ζ-functions of Fourier Integral Operators: gauged poly-log-homogeneous distributions

Tobias Hartung

King's College London

2018 / Aug / 27-30

Why do people care about operator ζ-functions?

Why do people care about operator ζ-functions?

- Why do people care about traces?

Why do people care about operator ζ-functions?

- Why do people care about traces?
- traces on an algebra $\Rightarrow$ invariants of the algebra

Why do people care about operator ζ-functions?

- Why do people care about traces?
- traces on an algebra ⇒ invariants of the algebra
- Atiyah-Singer index theorem: $\mathrm{tr}(e^{-tD^*D}) - \mathrm{tr}(e^{-tDD^*}) = \mathrm{ch}(V)\mathrm{Td}(X)[X]$

Why do people care about operator ζ-functions?

- Why do people care about traces?
- traces on an algebra $\Rightarrow$ invariants of the algebra
- Atiyah-Singer index theorem: $\operatorname{tr}(e^{-tD^*D}) - \operatorname{tr}(e^{-tDD^*}) = \operatorname{ch}(V)\operatorname{Td}(X)[X]$
- Heat trace: $\operatorname{tr} e^{-t|\Delta|} = (4\pi t)^{-\frac{\dim X}{2}} \sum_{k\in\mathbb{N}_0} A_k t^{\frac{k}{2}}$ where the A_k are called heat invariants ($A_0 = \operatorname{vol} X$, $A_2 = \text{total curvature}(X)$, ...)

Why do people care about operator ζ-functions?

- Why do people care about traces?
- traces on an algebra $\Rightarrow$ invariants of the algebra
- Atiyah-Singer index theorem: $\operatorname{tr}(e^{-tD^*D}) - \operatorname{tr}(e^{-tDD^*}) = \operatorname{ch}(V)\operatorname{Td}(X)[X]$
- Heat trace: $\operatorname{tr} e^{-t|\Delta|} = (4\pi t)^{-\frac{\dim X}{2}} \sum_{k\in\mathbb{N}_0} A_k t^{\frac{k}{2}}$ where the A_k are called heat invariants ($A_0 = \operatorname{vol} X$, $A_2 = \text{total curvature}(X)$, ...)
- Operator ζ-functions allow us to construct traces!

Why do people care about operator ζ-functions?

- Why do people care about traces?
- traces on an algebra $\Rightarrow$ invariants of the algebra
- Atiyah-Singer index theorem: $\operatorname{tr}(e^{-tD^*D}) - \operatorname{tr}(e^{-tDD^*}) = \operatorname{ch}(V)\operatorname{Td}(X)[X]$
- Heat trace: $\operatorname{tr} e^{-t|\Delta|} = (4\pi t)^{-\frac{\dim X}{2}} \sum_{k\in\mathbb{N}_0} A_k t^{\frac{k}{2}}$ where the A_k are called heat invariants ($A_0 = \operatorname{vol} X$, $A_2 = $ total curvature(X), ...)
- Operator ζ-functions allow us to construct traces!
- $\Rightarrow$ wave traces: $\operatorname{tr} e^{it\sqrt{|\Delta|}}$ (t-values of poles are lengths of closed geodesics)

Why do people care about operator ζ-functions?

- Why do people care about traces?
- traces on an algebra $\Rightarrow$ invariants of the algebra
- Atiyah-Singer index theorem: $\operatorname{tr}(e^{-tD^*D}) - \operatorname{tr}(e^{-tDD^*}) = \operatorname{ch}(V)\operatorname{Td}(X)[X]$
- Heat trace: $\operatorname{tr} e^{-t|\Delta|} = (4\pi t)^{-\frac{\dim X}{2}} \sum_{k\in\mathbb{N}_0} A_k t^{\frac{k}{2}}$ where the A_k are called heat invariants ($A_0 = \operatorname{vol} X$, $A_2 = $ total curvature(X), ...)
- Operator ζ-functions allow us to construct traces!
- $\Rightarrow$ wave traces: $\operatorname{tr} e^{it\sqrt{|\Delta|}}$ (t-values of poles are lengths of closed geodesics)
- physics: wave propagators are (closely related to) Fourier Integral Operators $\Rightarrow$ traces allow reconstruction of the QFT

Definition (Phase Function)

Let $N \in \mathbb{N}$. A function

$$\vartheta \in C\left(X \times X \times \mathbb{R}^N\right) \cap C^\infty\left(X \times X \times \left(\mathbb{R}^N \setminus \{0\}\right)\right)$$

is called a phase function if and only if it is positively homogeneous of degree 1 in the third argument, i.e.,

$$\forall x, y \in X \ \ \forall \xi \in \mathbb{R}^N \ \ \forall \lambda \in \mathbb{R}_{>0} : \ \vartheta(x, y, \lambda\xi) = \lambda\vartheta(x, y, \xi).$$

Example

Pseudo-differential phase function: $\vartheta(x, y, \xi) = \langle x - y, \xi \rangle_{\ell_2(N)}$ with $N = \dim X$.

Definition

Let $U \subseteq \mathbb{R}^n$ be open, $N \in \mathbb{N}$, and $m \in \mathbb{R}$. The Hörmander class $S^m(U \times U \times \mathbb{R}^N)$ is defined as the set of all $a \in C^\infty(U \times U \times \mathbb{R}^N)$ such that for every $K \subseteq_{\text{compact}} U^2$ and all multi-indices α, β, γ there exists a constant $c \in \mathbb{R}_{>0}$ such that

$$\forall (x,y) \in K \ \forall \xi \in \mathbb{R}^N \setminus B_{\mathbb{R}^N}(0,1) : \ \left|\partial_1^\alpha \partial_2^\beta \partial_3^\gamma a(x,y,\xi)\right| \leq c\left(1 + \|\xi\|_{\ell_2(N)}\right)^{m - \|\gamma\|_{\ell_1(N)}}$$

holds.

Definition

A Fourier Integral Operator $A\colon\ C_c^\infty(X)\to C_c^\infty(X)'$ on X is a linear operator whose Schwartz kernel $k\in C_c^\infty(X\times X)'$ is a locally finite sum of local representations of the form

$$k(x,y)=\int_{\mathbb{R}^N}e^{i\vartheta(x,y,\xi)}a(x,y,\xi)d\xi,$$

i.e.,

$$\forall\varphi,\psi\in C_c^\infty(X):\ A(\varphi)\psi=\sum_{i=1}^n\int_{X^2}k_i(x,y)\varphi(y)\psi(x)d\mathrm{vol}_{X^2}(x,y),$$

where, for each localization $U\subseteq X$, ϑ is a phase function and a is an element of some Hörmander class $S^m(U\times U\times\mathbb{R}^N)$. a is also called an amplitude or symbol.

Definition

Let ϑ be a phase function. Then, we call

$$C(\vartheta) := \left\{ (x,y,\xi) \in X \times X \times \left(\mathbb{R}^N \setminus \{0\} \right);\ \partial_3 \vartheta(x,y,\xi) = 0 \right\}$$

the critical set of ϑ.
ϑ is called non-degenerate if and only of the family of differentials

$$\left(d\partial_{3,j} \vartheta(x,y,\xi) \right)_{j \in \mathbb{N}_{\leq N}}$$

is linearly independent for every $(x,y,\xi) \in C(\vartheta)$ where $\partial_{3,j}$ denotes the derivative with respect to the j^{th} component of the third argument.

Definition

Let $\Lambda \subseteq T^*(X^2) \setminus 0$ be a Lagrangian manifold and A a Fourier Integral Operator of the form $A = \sum_{j=1}^{n} A_j$ where each A_j has a non-degenerate phase function ϑ_j defined on an open, conic subset $U_j \subseteq_{\mathrm{open}} X \times X \times (\mathbb{R}^{N_j} \setminus \{0\})$ such that

$$U_j \cap C(\vartheta_j) \ni (x,y,\xi) \mapsto (x,y,\partial_1\vartheta_j(x,y,\xi),\partial_2\vartheta_j(x,y,\xi))$$

is a diffeomorphism onto an open subset $U_j^\Lambda \subseteq_{\mathrm{open}} \Lambda$, and amplitude $a_j \in S^{m+\frac{\dim X - N_j}{2}}(X \times X \times \mathbb{R}^{N_j})$ with

$$\operatorname{spt} a_j \subseteq \{(x,y,t\xi) \in X \times X \times \mathbb{R}^{N_j};\ (x,y,\xi) \in K\ \wedge\ t \in \mathbb{R}_{>0}\}$$

for some $K \subseteq_{\mathrm{compact}} U_j$. Then, we say A is an element of $I^m(X \times X, \Lambda)$ (or more precisely, A has a kernel in $I^m(X \times X, \Lambda)$).

Definition (Canonical Relation)

Let $\Gamma \subseteq T_0^*X \times T_0^*X$ be a relation satisfying

(i) Γ is symmetric, i.e., $\forall (p,q) \in \Gamma:\ (q,p) \in \Gamma$,

(ii) Γ is transitive, i.e., $\forall (p,q),(q,r) \in \Gamma:\ (p,r) \in \Gamma$,

We will call any such Γ a canonical relation. Furthermore, we will assume that all canonical relations satisfy

(iii) the composition $\Gamma \circ \Gamma$ is clean, i.e., $\Gamma \times \Gamma$ intersects $T^*X \times \operatorname{diag}(T^*X \times T^*X) \times T^*X$ in a manifold whose tangent plane is precisely the intersection of the tangent planes of $\Gamma \times \Gamma$ and $T^*X \times \operatorname{diag}(T^*X \times T^*X) \times T^*X$ where $\operatorname{diag}(T^*X \times T^*X) := \{(x,y) \in T^*X \times T^*X;\ x = y\}$,

(iv) the projection $\mathrm{pr}_1:\ \Gamma \to T^*X;\ (p,q) \mapsto p$ is proper, i.e., pre-sets of compacta are compact.

Definition (Twisted Canonical Relation)

We will call the set

$$\Gamma' := \{((x,\xi),(y,\eta)) \in T_0^*X \times T_0^*X;\ ((x,\xi),(y,-\eta)) \in \Gamma\}$$

a twisted canonical relation.

Definition

Let $\Gamma \subseteq T_0^*X \times T_0^*X$ be a canonical relation. Γ is called a homogeneous canonical relation if and only if Γ' is a Lagrangian manifold.

Definition

Let $\Gamma \subseteq T_0^*X \times T_0^*X$ be a canonical relation. Γ is called a homogeneous canonical relation if and only if Γ' is a Lagrangian manifold.

Definition

Let $\Gamma \subseteq T_0^*X \times T_0^*X$ be a homogeneous canonical relation with $\Gamma \circ \Gamma = \Gamma$. Then, we call

$$\mathcal{A}_\Gamma := \bigcup_{m \in \mathbb{R}} I^m(X \times X, \Gamma')$$

the algebra of Fourier Integral Operators associated with Γ.

Lemma

Let A be a Fourier Integral Operator with kernel $k \in I^m(X \times X, \Lambda)$. If $m < -\dim X$, then A is of trace-class.

Lemma

Let A be a Fourier Integral Operator with kernel $k \in I^m(X \times X, \Lambda)$. If $m < -\dim X$, then A is of trace-class.

Lemma

Let $k(x,y) = \int_{\mathbb{R}^N} e^{i\vartheta(x,y,\xi)} a(x,y,\xi) d\xi$ be a localization of the Schwartz kernel of an $A \in \mathcal{A}_{\Lambda'}$ with $a \in S^m(U \times \mathbb{R}^N)$ for some $m < -N$ and $U \subseteq_{\text{open}} X^2$. Then, $k \in C(U)$.

Lemma

Let A be a Fourier Integral Operator with kernel $k \in I^m(X \times X, \Lambda)$. If $m < -\dim X$, then A is of trace-class.

Lemma

Let $k(x,y) = \int_{\mathbb{R}^N} e^{i\vartheta(x,y,\xi)} a(x,y,\xi) d\xi$ be a localization of the Schwartz kernel of an $A \in \mathcal{A}_{\Lambda'}$ with $a \in S^m(U \times \mathbb{R}^N)$ for some $m < -N$ and $U \subseteq_{\mathrm{open}} X^2$. Then, $k \in C(U)$.

Corollary

There exists a subalgebra $\mathcal{A}_{\Lambda',0} \subseteq \mathcal{A}_{\Lambda'}$ which consists of trace-class operators with continuous kernels. In particular, if k is the kernel of $A \in \mathcal{A}_{\Lambda',0}$, then

$$\mathrm{tr} A = \int_X k(x,x) \mathrm{dvol}_X(x) = \langle k, \delta_{\mathrm{diag}} \rangle.$$

ζ-regularization

- Let $\mathcal{A}$ be an operator algebra.
- Let $\mathcal{A}_0 \subseteq \mathcal{A}$ be a subalgebra.
- Let $\tau : \mathcal{A}_0 \to \mathbb{C}$ be a trace, i.e., linear functional such that $\forall x, y \in \mathcal{A}_0 : \tau(xy) = \tau(yx)$.

ζ-regularization

- Let $\mathcal{A}$ be an operator algebra.
- Let $\mathcal{A}_0 \subseteq \mathcal{A}$ be a subalgebra.
- Let $\tau : \mathcal{A}_0 \to \mathbb{C}$ be a trace, i.e., linear functional such that $\forall x, y \in \mathcal{A}_0 : \tau(xy) = \tau(yx)$.
- Instead of $A \in \mathcal{A}$, consider $\varphi : \mathbb{C} \to \mathcal{A}$ holomorphic such that $\varphi(0) = A$ and

$$\exists \Omega_0 \subseteq_{\text{open,connected}} \mathbb{C} : \varphi[\Omega_0] \subseteq \mathcal{A}_0.$$

ζ-regularization

- Let $\mathcal{A}$ be an operator algebra.
- Let $\mathcal{A}_0 \subseteq \mathcal{A}$ be a subalgebra.
- Let $\tau: \mathcal{A}_0 \to \mathbb{C}$ be a trace, i.e., linear functional such that $\forall x, y \in \mathcal{A}_0: \tau(xy) = \tau(yx)$.
- Instead of $A \in \mathcal{A}$, consider $\varphi: \mathbb{C} \to \mathcal{A}$ holomorphic such that $\varphi(0) = A$ and

$$\exists \Omega_0 \subseteq_{\text{open,connected}} \mathbb{C}: \varphi[\Omega_0] \subseteq \mathcal{A}_0.$$

- Let $\Omega \subseteq_{\text{open,connected}} \mathbb{C}$ be maximal satisfying $\Omega_0 \subseteq \Omega$ such that $\tau \circ \varphi: \Omega_0 \to \mathbb{C}$ has a holomorphic extension $\zeta(\varphi): \Omega \to \mathbb{C}$.

ζ-regularization

- Let $\mathcal{A}$ be an operator algebra.
- Let $\mathcal{A}_0 \subseteq \mathcal{A}$ be a subalgebra.
- Let $\tau : \mathcal{A}_0 \to \mathbb{C}$ be a trace, i.e., linear functional such that $\forall x, y \in \mathcal{A}_0 : \tau(xy) = \tau(yx)$.
- Instead of $A \in \mathcal{A}$, consider $\varphi : \mathbb{C} \to \mathcal{A}$ holomorphic such that $\varphi(0) = A$ and

$$\exists \Omega_0 \subseteq_{\text{open,connected}} \mathbb{C} : \varphi[\Omega_0] \subseteq \mathcal{A}_0.$$

- Let $\Omega \subseteq_{\text{open,connected}} \mathbb{C}$ be maximal satisfying $\Omega_0 \subseteq \Omega$ such that $\tau \circ \varphi : \Omega_0 \to \mathbb{C}$ has a holomorphic extension $\zeta(\varphi) : \Omega \to \mathbb{C}$.
- Is $\zeta(\varphi)$ holomorphic in a neighborhood of 0? (Want $\operatorname{tr} A := \zeta(\varphi)(0)$.)
- Does $\varphi(0) = \psi(0)$ imply $\zeta(\varphi)(0) = \zeta(\psi)(0)$?

ζ-regularization for pseudo-differential operators

- Let $\mathcal{A} = \Psi^{\mathrm{cl}}$ be the algebra of classical pseudo-differential operators on a compact manifold X without boundary.
- Let $\mathcal{A}_0 = \Psi^{\mathrm{cl}} \cap \mathcal{S}_1(L_2(X))$.
- Let $\tau : \mathcal{A}_0 \to \mathbb{C}$ be the trace tr in $\mathcal{S}_1(L_2(X))$.

ζ-regularization for pseudo-differential operators

- Let $\mathcal{A} = \Psi^{\mathrm{cl}}$ be the algebra of classical pseudo-differential operators on a compact manifold X without boundary.
- Let $\mathcal{A}_0 = \Psi^{\mathrm{cl}} \cap \mathcal{S}_1(L_2(X))$.
- Let $\tau : \mathcal{A}_0 \to \mathbb{C}$ be the trace tr in $\mathcal{S}_1(L_2(X))$.
- Instead of $A \in \mathcal{A}$, consider $\varphi : \mathbb{C} \to \mathcal{A}$ holomorphic such that $\varphi(0) = A$ and

$$\forall z \in \mathbb{C} : \varphi(z) \sim \sum_{j \in \mathbb{N}_0} a_{m-j+z}(z).$$

Then, $\Omega_0 = \mathbb{C}_{\Re(\cdot) < -\dim X - \Re(m)}$.

ζ-regularization for pseudo-differential operators

- Let $\mathcal{A} = \Psi^{\mathrm{cl}}$ be the algebra of classical pseudo-differential operators on a compact manifold X without boundary.
- Let $\mathcal{A}_0 = \Psi^{\mathrm{cl}} \cap \mathcal{S}_1(L_2(X))$.
- Let $\tau : \mathcal{A}_0 \to \mathbb{C}$ be the trace tr in $\mathcal{S}_1(L_2(X))$.
- Instead of $A \in \mathcal{A}$, consider $\varphi : \mathbb{C} \to \mathcal{A}$ holomorphic such that $\varphi(0) = A$ and

$$\forall z \in \mathbb{C} : \ \varphi(z) \sim \sum_{j \in \mathbb{N}_0} a_{m-j+z}(z).$$

  Then, $\Omega_0 = \mathbb{C}_{\Re(\cdot) < -\dim X - \Re(m)}$.
- $\mathrm{tr} \circ \varphi$ has meromorphic extension to $\mathbb{C}$ with isolated simple poles.

ζ-regularization for pseudo-differential operators

- Let $\mathcal{A} = \Psi^{\mathrm{cl}}$ be the algebra of classical pseudo-differential operators on a compact manifold X without boundary.
- Let $\mathcal{A}_0 = \Psi^{\mathrm{cl}} \cap \mathcal{S}_1(L_2(X))$.
- Let $\tau: \mathcal{A}_0 \to \mathbb{C}$ be the trace tr in $\mathcal{S}_1(L_2(X))$.
- Instead of $A \in \mathcal{A}$, consider $\varphi: \mathbb{C} \to \mathcal{A}$ holomorphic such that $\varphi(0) = A$ and

$$\forall z \in \mathbb{C}: \ \varphi(z) \sim \sum_{j \in \mathbb{N}_0} a_{m-j+z}(z).$$

  Then, $\Omega_0 = \mathbb{C}_{\Re(\cdot) < -\dim X - \Re(m)}$.
- $\mathrm{tr} \circ \varphi$ has meromorphic extension to $\mathbb{C}$ with isolated simple poles.
- Is $\zeta(\varphi)$ holomorphic in a neighborhood of 0?
- Does $\varphi(0) = \psi(0)$ imply $\zeta(\varphi)(0) = \zeta(\psi)(0)$?

ζ-regularization for pseudo-differential operators

- Let $\mathcal{A} = \Psi^{\mathrm{cl}}$ be the algebra of classical pseudo-differential operators on a compact manifold X without boundary.
- Let $\mathcal{A}_0 = \Psi^{\mathrm{cl}} \cap \mathcal{S}_1(L_2(X))$.
- Let $\tau : \mathcal{A}_0 \to \mathbb{C}$ be the trace tr in $\mathcal{S}_1(L_2(X))$.
- Instead of $A \in \mathcal{A}$, consider $\varphi : \mathbb{C} \to \mathcal{A}$ holomorphic such that $\varphi(0) = A$ and

  $$\forall z \in \mathbb{C} : \ \varphi(z) \sim \sum_{j \in \mathbb{N}_0} a_{m-j+z}(z).$$

  Then, $\Omega_0 = \mathbb{C}_{\Re(\cdot) < -\dim X - \Re(m)}$.
- $\mathrm{tr} \circ \varphi$ has meromorphic extension to $\mathbb{C}$ with isolated simple poles.
- $m \notin \mathbb{Z}_{\geq -\dim X} \Rightarrow \zeta(\varphi)$ holomorphic near 0
- Does $\varphi(0) = \psi(0)$ imply $\zeta(\varphi)(0) = \zeta(\psi)(0)$?

ζ-regularization for pseudo-differential operators

- Let $\mathcal{A} = \Psi^{\mathrm{cl}}$ be the algebra of classical pseudo-differential operators on a compact manifold X without boundary.
- Let $\mathcal{A}_0 = \Psi^{\mathrm{cl}} \cap \mathcal{S}_1(L_2(X))$.
- Let $\tau : \mathcal{A}_0 \to \mathbb{C}$ be the trace tr in $\mathcal{S}_1(L_2(X))$.
- Instead of $A \in \mathcal{A}$, consider $\varphi : \mathbb{C} \to \mathcal{A}$ holomorphic such that $\varphi(0) = A$ and

$$\forall z \in \mathbb{C} : \ \varphi(z) \sim \sum_{j \in \mathbb{N}_0} a_{m-j+z}(z).$$

  Then, $\Omega_0 = \mathbb{C}_{\Re(\cdot) < -\dim X - \Re(m)}$.
- $\mathrm{tr} \circ \varphi$ has meromorphic extension to $\mathbb{C}$ with isolated simple poles.
- $m \notin \mathbb{Z}_{\geq -\dim X} \Rightarrow \zeta(\varphi)$ holomorphic near 0
- $m \notin \mathbb{Z}_{\geq -\dim X} \ \wedge \ N \in \mathbb{N}_{>\dim X + \Re(m)}$
  $\Rightarrow \zeta(\varphi)(0) = \int_M \left(k(0) - \sum_{j=0}^{N} k_{m-j}(0)\right)(x,x) d\mathrm{vol}_X(x)$

- Let $\mathcal{A} = \mathcal{A}_\Gamma$ be an algebra of FIOs.
- Let $\mathcal{A}_0 = \mathcal{A}_\Gamma \cap \mathcal{S}_1(L_2(X))$.
- Let $\tau : \mathcal{A}_0 \to \mathbb{C}$ be the trace tr in $\mathcal{S}_1(L_2(M))$.

- Let $\mathcal{A} = \mathcal{A}_\Gamma$ be an algebra of FIOs.
- Let $\mathcal{A}_0 = \mathcal{A}_\Gamma \cap \mathcal{S}_1(L_2(X))$.
- Let $\tau: \mathcal{A}_0 \to \mathbb{C}$ be the trace tr in $\mathcal{S}_1(L_2(M))$.
- Instead of $A \in \mathcal{A}$, consider $\varphi: \mathbb{C} \to \mathcal{A}$ holomorphic such that $\varphi(0) = A$ and

$$\forall z \in \mathbb{C}: \ \varphi(z) \sim \sum_{j \in \mathbb{N}_0} a_{m-j+z}(z) \in S^{m+\Re(z)}(X^2 \times \mathbb{R}^N).$$

Then, $\Omega_0 = \mathbb{C}_{\Re(\cdot) < -\max\{\dim X, N\} - \Re(m)}$.

- Let $\mathcal{A} = \mathcal{A}_\Gamma$ be an algebra of FIOs.
- Let $\mathcal{A}_0 = \mathcal{A}_\Gamma \cap \mathcal{S}_1(L_2(X))$.
- Let $\tau : \mathcal{A}_0 \to \mathbb{C}$ be the trace tr in $\mathcal{S}_1(L_2(M))$.
- Instead of $A \in \mathcal{A}$, consider $\varphi : \mathbb{C} \to \mathcal{A}$ holomorphic such that $\varphi(0) = A$ and

$$\forall z \in \mathbb{C} : \varphi(z) \sim \sum_{j \in \mathbb{N}_0} a_{m-j+z}(z) \in S^{m+\Re(z)}(X^2 \times \mathbb{R}^N).$$

Then, $\Omega_0 = \mathbb{C}_{\Re(\cdot) < -\max\{\dim X, N\} - \Re(m)}$.

Let $k(z)$ be the kernel of $\varphi(z)$. Then, we want to show that

$$\Omega_0 \ni z \mapsto \langle k(z), \delta_{\mathrm{diag}} \rangle \in \mathbb{C}$$

has a meromorphic extension to $\mathbb{C}$.

The Black Box Magic Theorem

Theorem (Hörmander Thm 21.2.10)

Let S be a conic symplectic manifold of dimension $2n$ and V_1 and V_2 conic Lagrangian submanifolds intersecting cleanly at $\gamma \in S$.
Then, there are homogeneous symplectic coordinates (x, ξ) at γ such that $\gamma = (0, e_1)$, $e_1 = (1, 0, \ldots, 0)$, and near γ

$$\begin{aligned} V_1 =& \{(0, \xi)\} \\ V_2 =& \{(0, x'', \xi', 0)\} \end{aligned}$$

where $\xi' = (\xi_1, \ldots, \xi_k)$, $x'' = (x_{k+1}, \ldots, x_n)$, and $k = \dim V_1 \cap V_2$.

The Black Box Magic happening

- It is possible to write

$$k(z) = \int_{\mathbb{R}^k} e^{i\langle x',\xi'\rangle} a(z)(x'',\xi')d\xi'.$$

The Black Box Magic happening

- It is possible to write
$$k(z) = \int_{\mathbb{R}^k} e^{i\langle x',\xi'\rangle} a(z)(x'',\xi') d\xi'.$$
- There exists a pseudo-differential operator P such that $\delta_{\mathrm{diag}} = P\delta_0$.

The Black Box Magic happening

- It is possible to write

$$k(z) = \int_{\mathbb{R}^k} e^{i\langle x',\xi'\rangle} a(z)(x'',\xi')d\xi'.$$

- There exists a pseudo-differential operator P such that $\delta_{\mathrm{diag}} = P\delta_0$.
- Hence, there exists a polyhomogeneous $\alpha(z)$ such that

$$\langle k(z), \delta_{\mathrm{diag}}\rangle = \langle P^T k(z), \delta_0\rangle = \int_{\mathbb{R}^k} \alpha(z)(\xi)d\xi$$

The Black Box Magic happening

- It is possible to write

$$k(z) = \int_{\mathbb{R}^k} e^{i\langle x',\xi'\rangle} a(z)(x'',\xi')d\xi'.$$

- There exists a pseudo-differential operator P such that $\delta_{\mathrm{diag}} = P\delta_0$.
- Hence, there exists a polyhomogeneous $\alpha(z)$ such that

$$\langle k(z), \delta_{\mathrm{diag}}\rangle = \langle P^T k(z), \delta_0\rangle = \int_{\mathbb{R}^k} \alpha(z)(\xi)d\xi$$

Remark. This approach appears in various different forms in many publications by Duistermaat, Greenleaf, Guillemin, Hörmander, Melrose, Uhlmann, ...

The Black Box Magic happening

- It is possible to write

$$k(z) = \int_{\mathbb{R}^k} e^{i\langle x', \xi' \rangle} a(z)(x'', \xi') d\xi'.$$

- There exists a pseudo-differential operator P such that $\delta_{\mathrm{diag}} = P\delta_0$.
- Hence, there exists a polyhomogeneous $\alpha(z)$ such that

$$\langle k(z), \delta_{\mathrm{diag}} \rangle = \langle P^T k(z), \delta_0 \rangle = \int_{\mathbb{R}^k} \alpha(z)(\xi) d\xi$$

Remark. This approach appears in various different forms in many publications by Duistermaat, Greenleaf, Guillemin, Hörmander, Melrose, Uhlmann, ...
Remark (Zworski). For trace-class $A \in \mathcal{A}_\Gamma$ there exists a FIO F such that $\mathrm{tr} A = \mathrm{tr}(F^{-1}AF) = \int \alpha$.

Black Box Magic for pseudo-differential operators

Consider a trace-class pseudo-differential operator A with symbol σ. Then, we have

$$\operatorname{tr} A = \left\langle (x,y) \mapsto \int_{\mathbb{R}^{\dim X}} e^{i\langle x-y,\xi\rangle}\sigma(x,y,\xi)d\xi, \delta_{\mathrm{diag}} \right\rangle$$

Black Box Magic for pseudo-differential operators

Consider a trace-class pseudo-differential operator A with symbol σ. Then, we have

$$\begin{aligned}\operatorname{tr} A &= \left\langle (x,y) \mapsto \int_{\mathbb{R}^{\dim X}} e^{i\langle x-y,\xi\rangle}\sigma(x,y,\xi)d\xi, \delta_{\mathrm{diag}} \right\rangle \\ &= \int_X \int_{\mathbb{R}^{\dim X}} \sigma(x,x,\xi)d\xi d\mathrm{vol}_X(x)\end{aligned}$$

Black Box Magic for pseudo-differential operators

Consider a trace-class pseudo-differential operator A with symbol σ. Then, we have

$$\begin{aligned}
\operatorname{tr} A &= \left\langle (x,y) \mapsto \int_{\mathbb{R}^{\dim X}} e^{i\langle x-y,\xi\rangle}\sigma(x,y,\xi)d\xi, \delta_{\mathrm{diag}} \right\rangle \\
&= \int_X \int_{\mathbb{R}^{\dim X}} \sigma(x,x,\xi)d\xi d\mathrm{vol}_X(x) \\
&= \int_{\mathbb{R}^{\dim X}} \underbrace{\int_X \sigma(x,x,\xi)d\mathrm{vol}_X(x)}_{=:\alpha(\xi)} d\xi
\end{aligned}$$

A gauged poly-log-homogeneous distribution α is a holomorphic family $(\alpha(z))_{z\in\mathbb{C}}$ with an expansion

$$\alpha = \alpha_0 + \sum_{\iota\in I} \alpha_\iota$$

where

- $I \subseteq \mathbb{N}$
- $\alpha_0(z) \in L_1(\mathbb{R}_{\geq 1} \times M)$ for all z in an open neighborhood of $\mathbb{C}_{\Re \cdot \leq 0}$ where M is a compact, orientable, finite dimensional manifold without boundary
- $\forall \iota \in I\ \exists d_\iota \in \mathbb{C}\ \exists l_\iota \in \mathbb{N}_0\ \exists \tilde{\alpha}_\iota \in C^\omega(\mathbb{C}, L_1(M))\ \forall (r,\nu) \in \mathbb{R}_{\geq 1} \times M:$

$$\alpha_\iota(z)(r,\nu) = r^{d_\iota + z} (\ln r)^{l_\iota} \tilde{\alpha}_\iota(z)(\nu)$$

Furthermore (primarily if I is infinite)

- The family $(\Re(d_\iota))_{\iota\in I}$ is bounded from above.[1]
- The map $I \ni \iota \mapsto (d_\iota, l_\iota)$ is injective.
- There are only finitely many ι satisfying $d_\iota = d$ for any given $d \in \mathbb{C}$.
- The family $((d_\iota - \delta)^{-1})_{\iota\in I}$ is in $\ell_2(I)$ for any $\delta \in \mathbb{C} \setminus \{d_\iota;\ \iota \in I\}$.
- Each $\sum_{\iota\in I} \tilde{\alpha}_\iota(z)$ converges unconditionally in $L_1(M)$.[2]

[1] Note, we do not require $\Re(d_\iota) \to -\infty$. $\forall \iota \in I:\ \Re(d_\iota) = 42$ is entirely possible.

[2] Unconditional convergence of $\sum_{\iota\in I} \tilde{\alpha}_\iota(z)$ in $L_1(M)$ may also be replaced by the slightly weaker, though more artificial, condition $\sum_{\iota\in I} \|\tilde{\alpha}_\iota(z)\|_{L_1(M)}^2 < \infty$.

Example (Classical pseudo-differential operator)

- Let $\sigma \sim \sum_{j\in\mathbb{N}_0} a_{m-j}$ be a classical symbol.

Example (Classical pseudo-differential operator)

- Let $\sigma \sim \sum_{j\in\mathbb{N}_0} a_{m-j}$ be a classical symbol.
- gauging $\sigma \rightsquigarrow \sigma(z) \sim \sum_{j\in\mathbb{N}_0} a_{m-j+z}(z)$

Example (Classical pseudo-differential operator)

- Let $\sigma \sim \sum_{j\in\mathbb{N}_0} a_{m-j}$ be a classical symbol.
- gauging $\sigma \rightsquigarrow \sigma(z) \sim \sum_{j\in\mathbb{N}_0} a_{m-j+z}(z)$
- splitting into trace-class and non-trace-class: $I = \{j \in \mathbb{N}_0;\ \Re(m) - j \geq -\dim X\}$
- $M = \partial B_{\mathbb{R}^{\dim X}}$
- $\forall z \in \mathbb{C}\ \forall \iota \in I\ \forall (r,\nu) \in \mathbb{R}_{\geq 1} \times M:\ \alpha_\iota(z)(r,\nu) := \int_X a_{m-\iota+z}(z)(x,x,r\nu)\mathrm{dvol}_X(x)$
- $\alpha_0(z)(r,\nu) := \int_X \sigma(z)(x,x,r\nu) - \sum_{\iota\in I} a_{m-\iota+z}(z)(x,x,r\nu)\mathrm{dvol}_X(x)$

ζ-functions of gauged poly-log-homogeneous distributions

Formal computation:

$$\int_{\mathbb{R}_{\geq 1}\times M} \alpha(z) d\mathrm{vol}_{\mathbb{R}_{\geq 1}\times M}$$

ζ-functions of gauged poly-log-homogeneous distributions

Formal computation:

$$\int_{\mathbb{R}_{\ge 1}\times M}\alpha(z)d\mathrm{vol}_{\mathbb{R}_{\ge 1}\times M}=\int_{\mathbb{R}_{\ge 1}\times M}\alpha_0(z)d\mathrm{vol}_{\mathbb{R}_{\ge 1}\times M}+\sum_{\iota\in I}\int_{\mathbb{R}_{\ge 1}\times M}\alpha_\iota(z)d\mathrm{vol}_{\mathbb{R}_{\ge 1}\times M}$$

ζ-functions of gauged poly-log-homogeneous distributions

Formal computation:

$$\int_{\mathbb{R}_{\geq 1}\times M} \alpha(z) d\mathrm{vol}_{\mathbb{R}_{\geq 1}\times M} = \underbrace{\int_{\mathbb{R}_{\geq 1}\times M} \alpha_0(z) d\mathrm{vol}_{\mathbb{R}_{\geq 1}\times M}}_{=:\tau_0(z)\in\mathbb{C}} + \sum_{\iota\in I} \int_{\mathbb{R}_{\geq 1}\times M} \alpha_\iota(z) d\mathrm{vol}_{\mathbb{R}_{\geq 1}\times M}$$

ζ-functions of gauged poly-log-homogeneous distributions

Formal computation:

$$\int_{\mathbb{R}_{\geq 1}\times M}\alpha(z)d\mathrm{vol}_{\mathbb{R}_{\geq 1}\times M}=\underbrace{\int_{\mathbb{R}_{\geq 1}\times M}\alpha_0(z)d\mathrm{vol}_{\mathbb{R}_{\geq 1}\times M}}_{=:\tau_0(z)\in\mathbb{C}}+\sum_{\iota\in I}\int_{\mathbb{R}_{\geq 1}\times M}\alpha_\iota(z)d\mathrm{vol}_{\mathbb{R}_{\geq 1}\times M}$$

$$=\tau_0(z)+\sum_{\iota\in I}\int_{\mathbb{R}_{\geq 1}}\int_M\alpha_\iota(z)(r,\nu)r^{\dim M}d\mathrm{vol}_M(\nu)dr$$

ζ-functions of gauged poly-log-homogeneous distributions

Formal computation:

$$\int_{\mathbb{R}_{\geq 1}\times M}\alpha(z)d\mathrm{vol}_{\mathbb{R}_{\geq 1}\times M}=\underbrace{\int_{\mathbb{R}_{\geq 1}\times M}\alpha_0(z)d\mathrm{vol}_{\mathbb{R}_{\geq 1}\times M}}_{=:\tau_0(z)\in\mathbb{C}}+\sum_{\iota\in I}\int_{\mathbb{R}_{\geq 1}\times M}\alpha_\iota(z)d\mathrm{vol}_{\mathbb{R}_{\geq 1}\times M}$$

$$=\tau_0(z)+\sum_{\iota\in I}\int_{\mathbb{R}_{\geq 1}}\int_M\alpha_\iota(z)(r,\nu)r^{\dim M}d\mathrm{vol}_M(\nu)dr$$

$$=\tau_0(z)+\sum_{\iota\in I}\underbrace{\int_{\mathbb{R}_{\geq 1}}r^{\dim M+d_\iota+z}(\ln r)^{l_\iota}\,dr}_{=:c_\iota(z)}\underbrace{\int_M\tilde{\alpha}_\iota(z)d\mathrm{vol}_M}_{=:\mathrm{res}\alpha_\iota(z)\in\mathbb{C}}$$

ζ-functions of gauged poly-log-homogeneous distributions

Formal computation:

$$\int_{\mathbb{R}_{\geq 1}\times M}\alpha(z)d\mathrm{vol}_{\mathbb{R}_{\geq 1}\times M} = \underbrace{\int_{\mathbb{R}_{\geq 1}\times M}\alpha_0(z)d\mathrm{vol}_{\mathbb{R}_{\geq 1}\times M}}_{=:\tau_0(z)\in\mathbb{C}} + \sum_{\iota\in I}\int_{\mathbb{R}_{\geq 1}\times M}\alpha_\iota(z)d\mathrm{vol}_{\mathbb{R}_{\geq 1}\times M}$$

$$=\tau_0(z) + \sum_{\iota\in I}\int_{\mathbb{R}_{\geq 1}}\int_M \alpha_\iota(z)(r,\nu)r^{\dim M}d\mathrm{vol}_M(\nu)dr$$

$$=\tau_0(z) + \sum_{\iota\in I}\underbrace{\int_{\mathbb{R}_{\geq 1}} r^{\dim M + d_\iota + z}(\ln r)^{l_\iota}dr}_{=:c_\iota(z)}\underbrace{\int_M \tilde{\alpha}_\iota(z)d\mathrm{vol}_M}_{=:\mathrm{res}\alpha_\iota(z)\in\mathbb{C}}$$

$$=\tau_0(z) + \sum_{\iota\in I}c_\iota(z)\mathrm{res}\alpha_\iota(z)$$

ζ-functions of gauged poly-log-homogeneous distributions

Lemma

For $\Re(z) \ll 0: c_\iota(z) = (-1)^{l_\iota+1} l_\iota! \left(\dim M + d_\iota + z + 1\right)^{-(l_\iota+1)} =: \tilde{c}_\iota(z)$

Proof.

Use upper incomplete Γ-function Γ_{ui} to show

$$\left(\mathbb{R}_{>0} \ni y \mapsto \frac{-\Gamma_{ui}(l+1,-(d+1)\ln y)}{(-(d+1))^{l+1}} \in \mathbb{C}\right)'(x) = x^d (\ln x)^l$$

and then integrate

$$\int_{\mathbb{R}_{\geq 1}} r^{\dim M + d_\iota + z} (\ln r)^{l_\iota}\, dr.$$

□

Lemma

For every $z \in \mathbb{C} \setminus \{-\dim M - d_\iota - 1;\ \iota \in I\}$, $\sum_{\iota \in I} \tilde{c}_\iota(z) \operatorname{res} \alpha_\iota(z)$ *converges absolutely.*

Proof.

By assumption, $(\tilde{c}_\iota(z))_{\iota \in I} \in \ell_2(I)$ and $\sum_{\iota \in I} \tilde{\alpha}_\iota(z)$ uncond. conv. in $L_1(M)$. By

Theorem (Orlicz)

Let $p \in \mathbb{R}_{\geq 1}$, $q = \begin{cases} 2 & , p \in [1,2] \\ p & , p \in \mathbb{R}_{>2} \end{cases}$, *and* $\sum_{j \in \mathbb{N}} x_j$ *converges unconditionally in* L_p.
Then, $\sum_{j \in \mathbb{N}} \|x_j\|_{L_p}^q$ *converges.*

we have $(\operatorname{res} \alpha_\iota(z))_{\iota \in I} \in \ell_2(I)$, i.e., $(\tilde{c}_\iota(z) \operatorname{res} \alpha_\iota(z))_{\iota \in I} \in \ell_1(I)$.

$\square$

ζ-functions of gauged poly-log-homogeneous distributions

Definition

Let α be a gauged poly-log-homogeneous distribution. Then, we define the ζ-function $\zeta(\alpha)$ of α to be the meromorphic extension of

$$\zeta(\alpha)(z) := \int_{\mathbb{R}_{\geq 1}\times M} \alpha(z) d\mathrm{vol}_{\mathbb{R}_{\geq 1}\times M},$$

i.e., in an open neighborhood of $\mathbb{C}_{\Re(\cdot)\leq 0}$

$$\zeta(\alpha)(z) = \int_{\mathbb{R}_{\geq 1}\times M} \alpha_0(z) d\mathrm{vol}_{\mathbb{R}_{\geq 1}\times M} + \sum_{\iota\in I} \frac{(-1)^{l_\iota+1} l_\iota! \mathrm{res}\alpha_\iota(z)}{(\dim M + d_\iota + z + 1)^{l_\iota+1}}.$$

Theorem

$\zeta(\alpha)$ is a well-defined meromorphic function on an open neighborhood of $\mathbb{C}_{\Re(\cdot)\le 0}$ and has at most isolated poles of finite order in the set

$$\{-d_\iota - \dim M - 1;\ \iota \in I\}.$$

Theorem

$\zeta(\alpha)$ is a well-defined meromorphic function on an open neighborhood of $\mathbb{C}_{\Re(\cdot)\le 0}$ and has at most isolated poles of finite order in the set

$$\{-d_\iota - \dim M - 1;\ \iota \in I\}.$$

Example

For classical pseudo-differential operators: $\dim M = \dim X - 1$ and all l_ι vanish. Hence, ζ-functions of psudo-differential operators exist and have at most isolated simple poles in the set

$$\{-d_\iota - \dim X;\ \iota \in I\}.$$

Definition

Let $f(z) := \sum_{n\in\mathbb{Z}} a_n (z-z_0)^n$ be without essential singularity at z_0. Then we define:

- order of the initial Laurent coefficient: $\operatorname{oilc}_{z_0}(f) := \min\{n \in \mathbb{Z};\ a_n \neq 0\}$
- initial Laurent coefficient: $\operatorname{ilc}_{z_0}(f) := a_{\operatorname{oilc}_{z_0}(f)}$

Definition

Let $f(z) := \sum_{n\in\mathbb{Z}} a_n(z-z_0)^n$ be without essential singularity at z_0. Then we define:

- order of the initial Laurent coefficient: $\operatorname{oilc}_{z_0}(f) := \min\{n \in \mathbb{Z};\ a_n \neq 0\}$
- initial Laurent coefficient: $\operatorname{ilc}_{z_0}(f) := a_{\operatorname{oilc}_{z_0}(f)}$

Lemma

Let $\alpha = \alpha_0 + \sum_{\iota\in I} \alpha_\iota$ and $\beta = \beta_0 + \sum_{\iota\in I'} \beta_\iota$ be two gauged poly-log-homogeneous distributions with $\alpha(0) = \beta(0)$ and $\operatorname{res}\alpha_j(0) \neq 0$ if l_j is the maximal logarithmic order with $d_j = -\dim M - 1$.
Then, $\operatorname{oilc}_0(\zeta(\alpha)) = \operatorname{oilc}_0(\zeta(\beta)) = -l_j - 1$ and $\operatorname{ilc}_0(\zeta(\alpha)) = \operatorname{ilc}_0(\zeta(\beta))$.

Proof.

Since $\alpha(0) = \beta(0)$, we obtain that $z \mapsto \gamma(z) := \frac{\alpha(z)-\beta(z)}{z}$ is a gauged poly-log-homogeneous distribution again.

$\square$

Proof.

Since $\alpha(0)=\beta(0)$, we obtain that $z\mapsto\gamma(z):=\frac{\alpha(z)-\beta(z)}{z}$ is a gauged poly-log-homogeneous distribution again. Furthermore,

$$\operatorname{oilc}_0(\zeta(\gamma))\geq\min\{\operatorname{oilc}_0(\zeta(\alpha)),\operatorname{oilc}_0(\zeta(\beta))\}=:-l=-l_j-1$$

holds because each pair (d_ι,l_ι) in the expansion of γ appears in at least one of the expansions of α or β.

□

Proof.

Since $\alpha(0) = \beta(0)$, we obtain that $z \mapsto \gamma(z) := \frac{\alpha(z)-\beta(z)}{z}$ is a gauged poly-log-homogeneous distribution again. Furthermore,

$$\mathrm{oilc}_0(\zeta(\gamma)) \geq \min\{\mathrm{oilc}_0(\zeta(\alpha)), \mathrm{oilc}_0(\zeta(\beta))\} =: -l = -l_j - 1$$

holds because each pair (d_ι, l_ι) in the expansion of γ appears in at least one of the expansions of α or β.

This implies that $z \mapsto z^l \zeta(\gamma)(z) = z^{l-1}\left(\zeta(\alpha)(z) - \zeta(\beta)(z)\right)$ is holomorphic at zero (equality holds for $\Re(z)$ sufficiently small and, thence, in general by meromorphic extension).

□

Proof.

Since $\alpha(0)=\beta(0)$, we obtain that $z\mapsto\gamma(z):=\frac{\alpha(z)-\beta(z)}{z}$ is a gauged poly-log-homogeneous distribution again. Furthermore,

$$\operatorname{oilc}_0(\zeta(\gamma))\geq\min\{\operatorname{oilc}_0(\zeta(\alpha)),\operatorname{oilc}_0(\zeta(\beta))\}=:-l=-l_j-1$$

holds because each pair (d_ι,l_ι) in the expansion of γ appears in at least one of the expansions of α or β.

This implies that $z\mapsto z^l\zeta(\gamma)(z)=z^{l-1}\left(\zeta(\alpha)(z)-\zeta(\beta)(z)\right)$ is holomorphic at zero (equality holds for $\mathfrak{R}(z)$ sufficiently small and, thence, in general by meromorphic extension).

Hence, the highest order poles of $\zeta(\alpha)$ and $\zeta(\beta)$ at zero must cancel out which directly implies $\operatorname{oilc}_0(\zeta(\alpha))=\operatorname{oilc}_0(\zeta(\beta))$ and $\operatorname{ilc}_0(\zeta(\alpha))=\operatorname{ilc}_0(\zeta(\beta))$. □

Lemma

Let $\alpha = \alpha_0 + \sum_{\iota\in I}\alpha_\iota$ *and* $\beta = \beta_0 + \sum_{\iota\in I'}\beta_\iota$ *be two gauged poly-*log*-homogeneous distributions with* $\alpha(0) = \beta(0)$ *and* $\forall\iota\in I\cup I':\ d_\iota \neq -\dim M - 1$*. Then,* $\zeta(\alpha)(0) = \zeta(\beta)(0)$*.*

Lemma

Let $\alpha = \alpha_0 + \sum_{\iota\in I}\alpha_\iota$ *and* $\beta = \beta_0 + \sum_{\iota\in I'}\beta_\iota$ *be two gauged poly-log-homogeneous distributions with* $\alpha(0) = \beta(0)$ *and* $\forall \iota \in I \cup I' :\ d_\iota \neq -\dim M - 1$. *Then,* $\zeta(\alpha)(0) = \zeta(\beta)(0)$.

Proof.

Again, since $\alpha(0) = \beta(0)$, we obtain that $z \mapsto \gamma(z) := \frac{\alpha(z)-\beta(z)}{z}$ is a gauged poly-log-homogeneous distribution and $\mathrm{oilc}_0(\zeta(\gamma)) \geq 0$. Hence

$$\zeta(\alpha)(0) - \zeta(\beta)(0) = \mathrm{res}_0\left(z \mapsto \frac{\zeta(\alpha)(z) - \zeta(\beta)(z)}{z}\right) = \mathrm{res}_0\zeta(\gamma) = 0$$

where res_0 denotes the residue of a meromorphic function at zero.

□

Definition

Let $\alpha = \alpha_0 + \sum_{\iota\in I} \alpha_\iota$ be a gauged poly-log-homogeneous distribution and $I_{z_0} := \{\iota \in I;\ d_\iota = -\dim M - 1 - z_0\}$. Then, we define

$$\mathfrak{fp}_{z_0}(\alpha) := \alpha - \sum_{\iota\in I_{z_0}} \alpha_\iota = \alpha_0 + \sum_{\iota\in I\setminus I_{z_0}} \alpha_\iota .$$

Definition

Let $\alpha = \alpha_0 + \sum_{\iota\in I} \alpha_\iota$ be a gauged poly-log-homogeneous distribution and $I_{z_0} := \{\iota \in I;\ d_\iota = -\dim M - 1 - z_0\}$. Then, we define

$$\mathfrak{fp}_{z_0}(\alpha) := \alpha - \sum_{\iota\in I_{z_0}} \alpha_\iota = \alpha_0 + \sum_{\iota\in I\setminus I_{z_0}} \alpha_\iota .$$

Corollary

$\zeta(\mathfrak{fp}_0\alpha)(0)$ *is independent of the chosen gauge.*

Theorem (Laurent expansion of $\zeta(\mathfrak{fp}_0\alpha)$)

Let $\alpha = \alpha_0 + \sum_{\iota\in I} \alpha_\iota$ *be a gauged poly*-log-*homogeneous distribution with* $I_0 = \emptyset$. *Then,*

$$\zeta(\alpha)(z) = \sum_{n\in\mathbb{N}_0} \frac{\zeta(\partial^n \alpha)(0)}{n!} z^n$$

holds in a sufficiently small neighborhood of zero.

Theorem (Laurent expansion of $\zeta(\mathfrak{fp}_0\alpha)$)

Let $\alpha = \alpha_0 + \sum_{\iota\in I}\alpha_\iota$ be a gauged poly-log-*homogeneous distribution with $I_0 = \emptyset$. Then,*

$$\zeta(\alpha)(z) = \sum_{n\in\mathbb{N}_0} \frac{\zeta(\partial^n\alpha)(0)}{n!} z^n$$

holds in a sufficiently small neighborhood of zero.

The assertion is a direct consequence of the facts that the n^{th} Laurent coefficient of a holomorphic function f is given by $\frac{\partial^n f(0)}{n!}$ and

$$\partial^n\zeta(\alpha) = \partial^n \int_{\mathbb{R}_{\geq 1}\times M} \alpha \ \mathrm{dvol}_{\mathbb{R}_{\geq 1}\times M} = \int_{\mathbb{R}_{\geq 1}\times M} \partial^n\alpha \ \mathrm{dvol}_{\mathbb{R}_{\geq 1}\times M} = \zeta(\partial^n\alpha).$$

Theorem (Laurent expansion of $\zeta(\alpha)$)

Let $\alpha = \alpha_0 + \sum_{\iota\in I}\alpha_\iota$ *be a gauged poly*-log-*homogeneous distribution. Then, (in a sufficiently small neighborhood of zero)*

$$\begin{aligned}\zeta(\alpha)(z) =& \sum_{n\in\mathbb{N}_0}\sum_{\iota\in I_0}\frac{(-1)^{l_\iota+1}l_\iota!\int_M \partial^n\tilde{\alpha}_\iota(0)d\mathrm{vol}_M}{n!}z^{n-l_\iota-1}\\ &+\sum_{n\in\mathbb{N}_0}\frac{\int_{\mathbb{R}_{\geq 1}\times M}\partial^n\alpha_0(0)d\mathrm{vol}_{\mathbb{R}_{\geq 1}\times M}}{n!}z^n\\ &+\sum_{n\in\mathbb{N}_0}\sum_{\iota\in I\setminus I_0}\sum_{j=0}^{n}\frac{(-1)^{l_\iota+j+1}(l_\iota+j)!\int_M\partial^{n-j}\tilde{\alpha}_\iota(0)d\mathrm{vol}_M}{n!(\dim M+d_\iota+1)^{l_\iota+j+1}}z^n.\end{aligned}$$

$$
\begin{aligned}
\zeta(\mathfrak{A})(z) = & \sum_{n\in\mathbb{N}_0} \frac{\int_X \int_{B_{\mathbb{R}^N}(0,1)} e^{i\vartheta(x,x,\xi)} \partial^n a(0)(x,x,\xi)\ d\xi\ \mathrm{dvol}_X(x)}{n!} z^n \\
& + \sum_{n\in\mathbb{N}_0} \sum_{\iota\in I_0} \frac{(-1)^{l_\iota+1} l_\iota! \int_{\Delta(X)\times\partial B_{\mathbb{R}^N}} e^{i\vartheta} \partial^n \tilde{a}_\iota(0)\ \mathrm{dvol}_{\Delta(X)\times\partial B_{\mathbb{R}^N}}}{n!} z^{n-l_\iota-1} \\
& + \sum_{n\in\mathbb{N}_0} \frac{\int_{\mathbb{R}_{\geq 1}\times\partial B_{\mathbb{R}^N}} \int_X e^{i\vartheta(x,x,\xi)} \partial^n a_0(0)(x,x,\xi)\ \mathrm{dvol}_X(x)\ \mathrm{dvol}_{\mathbb{R}_{\geq 1}\times\partial B_{\mathbb{R}^N}}(\xi)}{n!} z^n \\
& + \sum_{n\in\mathbb{N}_0} \sum_{\iota\in I\setminus I_0} \sum_{j=0}^{n} \frac{(-1)^{l_\iota+j+1} (l_\iota+j)! \int_{\Delta(X)\times\partial B_{\mathbb{R}^N}} e^{i\vartheta} \partial^{n-j} \tilde{a}_\iota(0)\ \mathrm{dvol}_{\Delta(X)\times\partial B_{\mathbb{R}^N}}}{n!(N+d_\iota)^{l_\iota+j+1}} z^n
\end{aligned}
$$

where $\Delta(X) := \{(x,x) \in X^2;\ x \in X\}$.

Definition

If $a = a_0 + \sum_{\iota\in I} a_\iota$ is the amplitude of a gauged poly-log-homogeneous Fourier Integral Operator $\mathfrak{A}$ with phase function ϑ and $\mathfrak{A}_\iota$ the gauged Fourier Integral Operator with phase function ϑ and amplitude a_ι, then

$$\operatorname{res}\mathfrak{A}_\iota(z) := \int_{\partial B_{\mathbb{R}^N}} \int_X e^{i\vartheta(x,x,\xi)} \tilde{a}_\iota(z)(x,x,\xi)\ d\mathrm{vol}_X(x)\ d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\xi).$$

The Residue Trace (ψdo: Wodzicki 1984, Guillemin 1985; FIO: Guillemin 1993)

Theorem

Let A and B be polyhomogeneous Fourier Integral Operators. Let $\mathfrak{G}_1$ and $\mathfrak{G}_2$ be gauged Fourier Integral Operators with $\mathfrak{G}_1(0) = AB$ and $\mathfrak{G}_2(0) = BA$. Then,

$$\operatorname{res}_0\zeta(\mathfrak{G}_1) = \operatorname{res}_0\zeta(\mathfrak{G}_2),$$

i.e., the residue of the ζ-function is tracial and $A \mapsto \operatorname{res}_0\zeta(\mathfrak{A})$ is a well-defined trace where $\mathfrak{A}$ is any choice of gauge for A.

The Residue Trace (ψdo: Wodzicki 1984, Guillemin 1985; FIO: Guillemin 1993)

Proof.

This is a direct consequence of the following two facts.

(i) $\mathrm{res}_0\zeta(\mathfrak{G}_j) = -\sum_{\iota\in I_0}\mathrm{res}(\mathfrak{G}_j)_\iota(0)$ is independent of the gauge ($j \in \{1,2\}$).

(ii) $\zeta(\mathfrak{A}B) = \zeta(B\mathfrak{A})$ holds for any gauge $\mathfrak{A}$ of A because it is true for $\Re(z)$ sufficiently small.

Hence, $\mathrm{res}_0\zeta(\mathfrak{G}_1) = \mathrm{res}_0\zeta(\mathfrak{A}B) = \mathrm{res}_0\zeta(B\mathfrak{A}) = \mathrm{res}_0\zeta(\mathfrak{G}_2)$.

$\square$

The (generalized) Kontsevich-Vishik Trace (Kontsevich, Vishik 1994)

Theorem

Let A and B be Fourier Integral Operators. Let $\mathfrak{G}_1$ and $\mathfrak{G}_2$ be gauged Fourier Integral Operators with $\mathfrak{G}_1(0) = AB$, $\mathfrak{G}_2(0) = BA$, and $I_0 = \emptyset$. Then,

$$\zeta(\mathfrak{G}_1)(0) = \zeta(\mathfrak{G}_2)(0),$$

i.e., the constant Laurent coefficient of the ζ-function is tracial and $A \mapsto \zeta(\mathfrak{A})(0)$ is a well-defined trace where $\mathfrak{A}$ is any choice of gauge for A with $I_0 = \emptyset$.

The generalized Kontsevich-Vishik Trace

Definition

The generalized Kontsevich-Vishik trace is defined as

$$\operatorname{tr}_{\mathrm{KV}} : \ \{A \in \mathcal{A}_\Gamma; \ I_0 = \varnothing\} \subseteq \mathcal{A}_\Gamma \to \mathbb{C}; \ A \mapsto \zeta(\mathfrak{A})(0)$$

where $\mathfrak{A}$ is any choice of gauge for A.

- So far, we assumed amplitudes to be integrable on $X \times B_{\mathbb{R}^N}(0,1)$.

- So far, we assumed amplitudes to be integrable on $X \times B_{\mathbb{R}^N}(0,1)$.
- Many classical symbols are not.

- So far, we assumed amplitudes to be integrable on $X \times B_{\mathbb{R}^N}(0,1)$.
- Many classical symbols are not.
- How to get ζ-functions and Laurent expansion for gauged poly-log-homogeneous distributions on $\mathbb{R}_{>0} \times M$ instead of only $\mathbb{R}_{\geq 1} \times M$?

- So far, we assumed amplitudes to be integrable on $X \times B_{\mathbb{R}^N}(0,1)$.
- Many classical symbols are not.
- How to get ζ-functions and Laurent expansion for gauged poly-log-homogeneous distributions on $\mathbb{R}_{>0} \times M$ instead of only $\mathbb{R}_{\geq 1} \times M$?
- Idea: Approximation with distributions we can handle.

- So far, we assumed amplitudes to be integrable on $X \times B_{\mathbb{R}^N}(0,1)$.
- Many classical symbols are not.
- How to get ζ-functions and Laurent expansion for gauged poly-log-homogeneous distributions on $\mathbb{R}_{>0} \times M$ instead of only $\mathbb{R}_{\geq 1} \times M$?
- Idea: Approximation with distributions we can handle.
- We need convergence type such that
  - (i) sequence of meromorphic germs converges to a meromorphic germ
  - (ii) local properties are preserved taking limits

- So far, we assumed amplitudes to be integrable on $X \times B_{\mathbb{R}^N}(0,1)$.
- Many classical symbols are not.
- How to get ζ-functions and Laurent expansion for gauged poly-log-homogeneous distributions on $\mathbb{R}_{>0} \times M$ instead of only $\mathbb{R}_{\geq 1} \times M$?
- Idea: Approximation with distributions we can handle.
- We need convergence type such that
  - (i) sequence of meromorphic germs converges to a meromorphic germ
  - (ii) local properties are preserved taking limits
- Compact convergence on a punctured ball $B_{\mathbb{C}}(0,\varepsilon) \setminus \{0\}$ will do!

Suppose $\alpha = \alpha_\iota$, i.e., $\alpha_0 = 0$ and $\#I = 1$. We need to make sense of

$$\int_{(0,1)} r^{\dim M + d_\iota + z} (\ln r)^{l_\iota} dr.$$

Introducing a shift $h \in \mathbb{R}_{>0}$ gives

$$\begin{aligned} A_h :=& \int_{(0,1)} (r+h)^{\dim M + d_\iota + z} (\ln(r+h))^{l_\iota} dr \\ =& \int_{(0,1)} \partial^{l_\iota} \left(s \mapsto (r+h)^{\dim M + d_\iota + s} \right) (z) dr \\ =& \partial^{l_\iota} \left(s \mapsto \int_{(0,1)} (r+h)^{\dim M + d_\iota + s} dr \right) (z) \end{aligned}$$

$$
\begin{aligned}
A_h =&\partial^{l_\iota}\left(s \mapsto \frac{(1+h)^{\dim M + d_\iota + s + 1} - h^{\dim M + d_\iota + s + 1}}{\dim M + d_\iota + s + 1}\right)(z) \\
=&\sum_{j=0}^{l_\iota} \frac{(-1)^j j!}{(\dim M + d_\iota + z + 1)^{j+1}} (1+h)^{\dim M + d_\iota + z + 1} (\ln(1+h))^{l_\iota - j} \\
&- \sum_{j=0}^{l_\iota} \frac{(-1)^j j!}{(\dim M + d_\iota + z + 1)^{j+1}} h^{\dim M + d_\iota + z + 1} (\ln h)^{l_\iota - j}.
\end{aligned}
$$

$$
\begin{aligned}
A_h =& \partial^{l_\iota}\left(s \mapsto \frac{(1+h)^{\dim M + d_\iota + s + 1} - h^{\dim M + d_\iota + s + 1}}{\dim M + d_\iota + s + 1}\right)(z) \\
=& \sum_{j=0}^{l_\iota} \frac{(-1)^j j!}{(\dim M + d_\iota + z + 1)^{j+1}} (1+h)^{\dim M + d_\iota + z + 1} (\ln(1+h))^{l_\iota - j} \\
& - \sum_{j=0}^{l_\iota} \frac{(-1)^j j!}{(\dim M + d_\iota + z + 1)^{j+1}} h^{\dim M + d_\iota + z + 1} (\ln h)^{l_\iota - j}.
\end{aligned}
$$

- $(1+h)^{\dim M + d_\iota + z + 1}(\ln(1+h))^{l_\iota - j} \to \delta_{j,l_\iota}$ locally bounded

$$
\begin{aligned}
A_h =& \partial^{l_\iota}\left(s \mapsto \frac{(1+h)^{\dim M + d_\iota + s + 1} - h^{\dim M + d_\iota + s + 1}}{\dim M + d_\iota + s + 1}\right)(z) \\
=& \sum_{j=0}^{l_\iota} \frac{(-1)^j j!}{(\dim M + d_\iota + z + 1)^{j+1}} (1+h)^{\dim M + d_\iota + z + 1} (\ln(1+h))^{l_\iota - j} \\
&- \sum_{j=0}^{l_\iota} \frac{(-1)^j j!}{(\dim M + d_\iota + z + 1)^{j+1}} h^{\dim M + d_\iota + z + 1} (\ln h)^{l_\iota - j}.
\end{aligned}
$$

- $(1+h)^{\dim M + d_\iota + z + 1}(\ln(1+h))^{l_\iota - j} \to \delta_{j,l_\iota}$ locally bounded
- $h^{\dim M + d_\iota + z + 1}(\ln h)^{l_\iota - j} \to 0$ locally bounded

$$A_h = \partial^{l_\iota}\left(s \mapsto \frac{(1+h)^{\dim M + d_\iota + s + 1} - h^{\dim M + d_\iota + s + 1}}{\dim M + d_\iota + s + 1}\right)(z)$$

$$= \sum_{j=0}^{l_\iota} \frac{(-1)^j j!}{(\dim M + d_\iota + z + 1)^{j+1}} (1+h)^{\dim M + d_\iota + z + 1} (\ln(1+h))^{l_\iota - j}$$

$$- \sum_{j=0}^{l_\iota} \frac{(-1)^j j!}{(\dim M + d_\iota + z + 1)^{j+1}} h^{\dim M + d_\iota + z + 1} (\ln h)^{l_\iota - j}.$$

- $(1+h)^{\dim M + d_\iota + z + 1}(\ln(1+h))^{l_\iota - j} \to \delta_{j,l_\iota}$ locally bounded
- $h^{\dim M + d_\iota + z + 1}(\ln h)^{l_\iota - j} \to 0$ locally bounded

$$A_h \to \frac{(-1)^{l_\iota} l_\iota!}{(\dim M + d_\iota + z + 1)^{l_\iota + 1}} \qquad \text{pointwise and locally bounded}$$

Theorem (Vitali)

Let $\Omega \subseteq_{\mathrm{open,connected}} \mathbb{C}$, $f \in C^{\omega}(\Omega)^{\mathbb{N}}$ *locally bounded, and let*

$$\{z \in \Omega;\ (f_n(z))_{n\in\mathbb{N}} \text{ converges}\}$$

have an accumulation point in Ω. *Then,* f *is compactly convergent.*

Theorem (Vitali)

Let $\Omega \subseteq_{\text{open,connected}} \mathbb{C}$, $f \in C^\omega(\Omega)^{\mathbb{N}}$ *locally bounded, and let*

$$\{z \in \Omega;\ (f_n(z))_{n\in\mathbb{N}} \text{ converges}\}$$

have an accumulation point in Ω*. Then,* f *is compactly convergent.*

Corollary

A_h *converges compactly to* $z \mapsto \frac{(-1)^{l_\iota} l_\iota!}{(\dim M + d_\iota + z + 1)^{l_\iota + 1}}$.

Theorem

$$\sum_{\iota\in I}\int_{(0,1)}(h_\iota+r)^{\dim M+d_\iota+z}(\ln(h_\iota+r))^{l_\iota}dr$$

is compactly convergent for $h := (h_\iota)_{\iota\in I}\in\ell_\infty(I;\mathbb{R}_{>0})$ *and* $h\searrow 0$ *in* $\ell_\infty(I)$ *such that*

$$Z_\iota(z) := l_\iota\sum_{j=0}^{l_\iota}\left|\zeta_H(l_\iota-j-d_\iota-z;h_\iota)-\zeta_H(l_\iota-j-d_\iota-z;1+h_\iota)\right|$$

is uniformly bounded on an exhausting family of compacta as $h\searrow 0$.

$$\zeta(\mathfrak{A})(z) = \sum_{n\in\mathbb{N}_0} \frac{\int_X \int_{B_{\mathbb{R}^N}(0,1)} e^{i\vartheta(x,x,\xi)} \partial^n a(0)(x,x,\xi)\ d\xi\ \mathrm{dvol}_X(x)}{n!} z^n$$
$$+ \sum_{n\in\mathbb{N}_0} \sum_{\iota\in I_0} \frac{(-1)^{l_\iota+1} l_\iota! \int_{\Delta(X)\times\partial B_{\mathbb{R}^N}} e^{i\vartheta} \partial^n \tilde{a}_\iota(0)\ \mathrm{dvol}_{\Delta(X)\times\partial B_{\mathbb{R}^N}}}{n!} z^{n-l_\iota-1}$$
$$+ \sum_{n\in\mathbb{N}_0} \frac{\int_{\mathbb{R}_{\geq 1}\times\partial B_{\mathbb{R}^N}} \int_X e^{i\vartheta(x,x,\xi)} \partial^n a_0(0)(x,x,\xi)\ \mathrm{dvol}_X(x)\ \mathrm{dvol}_{\mathbb{R}_{\geq 1}\times\partial B_{\mathbb{R}^N}}(\xi)}{n!} z^n$$
$$+ \sum_{n\in\mathbb{N}_0} \sum_{\iota\in I\setminus I_0} \sum_{j=0}^{n} \frac{(-1)^{l_\iota+j+1}(l_\iota+j)! \int_{\Delta(X)\times\partial B_{\mathbb{R}^N}} e^{i\vartheta} \partial^{n-j} \tilde{a}_\iota(0)\ \mathrm{dvol}_{\Delta(X)\times\partial B_{\mathbb{R}^N}}}{n!(N+d_\iota)^{l_\iota+j+1}} z^n$$

$$\zeta(\mathfrak{A})(z) = \sum_{n\in\mathbb{N}_0} \frac{\int_X \mathrm{pv}\int_{B_{\mathbb{R}^N}(0,1)} e^{i\vartheta(x,x,\xi)}\partial^n a(0)(x,x,\xi)\ d\xi\ \mathrm{dvol}_X(x)}{n!} z^n$$

$$+ \sum_{n\in\mathbb{N}_0}\sum_{\iota\in I_0} \frac{(-1)^{l_\iota+1} l_\iota! \int_{\Delta(X)\times\partial B_{\mathbb{R}^N}} e^{i\vartheta}\partial^n \tilde{a}_\iota(0)\ \mathrm{dvol}_{\Delta(X)\times\partial B_{\mathbb{R}^N}}}{n!} z^{n-l_\iota-1}$$

$$+ \sum_{n\in\mathbb{N}_0} \frac{\int_{\mathbb{R}_{\ge1}\times\partial B_{\mathbb{R}^N}}\int_X e^{i\vartheta(x,x,\xi)}\partial^n a_0(0)(x,x,\xi)\ \mathrm{dvol}_X(x)\ \mathrm{dvol}_{\mathbb{R}_{\ge1}\times\partial B_{\mathbb{R}^N}}(\xi)}{n!} z^n$$

$$+ \sum_{n\in\mathbb{N}_0}\sum_{\iota\in I\setminus I_0}\sum_{j=0}^{n} \frac{(-1)^{l_\iota+j+1}(l_\iota+j)! \int_{\Delta(X)\times\partial B_{\mathbb{R}^N}} e^{i\vartheta}\partial^{n-j}\tilde{a}_\iota(0)\ \mathrm{dvol}_{\Delta(X)\times\partial B_{\mathbb{R}^N}}}{n!(N+d_\iota)^{l_\iota+j+1}} z^n$$

Example: $|\partial|^z$ on $\mathbb{R}/2\pi\mathbb{Z}$

The operator $|\partial|^z$ has kernel

$$k(z)(x,y) = \sum_{n\in\mathbb{Z}} \int_{\mathbb{R}} e^{i(x-y-2\pi n)\xi} \frac{|\xi|^z}{2\pi} d\xi$$

and spectrum $\sigma(|\partial|^z) = \{|n|^z;\ n \in \mathbb{Z}\}$ counting multiplicities.

Example: $|\partial|^z$ on $\mathbb{R}/2\pi\mathbb{Z}$

The operator $|\partial|^z$ has kernel

$$k(z)(x,y) = \sum_{n\in\mathbb{Z}} \int_{\mathbb{R}} e^{i(x-y-2\pi n)\xi} \frac{|\xi|^z}{2\pi} d\xi$$

and spectrum $\sigma(|\partial|^z) = \{|n|^z;\ n \in \mathbb{Z}\}$ counting multiplicities. Hence, for $\Re(z) < -1$

$$\operatorname{tr} |\partial|^z = \sum_{n\in\mathbb{Z}} |n|^z$$

Example: $|\partial|^z$ on $\mathbb{R}/2\pi\mathbb{Z}$

The operator $|\partial|^z$ has kernel

$$k(z)(x,y) = \sum_{n\in\mathbb{Z}} \int_{\mathbb{R}} e^{i(x-y-2\pi n)\xi} \frac{|\xi|^z}{2\pi} d\xi$$

and spectrum $\sigma(|\partial|^z) = \{|n|^z;\ n \in \mathbb{Z}\}$ counting multiplicities. Hence, for $\Re(z) < -1$

$$\operatorname{tr} |\partial|^z = \sum_{n\in\mathbb{Z}} |n|^z \quad \Rightarrow \quad \zeta(s \mapsto |\partial|^s)(z) = 2\zeta_R(-z)$$

where ζ_R denotes the Riemann ζ-function.

Example: Mollifying $|\partial|^z$ on $\mathbb{R}/2\pi\mathbb{Z}$

Let $h \in (0,1)$. Then, $(h+|\partial|)^z$ has kernel

$$k_h(z)(x,y) = \sum_{n\in\mathbb{Z}} \int_{\mathbb{R}} e^{i(x-y-2\pi n)\xi} \frac{(h+|\xi|)^z}{2\pi} d\xi$$

and spectrum $\sigma((h+|\partial|)^z) = \{(h+|n|)^z;\ n \in \mathbb{Z}\}$ counting multiplicities.

Example: Mollifying $|\partial|^z$ on $\mathbb{R}/2\pi\mathbb{Z}$

Let $h \in (0,1)$. Then, $(h+|\partial|)^z$ has kernel

$$k_h(z)(x,y) = \sum_{n\in\mathbb{Z}} \int_{\mathbb{R}} e^{i(x-y-2\pi n)\xi} \frac{(h+|\xi|)^z}{2\pi} d\xi$$

and spectrum $\sigma((h+|\partial|)^z) = \{(h+|n|)^z;\ n\in\mathbb{Z}\}$ counting multiplicities. Hence, for $\Re(z) < -1$

$$\operatorname{tr}(h+|\partial|)^z = \sum_{n\in\mathbb{Z}} (h+|n|)^z$$

Example: Mollifying $|\partial|^z$ on $\mathbb{R}/2\pi\mathbb{Z}$

Let $h \in (0,1)$. Then, $(h+|\partial|)^z$ has kernel

$$k_h(z)(x,y) = \sum_{n\in\mathbb{Z}} \int_{\mathbb{R}} e^{i(x-y-2\pi n)\xi} \frac{(h+|\xi|)^z}{2\pi} d\xi$$

and spectrum $\sigma((h+|\partial|)^z) = \{(h+|n|)^z;\ n\in\mathbb{Z}\}$ counting multiplicities. Hence, for $\Re(z) < -1$

$$\operatorname{tr}(h+|\partial|)^z = \sum_{n\in\mathbb{Z}} (h+|n|)^z \quad \Rightarrow \quad \zeta(s\mapsto (h+|\partial|)^s)(z) = \zeta_H(-z;h) + \zeta_H(-z;1+h)$$

where ζ_H denotes the Hurwitz ζ-function.

Example: Mollifying $|\partial|^z$ on $\mathbb{R}/2\pi\mathbb{Z}$ - convergence

Theorem

Both $\zeta_H(\cdot;h)$ and $\zeta_H(\cdot;1+h)$ converge compactly on $\mathbb{C}\setminus\{1\}$ to ζ_R for $h \searrow 0$.

Example: Mollifying $|\partial|^z$ on $\mathbb{R}/2\pi\mathbb{Z}$ - convergence

Theorem

Both $\zeta_H(\cdot;h)$ and $\zeta_H(\cdot;1+h)$ converge compactly on $\mathbb{C}\setminus\{1\}$ to ζ_R for $h\searrow 0$.

Theorem

$\zeta(s\mapsto(h+|\partial|)^s)$ converges compactly to $\zeta(s\mapsto|\partial|^s)$ on $\mathbb{C}\setminus\{-1\}$ for $h\searrow 0$.

Let $A \in \mathcal{A}_\Gamma$ and $\mathfrak{A}$ a gauged Fourier Integral Operator with $\mathfrak{A}(0) = A$. Then, we define the generalized Kontsevich-Vishik trace $\operatorname{tr}_{\mathrm{KV}} A$ of A to be the constant Laurent coefficient $c_0(\zeta(\mathfrak{A}), 0)$ of $\zeta(\mathfrak{A})$ in zero, i.e.,

$$\begin{aligned}
\operatorname{tr}_{\mathrm{KV}} A = &\int_X \mathrm{pv} \int_{B_{\mathbb{R}^N}(0,1)} e^{i\vartheta(x,x,\xi)} a(x,x,\xi)\ d\xi\ d\mathrm{vol}_X(x) \\
&+ \sum_{\iota \in I_0} \frac{(-1)^{l_\iota+1} l_\iota! \int_{\Delta(X)\times\partial B_{\mathbb{R}^N}} e^{i\vartheta} \partial^{l_\iota+1} \tilde{a}_\iota(0)\ d\mathrm{vol}_{\Delta(X)\times\partial B_{\mathbb{R}^N}}}{(l_\iota+1)!} \\
&+ \int_{\mathbb{R}_{\geq 1}\times\partial B_{\mathbb{R}^N}} \int_X e^{i\vartheta(x,x,\xi)} a_0(x,x,\xi)\ d\mathrm{vol}_X(x)\ d\mathrm{vol}_{\mathbb{R}_{\geq 1}\times\partial B_{\mathbb{R}^N}}(\xi) \\
&+ \sum_{\iota \in I\setminus I_0} \frac{(-1)^{l_\iota+1} l_\iota! \int_{X\times\partial B_{\mathbb{R}^N}} e^{i\vartheta(x,x,\xi)} \tilde{a}_\iota(x,x,\xi)\ d\mathrm{vol}_{X\times\partial B_{\mathbb{R}^N}}(x,\xi)}{(N+d_\iota)^{l_\iota+1}}.
\end{aligned}$$

Example

For a classical pseudo-differential operators A without critical degree of homogeneity, we observe

$$\int_X \mathrm{pv}\int_{B_{\mathbb{R}^N}(0,1)} a_\iota(x,x,\xi)\ d\xi\ d\mathrm{vol}_X(x) + \frac{-\int_{X\times\partial B_{\mathbb{R}^N}} \tilde{a}_\iota(x,x,\xi)\ d\mathrm{vol}_{X\times\partial B_{\mathbb{R}^N}}(x,\xi)}{N+d_\iota}$$

$$= \int_{X\times\partial B_{\mathbb{R}^N}} \tilde{a}_\iota(x,x,\xi)\ d\mathrm{vol}_{X\times\partial B_{\mathbb{R}^N}}(x,\xi) \lim_{h\searrow 0}\int_h^{1+h} r^{N+d_\iota-1} dt$$

$$- \frac{\int_{X\times\partial B_{\mathbb{R}^N}} \tilde{a}_\iota(x,x,\xi)\ d\mathrm{vol}_{X\times\partial B_{\mathbb{R}^N}}(x,\xi)}{N+d_\iota}$$

$$= 0$$

Example (continued)

For a classical pseudo-differential operators A without critical degree of homogeneity, we hence obtain

$$\operatorname{tr}_{\mathrm{KV}} A = \int_X \int_{\mathbb{R}^N} e^{i\vartheta(x,x,\xi)} a_0(x,x,\xi)\ d\xi\ d\mathrm{vol}_X(x)$$
$$= \int_X \left(k - \sum_{j=1}^{J} k_{m-j} \right)(x,x) d\mathrm{vol}_X(x)$$

for any $J \in \mathbb{N}_{>\Re(m)+\dim X}$.

This can fail spectacularly for FIOs

This does not happen for general Fourier Integral Operators. Consider phase $\langle \Theta(x,y),\xi\rangle$ such that $x\mapsto\Theta(x,x)\in C(X)$ has no zeros. Then,

$$\int_X\int_{\mathbb{R}^N} e^{i\langle\Theta(x,x),\xi\rangle}a(x,x,\xi)d\xi d\mathrm{vol}_X(x)=\int_X\mathcal{F}(a(x,x,\cdot))(-\Theta(x,x))d\mathrm{vol}_X(x)$$

is well-defined. Choose Θ and a such that $x\mapsto\mathcal{F}(a(x,x,\cdot))(-\Theta(x,x))$ is pointwise positive to construct a counterexample.

The critical case - explicitly

- Consider polyhomogeneous multiplicative gauge $\mathfrak{A}(z) = BQ^z$.

The critical case - explicitly

- Consider polyhomogeneous multiplicative gauge $\mathfrak{A}(z) = BQ^z$.
- Then $Q^0 = 1 - 1_{\{0\}}(Q)$ where $1_{\{0\}}(Q) = \frac{1}{2\pi i}\int_{\partial B_{\mathbb{C}}(0,\varepsilon)}(\lambda - Q)^{-1}d\lambda$ and $B_{\mathbb{C}}[0,\varepsilon] \cap \sigma(Q) = \{0\}$.

The critical case - explicitly

- Consider polyhomogeneous multiplicative gauge $\mathfrak{A}(z) = BQ^z$.
- Then $Q^0 = 1 - 1_{\{0\}}(Q)$ where $1_{\{0\}}(Q) = \frac{1}{2\pi i}\int_{\partial B_{\mathbb{C}}(0,\varepsilon)}(\lambda - Q)^{-1}d\lambda$ and $B_{\mathbb{C}}[0,\varepsilon] \cap \sigma(Q) = \{0\}$.
- For $I_0 = \varnothing$

$$\begin{aligned}\forall k \in \mathbb{N}_0:\ \partial^k\zeta(\mathfrak{A})(0) =& \zeta(\partial^k\mathfrak{A})(0) = \operatorname{tr}_{KV}(B(\ln Q)^k Q^0)\\ =& \operatorname{tr}_{KV}(B(\ln Q)^k) - \operatorname{tr}_{KV}(B(\ln Q)^k 1_{\{0\}}(Q))\end{aligned}$$

The critical case - explicitly

- Consider polyhomogeneous multiplicative gauge $\mathfrak{A}(z) = BQ^z$.
- Then $Q^0 = 1 - 1_{\{0\}}(Q)$ where $1_{\{0\}}(Q) = \frac{1}{2\pi i}\int_{\partial B_{\mathbb{C}}(0,\varepsilon)}(\lambda - Q)^{-1}d\lambda$ and $B_{\mathbb{C}}[0,\varepsilon] \cap \sigma(Q) = \{0\}$.
- For $I_0 = \emptyset$

$$\begin{aligned}\forall k \in \mathbb{N}_0:\ \partial^k\zeta(\mathfrak{A})(0) =& \zeta(\partial^k\mathfrak{A})(0) = \operatorname{tr}_{\mathrm{KV}}(B(\ln Q)^k Q^0)\\ =& \operatorname{tr}_{\mathrm{KV}}(B(\ln Q)^k) - \operatorname{tr}_{\mathrm{KV}}(B(\ln Q)^k 1_{\{0\}}(Q))\end{aligned}$$

- For $I_0 \neq \emptyset$: $\mathfrak{fp}\mathfrak{A} := \mathfrak{A} - \sum_{\iota\in I_0}\mathfrak{A}_\iota$, $\mathfrak{fp}\zeta(\mathfrak{A}) := \zeta(\mathfrak{fp}\mathfrak{A}) + \sum_{\iota\in I_0}\int_X\int_{B(0,1)} e^{i\vartheta}a_\iota$, and $\operatorname{tr}_{\mathfrak{fp}}(\cdot) = \mathfrak{fp}\zeta(\cdot)(0)$

- Let $c_k(Q)$ be the coefficient of the $\frac{z^k}{k!}$ term in the Laurent expansion of $\zeta(\mathfrak{A})$.

- Let $c_k(Q)$ be the coefficient of the $\frac{z^k}{k!}$ term in the Laurent expansion of $\zeta(\mathfrak{A})$.
- Then

$$\begin{aligned} c_k(Q) =& \mathfrak{fp}\zeta\left(\partial^k \mathfrak{A}\right)(0) - \frac{1}{k+1}\operatorname{res}\left(\partial^{k+1}\mathfrak{A}\right)(0) \\ =& \operatorname{tr}_{\mathfrak{fp}}\left(B(\ln Q)^k Q^0\right) - \frac{1}{k+1}\operatorname{res}\left(B(\ln Q)^{k+1} Q^0\right) \\ =& \operatorname{tr}_{\mathfrak{fp}}\left(B(\ln Q)^k\right) - \operatorname{tr}_{\mathfrak{fp}}\left(B(\ln Q)^k 1_{\{0\}}(Q)\right) - \frac{1}{k+1}\operatorname{res}\left(B(\ln Q)^{k+1}\right). \end{aligned}$$

- Let $c_k(Q)$ be the coefficient of the $\frac{z^k}{k!}$ term in the Laurent expansion of $\zeta(\mathfrak{A})$.
- Then

$$\begin{aligned}
c_k(Q) =& \mathfrak{fp}\zeta\left(\partial^k\mathfrak{A}\right)(0) - \frac{1}{k+1}\operatorname{res}\left(\partial^{k+1}\mathfrak{A}\right)(0)\\
=& \operatorname{tr}_{\mathfrak{fp}}\left(B(\ln Q)^k Q^0\right) - \frac{1}{k+1}\operatorname{res}\left(B(\ln Q)^{k+1}Q^0\right)\\
=& \operatorname{tr}_{\mathfrak{fp}}\left(B(\ln Q)^k\right) - \operatorname{tr}_{\mathfrak{fp}}\left(B(\ln Q)^k 1_{\{0\}}(Q)\right) - \frac{1}{k+1}\operatorname{res}\left(B(\ln Q)^{k+1}\right).
\end{aligned}$$

- If Q is invertible and Q' is another invertible and admissible, then

$$c_0(Q) - c_0(Q') = -\operatorname{res}\left(B\left(\ln Q - \ln Q'\right)\right)$$

and since $\zeta(\mathfrak{A})(0) = 0$

$$\operatorname{tr}_{\mathfrak{fp}}([B, CQ^z])\Big|_{z=0} = \operatorname{res}\left([B, C\ln Q]\right).$$

Theorem (Guillemin)

Let $\mathcal{A}_\Gamma$ be an algebra of classical Fourier Integral Operators. Then, the residue-trace of $A \in \mathcal{A}_\Gamma$ vanishes if and only if A is a smoothing operator plus a sum of commutators $[P_i, A_i]$ where the P_i are pseudo-differential operators and the $A_i \in \mathcal{A}_\Gamma$.

Theorem (Guillemin)

Let Γ be connected. Then, the commutator of $\mathcal{A}_\Gamma$ is of co-dimension one in $\mathcal{A}_\Gamma$ modulo smoothing operators.

These theorems (and the corresponding theorems for smoothing operators) yield the following table assuming that the residue trace $\operatorname{res}_0 \circ \zeta$ is non-trivial and unique, and $\mathcal{A}_\Gamma = \langle \mathfrak{A} \rangle + \langle [\mathcal{A}_\Gamma, \mathcal{A}_\Gamma] \rangle + \{\text{smoothing operators}\}$ for some $\mathfrak{A} \in \mathcal{A}_\Gamma$ with $\operatorname{res}_0 \zeta(\mathfrak{A}) \neq 0$.

<table>
<tr><th colspan="2">$I_0 \neq \emptyset$</th><th colspan="2">$I_0 = \emptyset$</th></tr>
<tr><td>$\operatorname{res}_0 \zeta(A) \neq 0$</td><td>$\operatorname{res}_0 \zeta(A) = 0$</td><td>$\zeta(A)(0) \neq 0$</td><td>$\zeta(A)(0) = 0$</td></tr>
<tr><td>$A = \alpha\mathfrak{A} + S + \sum_{i=1}^k C_i$
$C_i \in [\mathcal{A}_\Gamma, \mathcal{A}_\Gamma]$
$\alpha = (\operatorname{res}_0 \zeta(\mathfrak{A}))^{-1} \operatorname{res}_0 \zeta(A)$
S smoothing</td><td colspan="2">$A = S + \sum_{i=1}^k C_i$
$C_i \in [\mathcal{A}_\Gamma, \mathcal{A}_\Gamma]$
S smoothing</td><td>$A = \sum_{i=1}^k C_i$
C_i commutators</td></tr>
</table>

- Want to compute $I(x,y,r) := \int_{\partial B_{\mathbb{R}^N}} e^{ir\vartheta(x,y,\xi)}\tilde{a}(x,y,\xi)d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\xi)$

- Want to compute $I(x,y,r) := \int_{\partial B_{\mathbb{R}^N}} e^{ir\vartheta(x,y,\xi)}\tilde{a}(x,y,\xi)d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\xi)$
- (x,y) off critical manifold $\iff$ $\forall \xi \in \partial B_{\mathbb{R}^N}:\ \partial_3\vartheta(x,y,\xi) \neq 0$

- Want to compute $I(x,y,r) := \int_{\partial B_{\mathbb{R}^N}} e^{ir\vartheta(x,y,\xi)}\tilde{a}(x,y,\xi)d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\xi)$
- (x,y) off critical manifold $\iff$ $\forall \xi \in \partial B_{\mathbb{R}^N} :\ \partial_3\vartheta(x,y,\xi) \neq 0$
- $\partial_3 e^{ir\vartheta(x,y,\xi)} = ire^{ir\vartheta(x,y,\xi)}\partial_3\vartheta(x,y,\xi) \Rightarrow e^{ir\vartheta}\tilde{a} = \frac{\langle \partial_3 e^{ir\vartheta}, \tilde{a}\partial_3\vartheta\rangle_{\ell_2(N)}}{ir\|\partial_3\vartheta\|^2_{\ell_2(N)}}$

- Want to compute $I(x,y,r) := \int_{\partial B_{\mathbb{R}^N}} e^{ir\vartheta(x,y,\xi)}\tilde{a}(x,y,\xi)d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\xi)$
- (x,y) off critical manifold $\iff \forall \xi \in \partial B_{\mathbb{R}^N}: \partial_3\vartheta(x,y,\xi) \neq 0$
- $\partial_3 e^{ir\vartheta(x,y,\xi)} = ire^{ir\vartheta(x,y,\xi)}\partial_3\vartheta(x,y,\xi) \Rightarrow e^{ir\vartheta}\tilde{a} = \frac{\langle \partial_3 e^{ir\vartheta}, \tilde{a}\partial_3\vartheta\rangle_{\ell_2(N)}}{ir\|\partial_3\vartheta\|^2_{\ell_2(N)}}$
- Let $\mathcal{D}\tilde{a}(x,y,\xi) := \partial_3^* \frac{\tilde{a}(x,y,\xi)\partial_3\vartheta(x,y,\xi)}{\|\partial_3\vartheta(x,y,\xi)\|^2_{\ell_2(N)}}$. Then

$$\forall n \in \mathbb{N}: \ I(x,y,r) = \frac{1}{(ir)^n}\int_{\partial B_{\mathbb{R}^N}} e^{ir\vartheta(x,y,\xi)}\mathcal{D}^n\tilde{a}(x,y,\xi)d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\xi)$$
$$\Rightarrow |I(x,y,r)| \leq \frac{\|\mathcal{D}^n a\|_{L_\infty(X\times X\times\partial B_{\mathbb{R}^N})}}{r^n}$$

- Hence, kernel off critical manifold: $k = \int_{\mathbb{R}_{>0}} r^{N+d-1}I(\cdot,\cdot,r)dr \in C^\infty$

Lemma (Morse' Lemma)

Let $(x_0,y_0,\xi_0)\in X\times X\times\partial B_{\mathbb{R}^N}$ *be stationary* ($\partial_{\partial B}\vartheta(x_0,y_0,\xi_0)=0$) *and*

$$\partial_{\partial B}^2\vartheta(x_0,y_0,\xi_0)=\partial_3^2\left(\vartheta|_{X\times X\times\partial B_{\mathbb{R}^N}}\right)(x_0,y_0,\xi_0)\in GL\left(\mathbb{R}^{N-1}\right).$$

Then, there are neighborhoods $U\subseteq_{\mathrm{open}}X\times X$ *of* (x_0,y_0) *and* $V\subseteq_{\mathrm{open}}\partial B_{\mathbb{R}^N}$ *of* ξ_0 *and a function* $\hat\xi\in C^\infty(U,V)$ *such that*

$$\forall(x,y,\xi)\in U\times V:\ \partial_{\partial B}\vartheta(x,y,\xi)=0\iff\xi=\hat\xi(x,y).$$

Furthermore, there is a function $\eta\in C^\infty\left(U\times V,\mathbb{R}^N\right)$ *such that*

$$\forall(x,y,\xi)\in U\times V:\ \eta(x,y,\xi)-\left(\xi-\hat\xi(x,y)\right)\in O\left(\left\|\xi-\hat\xi(x,y)\right\|_{\ell_2(N)}^2\right)$$

and $\partial_3\eta\left(x,y,\hat\xi(x,y)\right)=1=\mathrm{id}_{\mathbb{R}^N}$.

Corollary

Let ϑ be as in Morse' Lemma. Then, stationary points of $\vartheta(x,y,\cdot)$ are isolated in $\partial B_{\mathbb{R}^N}$. In particular, there are only finitely many.

Corollary

Let ϑ be as in Morse' Lemma. Then, stationary points of $\vartheta(x,y,\cdot)$ are isolated in $\partial B_{\mathbb{R}^N}$. In particular, there are only finitely many.

Hence, we may assume that

$$k(x,y)=\sum_{s=0}^{S}\int_{\mathbb{R}^N}e^{i\vartheta(x,y,\xi)}a^s(x,y,\xi)d\xi$$

where a^0 has no stationary points in its support and each of the a^s has exactly one branch $(x,y,\hat{\xi}^s(x,y))$ in its support. As we have already treated the a^0 case, we will assume, without loss of generality, that a is of the form of one of the a^s.

- Define $\hat{\vartheta} := \vartheta\left(x,y,\hat{\xi}(x,y)\right)$, $\Theta(x,y) := \partial_{\partial B}^2\vartheta\left(x,y,\hat{\xi}(x,y)\right)$ and $\eta_{\partial B}$ the spherical part of η (polar coordinates). Then, (Corollary of Morse' Lemma Proof)

$$\begin{aligned}I(x,y,r) =& \int_{\partial B_{\mathbb{R}^N}} e^{ir\vartheta(x,y,\xi)} a(x,y,\xi) d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\xi)\\ =& e^{ir\hat{\vartheta}} \int_{\partial B_{\mathbb{R}^N}} e^{i\frac{r}{2}\langle \Theta(x,y)\eta_{\partial B}(x,y,\xi), \eta_{\partial B}(x,y,\xi)\rangle_{\mathbb{R}^{N-1}}} a(x,y,\xi) d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\xi).\end{aligned}$$

- Define $\hat\vartheta := \vartheta\left(x,y,\hat\xi(x,y)\right)$, $\Theta(x,y) := \partial^2_{\partial B}\vartheta\left(x,y,\hat\xi(x,y)\right)$ and $\eta_{\partial B}$ the spherical part of η (polar coordinates). Then, (Corollary of Morse' Lemma Proof)

$$\begin{aligned} I(x,y,r) &= \int_{\partial B_{\mathbb{R}^N}} e^{ir\vartheta(x,y,\xi)} a(x,y,\xi) d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\xi) \\ &= e^{ir\hat\vartheta} \int_{\partial B_{\mathbb{R}^N}} e^{i\frac{r}{2}\langle \Theta(x,y)\eta_{\partial B}(x,y,\xi), \eta_{\partial B}(x,y,\xi)\rangle_{\mathbb{R}^{N-1}}} a(x,y,\xi) d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\xi). \end{aligned}$$

- Let $\sigma:\ \mathbb{R}^{N-1} \to \partial B_{\mathbb{R}^N}$ be a stereographic projection with pole $-\hat\xi(x,y)$, $\eta_\sigma(x,y,\xi) := \eta_{\partial B}(x,y,\sigma(\xi))$, and $a_\sigma(x,y,\xi) := a(x,y,\sigma(\xi))\sqrt{\det\left(\sigma'(\xi)^*\sigma'(\xi)\right)}$. Then,

$$\begin{aligned} I(x,y,r) &= e^{ir\hat\vartheta} \int_{\partial B_{\mathbb{R}^N}} e^{i\frac{r}{2}\langle \Theta(x,y)\eta_{\partial B}(x,y,\xi), \eta_{\partial B}(x,y,\xi)\rangle_{\ell_2(N-1)}} a(x,y,\xi) d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\xi) \\ &= e^{ir\hat\vartheta} \int_{\mathbb{R}^{N-1}} e^{i\frac{r}{2}\langle \Theta(x,y)\eta_\sigma(x,y,\xi), \eta_\sigma(x,y,\xi)\rangle_{\ell_2(N-1)}} a_\sigma(x,y,\xi) d\xi \end{aligned}$$

- $\partial_3\eta_\sigma(x,y,\xi) = \partial_3\eta_{\partial B}(x,y,\sigma(\xi))\sigma'(\xi)$ and $\partial_3\eta\left(x,y,\hat{\xi}(x,y)\right) = 1 = \mathrm{id}_{\mathbb{R}^N} \Rightarrow$ $\eta_\sigma(x,y,\cdot)$ invertible in neighborhood of $\sigma^{-1}\left(\hat{\xi}(x,y)\right) = 0$

- $\partial_3\eta_\sigma(x,y,\xi) = \partial_3\eta_{\partial B}(x,y,\sigma(\xi))\sigma'(\xi)$ and $\partial_3\eta\left(x,y,\hat{\xi}(x,y)\right) = 1 = \mathrm{id}_{\mathbb{R}^N} \Rightarrow$ $\eta_\sigma(x,y,\cdot)$ invertible in neighborhood of $\sigma^{-1}\left(\hat{\xi}(x,y)\right) = 0$
- Assume a_σ has support in such a neighborhood and define

$$\tilde{a}(x,y,\xi) := a_\sigma(x,y,\eta_\sigma(x,y)^{-1}(\xi))\sqrt{\det\left(\left(\eta_\sigma(x,y)^{-1}\right)'(\xi)^*\left(\eta_\sigma(x,y)^{-1}\right)'(\xi)\right)}.$$

Then

$$I(x,y,r) = e^{ir\hat{\vartheta}}\int_{\mathbb{R}^{N-1}} e^{i\frac{r}{2}\langle\Theta(x,y)\xi,\xi\rangle_{\ell_2(N-1)}}\tilde{a}(x,y,\xi)d\xi.$$

- $\partial_3\eta_\sigma(x,y,\xi) = \partial_3\eta_{\partial B}(x,y,\sigma(\xi))\sigma'(\xi)$ and $\partial_3\eta\left(x,y,\hat{\xi}(x,y)\right) = 1 = \mathrm{id}_{\mathbb{R}^N} \Rightarrow$ $\eta_\sigma(x,y,\cdot)$ invertible in neighborhood of $\sigma^{-1}\left(\hat{\xi}(x,y)\right) = 0$
- Assume a_σ has support in such a neighborhood and define

$$\tilde{a}(x,y,\xi) := a_\sigma(x,y,\eta_\sigma(x,y)^{-1}(\xi))\sqrt{\det\left(\left(\eta_\sigma(x,y)^{-1}\right)'(\xi)^*\left(\eta_\sigma(x,y)^{-1}\right)'(\xi)\right)}.$$

Then

$$I(x,y,r) = e^{ir\hat{\vartheta}}\int_{\mathbb{R}^{N-1}} e^{i\frac{r}{2}\langle\Theta(x,y)\xi,\xi\rangle_{\ell_2(N-1)}}\tilde{a}(x,y,\xi)d\xi.$$

- $\mathcal{F}\left(\xi \mapsto e^{i\frac{1}{2}\langle r\Theta(x,y)\xi,\xi\rangle}\right)(\xi) = r^{\frac{1-N}{2}}\left|\det\Theta(x,y)\right|^{-\frac{1}{2}} e^{\frac{i\pi}{4}\mathrm{sgn}(\Theta(x,y))}e^{-i\frac{1}{2}\left\langle(r\Theta(x,y))^{-1}\xi,\xi\right\rangle}$ where $\mathrm{sgn}(\Theta(x,y))$ is the number of positive eigenvalues minus the number of negative eigenvalues of $\Theta(x,y)$.

$$\begin{aligned}\int_{\mathbb{R}^{N-1}} e^{i\frac{r}{2}\langle\Theta\xi,\xi\rangle}\tilde{a}(\xi)d\xi =&\text{const.}\int_{\mathbb{R}^{N-1}} e^{-\frac{i}{2}\langle(r\Theta)^{-1}\xi,\xi\rangle}\mathcal{F}_3\tilde{a}(\xi)d\xi\\ =&\text{const.}\sum_{j\in\mathbb{N}_0}\frac{r^{-j}}{j!}\int_{\mathbb{R}^{N-1}}\left(-\frac{i}{2}\left\langle\Theta^{-1}\xi,\xi\right\rangle\right)^j\mathcal{F}_3\tilde{a}(\xi)d\xi\\ =&\text{const.}\sum_{j\in\mathbb{N}_0}\frac{r^{-j}}{j!}\int_{\mathbb{R}^{N-1}}\mathcal{F}_3\left(\left(-\frac{i}{2}\left\langle\Theta^{-1}\partial_3,\partial_3\right\rangle\right)^j\tilde{a}\right)(\xi)d\xi\end{aligned}$$

and with

$$\int_{\mathbb{R}^n}\mathcal{F}f(\xi)d\xi=\int_{\mathbb{R}^n}e^{i\langle 0,\xi\rangle}\mathcal{F}f(\xi)d\xi=(2\pi)^{\frac{n}{2}}\mathcal{F}^{-1}(\mathcal{F}f)(0)=(2\pi)^{\frac{n}{2}}f(0)$$

we obtain

$$\int_{\mathbb{R}^{N-1}} e^{i\frac{1}{2}\langle r\Theta\xi,\xi\rangle}\tilde{a}(\xi)d\xi=\left(\frac{2\pi}{r}\right)^{\frac{N-1}{2}}|\det\Theta|^{-\frac{1}{2}}e^{\frac{i\pi}{4}\operatorname{sgn}\Theta}\sum_{j\in\mathbb{N}_0}\frac{(-i)^j r^{-j}}{j!2^j}\left\langle\Theta^{-1}\partial_3,\partial_3\right\rangle^j\tilde{a}(0).$$

Hence, defining

$$h_j(x,y) := \frac{(2\pi)^{\frac{N-1}{2}} \left|\det\Theta(x,y)\right|^{-\frac{1}{2}} e^{\frac{i\pi}{4}\operatorname{sgn}\Theta(x,y)}}{j!(2i)^j} \left\langle \Theta(x,y)^{-1}\partial_3, \partial_3 \right\rangle^j \tilde{a}(x,y,0)$$

we obtain

$$\begin{aligned} k(x,y) = &\int_{\mathbb{R}_{>0}} r^{N+d-1} (\ln r)^l \int_{\partial B_{\mathbb{R}^N}} e^{ir\vartheta(x,y,\xi)} a^0(x,y,\xi) d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\xi)\ dr \\ &+ \sum_{s=1}^{S} \sum_{j\in\mathbb{N}_0} h_j^s(x,y) \int_{\mathbb{R}_{>0}} r^{d+\frac{N-1}{2}-j} (\ln r)^l e^{ir\hat{\vartheta}^s(x,y)}\ dr. \end{aligned}$$

Remark. The evaluation of $\left\langle \Theta(x,y)^{-1}\partial_3, \partial_3 \right\rangle^j \tilde{a}(x,y,\cdot)$ at zero yields an evaluation at $\hat{\xi}(x,y)$ undoing all the changes of variables (stereographic proj. with pole $-\hat{\xi}$).

For $l=0$:

$$\forall q\in\mathbb{C}_{\Re(\cdot)>-1}\ \forall s\in\mathbb{C}_{\Re(\cdot)>0}:\ \int_{\mathbb{R}_{>0}} t^q e^{-st}dt=\Gamma(q+1)s^{-q-1}$$

and meromorphic extension

$$\int_{\mathbb{R}_{>0}} r^{d+\frac{N-1}{2}-j}e^{ir\hat{\vartheta}^s(x,y)}dr=\Gamma\left(d+\frac{N+1}{2}-j\right)i^{d+\frac{N+1}{2}-j}\left(\hat{\vartheta}^s(x,y)+i0\right)^{-d-\frac{N+1}{2}+j}$$

whenever $d+\frac{N+1}{2}-j\in\mathbb{C}\setminus(-\mathbb{N}_0)$ and, for $l\in\mathbb{N}_0$,

$$\begin{aligned}\int_{\mathbb{R}_{>0}} r^q(\ln r)^l e^{ir\hat{\vartheta}^s(x,y)}dr=&\partial^l\left(z\mapsto\int_{\mathbb{R}_{>0}} r^{q+z}e^{ir\hat{\vartheta}^s(x,y)}dr\right)(0)\\=&\partial^l\left(z\mapsto\Gamma(q+1+z)\,i^{q+1+z}\left(\hat{\vartheta}^s(x,y)+i0\right)^{-q-1-z}\right)(0).\end{aligned}$$

For $c \in \mathbb{R}_{>0}$, $q \in -\mathbb{N}$, and $l \in \mathbb{N}_0$, we obtain (lots and lots of fun with the Laplace transform later)

$$\int_{\mathbb{R}_{>0}} r^q (\ln r)^l e^{-sr} dr \Big|_{s=-i\hat{\vartheta}^s(x,y)+0}$$

$$= \partial^l \left(z \mapsto \frac{-\Gamma(z+1)}{2\pi i(-q-1)!} \int_{c+i\mathbb{R}} (-\sigma)^{-q-1} (c_{\ln} + \ln\sigma)(s-\sigma)^{-z-1} d\sigma \right)(0) \Big|_{s=-i\hat{\vartheta}^s(x,y)+0}.$$

Theorem

Let all assumptions above be satisfied and

$$g_{j,\iota}^s(x,y) := \begin{cases} \partial^{l_\iota}\left(z \mapsto \Gamma(q+1+z)\, i^{q+1+z}\left(\hat{\vartheta}^s(x,y)+i0\right)^{-q-1-z}\right)(0) & , \ q \in \mathbb{C} \setminus (-\mathbb{N}_0) \\ \partial^{l_\iota}\left(z \mapsto \frac{-\Gamma(z+1)}{2\pi i \ (-q)!} \int_{c+i\mathbb{R}} \frac{(-\sigma)^{-q}(c_{\ln}+\ln\sigma)}{\left(-i\hat{\vartheta}^s(x,y)+0-\sigma\right)^{z+1}} d\sigma\right)(0) & , \ q \in -\mathbb{N}_0 \end{cases}$$

with $q := d_\iota + \frac{N+1}{2} - j$, $c \in \mathbb{R}_{>0}$, *and some constant* $c_{\ln} \in \mathbb{C}$. *Then,*

$$k(x,y) = \int_{\mathbb{R}^N} e^{i\vartheta(x,y,\xi)} a^0(x,y,\xi)d\xi + \sum_{\iota \in \tilde{I}} \sum_{s=1}^{S} \sum_{j \in \mathbb{N}_0} h_{j,\iota}^s(x,y) g_{j,\iota}^s(x,y).$$