Introduction to Microlocal Analysis

Fourth lecture: Lagrangian distributions

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1. Fourier integral operators

2. Pseudodifferential operators of real principal type, II

3. Weyl asymptotics

"Calculus" of wave front sets

Let $A\colon \mathscr{C}_c^\infty(Y) \to \mathscr{D}'(X)$ be a linear operator and $K \in \mathscr{D}'(X \times Y)$ be its distributional kernel. Identify $T^*(X \times Y) \cong T^*X \times T^*Y$. Define

- $\mathrm{WF}'(A) = \{(x,\xi,y,\eta) \in \dot{T}^*X \times \dot{T}^*Y \mid (x,y,\xi,-\eta) \in \mathrm{WF}(K)\}$ (wave front relation),
- $\mathrm{WF}_X(A) = \{(x,\xi) \in \dot{T}^*X \mid \exists y \in Y\colon (x,y,\xi,0) \in \mathrm{WF}(K)\}$,
- $\mathrm{WF}'_Y(A) = \{(y,\eta) \in \dot{T}^*X \mid \exists x \in X\colon (x,y,0,-\eta) \in \mathrm{WF}(K)\}$.

Proposition

Let $A\colon \mathscr{C}_c^\infty(Y) \to \mathscr{D}'(X)$ and $B\colon \mathscr{C}_c^\infty(Z) \to \mathscr{D}'(Y)$. Suppose that

- $\mathrm{WF}'_Y(A) \cap \mathrm{WF}_Y(B) = \emptyset$,
- *the canonical projection* $(\operatorname{supp} K_A \times \operatorname{supp} K_B) \cap (X \times \Delta_Y \times Z) \to X \times Z$ *is proper.*

Then $A \circ B\colon \mathscr{C}_c^\infty(Z) \to \mathscr{D}'(X)$ is defined. Moreover,

$$\mathrm{WF}'(A \circ B) \subseteq \mathrm{WF}'(A) \circ \mathrm{WF}'(B) \cup (\mathrm{WF}_X(A) \times 0_Z) \cup \big(0_X \times \mathrm{WF}'_Z(B)\big).$$

An example: Pullbacks

Let $f\colon X \to Y$ be $\mathscr{C}^\infty$. Then $f^*\colon \mathscr{C}^\infty(Y) \to \mathscr{C}^\infty(X)$, $u \mapsto u \circ f$, has kernel

$$K(x,y) = \delta(y - f(x))$$

and, therefore,

$$\mathrm{WF}(K) = \{(x,y,\xi,\eta) \mid y = f(x),\ \xi + {}^t\mathrm{d}f(x)\eta = 0,\ \eta \neq 0\}.$$

We conclude that

$$\mathrm{WF}'(f^*) = \{(x, {}^t\mathrm{d}f(x)\eta, f(x), \eta) \mid {}^t\mathrm{d}f(x)\eta \neq 0\},$$
$$\mathrm{WF}'_Y(f^*) = \{(f(x), \eta) \mid {}^t\mathrm{d}f(x)\eta = 0,\ \eta \neq 0\},$$

and $\mathrm{WF}_X(f^*) = \emptyset$.

Fourier integral operators

Let $\Lambda \subset \dot{T}^*(X \times Y)$ be a conic Lagrangian submanifold such that $C = \Lambda' \subset \dot{T}^*X \times \dot{T}^*Y$ (homogeneous canonical relation).

Example

Let $\chi\colon \dot{T}^*X \to \dot{T}^*Y$ be a homogeneous canonical transformation. Then $C = \operatorname{graph}\chi$ is a homogeneous canonical relation.

Definition

$A\colon \mathscr{C}_c^\infty(Y) \to \mathscr{C}^\infty(X)$ is said to be a classical Fourier integral operator associated with C, of order $\mu \in \mathbb{C}$, if $K \in I_{\mathrm{cl}}^\mu(X \times Y, \Lambda)$.

We write $A \in I_{\mathrm{cl}}^\mu(X, Y, C)$.

Microlocally,

$$Au(x) = \int e^{i\varphi(x,y,\theta)} a(x,y,\theta) u(y)\, dy\, đ\theta,$$

where $\Lambda = \Lambda_\varphi$ and $a \in S_{\mathrm{cl}}^{\mu+(n_X+n_Y-2N)/4}(X \times Y \times \mathbb{R}^N)$.

Fourier integral operators, II

Remark

For $u \in \mathscr{E}'(Y)$, $\mathrm{WF}(Au) \subseteq C \circ \mathrm{WF}(u)$.

Examples

- $\Psi^\mu_{\mathrm{cl}}(X) = I^\mu_{\mathrm{cl}}(X, X, \Delta_{\dot{T}^*(X)})$.

Let $\{U(t,s)\}_{0 \le t,s \le T}$ be the propagator of $D_t - a(t,x,D_x)$, where $a \in \mathscr{C}^\infty([0,T]; \Psi^1_{\mathrm{cl}}(X))$ and $\sigma^1(a)(t,x,\xi)$ is real-valued. Then:

- $U(t,s) \in I^0_{\mathrm{cl}}(X, X, \operatorname{graph} \chi_{t,s})$, where $\chi_{t,s}$ is the flow of the time-dependent Hamilton vector field $H_{\sigma^1(a)}$ from time t to time s.
- $U(\cdot, 0) \in I^{-1/4}_{\mathrm{cl}}((0,T) \times X, X, C)$, where $C = \{(t, x, \tau, \xi, y, \eta \mid \tau = \sigma^1(a)(t,x,\xi),\ \chi_{t,0}(x,\xi) = (y,\eta))\}$.

Principal symbol

- $A \in I^{\mu}_{\mathrm{cl}}(X, Y, C)$ possesses a principal symbol $\sigma^{\mu}(A)$ which is a function on C (ignoring density factors and contributions from the Keller-Maslov bundle). In local coordinates,

$$\sigma^{\mu}(A)(x, \xi, y, \eta) = a_{(\mu+(n_X+n_Y-2N)/4)}(x, y, \theta)$$

where $\varphi'_{\theta}(x, y, \theta) = 0$, $\xi = \varphi'_x(x, y, \theta)$, and $\eta = -\varphi'_y(x, y, \theta)$.

Remark

One has a notion of ellipticity if C is the graph of a homogeneous canonical transformation. For an elliptic Fourier integral operator, there is a parametrix as before (see also below).

Compositions

Now let $A \in I_{\mathrm{cl}}^{\mu}(X, Y, C_0)$ and $B \in I_{\mathrm{cl}}^{\mu'}(Y, Z, C_1)$. We make the following assumptions:

- $C_0 \times C_1$ and $\dot{T}^*X \times \Delta_{\dot{T}^*Y} \times \dot{T}^*Z$ intersect cleanly in a manifold C, with excess e,
- the fibers of the canonical map $C \to C_0 \circ C_1$ are connected and compact,
- the canonical projection $(\operatorname{supp} K_A \times \operatorname{supp} K_B) \cap (X \times \Delta_Y \times Z) \to X \times Z$ is proper.

Then $C_0 \circ C_1$ is a homogeneous canonical relation.

Theorem

Under these conditions, $A \circ B \in I_{\mathrm{cl}}^{\mu+\mu'+e/2}(X, Z, C_0 \circ C_1)$ and $\sigma^{\mu+\mu'+e/2}(A \circ B)$ is computable in terms of $\sigma^{\mu}(A)$ and $\sigma^{\mu'}(B)$.

Mapping properties

Theorem

Let $A \in I^m(X, Y, C)$, where C is the graph of a homogeneous canonical transformation. Then

$$A\colon H_{\mathrm{c}}^{\sigma+m}(Y) \to H_{\mathrm{loc}}^{\sigma}(X)$$

for all $\sigma \in \mathbb{R}$.

Theorem

Let $A \in I^m(X, Y, C)$. Suppose that the radial vector fields of $\dot{T}^*X$ and $\dot{T}^*Y$ are nowhere tangent to C. Then

$$A\colon H_{\mathrm{c}}^{\sigma+m}(Y) \to H_{\mathrm{loc}}^{\sigma-r}(X),$$

where $r \geq \operatorname{corank} \omega_C/4$ and $\omega_C = \left(\pi_X\big|_C\right)^* \omega_X = \left(\pi_Y\big|_C\right)^* \omega_Y$.

Egorov's theorem, revised

Theorem

Let C be the graph of the homogeneous canonical transformation $\chi\colon \dot{T}^*Y \to \dot{T}^*X$. Let $A \in I^{\nu}_{\mathrm{cl}}(X, Y, C)$, $B \in I^{-\nu}_{\mathrm{cl}}(Y, X, C^{-1})$, and $P \in \Psi^{\mu}_{\mathrm{cl}}(X)$, where at least two of these three operators are properly supported. Then $Q = BPA \in \Psi^{\mu}_{\mathrm{cl}}(Y)$.
Moreover,

$$\sigma^{\mu}(Q) = (\sigma^{\mu}(P) \circ \chi)\, \sigma^{0}(BA).$$

- Note that $BA \in \Psi^{0}_{\mathrm{cl}}(Y)$.
- Usually one chooses A to be elliptic and B to be a parametrix to A. Then, in particular, $\sigma^{0}(BA) = 1$.
- **Interpretation.** In microlocal analyis, pseudodifferential operators assume the role of cut-off functions, while elliptic Fourier integral operators take the part of coordinate changes (e.g., in order to arrange some normal form).

Pseudodifferential operators of real principal type

Definition

A pseudodifferential operator $P \in \Psi^m_{\mathrm{cl}}(X)$ is said to be of real principal type if

- $p = \sigma^m(P)$ is real-valued,
- no complete null bicharacteristics of P stays over a compact set in X.

Furthermore, X is said to be pseudo-convex with respect to P if, in addition,

- for any compact set $K \subseteq X$ there exists a compact set $K' \subseteq X$ with the property that each null bicharacteristics of P with endpoints over K stays entirely over K'.

Example. Let (X, h) a globally hyperbolic space-time. Then the d'Alembertian $\Box_h$ is of real principal type and X is pseudo-convex with respect to $\Box_h$.

Bicharacteristic relation

Remark

Apart from having a real principal symbol, the microlocal condition for P to be of real principal type is that dp and $\alpha = \xi\, dx$ are nowhere collinear.

Lemma

Let $P \in \Psi^m_{\mathrm{cl}}(X)$ be of real principal type and X be pseudo-convex with respect to P. Then the bicharacteristic relation

$$C = \big\{(x,\xi,y,\eta) \in \dot{T}^*X \times \dot{T}^*X \mid \\ (x,\xi) \text{ and } (y,\eta) \text{ lie on the same null bicharacteristics of } P\big\}$$

is a homogeneous canonical relation.

Distinguished parametrices

Now choose, for each connected component of $\operatorname{Char} P$, an orientation in the sense that (x, ξ) should lie either before or after (y, η) on the null bicharacteristics that contains both of them and then form the homogeneous canonical relation D, with nonempty boundary ∂D, that contains the (x, ξ, y, η) determined that way. (In a sense, D is “half” of C.)

Theorem (Duistermaat-Hörmander, ’72)

Under these conditions, P possesses a parametrix E such that

$$\mathrm{WF}'(E) = \Delta_{\dot{T}^*X} \cup D.$$

Moreover, E is unique up to smoothing.

Paired Lagrangian distributions

In fact, the kernel of E is a one-sided paired Lagrangian distribution (Melrose-Uhlmann, '79, Joshi, '98). In the special case considered here, the local model for the latter is $u \in I_{\mathrm{cl}}^{\mu-1/2,\mu}(\mathbb{R}^n, \Lambda_0, \Lambda_1)$, where $x = (x_1, x'') \in \mathbb{R} \times \mathbb{R}^{n-1}$, $\Lambda_0 = \dot{T}_0^*\mathbb{R}^n = \{(0,\xi) \mid \xi \neq 0\}$, $\Lambda_1 = \{(x_1, 0, 0, \xi'') \mid x_1 \geq 0,\ \xi'' \neq 0\}$, and

$$u(x) = \int_0^\infty \int_{\mathbb{R}^n} e^{i\left((x_1 - s)\xi_1 + x''\cdot\xi''\right)} a(s, x_1, \xi)\, d\xi\, ds,$$

where $a \in S_{\mathrm{cl}}^{\mu+1/2-n/4}([0,\infty) \times \mathbb{R}^n \times \mathbb{R}^n)$.

Note that $u \in I_{\mathrm{cl}}^{\mu-1/2}(\mathbb{R}^n, \Lambda_0 \setminus \Lambda_1)$ and $u \in I_{\mathrm{cl}}^{\mu}(\mathbb{R}^n, \Lambda_1 \setminus \Lambda_0)$.

Moreover,

- $\sigma_{\Lambda_0}^{\mu-1/2}(u) = \dfrac{a_{(\mu+1/2-n/4)}(0,0,\xi)}{i\xi_1}$ on $\Lambda_0 \setminus \Lambda_1$,
- $\sigma_{\Lambda_1}^{\mu}(u) = a_{(\mu+1/2-n/4)}(x_1, (x_1, 0), (0, \xi''))$ on $\Lambda_1 \setminus \Lambda_0$.

Further observe that there is the compatibity condition

$$i\xi_1\, \sigma_{\Lambda_0}^{\mu-1/2}(u) = \sigma_{\Lambda_1}^{\mu}(u) \quad \text{on } \partial\Lambda_1 = \Lambda_0 \cap \Lambda_1.$$

The wave trace

- X – $\mathscr{C}^\infty$ closed manifold, $\dim X = n$,
- $P \in \Psi^1(X; \Omega^{1/2})$, $P = P^* > 0$,
- P has purely discrete spectrum (as an unbounded operator in $L^2(X; \Omega^{1/2})$),
- $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots$ – eigenvalues of P, with eigenfunctions $\phi_j \in \mathscr{C}^\infty(X; \Omega^{1/2})$ that are chosen to form an orthonormal basis in $L^2(X; \Omega^{1/2})$,
- $\lambda_j \sim c\, j^{1/n}$ as $j \to \infty$ by a rough version of Weyl's law.

The wave trace is

$$w(t) = \sum_{j=1}^{\infty} e^{i\lambda_j t}, \quad t \in \mathbb{R}.$$

which is formally the trace of the wave group $\{e^{itP}\}_{t\in\mathbb{R}}$.

The wave kernel

Remark

It holds $e^{itP} = \int_0^\infty e^{it\lambda}\, dE_\lambda$, where $\{dE_\lambda\}_{\lambda\ge 0}$ is the spectral measure of P. Taking traces on both sides, one obtains $w(t) = \int_0^\infty e^{it\lambda}\, dN(\lambda)$, where $N(\lambda) = \#\{j \mid \lambda_j \le \lambda\}$ is the counting function of P. In particular, the wave trace w is (essentially) the Fourier transform of the (counting) measure dN.

Let $U\colon \mathscr{C}^\infty(X;\Omega^{1/2}) \to \mathscr{C}^\infty(\mathbb{R}\times X;\Omega^0 \boxtimes \Omega^{1/2})$ be the solution operator of the Cauchy problem for $D_t - P$. We already know that $U \in I_{\mathrm{cl}}^{-1/4}(\mathbb{R}\times X, X, C;\Omega^{1/2})$, where

$$C = \{(t,x,\tau,\xi,y,\eta) \mid \tau = p(x,\xi),\ (x,\xi) = \chi_t(y,\eta)\},$$

$p = \sigma^1(P)$, and $\{\chi_t\}_{t\in\mathbb{R}}$ is the flow of H_p.

The kernel of U is

$$U(t,x,y) = \sum_j e^{it\lambda_j}\phi_j(x)\overline{\phi_j(y)}.$$

Singularities of the wave trace

Hence, $w = \pi_* \Delta^* U$, where

- $\Delta \colon \mathbb{R} \times X \to \mathbb{R} \times X \times X$ is the diagonal map, $\Delta^* \colon \mathscr{C}^\infty(\mathbb{R} \times X \times X; \Omega^0 \boxtimes \Omega^{1/2} \boxtimes \Omega^{1/2}) \to \mathscr{C}^\infty(\mathbb{R} \times X; \Omega^0 \boxtimes \Omega^1)$,
- $\pi_* \colon \mathscr{C}^\infty(\mathbb{R} \times X; \Omega^0 \boxtimes \Omega^1) \to \mathscr{C}^\infty(\mathbb{R})$ is integration along fibers.

Note that $\pi_* \Delta^* \in I^0_{\mathrm{cl}}(\mathbb{R}, \mathbb{R} \times X \times X, C_0; \Omega^{1/2})$, where

$$C_0 = \{(t, \tau, t, x, x, \tau, \xi, -\xi) \mid \tau \neq 0\}.$$

Lemma

It holds $w \in \mathscr{S}'(\mathbb{R}; \Omega^{1/2})$ and

$$\mathrm{WF}(w) \subseteq \{(t, \tau) \mid \exists\, (x, \xi) \in \dot{T}^* X \colon \chi_t(x, \xi) = (x, \xi),\ \tau > 0\}.$$

Weyl's law with remainder estimate

Now a careful analysis of the "big" singularity of the wave trace w at $t = 0$ yields Weyl's law together with a sharp remainder estimate:

Theorem (Hörmander, '68)

Let $P \in \Psi^m(X; \Omega^{1/2})$, $m > 0$, *and* $P = P^* > 0$. *Then*

$$N_P(\lambda) = \#\{j \mid \lambda_j \leq \lambda\} = c\,\lambda^{n/m} + O(\lambda^{(n-1)/m}) \quad \text{as } \lambda \to \infty,$$

where

$$c = \int_{p(x,\xi) \leq 1} dx đ\xi.$$

Proof. Work with $P^{1/m} \in \Psi^1_{\mathrm{cl}}(X; \Omega^{1/2})$. $\square$