Introduction to Microlocal Analysis

Second lecture: Pseudodifferential operators on $\mathbb{R}^n$

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1. Oscillatory integrals

2. Pseudodifferential operators on $\mathbb{R}^n$

Oscillatory integrals – Intro

Let φ and a be smooth functions on $X \times \mathbb{R}^N$, $X \subseteq \mathbb{R}^n$ open. If a is compactly supported in the θ variable, and $\Im\phi \geq 0$, then

$$u(x) = \int e^{i\varphi(x,\theta)} a(x,\theta) d\theta$$

is a function. Goal: Give this expression meaning as a distribution

$$\mathscr{C}_c^\infty(X) \ni \phi \mapsto \int e^{i\varphi(x,\theta)} a(x,\theta)\phi(x) dx d\theta$$

for more general a and suitable φ.

Example

$\int e^{ix\theta} đ\theta = \delta_0(x)$.

Phase functions and symbols

Definition (phase function)

Let $X \subseteq \mathbb{R}^n$ be open, let Γ be an open cone in $X \times \dot{\mathbb{R}}^N$) (i.e. conic w.r.t. the second set of variables). A function $\varphi \in \mathscr{C}^\infty(\Gamma)$ is called a phase function if
a) $\Im\varphi \geq 0$, b) $\varphi(x, \lambda\theta) = \lambda\varphi(x, \theta)$ for all $\lambda > 0$ (homogeneity), and
c) $\varphi'(x, \theta) \neq 0$ (non-degeneracy).

Definition (Symbols)

Let m be real, $\rho \in (0, 1]$ and $\delta \in [0, 1)$. Then

$$S^m_{\rho,\delta}(X \times \mathbb{R}^N) = \{a \in \mathscr{C}^\infty(X \times \mathbb{R}^N) \mid \text{for any } \alpha, \beta \text{ and cpt } K \subset X,\ \exists\, C \text{ s.t. } |\partial_x^\beta \partial_\theta^\alpha a(x, \theta)| \leq C \langle\theta\rangle^{m - \rho|\alpha| + \delta|\beta|}\ \}$$

is called the space of symbols of order m and type ρ, δ. Topology on $S^m_{\rho,\delta}$: given by optimal constants (which give a family of seminorms).

Oscillatory integrals

Theorem

Given a phase function φ on Γ and a closed cone $F \subset \Gamma \cup (X \times \{0\})$, there is a unique way to define $I_\varphi(a) \in \mathscr{D}'(X)$ for all $a \in \bigcup_{m,\rho,\delta} S^m_{\rho,\delta}$ with support in F, such that

a) *If $\int e^{i\varphi(x,\theta)} a(x,\theta) d\theta$ is absolutely convergent, it is equal to $I_\varphi(a)$*

b) *for every m, ρ, δ fixed, the map $S^m_{\rho,\delta} \ni a \mapsto I_\varphi(a) \in \mathscr{D}'(X)$ is continuous and linear.*

$I_\varphi(a)$ is called an oscillatory integral, formally denoted by the integral above.

One possible way to prove this is a partition of unity argument: $\sum_j \psi_j(\theta) = 1$ with compactly supported ψ_j. Set $\langle I_\varphi(a), \phi\rangle = \sum_j \langle I_\varphi(\psi_j a), \phi\rangle$, show that the r.h.s. converges and has the desired properties.

Why these symbol classes? We will look at this question later (slide 8).

WF of $I_\varphi(a)$

Theorem

With definitions as above, one finds

$$WF(I_\varphi(a)) \subset \Lambda_\varphi$$

where

$$\Lambda_\varphi = \{(x, \varphi'_x(x,\theta)) | (x,\theta) \in F \text{ and } \varphi'_\theta(x,\theta) = 0\} \subset X \times \mathbb{R}^n$$

is the manifold of stationary phase (of φ).

Proof idea: estimate $\widehat{\phi I_\varphi(a)}(\xi)$ using again the partition of unity,

$$\sum \int \int e^{i(\varphi(x,\theta) - x\xi)} \, \phi(x) \, \chi_j(\theta) \, a(x,\theta) \, dx d\theta$$

Intuition: when $\varphi'_\theta(x,\theta) = 0$, the oscillations are too slow to control the θ-integration.

Pseudodifferential operators on $\mathbb{R}^n$ – definition

Definition

For $m \in \mathbb{R}$, $\rho \in (0,1]$ and $\delta \in [0,1)$, $S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$ is the set of all $a \in \mathscr{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ s.t. for any α, β $\exists C$ s.t.

$$|\partial_x^\beta \partial_\xi^\alpha a(x,\xi)| \leq C \langle \xi \rangle^{m-\rho|\alpha|+\delta|\beta|}$$

Let $\phi \in \mathscr{S}(\mathbb{R}^n)$, $a \in S^m_{\rho,\delta}$, then

$$(Op(a)\phi)(x) := a(x,D)\phi(x) := \int e^{ix\xi} a(x,\xi)\hat{\phi}(\xi) đ\xi \ .$$

is called the pseudodifferential operator of the symbol a. Denote by $\Psi^m_{\rho,\delta}(\mathbb{R}^n)$ all pseudodiff. op's of symbols $a \in S^m_{\rho,\delta}$.

Modifications... e.g. $\langle x \rangle^\mu$ for fixed μ on the r.h.s. Important special case: $S^m := S^m_{1,0}$.
Example: $S^m \ni a(x,\xi) = \sum_{|\alpha| \leq m} c_\alpha(x)\xi^\alpha$, $c_\alpha \in \mathscr{C}^\infty_b$, and $a(x,D)\phi(x) = \sum c_\alpha(x) D^\alpha_x \phi$

Remarks

Rewrite

$$(Op(a)\phi)(x) = \int e^{ix\xi} a(x,\xi)\hat{\phi}(\xi)đ\xi$$

in terms of its Schwartz kernel:

$$(Op(a)\phi)(x) = \int K(x, x-y)\phi(y)dy\,, \quad K(x, x-y) = \int e^{i(x-y)\xi} a(x,\xi)đ\xi\,.$$

Observe:

As long as $\rho > 0$, K is $\mathscr{C}^\infty$ away from the diagonal $x = y$ and rapidly decreasing for $|x-y| \to \infty$. Reason:

$$D_x^\beta D_z^\gamma z^\alpha K(x,z) = \int \underbrace{D_x^\beta D_\xi^\alpha a(x,\xi)\ \xi^\gamma}_{\in S^{m-\rho|\alpha|+\delta|\beta|+\gamma|}} e^{iz\xi}đ\xi$$

so, given β and γ, the integrand is integrable for α big enough.

We specialize to $\rho = 1$, $\delta = 0$

Definition (Sobolev space)

Let s be real. The L^2-based Sobolev space H^s is the space of all $u \in \mathscr{S}'(\mathbb{R}^n)$, s.t. $\hat{u}$ is a function and $\langle\xi\rangle^s \hat{u} \in L^2$. H^s is endowed with the norm $\|u\|_{H^s}^2 = \int \langle\xi\rangle^{2s} |\hat{u}(\xi)|^2 \, đ\xi$.

Example: $\delta \in H^s(\mathbb{R}^n)$ for $s < -n/2$.

Observe (Sobolev embedding theorem): $u \in H^s(\mathbb{R}^n)$ for $s > n/2$ then u is continuous.

Theorem

Let $A \in \Psi^m_{1,0}(\mathbb{R}^n)$.

- $A : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$ *is continuous.*
- $A : H^s(\mathbb{R}^n) \to H^{s-m}(\mathbb{R}^n)$ *is bounded.*

Asymptotic expansion

Definition

$a \in S^m = S^m_{1,0}$ has the asymptotic expansion $a \sim \sum_{j=0}^{\infty} a_{(m-j)}$ in S^m if the $a_{(m-j)}$ are in S^{m-j}, and for all M, we have

$$a - \sum_{j<M} a_{(m-j)} \in S^{m-M} \, .$$

a_m is called the principal symbol. A symbol a is called classical if $a \sim \sum_j a_{(m-j)}$ where each $a_{(m-j)}$ is positively homogeneous of degree $m-j$ in ξ for $|\xi| \geq 1$.

Example: $a(x,\xi) = \langle \xi \rangle = (1+|\xi|^2)^{\frac{1}{2}} \in S^1$. For $|\xi| \geq 1$, convergent series representation $\langle \xi \rangle = |\xi|(1 + \frac{1}{2}|\xi|^{-2} - \frac{1}{8}|\xi|^{-4} + \ldots)$ Asymptoptic expansion: $\langle \xi \rangle \sim \chi(\xi)|\xi| + \frac{1}{2}\chi(\xi)|\xi|^{-1} - \frac{1}{8}\chi(\theta)|\xi|^{-3} + \ldots$ where $\chi \in \mathscr{C}^\infty$ smoothly cuts out the singularity in 0

$$\chi(\xi) = \begin{cases} 0 & \text{for } |\xi| \leq \frac{1}{2} \\ 1 & \text{for } |\xi| \geq 1 \end{cases}$$

Reconstruction

Often, we only have asymptotic formulas (see below). However, by the following theorem, they are essentially as good as the real thing:

Lemma (“reconstruction”)

Let $m \in \mathbb{R}$. Given $a_{(m-j)} \in S^{m-j}$ for $j = 0, 1, 2, \ldots$ there is a symbol in $a \in S^m$ such that $a \sim \sum_j a_{(m-j)}$ in S^m

Proposition (application of the lemma)

Let $a \in S^m$, $b \in S^k$. Then there is $p =: a\#b \in S^{m+k}$ such that $Op(a) \circ Op(b) = Op(p)$. It has the asymptotic expansion

$$p(x,\xi) \sim \sum_\alpha \frac{1}{\alpha!} \partial_\xi^\alpha a(x,\xi) D_x^\alpha b(x,\xi)$$

Moreover, $\# : S^m \times S^k \to S^{m+k}$ is continuous.

Similarly, the asymptotic expansion for the formal adjoint of a symbol is known.

Parametrices

Another important application: Existence of “inverses” for $Op(a)$:

Theorem

Let $a \in S^m$ be elliptic of order m i.e. there is $R \geq 0$ and C s.t. $a(x,\xi)$ is invertible for all $|\xi| \geq R, x \in \mathbb{R}^n$ and $|a(x,\xi)^{-1}| \leq C|\xi|^{-m}$.
Then there is a symbol $b \in S^{-m}$ (called parametrix for a) s.t.

$$a\#b - 1 =: r_1 \qquad \text{and} \qquad b\#a - 1 =: r_2$$

with $r_j \in S^{-\infty}$.

Proof idea: iteratively construct a suitable asymptotic series.

Elliptic regularity

Theorem

Let $P \in \Psi^m_{\rho,\delta}$, $\rho \in (0,1]$ and $\delta \in [0,1)$, then for any $u \in \mathscr{D}'(\mathbb{R}^n)$,

$$WF(Pu) \subseteq WF(u) \subseteq WF(Pu) \cup \operatorname{char} P\,,$$

where $\operatorname{char} P$ is the set of all $(x,\xi) \in \mathbb{R}^n \times \dot{\mathbb{R}}^n$ where the principal symbol cannot be inverted with inverse in $S^{-m}_{\rho,\delta}$.

The first inclusion was discussed in the first lecture for differential operators.

Corollary

If P as above is elliptic, then

$$WF(u) = WF(Pu)$$