Introduction to Microlocal Analysis

First lecture: Basics

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Distributions

Definition (Topology on $\mathscr{C}_c^\infty(X)$, the space of test functions)

Let $X \subseteq \mathbb{R}^n$ be open. A sequence $(\phi_j)_j \subset \mathscr{C}_c^\infty(X)$ is said to converge to 0 in $\mathscr{C}_c^\infty(X)$ iff there is a fixed compact $K \subset X$ s.t. $\operatorname{supp} \phi_j \subseteq K$ for all j and all derivatives $\partial^\alpha \phi_j$ converge uniformly to 0 as j tends to ∞.

Definition

The space of distributions $\mathscr{D}'(X)$ is the topological dual of $\mathscr{C}_c^\infty(X)$.

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Definition

The space of distributions $\mathscr{D}'(X)$ is the topological dual of $\mathscr{C}_c^\infty(X)$.

Topological dual = (sequ.) continuous linear functionals $u : \mathscr{C}_c^\infty(X) \to \mathbb{C}$

$$\mathscr{C}_c^\infty(X) \ni \phi \mapsto u(\phi) =: \langle u, \phi \rangle$$

Examples: the bump function ϕ is in $\mathscr{C}_c^\infty(\mathbb{R}^n)$. Any $f \in L^1_{loc}(\mathbb{R}^n)$ defines a distribution,

$$\langle f, \phi \rangle = \int f(x)\phi(x)\mathrm{d}x \ ,$$

$\phi \mapsto \delta_{x_0}(\phi) = \phi(x_0)$ is a distribution. Hence the name “generalized functions”.

Distributions II

Observe (not obvious!):

A linear functional on $\mathscr{C}_c^\infty(\mathbb{R}^n)$ is in $\mathscr{D}'(\mathbb{R}^n)$ iff for every compact $K \subset \mathbb{R}^n$, there exist C and k such that

$$|\langle u, \phi \rangle| \leq C \sum_{|\alpha| \leq k} \sup |\partial^\alpha \phi(x)| \ , \text{ for all } \phi \in \mathscr{C}_c^\infty(K)$$

Definition (order of a distribution)

If the same k can be chosen for all K, u is said to be of order $\leq k$.

Example

$\phi \mapsto \partial^\alpha \phi(x_0)$ for fixed $x_0 \in \mathbb{R}^n$ is of order $|\alpha|$.

Localization and (singular) support

Remark (localization) Let $Y \subseteq X \subseteq \mathbb{R}^n$ be open.

- Then $\mathscr{C}_c^\infty(Y) \subseteq \mathscr{C}_c^\infty(X) \subseteq \mathscr{C}_c^\infty(\mathbb{R}^n)$
- Let $u \in \mathscr{D}'(X)$, then

$$u_Y \colon \ \mathscr{C}_c^\infty(Y) \ni \phi \mapsto \langle u, \phi \rangle$$

  is a distribution on Y, called the restriction (localization) of u to Y.

Definition

Let $u \in \mathscr{D}'(X)$, then the support of u is the set of all x such that

there is no open nbhd Y of x s.t. $u_Y = 0$

Its singular support $\operatorname{singsupp} u$ is the set of all x such that

there is no open nbhd Y of x s.t. $u_Y \in \mathscr{C}^\infty$

Compactly supported distributions

Definition (Topology on $\mathscr{C}^\infty(X)$)

A sequence $(\phi_j)_j \subset \mathscr{C}^\infty(X)$ is said to converge to 0 in $\mathscr{C}^\infty(X)$ iff all derivatives $\partial^\alpha \phi_j$ converge to 0 uniformly on any compact set $K \subset X$ as j tends to ∞.

Definition

The topological dual of $\mathscr{C}^\infty(X)$ (with this topology) is denoted $\mathscr{E}'(X)$.

Proposition

$\mathscr{E}'(X)$ is identical with the set of distributions in X with compact support.

Tempered distributions

Schwartz space $=$ the space of rapidly decreasing smooth functions,
$\mathscr{S}(\mathbb{R}^n) = \{\phi \in \mathscr{C}^\infty(\mathbb{R}^n) \mid \|\phi\|_{\alpha,\beta} := \sup|x^\alpha \partial^\beta \phi(x)| < \infty$ for all $\alpha, \beta\}$

Definition (Topology on $\mathscr{S}(\mathbb{R}^n)$)

A sequence of Schwartz functions $(\phi_j)_j \subset \mathscr{S}(\mathbb{R}^n)$ is said to converge to 0 in $\mathscr{S}(\mathbb{R}^n)$ iff $\|\phi_j\|_{\alpha,\beta} \to 0$ as j tends to ∞ for all α, β.

Definition (Tempered distributions)

The space of tempered distributions $\mathscr{S}'(\mathbb{R}^n)$ is the topological dual of $\mathscr{S}(\mathbb{R}^n)$.

Examples: $\mathscr{C}_c^\infty(\mathbb{R}^n)$ is dense in $\mathscr{S}(\mathbb{R}^n)$. $\mathrm{e}^{-x^2} \in \mathscr{S}(\mathbb{R}^n)$.
Restricting $u \in \mathscr{S}'(\mathbb{R}^n)$ to $\mathscr{C}_c^\infty(\mathbb{R}^n)$ gives a distributions. By the denseness above, $\mathscr{S}'(\mathbb{R}^n)$ can be identified with a subspace of $\mathscr{D}'(\mathbb{R}^n)$. Moreover, $\mathscr{E}(\mathbb{R}^n) \subset \mathscr{S}'(\mathbb{R}^n)$.
Any polynomially bounded continuous function f is a tempered distribution with

$$\phi \mapsto \int f(x)\phi(x)dx =: \langle f, \phi \rangle$$

Operations on distributions

Definition (Topology on $\mathscr{D}'(X)$ and $\mathscr{S}'(\mathbb{R}^n)$: pointwise convergence)

A sequence of (tempered) distributions (u_j) converges to 0 as j tends to ∞ iff $\langle u_j, \phi\rangle \to 0$ for all $\phi \in \mathscr{C}_c^\infty(X)$ (or $\mathscr{S}(\mathbb{R}^n)$, resp.)

Corollary (Extending operations by duality)

Let g be a linear continuous map $\mathscr{C}_c^\infty(Y) \to \mathscr{C}_c^\infty(X)$ (or $\mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$), then the transpose g^t defined by

$$\langle g^t(u), \phi\rangle := \langle u, g(\phi)\rangle$$

is a (seq.) continuous map $\mathscr{D}'(Y) \to \mathscr{D}'(X)$ (or $\mathscr{S}'(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n)$).

To extend a contin. operation f on $\mathscr{C}_c^\infty(X)$ to one in $\mathscr{D}'(\mathbb{R}^n)$, calculate $\langle fu, \phi\rangle = \langle u, f^t\phi\rangle$ for $u \in \mathscr{C}_c^\infty(X)$ and set $g = f^t$.

Operations on distributions – Examples

- The derivative $\partial^\alpha : \mathscr{D}'(X) \to \mathscr{D}'(X)$ is defined as the transpose of $(-1)^{|\alpha|}\partial^\alpha : \mathscr{C}_c^\infty(X) \to \mathscr{C}_c^\infty(X)$,

$$\langle \partial^\alpha u, \phi \rangle = \langle u, (-1)^{|\alpha|}\partial^\alpha \phi \rangle$$

For $u \in \mathscr{C}_c^\infty(X)$, this is the usual derivative (by partial integration).

Operations on distributions – Examples

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$$\langle \partial^\alpha u, \phi \rangle = \langle u, (-1)^{|\alpha|}\partial^\alpha \phi \rangle$$

  For $u \in \mathscr{C}_c^\infty(X)$, this is the usual derivative (by partial integration). Similarly, one defines:
- Given a diffeomorphism $f : X \to Y$, the pullback of $u \in \mathscr{D}'(Y)$ along f is

$$\langle f^* u, \phi \rangle = \langle u, g^* \phi \, |\det g'| \rangle \quad \text{with } g = f^{-1}, g^* \phi = \phi \circ g$$

  This way, $f^* u = u \circ f$ for $u \in \mathscr{C}_c^\infty(Y)$ (by the transf. formula).
- Multiplication with $\psi \in \mathscr{C}^\infty(X)$:

$$\langle \psi u, \phi \rangle = \langle u, \psi \phi \rangle$$

  Observe: $\operatorname{supp} \psi u \subset \operatorname{supp} \psi$.
- Analogously for $\mathscr{S}'(\mathbb{R}^n)$ (with $X = Y = \mathbb{R}^n$ and ψ polyn. bdd)

Operations on distributions – Fourier transform

- The Fourier transform

$$\mathscr{F} : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n) \,, \qquad \mathscr{F}\phi(\xi) = \int e^{-ix\xi}\phi(x)dx =: \hat{\phi}(\xi)$$

is a continuous isomorphism, with inverse

$$\mathscr{F}^{-1}\phi(x) = \int \phi(\xi)e^{ix\xi}đ\xi \,, \quad \text{where } đ\xi = (2\pi)^{-n}d\xi$$

Hence, $\mathscr{F} : \mathscr{S}'(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n)$, $u \mapsto \hat{u}$,

$$\langle \hat{u}, \phi \rangle := \langle u, \hat{\phi} \rangle$$

is continuous.

Again, if $u \in \mathscr{S}(\mathbb{R}^n)$, we recover the ordinary Fourier transform.

Properties of the Fourier transform

Proposition

$\mathscr{F} : \mathscr{S}'(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n)$ is an isomorphism.

It holds ($D = -\mathrm{i}\partial$):

- $\widehat{D^\alpha u} = \xi^\alpha \hat{u}$ *and* $\widehat{x^\alpha u} = (-1)^{|\alpha|} D^\alpha \hat{u}$
- *If u has compact support, its Fourier transform is a smooth function, and*

$$\hat{u}(\xi) = \langle u, \mathrm{e}^{-\mathrm{i}\xi\cdot} \rangle = \langle u(x), \mathrm{e}^{-\mathrm{i}\xi x} \rangle \overset{\text{notation}}{=} \int u(x) e^{-\mathrm{i}\xi x} dx$$

Example: For the δ distribution, $\delta_{x_0}(\phi) = \phi(x_0)$, we have

$$\langle \widehat{\delta_{x_0}}, \phi \rangle = \langle \delta_{x_0}, \hat{\phi} \rangle = \hat{\phi}(x_0) = \int \mathrm{e}^{-\mathrm{i}x_0\xi} \phi(\xi) d\xi = \langle \mathrm{e}^{-\mathrm{i}x_0\cdot}, \phi \rangle ,$$

so

$$\widehat{\delta_{x_0}}(\xi) = e^{-\mathrm{i}x_0\xi} = \langle \delta_{x_0}, e^{-\mathrm{i}\xi\cdot} \rangle \overset{\text{notation}}{=} \int \delta_{x_0}(x) e^{-\mathrm{i}\xi x} dx$$

Tensorproduct

Let $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$ be open.

Theorem (tensor product)

Let $u \in \mathscr{D}'(X)$, $v \in \mathscr{D}'(Y)$. Then there is a unique element of $\mathscr{D}'(X \times Y)$, called the tensor product $u \otimes v$, such that

$$\langle u \otimes v, \phi \otimes \psi \rangle = \langle u, \phi \rangle \langle v, \psi \rangle \quad \text{for } \phi \in \mathscr{C}_c^\infty(X)\,,\ \psi \in \mathscr{C}_c^\infty(Y).$$

The proof relies on the fact that $\{\phi \otimes \psi | \phi \in \mathscr{C}_c^\infty(X), \psi \in \mathscr{C}_c^\infty(Y)\}$ is dense in $\mathscr{C}_c^\infty(X \times Y)$.

For $u \in \mathscr{C}_c^\infty(X)$, $v \in \mathscr{C}_c^\infty(Y)$, $u \otimes v$ as above is the tensor product of functions, $u \otimes v(x, y) = u(x)v(y)$.

Convolution

Proposition

Let $u, v \in \mathscr{D}'(\mathbb{R}^n)$, v compactly supported. Then

$$\langle u * v, \phi \rangle := \langle u \otimes v(x,y), \phi(x+y) \rangle$$

defines a distribution. It holds
$\partial^\alpha (u * v) = u * \partial^\alpha v = \partial^\alpha u * v$ *and* $\operatorname{supp} u * v \subset \operatorname{supp} u + \operatorname{supp} v$.

If $v = \rho \in \mathscr{C}_c^\infty(\mathbb{R}^n)$, then $u * \rho \in \mathscr{C}^\infty(\mathbb{R}^n)$ (regularization).
If $u \in \mathscr{C}^\infty(\mathbb{R}^n)$, then $u * \rho$ is the convolution.

Notation: $\langle u * v, \phi \rangle = \int u(x) v(y) \phi(x+y) \mathrm{d}x \mathrm{d}y$.

$u * v$ is still defined by the above for general $u, v \in \mathscr{D}'(\mathbb{R}^n)$ if $(x, y) \mapsto x + y$ is proper on $\operatorname{supp} u \times \operatorname{supp} v$.

$\delta * u = u$ for all $u \in \mathscr{D}'(\mathbb{R}^n)$.

Schwarz kernel theorem

An integral transform $f \mapsto \int_Y K(x,y)f(y)dy$ with kernel K (function on $X \times Y$) maps a suitable class of functions on Y to functions on X. Generalization to distributions:

Theorem

There is an one-to-one correspondence between linear continuous operators $A\colon \mathscr{C}^\infty_\mathrm{c}(Y) \to \mathcal{D}'(X)$ and kernels $K \in \mathscr{D}'(X \times Y)$. For $\phi \in \mathscr{C}^\infty_c(X)$ and $\psi \in \mathscr{C}^\infty_c(Y)$,

$$\langle A\psi, \phi\rangle = \langle K, \phi \otimes \psi\rangle.$$

Proof idea "$\supseteq$" For $K \subset X$, $K' \subset Y$ compact, there are C and N s.t.

$$|\langle K, \phi \otimes \psi\rangle| \leq C \sum_{|\alpha|\leq N} \|\partial^\alpha \phi\|_\infty \sum_{|\beta|\leq N} \|\partial^\beta \psi\|_\infty \quad \text{for all } \phi \in \mathscr{C}^\infty_c(K), \psi \in \mathscr{C}^\infty_c(K')$$

"$\subseteq$" uniqueness: from denseness of $\{\phi \otimes \psi\}$ in $\mathscr{C}^\infty_c(X \times Y)$. Existence: show that estimate above holds for $B(\psi, \phi) := \langle A\psi, \phi\rangle$ and that this suffices for B to extend to a distribution on $X \times Y$.

High frequency cone

- Analysis on T^*X (phase space): in addition to local information (x-space) analyse behaviour w.r.t. covariables (ξ-space).
- Main idea: Fourier transform of a compactly supported smooth function is a Schwartz function; Fourier transform of a compactly supported distribution v is smooth. Hence: if there are directions where $\mathscr{F}(v)$ does not decay quickly, there are singularities.

Definition

Let v be cptly supported distribution. Its high frequency cone $\Sigma(v)$ is the set of all $\eta \neq 0$ that have no conic nbhd V_η such that

$$\text{for all } N = 1, 2, \ldots \ \exists C_N \text{ s.t. } |\hat{v}(\xi)| \leq C_N \langle \xi \rangle^{-N} \text{ for all } \xi \in V_\eta$$

where $\langle \xi \rangle = (1 + |\xi|^2)^{\frac{1}{2}}$ is the Japanese bracket.

$\Sigma(v)$ is a closed cone. Example: $\Sigma(\delta) = \mathbb{R}^n \setminus 0$

Wavefront set

For $u \in \mathscr{D}'(X)$ localize:

For $x \in X$, set $\Sigma_x = \bigcap_\phi \Sigma(\phi u)$ where $\phi \in \mathscr{C}_c^\infty(X)$ with $\phi(x) \neq 0$.

Definition

The wavefront set of $u \in \mathscr{D}'(X)$ is

$$WF(u) = \{(x, \xi) \in \dot{T}^* X = X \times (\mathbb{R}^n \setminus 0) \,|\, \xi \in \Sigma_x(u)\}$$

The projection of $WF(u)$ onto the first variables is the singular support of u.

Previous constructions revisited

- For a linear differential operator $P = \sum_{|\alpha|\leq m} a_\alpha D^\alpha$ be with smooth coefficients a_α on X, we have

$$WF(Pu) \subseteq WF(u)$$

- If the set of normals of a smooth map $f : Y \to X$ has empty intersection with $WF(u)$, the pullback of u along f can be uniquely defined and $WF(f^*u) \subseteq f^*WF(u)$.
- For contin. lin. $A \colon \mathscr{C}_\mathrm{c}^\infty(Y) \to \mathcal{D}'(X)$ and its kernel $K \in \mathscr{D}'(X \times Y)$ from Schwartz' kernel theorem, we have

$$\mathrm{WF}(A\psi) \subset \{(x,\xi)\,|\,(x,y,\xi,0) \in WF(K) \text{ for some } y \in \operatorname{supp}\psi\}$$

Fundamental solutions

Definition

A fundamental solution for a differential operator $P(D) = \sum_{|\alpha| \leq m} a_\alpha D^\alpha$ on $\mathbb{R}^n$ is a distribution $E \in \mathscr{D}'(\mathbb{R}^n)$, s.t. $P(D)E = \delta$.

Why is this interesting? If E exists, the existence of smooth solutions to $P(D)u = f$ with $f \in \mathscr{C}_c^\infty(\mathbb{R}^n)$ is guaranteed (and can be calculated explicitly if E is known explicitly):

$u := E * f$ is a smooth function with

$$P(D)u = (P(D)E) * f = \delta * f = f$$

Example

On $\mathbb{R}^3$, $E(x) = -\frac{1}{4\pi|x|}$ is a fundamental sol'n to Poisson's equation $\Delta u = f$.

Malgrange-Ehrenpreis theorem

Theorem

Every constant coefficient differential operator on $\mathbb{R}^n$ possesses a fundamental solution.

Remarks:

Classic proof idea relies on Hahn-Banach argument, were not constructive.

The theorem fails in general for nonconstant smooth coefficients (Lewy's example: there is an equation $P(D)u = f$ (everything smooth) that does not have a smooth solution).

Structure theorems

Theorem

Every tempered distribution is a derivative of finite order of some continuous function of polynomial growth.

Theorem

If $u \in \mathscr{D}'(X)$ then there are continuous functions f_α, such that

$$u = \sum_{\alpha \geq 0} \partial^\alpha f_\alpha$$

where the sum is locally finite, i.e. on any compact set $K \subset X$, only a finite number of the f_α do not vanish identically.

Theorem

If $\operatorname{supp} u = \{x_0\}$*, then there are $N \in \mathbb{N}_0$ and $c_\alpha \in \mathbb{C}$, s.t.*

$$u = \sum_{|\alpha| \leq N} c_\alpha \, \partial^\alpha \delta_{x_0}$$