_R(y,\xi)\mathrm{e}^{i(x-y,\xi)} f(y)\mathrm{d}y\mathrm{d}\xi.$$

*In particular, the class of pseudo-differential operators is closed under adjoints.*

**Definition 3.10.** A symbol $a \in S^m(U \times \mathbb{R}^d)$ is said to be classical, if there exists functions $a_j \in C^\infty(U \times \mathbb{R}^d)$ homogenous away from $\theta = 0$ of order $m - j$ with

$$a \sim \sum_{j=0}^{\infty} a_j.$$

We denote the associated space of operators by $\Psi^m_{cl}(U)$.

*Remark* 3.11. For a symbol $a \in S^m$, its principal symbol is the associated class in $S^m/S^{m-1}$ is often denoted by $\sigma_m(a)$. If $a$ is classical, $\sigma_m(a)$ is a well defined element of $C^\infty(U \times S^{d-1})$. This in fact fits into a short exact sequence

$$0 \to \Psi^{m-1}_{cl}(U) \to \Psi^m_{cl}(U) \to C^\infty(U \times S^{d-1}) \to 0.$$

## 4. Operators on manifolds

4.1. **Changing coordinates.** To ensure that operators on manifolds behave well under coordinate changes, we will make use of the following Theorem.

**Theorem 4.1.** *Let $U, V \subseteq \mathbb{R}^d$ be open subsets, $\kappa : U \to V$ a diffeomorphism and $A \in \Psi^m(U)$. Then the operator*

$$\kappa^* A : C^\infty_c(V) \to C^\infty(V), \quad f \mapsto (A(f \circ \kappa)) \circ \kappa^{-1},$$

*is a pseudodifferential operator of order $m$ which is properly supported if and only if $A$ is.*

*Sketch of proof.* We can assume $A = Op(a)$. It holds that

$$\begin{aligned}
[\kappa^* A]f(x) &= \int_V \int_{\mathbb{R}^d} \mathrm{e}^{i(\kappa^{-1}(x)-y,\xi)} a(\kappa^{-1}(x), y, \xi) f(\kappa(y))\mathrm{d}\xi\mathrm{d}y \\
&\{z = \kappa(y)\} \\
&= \int_U \int_{\mathbb{R}^d} \mathrm{e}^{i(\kappa^{-1}(x)-\kappa^{-1}(z),\xi)} a(\kappa^{-1}(x), \kappa^{-1}(z), \xi) f(\kappa(y)) \frac{\mathrm{d}\xi\mathrm{d}z}{|\det \kappa'(\kappa^{-1}(z))|} \\
&\{\kappa^{-1}(x) - \kappa^{-1}(z) = A(x,z)(x-z), \quad \eta = A^t(x,z)\xi\} \\
&= \int_U \int_{\mathbb{R}^d} \mathrm{e}^{i(x-z,\eta)} \frac{a(\kappa^{-1}(x), \kappa^{-1}(z), \eta)}{|\det \kappa'(\kappa^{-1}(z))||\det A^t(x,z)|} f(\kappa(y))\mathrm{d}\eta\mathrm{d}z.
\end{aligned}$$

The proof follows from the next exercise. □

*Exercise* 4.1. Show that

$$a_\kappa(x,z,\eta) := \frac{a(\kappa^{-1}(x),\kappa^{-1}(z),\eta)}{|\det\kappa'(\kappa^{-1}(z))||\det A^t(x,z)|}$$

is a symbol.

4.2. **Pseudodifferential operators on manifolds.** From now on, let $M$ denote a $d$-dimensional manifold.

**Corollary 4.2.** *Let $U \subseteq M$ be a coordinate neighborhood, $\alpha_i : U \to U_i \subseteq \mathbb{R}^d$, $i = 1,2$ two different choices of coordinates charts and $A : C_c^\infty(U) \to C^\infty(U)$ a linear operator. Then $\alpha_1^* A : C_c^\infty(U_1) \to C^\infty(U_1)$ is a pseudo-differential operator of order $m$ if and only if $\alpha_2^* A : C_c^\infty(U_2) \to C^\infty(U_2)$ is a pseudo-differential operator of order $m$.*

If the conditions in the corollary holds, we say that $A$ is a pseudodifferential operator of order $m$. The above corollary guarantees that the following definition makes sense.

**Definition 4.3.** A linear operator $A : C_c^\infty(M) \to C^\infty(M)$ is called a pseudodifferential operator of order $m$ if for any coordinate neighborhood $U \subseteq M$ and $\chi,\chi' \in C_c^\infty(U)$ the operator $\chi A \chi'$ is a pseudodifferential operator of order $m$.

*Remark* 4.4. If $E,E' \to M$ are vector bundles, $\Psi^m(M;E,E')$ is defined similarly, in local coordinates $\chi P \chi' = (\chi P_{ij} \chi') : C_c^\infty(U,\mathbb{C}^n) \to C_c^\infty(U,\mathbb{C}^m)$ is a matrix of pseudo-differential operators.

*Remark* 4.5. If $(U_\alpha)_{\alpha\in I}$ is a cover of coordinate neighborhoods, $A$ is a pseudodifferential operator of order $m$ if and only if $\chi A \chi'$ is a pseudodifferential operator of order $m$ for any $\chi,\chi' \in C_c^\infty(U_\alpha)$ **and** $\chi A \chi'$ is smoothing whenever $\chi,\chi' \in C_c^\infty(M)$ satisfies $\chi \cdot \chi' = 0$.

**Proposition 4.6.** *Given a cover $(U_\alpha)_{\alpha\in I}$ and a pseudodifferential operator $A$ of order $m$, there exists a refinement $(V_\beta)_{\beta\in J}$ with $|J| \leq |I|^2$ and pseudodifferential operators $A_\beta$ of order $m$ compactly supported in $V_\beta$ such that*

$$A \sim \sum_\beta A_\beta.$$

*Remark* 4.7. On a manifold, the principal symbol of a classical operator is a smooth function on $S^*M$. The symbol mapping fits into a short exact sequence:

$$0 \to \Psi_{cl}^{m-1}(M;E,E') \to \Psi_{cl}^m(M;E,E') \to C^\infty(S^*M;\pi^*\mathrm{Hom}(E,E')) \to 0.$$

4.3. **Global quantization.** We will now turn a procedure of constructing operators from symbols on manifolds. For any coordinate neighborhood $U \subseteq M$ we can define the Frechet space of symbols $S^m(T^*U)$ by means of a trivialization $T^*U \cong U \times \mathbb{R}^d$. We define the Frechet space

$$S^m(M) := \{a \in C^\infty(T^*M) : a\cdot\chi \in S^m(T^*U) \text{ for any chart } U \subseteq M \text{ and } \chi \in C_c^\infty(U)\}.$$

We let $N^*M \to M$ denote the normal bundle of the diagonal $M \cong \Delta_M \subseteq M \times M$. As a manifold, $N^*M$ is diffeomorphic to an open neighborhood of the diagonal $\Delta_M \subseteq M \times M$. Such a diffeomorphism can be constructed by for instance parallel transport along a connection. We also let $\Omega M \to M$ denote the $\mathbb{R}_{>0}$-bundle of nonvanishing densities on $M$. The bundle of non vanishing densities is trivializable since $\mathbb{R}_{>0}$ is homotopic to the trivial group.

**Theorem 4.8.** *Given a diffeomorphism $\phi : N^*M \xrightarrow{\sim} U \supseteq \Delta_M$ onto an open neighborhood of the diagonal and a trivialization $\omega$ of $\Omega M$, there is a linear mapping*

$$Op_{\phi,\omega} : S^m(M) \to \Psi^m_{prop}(M),$$

*coinciding with $Op$ in local coordinates, which induces an isomorphism*

$$S^m(M)/S^{-\infty}(M) \cong \Psi^m_{prop}(M)/\Psi^{-\infty}_{prop}(M).$$

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