# TUTORIAL ON PSEUDO-DIFFERENTIAL OPERATORS

PART II: ALGEBRAIC AND ANALYTIC PROPERTIES
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## 1. Compositions and Parametrices

**Disclaimer:** these notes are a rough outline of what was discussed during the tutorial. They were not proof-read in detail. Any pointing out of errors or any suggestions are very welcome (by mail).

Notation: In the following, let $X$ denote a closed $\mathscr{C}^\infty$-manifold or a subset of $\mathbb{R}^d$. The symbol $\dot{T}^*X$ denotes $T^*X \setminus \{0\}$.
*Recall:* In the previous part of this tutorial by Magnus Goffeng, we have seen the definition of a class of pseudo-differential operators $\Psi(X)$, which generalizes that of differential operators.
In the following, we will encounter elements of the *calculus* of these operators, in particular we will study compositions and (pseudo-)inverses.
In order to do spectral theory of these operators, which is our initial motivation, we will also study how these classes of operators may be realized as operators between certain Hilbert spaces.

### 1.1. **Properly supported operators.**

Recall that $A \in \Psi(X)$ maps $\begin{cases} \mathscr{D}(X) \to \mathscr{E}(X) \\ \mathscr{E}'(X) \to \mathscr{D}'(X) \end{cases}$. Therefore, it is in general not possible to compose $\Psi$DOs, unless one makes the following technical assumption:

**Definition 1.1** (Properly supported operators)**.** Let $A \in \Psi(X)$, identified with its Schwartz kernel $K_A \in \mathscr{D}'(X \times X)$. $A$ is called *properly supported* if the projections $\pi_1$ and $\pi_2$ on $\mathrm{supp}(K_A) \to X$ are proper maps.[1]

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[1]Recall that a map is called *proper* if the pre-images of compact sets are compact.

The condition of being properly supported may be visualized as $K_A$ "being supported sufficiently close to the diagonal" $\Delta_X \subset X \times X$. Since the singularities of $K_A$ lie on the diagonal, i.e. $\operatorname{singsupp}(K_A) \subset \Delta_X$, we have by application of some localizer supported around $\Delta_X$:

**Lemma 1.2.** *Every $A \in \Psi^m(X)$ has a decomposition $A = A_{\mathrm{pr}} + R$, where $A_{\mathrm{pr}} \in \Psi^m(X)$ is properly supported and $R$ is smoothing.*

If $A$ is properly supported, we have

$$A : \begin{cases} \mathscr{D}(X) \to \mathscr{D}(X) \\ \mathscr{D}'(X) \to \mathscr{D}'(X) \\ \mathscr{E}(X) \to \mathscr{E}(X) \\ \mathscr{E}'(X) \to \mathscr{E}'(X). \end{cases}$$

Consequently, given a properly supported ΨDO, we may hope to compose it with other ΨDOs. We will now study that this is indeed possible and that the composition remains in the class of ΨDOs.

1.2. **Composition ΨDOs.**
First: If $R$ is regularizing $\mathscr{E}'(X) \to \mathscr{C}^\infty(X)$, then if $A$ is properly supported, we have that $A \circ R$ and $R \circ A$ is regularizing, by the mapping properties of properly supported operators.

**Proposition 1.3.** *Let $A_j \in \Psi^{m_i}_{(\mathrm{cl})}(\mathbb{R}^d)$, $j \in \{1,2\}$, one of them properly supported. Then $A_1 \circ A_2 \in \Psi^{m_1+m_2}_{(\mathrm{cl})}(\mathbb{R}^d)$ and we have*

$$\sigma_{A_1 \circ A_2}(x,\xi) \sim \sum_\gamma \frac{(-i)^{|\gamma|}}{\gamma!} \partial_\xi^\gamma a_1(x,\xi) \partial_x^\gamma a_2(x,\xi).$$

*If both operators are properly supported, then $A_1 \circ A_2$ is properly supported as well. Consequently, $\Psi(X)/\sim$, meaning modulo regularizing operators, is an algebra.*

*Remark* 1.4. In particular we have

$$\begin{aligned} \sigma^{m_1+m_2}(A_1 \circ A_2) &= \sigma^{m_1}(A_1) \cdot \sigma^{m_2}(A_2) \\ \sigma^{m_1+m_2}([A_1, A_2]) &= 0 \\ \sigma^{m_1+m_2-1}([A_1, A_2]) &= \{a_1, a_2\} = \sum_{j=1}^n \big(\partial_{\xi_j} a_1 \partial_{x_j} a_2 - \partial_{\xi_j} a_2 \partial_{x_j} a_1\big). \end{aligned}$$

*Idea of proof.* After using a partition of unity to localize everything, use partial integration and the symbol estimates.
The example where both ΨDOs are actually differential operators already highlights the essence of the proof:

$$
\begin{aligned}
a_1(x,D)\circ a_2(x,D) &= \left(\sum_\alpha a_1^\alpha(x)D_x^\alpha\right)\circ\left(\sum_\beta a_2^\beta(x)D_x^\beta\right)\\
&= \sum_{\alpha,\beta} a_1^\alpha(x)D_x^\alpha\big(a_2^\beta(x)D_x^\beta\big)\\
&= \sum_{\alpha,\beta}\sum_\gamma c_\gamma a_1^\alpha(x)(D_x^\gamma a_2^\beta(x))D_x^{\beta+\alpha-\gamma}\\
&\hat{=} \sum_{\alpha,\beta}\sum_\gamma c_\gamma a_1^\alpha(x)\frac{(-i)^{|\gamma|}}{\gamma!}(\partial_\xi^\gamma\xi^\alpha)(D_x^\gamma a_2^\beta(x))\xi^\beta.
\end{aligned}
$$

□

### 1.3. **Ellipticity and (Quasi-)Inversion.**

Motivation: earlier we have seen that $\sigma^{m_1+m_2}(A_1\circ A_2)=\sigma^{m_1}(A_1)\sigma^{m_2}(A_2)$. In order to invert $A$, we therefore look for a properly supported ΨDO with an amplitude $a_1(x,\xi)^{-1}$.
To ensure existence, we make the following definition:

**Definition 1.5** (Elliptic symbols). $a\in S^m(X\times\mathbb{R}^d)$ is called *elliptic* at $(x_0,\xi_0)\in\dot{T}^*X$ if there exists $C>0$ and a conical neighbourhood $\Gamma$ of $(x_0,\xi_0)$ such that

$$|a(x,\xi)|\geq\frac{1}{C}\langle\xi\rangle^m \text{ for } (x,\xi)\in\Gamma,\ |\xi|>C.$$

For $a\in S^m_{\mathrm{cl}}(X\times\mathbb{R}^d)$ this is equivalent to $\sigma^m(a)(x_0,\xi_0)\neq 0$.
A symbol is elliptic if it is elliptic on all of $\dot{T}^*X$. We say that a ΨDO $A\in\Psi^m(X)$ is elliptic if it is given by an elliptic amplitude.

**Example 1.6.** $-\Delta$ on $\mathbb{R}^d$, since in this case we have $\sigma(-\Delta)=|\xi|^2$.

By the previous considerations, we may thus find an operator $P_0$ with $\sigma^{-m}(P)=\chi(\theta)\sigma^m(A)^{-1}$ for some excision function $\chi$. We may then compute $A\circ P=I+R_1$ with $R_1\in\Psi^{-1}$. The idea is then to compose again with $(I-R_1)$ to obtain

$$(A\circ P)(I-R_1)=(I+R_1)(I-R_1)=I-R_1\circ R_1=:I+R_2$$

with $R_2 \in \Psi^{-2}$. Repeating this - essentially repeating the Neumann sum construction - and using asymptotic completeness, one obtains

**Theorem 1.7.** *If $A \in \Psi^m_{(\mathrm{cl})}(X)$ is elliptic, then $\exists P \in \Psi^{-m}_{(\mathrm{cl})}(X)$, properly supported, such that*

$$A \circ P = I + R_1$$
$$P \circ A = I + R_2,$$

*where the $R_j$ are smoothing.*
*Moreover, $P$, which is called a* parametrix *or* quasi-inverse *to $A$, is unique modulo smoothing operators.*

**Corollary 1.8.** *If $A \in \Psi^m(X)$ is elliptic and properly supported, then it induces an isomorphism on $\mathscr{D}'(X)/\mathscr{C}^\infty(X)$.*
*Moreover* $\operatorname{singsupp}(Au) = \operatorname{singsupp}(u)$.

**Example 1.9.** In $\mathbb{R}$, the equation $\Delta u = \delta_0 \mod \mathscr{C}^\infty$ has as solution $u = \frac{1}{2}|u| \mod \mathscr{C}^\infty$.

We now want to realize ΨDOs as operators between Hilbert spaces.

## 2. Realization on Hilbert spaces

### 2.1. $L^2$-continuity.

**Theorem 2.1.** *Let $A \in \Psi^0(\mathbb{R}^d)$ with compactly supported Schwartz kernel, in particular $M := \sup_{x\in\mathbb{R}^d}\sup_\xi |\sigma_A(x,\xi)| < \infty$. Then $A : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is continuous.*

*Idea of proof.* One proves $A^*A \le (M+\epsilon)^2 I$ (in the sense of self-adjoint operators) by constructing an approximate square root to $(M+\epsilon)^2 I - A^*A$.
This is achieved on a symbolic level. □

During the proof, one obtains

**Corollary 2.2.** *Let $A \in \Psi^0(\mathbb{R}^d)$ with compactly supported Schwartz kernel, and $\lim_{|\xi|\to\infty}\sup_{x\in\mathbb{R}^d}|\sigma_A(x,\xi)| = 0$. Then $A : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is compact.*

General operators, like $-\Delta$, are of course not bounded on $L^2$. We therefore need spaces with additional regularity.

## 2.2. **Sobolev spaces.**

**Definition 2.3.** Let $s \in \mathbb{R}$. Define

$$H^s(\mathbb{R}^d) = \left\{ u \in \mathscr{S}'\left(\mathbb{R}^d\right) \,\middle|\, \hat{u} \in L^2_{loc}(\mathbb{R}^d) \text{ and } \|u\|_s < \infty \right\}$$

where $\|u\|_s^2 = \frac{1}{(2\pi)^d} \int |\hat{u}(\xi)|^2 \langle \xi \rangle^{2s} d\xi$.

**Example 2.4.** $H^0(\mathbb{R}^d) = L^2(\mathbb{R}^d)$.

**Proposition 2.5** (Properties of Sobolev spaces)**.**

- *$H^s(\mathbb{R}^d)$ is a Hilbert space with*

$$(u, v)_{H^s} = \frac{1}{(2\pi)^d} \int \hat{u}(\xi) \overline{\hat{v}(\xi)} \langle \xi \rangle^{2s} d\xi.$$

- *The dual of $H^s(\mathbb{R}^d)$ with respect to the $L^2$-pairing $\langle \cdot, \cdot \rangle_{L^2}$ is $H^{-s}(\mathbb{R}^d)$.*[2]
- *We obtain that $H^s(\mathbb{R}^d) = (1 + |D|^2)^{-s/2} L^2(\mathbb{R}^d)$.*
- *For $\phi \in \mathscr{C}_c^\infty(\mathbb{R}^d)$ we have $u \in H^s(\mathbb{R}^d) \Rightarrow \phi u \in H^s(\mathbb{R}^d)$. Consequently, we may define for open $X \subset \mathbb{R}^d$*

$$H^s_{loc}(X) := \{ u \in \mathscr{D}'(X) \,|\, \forall \phi \in \mathscr{C}_c^\infty(X) : \varphi u \in H^s(\mathbb{R}^d) \}$$

  *which may be carried over to manifolds.*
- *We obtain the compactly supported*

$$H^s_c(X) := H^s_{loc}(X) \cap \mathscr{E}'(X) = \left( H^{-s}_{loc}(X) \right)'.$$

- *For $K \subset \mathbb{R}^d$ compact, one may define $H^s(K) := H^s(\mathbb{R}^d) \cap \mathscr{E}'(K)$. One then obtains that for $s' < s$ the embedding $H^{s'}(K) \hookrightarrow H^s(K)$ is compact.*

**Theorem 2.6** (Sobolev continuity)**.** *Let $A \in \Psi^m(X)$ properly supported. Then*

$$A : \begin{cases} H^s_{loc}(X) \to H^{s-m}_{loc}(X) \\ H^s_c(X) \to H^{s-m}_c(X) \end{cases}$$

*Proof.* The delicate part of the proof is taking care of the proper localizations. Apart from that, for $u \in H^s(\mathbb{R}^d)$ write $u = (1 + |D|^2)^{-s/2} f$ for $f \in L^2(\mathbb{R}^d)$ and use

[2]By the Riesz representation theorem, we know that every Hilbert space may be identified with its dual, meaning $v \in H^{-s}(\mathbb{R}^d)$, acting by $\langle v, \cdot \rangle$ may be represented by $(w, \cdot)_{H^s}$ for a unique $w \in H^s(\mathbb{R}^d)$. We obtain explicitly $w = (1 + |D|^2)^s v$.

$L^2$-continuity[3]

$$\big(\underbrace{(1+|D|^2)^{(s-m)/2}}_{\in\Psi^{s-m}}\circ\underbrace{A}_{\in\Psi^m}\circ\underbrace{(1+|D|^2)^{-s/2}}_{\in\Psi^{-s}}\big)f\in L^2(\mathbb{R}^d)$$

□

Having these spaces at hand, we may now look at elliptic operators. By use of the parametrix construction, we may actually go back and forth between these spaces.

**Lemma 2.7** (Elliptic regularity)**.** *Let $A$ elliptic and properly supported. Then $\forall u\in\mathscr{D}'(X)$ we may conclude*

$$u\in H^s_{loc}(X)\Leftrightarrow Au\in H^{s-m}_{loc}(X).$$

*For the special case of $Au=0$, we obtain $u\in\mathscr{C}^\infty(X)$.*

$H^s_{loc}$ is usually not a Hilbert space. If $X$ is a closed Riemannian $\mathscr{C}^\infty$-manifold, however, we may still define $H^s(X)=H^s_c(X)$ for $s\in\mathbb{R}_{\geq 0}$ as the closure of $\mathscr{C}^\infty(X)$ in the topology induced by the scalar product

$$(f,g)_{H^s}=(f,g)_{L^2}+((I-\Delta)^{s/2}f,(I-\Delta)^{s/2}f),$$

and for $s<0$ by duality.

**Proposition 2.8.** *Let $X$ a closed manifold, $A\in\Psi^m$ elliptic. Then*

- *$A|\mathscr{C}^\infty_c(X)$ admits a closed extension to an operator $L^2(X)$ with domain $H^m(X)$.*
- *Let $m>0$. Then the spectrum of this extension is either $\mathbb{C}$ or discrete. Furthermore, if it exists, the resolvent is compact on $L^2$.*

2.3. **Fredholm property.**
Let $\mathscr{H}_j$, $j\in\{1,2\}$, Hilbert spaces. Recall that a bounded $T:\mathscr{H}_1\to\mathscr{H}_2$ is called *Fredholm operator*, if $\dim(\ker T)<\infty$ and $\dim(\mathrm{coker}T)<\infty$ (and $T(\mathscr{H}_1)$ is closed).
The *index* of a Fredholm operator is defined as

$$\mathrm{ind}T:=\dim(\ker T)-\dim\mathrm{coker}T.$$

**Example 2.9.** Let $K:\mathscr{H}\to\mathscr{H}$ compact, then $\mathrm{id}_{\mathscr{H}}+K$ is Fredholm.

[3] After localization, wherein compact distributional kernels may be achieved.

**Proposition 2.10.** *If $T \in \mathscr{B}(H_1, H_2)$ and $S \in \mathscr{B}(H_2, H_1)$ such that*

$$TS = I_2 + K_2$$
$$ST = I_1 + K_1$$

*with $K_j$ compact, then $T$ and $S$ are Fredholm operators and* $\operatorname{ind}(T) = -\operatorname{ind}(S)$.

In the previous case where $A \in \Psi^m(X)$ is an elliptic ΨDO, we have such an operator $P$ by the parametrix construction, since on closed $X$ every $R \in \Psi^{-\infty}$ is compact. Therefore if $A \in \Psi^m(X)$ is elliptic, it is Fredholm $H^s(X) \to H^{s-m}(X)$ for any $s \in \mathbb{R}$. Furthermore, the index does not depend on $s$.

## 3. Wave front sets

As an application of pseudodifferential calculus, we want to use ΨDOs to obtain more information on singularities of $u \in \mathscr{D}'(X)$. Consider first $X \subset \mathbb{R}^d$. Then

$$\begin{aligned} x_0 \notin \operatorname{singsupp}(u) &\Leftrightarrow \exists \phi \in \mathscr{C}_c^\infty(\mathbb{R}^d) \text{ s.t. } \phi(x_0) = 1 \text{ and } \phi u \in \mathscr{C}_c^\infty(\mathbb{R}^d) \\ &\Leftrightarrow \exists \phi \in \mathscr{C}_c^\infty(\mathbb{R}^d) \text{ s.t. } \phi(x_0) = 1 \text{ and } \underbrace{\widehat{\phi u}}_{\in \mathscr{C}^\infty(\mathbb{R}^d)} \in \mathscr{S}(\mathbb{R}^d) \end{aligned}$$

This is the starting point of microlocalization. Using $x_0 \in \operatorname{singsupp}(u) \Leftrightarrow \exists \phi \in \mathscr{C}_c^\infty(\mathbb{R}^d)$ s.t. $\phi(x_0) = 1$ and $\widehat{\phi u} \notin \mathscr{S}(\mathbb{R}^d)$ one obtains more information on a singularity at $x_0$ by localizing in the latter space. Let $(x_0, \xi_0) \in \dot{T}^*\mathbb{R}^d \cong \mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\})$. Then $u \in \mathscr{D}'(\mathbb{R}^d)$ is microlocally $\mathscr{C}^\infty$-regular at $(x_0, \xi_0)$ if

(1) $\exists \phi \in \mathscr{C}_c^\infty(\mathbb{R}^d)$ s.t. $\phi(x_0) = 1$
(2) $\widehat{\phi u}$ is rapidly decaying in a conic neighbourhood of $\xi_0$

Equivalently to the second condition, we may find a conic localizer $\psi u$ with $\psi = 1$ for $\xi$ in a neighbourhood of $\xi_0$ satisfying $|\xi| > C$ for some $C > 0$ such that $\psi\widehat{(\phi u)} \in \mathscr{S}\left(\mathbb{R}^d\right)$.
This is yet again equivalent to $\psi(D)(\phi u) \in \mathscr{S}\left(\mathbb{R}^d\right)$ and (which requires some thought) to the existence of such $\phi$ and $\psi$ such that $\psi(D)(\phi u) \in \mathscr{C}^\infty(\mathbb{R}^d)$.
By some (localized) parametrix construction, this is finally equivalent to the existence of a general $A \in \Psi^d_{\mathrm{cl}}(\mathbb{R}^d)$, elliptic at $(x_0, \xi_0)$, such that $Au \in \mathscr{C}^\infty(\mathbb{R}^d)$.
Consequently, we set

**Definition 3.1.** For $u \in \mathscr{D}'(X)$, we define the *wave front set*

$$\mathrm{WF}(u) := \bigcap_{\substack{A \in \Psi^0_{\mathrm{cl}}(X) \\ Au \in \mathscr{C}^\infty(X)}} \mathrm{char}(A) \subset \dot{T}^*X.$$

We further collect for closed conic $\Gamma \subset \dot{T}^*X$

$$\mathscr{D}'_\Gamma(X) := \{u \in \mathscr{D}(X) \,|\, \mathrm{WF}(u) \subset \Gamma\}$$

and it is possible to endow this space with a suiting notion of convergence.

**Proposition 3.2** (Properties of WF)**.**

- $\mathrm{WF}(u) \subset \dot{T}^*X$ *is closed.*
- $\pi_1(\mathrm{WF}(u)) = \mathrm{singsupp}(u)$.
- $\mathrm{WF}(u_1 + u_2) \subset \mathrm{WF}(u_1) \cup \mathrm{WF}(u_2)$.
- *Let* $f \in \mathscr{C}^\infty(\mathbb{R}^d)$. *Then* $\mathrm{WF}(fu) \subset \mathrm{WF}(u) \cap \big(\mathrm{supp}(f) \times (\mathbb{R}^d \setminus \{0\})\big)$.

**Example 3.3.** Let $x_0 \in \mathbb{R}^d$, $f \in \mathscr{C}^\infty(\mathbb{R}^{d'})$.

$$\begin{aligned} \mathrm{WF}(\delta_{x_0}) &= \{x_0\} \times (\mathbb{R}^d \setminus \{0\}) \\ \mathrm{WF}(\delta_{x_0} \otimes f) &= \big\{(x_0, y, \xi, 0) \,\big|\, \xi \in (\mathbb{R}^d \setminus \{0\}),\, y \in \mathrm{supp}(f)\big\} \end{aligned}$$

There exists an elaborate machinery to relate the wave front set of the kernel $K_A$ of an operator $A$ to mapping properties of this operator, i.e. to compute $\mathrm{WF}(Au)$. Furthermore, given conditions on the wave front set of the kernel of an operator $\mathscr{C}^\infty_c(\mathbb{R}^d) \to \mathscr{D}'(\mathbb{R}^d)$, it is possible to extend the operator to certain spaces of distributions with assigned wave front set, i.e. to an operator of the form

$$\mathscr{D}'_{\Gamma_1}(X) \to \mathscr{D}'_{\Gamma_2}(X).$$

With this machinery, it is then possible to extend operations such as products and pull-backs by smooth maps to distributions.

### 3.1. **Wave front sets and pseudo-differential operators.**

For ΨDOs instead of general operators, the situation is much simpler. From the definition one may quickly deduce that for $A \in \Psi^m(X)$ we have

$$\mathrm{WF}(Au) \subset \mathrm{WF}(u).$$

This may be used to prove for elliptic operators, by use of a parametrix, that $\mathrm{WF}(Au) = \mathrm{WF}(u)$, since

$$\mathrm{WF}(u) = \mathrm{WF}((I+R)u) = \mathrm{WF}(PAu) \subset \mathrm{WF}(Au).$$

For general ΨDO, by use of a localized parametrix construction, one may obtain in the same way that $\mathrm{WF}(u) \subset \mathrm{WF}(Au) \cup \mathrm{char}(A)$.

3.2. **Conormal distributions.** The fact that ΨDOs fulfill

$$\mathrm{WF}(Au) \subset \mathrm{WF}(u)$$

stems from the fact that $\mathrm{WF}(K_A) \subset \{(x,x,\xi,-\xi) \,|\, (x,\xi) \in \dot{T}^*X\}$, meaning their wave front set is contained in the *conormal bundle to the diagonal* in $X \times X$.
The possibility to obtain a calculus of ΨDOs was mainly due to their exact (local) form, in particular their symbolic properties. Since we generally want to pass from operators to distributions, it is useful to elaborate on these facts.
For that we note that the kernel of a (reduced) ΨDO is given, after a change of variables $x' = (x-y)$ $x'' = x$

$$K_A(x,y) = (2\pi)^{-d} \int e^{i(x-y)\theta} a(x,\theta)\, d\theta K_A(x',x'') \quad = (2\pi)^{-d} \int e^{ix'\theta} a(x'',\theta)\, d\theta.$$

Distributions that are of the latter form for a submanifold $Y \subset X \subset \mathbb{R}^d$ locally given by $x' = 0$ are called conormal distributions. We mention that there also exists a global way of defining these in terms of the action of vector fields parallel to $Y$, see [7, Chap. XVIII].
To close things up: a generalization to these conormal distributions are then given by Lagrangian distributions, where the underlying singularities are not conormal bundles, but Lagrangian submanifolds of $\dot{T}^*X$. The associated class of operators are then called Fourier Integral Operators (FIOs). Locally, these are given as oscillatory integrals with more general phases, which brings us back to the first definitions of this tutorial...

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