ζ-functions of Fourier Integral Operators: Path integral regularization

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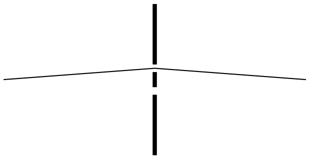
Feynman's alternative formulation of quantum mechanics

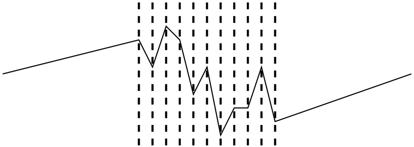
the double slit experiment



- interference pattern
- superposition principle
- probability density $P = |\Phi_1 + \Phi_2|^2$ (Φ_j quantum mechanical amplitudes)

Feynman's alternative formulation of quantum mechanics

more slits



probability density $P = \left|\sum_j \Phi_j\right|^2 = \left|\sum_{p \in \text{paths}} \Phi_p\right|^2$

Feynman's alternative formulation of quantum mechanics

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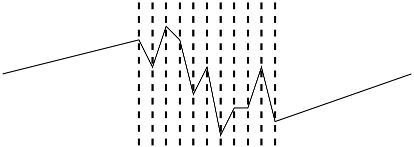

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Feynman

Let S_{cl} be the classical action. Then

$$\forall p \in \text{paths}: \;\; \Phi_p = \exp\left(\frac{i}{\hbar} S_{\text{cl}}(p)\right)$$

Feynman's alternative formulation of quantum mechanics

- number of slits $\to \infty$
- size of / distance between slits $\to 0$
- continuous paths
- $\rightsquigarrow$ inductive limit

Feynman path integral

Propagator of a particle moving from (t_0, x_0) to (t_1, x_1):

$$K(t_1, x_1; t_0, x_0) = \int_{p \in \mathrm{paths}((t_0,x_0)\to(t_1,x_1))} \exp\left(\frac{i}{\hbar} S_{\mathrm{cl}}(p)\right) \mathcal{D}p$$

Schrödinger equation and time evolution

- time evolution:

$$K(t,x;t',x') = \langle \delta_x, U(t,t')\delta_{x'} \rangle$$

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- Hamiltonian formulation (state vector ψ)

$$\psi(t) = U(t,t')\psi(t') = \operatorname{texp}\left(\frac{-i}{\hbar} \int_{t'}^{t} H(s)ds \right) \psi(t')$$

Partition function

Consider a time torus of length T. Then (for the state vector ψ)

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$\Rightarrow$ partition function (statistical properties of thermal equilibrium)

$$\mathcal{Z}_T = \mathrm{tr} U(T,0).$$

Observables and their vacuum expectation values

Given an observable Ω, its toroidal vacuum expectation is given by

$$\begin{aligned}\langle\Omega\rangle_T :=& \frac{\operatorname{tr}\ (U(T,0)\Omega)}{\mathcal{Z}_T}\\ =& \frac{\operatorname{tr}\ (U(T,0)\Omega)}{\operatorname{tr}\ U(T,0)}.\end{aligned}$$

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Recover vacuum expectation of Ω using the thermal limit

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a new path integral

Definition

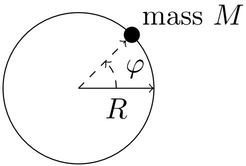
Let $\mathfrak{G}_T$ be a gauged family of operators with

$$\mathfrak{G}_T(0) = U(T,0).$$

Then we define

$$\langle\Omega\rangle_{\mathfrak{G}} := \lim_{T\to\infty} \frac{\zeta(\mathfrak{G}_T\Omega)}{\zeta(\mathfrak{G}_T)}(0).$$

Topological rotor



- $J := MR^2$
- generalized momentum: $p = J\partial_0\varphi$
- Hamiltonian: $H = \frac{p^2}{2J}$
- topological charge: $Q = \frac{1}{2\pi}\int_0^T \frac{p}{J}$
- topological susceptibility:
  $\chi_{\mathrm{top}} = \lim_{T\to\infty} \frac{\langle Q^2\rangle_T}{-iT}$

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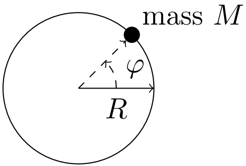

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Energy gap: $$\Delta E = 2\pi^2\chi_{\text{top}} = \frac{1}{2J}$$

Spontaneous Symmetry Breaking

- scalar fields $\varphi^1, \dots, \varphi^k$
- Hamiltonian: $H(\varphi^1, \dots, \varphi^k)$
- partition function: $\mathcal{Z}_T(\varphi) = \zeta\left(\exp\left(\frac{-i}{\hbar}\int_0^T H(\varphi)(t)dt\right)\mathfrak{g}\right)(0)$

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- we are looking for constant φ_0^j that locally minimize

$$V_e^T(\varphi_0) = \frac{\mathcal{Z}_T(\varphi)}{-iT\mathrm{vol}(X)} = \frac{\zeta\left(\exp\left(\frac{-i}{\hbar}\int_0^T H(\varphi_0)(t)dt\right)\mathfrak{g}\right)(0)}{-iT\mathrm{vol}(X)}$$

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- φ_0^j are vacuum expectation values of the φ^j (in the limit $T \to \infty$)
- $(\partial_i\partial_j V_e^T(\varphi_0))_{i,j}$ self-adjoint $\rightsquigarrow$ eigenvalues are squared field masses (in the limit $T \to \infty$)

Spontaneous Symmetry Breaking - the φ^4 model

- Hamiltonian: $H = \int_X \frac{p^2}{2} - \frac{\varphi\Delta\varphi}{2} - \frac{\mu^2}{2}\varphi^2 + \frac{\lambda}{4!}\varphi^4 dx$
- vacuum expectation values: $\varphi_0 = \pm\sqrt{\frac{6}{\lambda}}\mu$
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gauge dependence of $\mathcal{Z}_T$

The partition function

$$\mathcal{Z}_T(\varphi) = \zeta\left(\exp\left(\frac{-i}{\hbar}\int_0^T H(\varphi)(t)dt\right)\mathfrak{g}\right)(0),$$

and thus V_e^T, is independent of the particular choice of gauge $\mathfrak{g}$.

Free Radiation Field (QED without matter)

- electromagnetic field strength tensor $F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu$
- Lagrangian $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}(\partial_\mu A^\mu)^2$
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- conjugate momenta: $\pi^0 = -\partial_\mu A^\mu$, $\pi^j = \partial^j A^0 - \partial^0 A^j$
- quantization of field and momentum on $(\mathbb{R}/X\mathbb{Z})^3$

$$A_\mu(x) = \sum_{p\in\mathbb{Z}^3\setminus\{0\}} \sqrt{\frac{X}{2\pi\,\|p\|_{\ell_2(3)}}} \sum_{\lambda\in 4} \varepsilon_\mu^\lambda(p)\left(a^\lambda(p)e^{\frac{2\pi i}{X}\langle p,x\rangle_{\ell_2(3)}} + a^\lambda(p)^\dagger e^{-\frac{2\pi i}{X}\langle p,x\rangle_{\ell_2(3)}}\right)$$

$$\pi^\mu(x) = \sum_{p\in\mathbb{Z}^3} i\sqrt{\frac{2\pi\,\|p\|_{\ell_2(3)}}{X}} \sum_{\lambda\in 4} (\varepsilon^\mu)^\lambda(p)\left(a^\lambda(p)e^{\frac{2\pi i}{X}\langle p,x\rangle_{\ell_2(3)}} - a^\lambda(p)^\dagger e^{-\frac{2\pi i}{X}\langle p,x\rangle_{\ell_2(3)}}\right)$$

where a^λ, $a^{\lambda\dagger}$ annihilation/creation op.s, and ε^λ are polarization 4-vectors

- Choose ε^0 timelike, ε^3 longitudinal, and ε^1, ε^2 transversal $(\forall \lambda \in \{1,2\}: \ \varepsilon_\mu^\lambda p^\mu = 0)$, i.e., for 4-momenta $(\|p\|, p) \propto (1,0,0,1)$: $\varepsilon^\lambda = (\delta_{\lambda,\mu})_{\mu\in 4}$ and all other cases arise using the corresponding Lorentz transform

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$$A^+_\mu(x) = \sum_{p\in\mathbb{Z}^3\setminus\{0\}} \sqrt{\frac{X}{2\pi\|p\|}} \sum_{\lambda\in 4} \varepsilon^\lambda_\mu(p) a^\lambda(p) e^{\frac{2\pi i}{X}\langle p,x\rangle}$$

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- state $|\psi\rangle$ is physical $:\Longleftrightarrow \quad \partial^\mu A^+_\mu |\psi\rangle = 0$
- this ensures $\forall \varphi, \psi$ physical: $\langle\varphi|\,\partial_\mu A^\mu\,|\psi\rangle = 0$

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- Hamiltonian

$$\begin{aligned} H_0 &= \int_{(\mathbb{R}/X\mathbb{Z})^3} \pi^\mu \partial_0 A_\mu - \mathcal{L} d\mathrm{vol} \\ &= \sum_{p\in\mathbb{Z}^3\setminus\{0\}} \frac{2\pi\,\|p\|}{X} \left(-a^0(p)^\dagger a^0(p) + \sum_{j=1}^{3} a^j(p)^\dagger a^j(p) \right) \\ &= \sum_{p\in\mathbb{Z}^3\setminus\{0\}} \frac{2\pi\,\|p\|}{X} \sum_{j=1}^{2} a^j(p)^\dagger a^j(p) \end{aligned}$$

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$$\psi_{p,1}(x) = \frac{1}{\sqrt{X^3}} \exp\left(\frac{2\pi i}{X}\langle p,x\rangle_{\ell_2(3)}\right)\begin{pmatrix}1\\0\end{pmatrix}$$

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- $\Rightarrow$ up-to-one-particle Hamiltonian

$$H_{\leq 1} = |\nabla|\,\mathrm{id}_{\mathbb{C}^2}$$

- up to N particles: $|(p_1, j_1), \ldots, (p_N, j_N)\rangle = |p_1, j_1\rangle \otimes \ldots \otimes |p_N, j_N\rangle$ in $S \bigotimes_{j=1}^{N} \mathcal{H}_1$

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- Hamiltonian acts diagonally

$$H_{\leq N} = \sum_{k=1}^{N} \left(\bigotimes_{m=1}^{k-1} \mathrm{id}_{\mathcal{H}_1} \right) \otimes H_{\leq 1} \otimes \left(\bigotimes_{m=k+1}^{N} \mathrm{id}_{\mathcal{H}_1} \right)$$

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- Symbol of $H_{\leq N}$: $\sum_{n=1}^N \|\xi_n\|_{\ell_2(3)}\mathrm{id}_{\mathbb{C}^2}$
- time evolution: $U = e^{-iTH_{\leq N}} \sim e^{-iT\sum_{n=1}^N\|\xi_n\|_{\ell_2(3)}}\mathrm{id}_{\mathbb{C}^2}$

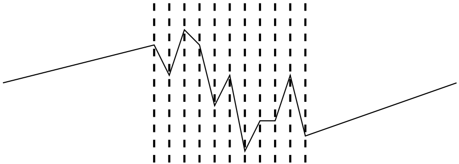
$$\lim_{N\to\infty}\langle H_{\le N}\rangle_{\mathfrak{G}}(z) = \lim_{N\to\infty}\lim_{T\to\infty}\frac{\int_{(\mathbb{R}^3)^N}\operatorname{tr}\left(e^{-iT\sum_{n=1}^N\|\xi_n\|}\sum_{n=1}^N\|\xi_n\|\prod_{m=1}^N\|\xi_m\|^z\operatorname{id}_{\mathbb{C}^2}\right)d\xi}{\int_{(\mathbb{R}^3)^N}\operatorname{tr}\left(e^{-iT\sum_{n=1}^N\|\xi_n\|}\prod_{m=1}^N\|\xi_m\|^z\operatorname{id}_{\mathbb{C}^2}\right)d\xi}$$

$$= \lim_{N\to\infty}\lim_{T\to\infty}\frac{\sum_{n=1}^N\prod_{m=1}^N\int_{\mathbb{R}^3}e^{-iT\|\xi_m\|_{\ell_2(3)}}\|\xi_m\|_{\ell_2(3)}^{z+\delta_{mn}}d\xi_m}{\prod_{m=1}^N\int_{\mathbb{R}^3}e^{-iT\|\xi_m\|_{\ell_2(3)}}\|\xi_m\|_{\ell_2(3)}^{z}d\xi_m}$$

$$= \lim_{N\to\infty}\lim_{T\to\infty}\frac{N\int_{\mathbb{R}_{>0}}e^{-iTr}r^{z+3}dr}{\int_{\mathbb{R}_{>0}}e^{-iTr}r^{z+2}dr}$$

$$= \lim_{N\to\infty}\lim_{T\to\infty}\underbrace{\frac{N\Gamma(z+4)(iT)^{-z-4}}{\Gamma(z+3)(iT)^{-z-3}}}_{\propto\frac{1}{T}} = 0.$$

Lattice QFT



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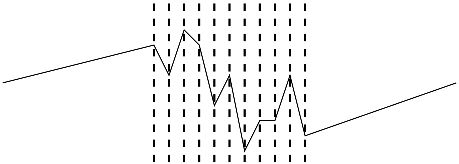

$$\mathcal{Z}_T^\Omega = \int_{\mathbb{R}^n} \exp\left(\frac{i}{\hbar}\int \text{Lagrangian}(p)\ d\text{vol}_{\text{universe}}\right) a^\Omega(p)dp$$

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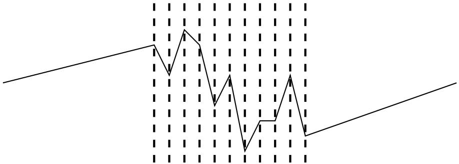

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Lattice QFT: Wick Rotation

$$\mathcal{Z}_T^{\Omega,\text{Wick}} = \int_{\mathbb{R}^n} \exp\left(-\frac{1}{\hbar}\int \text{Lagrangian}(p)\ d\text{vol}_{\text{universe}}\right) a^\Omega(p)dp$$

"ζ-Lattice QFT"

(i) write down your LQFT

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(ii) and gauge it

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"ζ-Lattice QFT"

(i) write down your LQFT

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(iii) construct distributionally equivalent family of Fourier Integral Operator "traces"

$$\mathcal{Z}_T^{\Omega}(z) = \text{holomorphic} + \sum_{\iota\in I}\int_{\partial B_{\mathbb{R}^n}}\int_{\mathbb{R}_{>0}} e^{ir\vartheta(\xi)} r^{d_\iota+z}\alpha_\iota^{\Omega}(z)(\xi)\ dr\ d\xi$$

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(vi) and enjoy the remaining (fully regular) integrals

$$\mathcal{Z}_T^\Omega(0) = \text{holomorphic}(0) + \sum_{\iota \in I} \Gamma(d_\iota + 1) \int_{\partial B_{\mathbb{R}^n}} \frac{\alpha_\iota^\Omega(0)(\xi)}{(-i\vartheta(\xi))^{d_\iota + 1}} d\xi$$

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- For $U\mathfrak{G}(z)A$ and $U\mathfrak{G}(z)$ trace-class

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- consider densely embedded Hilbert space $\mathcal{H}_1$ in $\mathcal{H}$ (e.g., W_2^k in L_2) with $\forall n \in \mathbb{N}:\ P_n[\mathcal{H}] \subseteq \mathcal{H}_1$ and let Q_n be the orthogonal projector onto $P_n[\mathcal{H}]$ in $\mathcal{H}_1$.

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- discretized Hamiltonian $H_n := P_n H Q_n$, observable $A_n := P_n A Q_n$, time evolution $U_n := \operatorname{texp}\left(-i\int_0^T P_n H(s) Q_n ds\right)$

Lemma

Let $A \in L(\mathcal{H}_1, \mathcal{H})$*. Then*

$$\forall x \in \mathcal{H}_1 : \lim_{n\to\infty} P_n A Q_n x = Ax.$$

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Let $B \in L(\mathcal{H})$, $A \in L(\mathcal{H}_1, \mathcal{H})$. *Then*

$$\forall x \in \mathcal{H}_1 : \lim_{n\to\infty} P_n B Q_n P_n A Q_n x = BAx.$$

Theorem

Let H be the Hamiltonian, i.e., $\forall s \in [0,T]:$ $H(s)$ is a self-adjoint operator, each $-H(s)$ generates a C_0-semigroup, and there exists $E_0 := \min \sigma(H(s))$ such that E_0 is in the point spectrum, $\ker(H(s) - E_0)$ is independent of s, and $\exists \varepsilon \in \mathbb{R}_{>0}\ \forall s \in [0,T]:$ $B(E_0,\varepsilon) \cap \sigma(H(s)) = \{E_0\}$ (in other words, the QFT has a mass gap). Then the following are true.

(i) *Let A be self-adjoint in $\mathcal{H}$. Then $P_n A Q_n$ is self-adjoint on $(P_n[\mathcal{H}], \langle\cdot,\cdot\rangle_{\mathcal{H}})$. In particular, $P_n H(s) Q_n$ is self-adjoint.*

(ii) *Let ψ be the vacuum state (i.e., an eigenvector with respect to E_0) and ψ_n the vacuum state of $P_n H(s) Q_n$ in $P_n[\mathcal{H}]$. Then*

$$\lim_{n\to\infty} \langle \psi_n, P_n H(s) Q_n \psi_n \rangle_{\mathcal{H}} = \lim_{n\to\infty} \langle \psi, P_n H(s) Q_n \psi \rangle_{\mathcal{H}} = \langle \psi, H(s)\psi \rangle_{\mathcal{H}} = E_0.$$

Theorem (continued)

(iii) *Let* $\hat{H} := H(s) - E_0$. *If the vacuum is non-degenerate, i.e.,* $\ker \hat{H} = \operatorname{lin}\{\psi\}$, *then* $\langle \psi_n, \psi \rangle_{\mathcal{H}} \psi_n \to \psi$ *in* $\mathcal{H}$.

(iv) *Let the vacuum be non-degenerate and* $A \in L(\mathcal{H}_1, \mathcal{H})$ *be such that the sequence* $(\|A\psi_n\|_{\mathcal{H}})_{n\in\mathbb{N}}$ *is bounded and* $\psi \in D(A^*)$ *where* A^* *is the adjoint of* A *as an unbounded operator in* $\mathcal{H}$. *Then*

$$\langle \psi_n, A\psi_n \rangle_{\mathcal{H}} = \langle \psi_n, P_n A Q_n \psi_n \rangle_{\mathcal{H}} \to \langle \psi, A\psi \rangle_{\mathcal{H}}$$

and

$$\langle \psi, P_n A Q_n \psi \rangle_{\mathcal{H}} \to \langle \psi, A\psi \rangle_{\mathcal{H}}.$$

Lemma

Let $p \in \mathbb{R}_{\geq 1} \cup \{\infty\}$ *and* $A \in \mathcal{S}_p(\mathcal{H}_1, \mathcal{H})$. *Then* $\|P_n A Q_n - A\|_{\mathcal{S}_p(\mathcal{H}_1,\mathcal{H})} \to 0$ *and hence* $\|P_n A Q_n - A\|_{L(\mathcal{H}_1,\mathcal{H})} \to 0$ *as well.*
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Lemma

Let the Hamiltonian H *satisfy the conditions necessary for the time-dependent Hille-Yosida theorem and* $\mathcal{H}_1$ *such that* $\forall s \in [0,T]:\ H(s) \in \mathcal{S}_1(\mathcal{H}_1, \mathcal{H})$*, as well as* $A, B \in \mathcal{S}_1(\mathcal{H})$. *Then*

$$\frac{\operatorname{tr}(U_n P_n A Q_n)}{\operatorname{tr}(U_n P_n B Q_n)} \to \frac{\operatorname{tr}(UA)}{\operatorname{tr}(UB)}.$$

Let $e_j(x) := \sin(j^2 x)$ for $j \in \mathbb{N}$ and consider

$$\mathcal{W} := \overline{\operatorname{lin}\{e_j;\ j \in \mathbb{N}\}}^{W_{2,0}^1([0,2\pi])} \text{ and } \mathcal{H} := \overline{\operatorname{lin}\{e_j;\ j \in \mathbb{N}\}}^{L_2([0,2\pi])}$$

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Let $(z \mapsto \|\mathfrak{G}(z)A\psi_n\|_{\mathcal{H}})_{n\in\mathbb{N}}$ and $(z \mapsto \|\mathfrak{G}(z)\psi_n\|_{\mathcal{H}})_{n\in\mathbb{N}}$ be locally bounded in $C(\mathbb{C})$.

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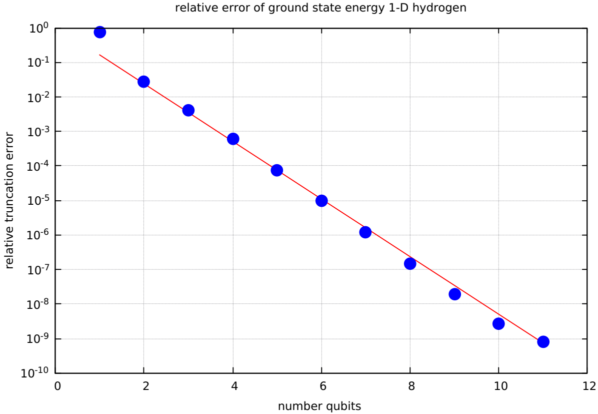
$\left(z \mapsto \frac{\langle \psi_n, \mathfrak{G}(z)A\psi_n\rangle_{\mathcal{H}}}{\langle \psi_n, \mathfrak{G}(z)\psi_n\rangle_{\mathcal{H}}}\right)_{n\in\mathbb{N}}$ *is compactly convergent to* $\langle A\rangle_{\mathfrak{G}}$ *on* $\mathbb{C} \setminus [\{0\}]\mathcal{D}$.

Corollary

Since $0 \notin [\{0\}]\mathcal{D}$

$$\langle A\rangle = \lim_{n\to\infty}\langle\psi_n, A\psi_n\rangle_{\mathcal{H}} = \lim_{n\to\infty}\frac{\langle\psi_n, \mathfrak{G}(0)A\psi_n\rangle_{\mathcal{H}}}{\langle\psi_n, \mathfrak{G}(0)\psi_n\rangle_{\mathcal{H}}} = \langle A\rangle_{\mathfrak{G}}(0).$$

Ground state energy of 1-D hydrogen atom - convergence - QVM



Regression:

- q = number qubits
- $E(q)$ = rel. error of ground state energy

$$E(q) \approx 1.14 \cdot e^{-1.92q}$$