THE DISCRIMINANT OF A CUBIC SURFACE

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Abstract. We construct explicit examples of cubic surfaces over $\mathbb{Q}$ such that the 27 lines are acted upon by the index two subgroup of the maximal possible Galois group. This is the simple group of order 25 920. Our examples are given in pentahedral normal form with rational coefficients. For such cubic surfaces, we study the discriminant and show its relation to the index two subgroup. On the corresponding parameter space, we search for rational points, discuss their asymptotic, and construct an accumulating subvariety.

1. Introduction

1.1. Let $\mathscr{S} \subset \mathbf{P}^3$ be a smooth cubic surface over an algebraically closed field. It is well known that there are exactly 27 lines on $\mathscr{S}$. The intersection matrix of these lines is essentially the same for every smooth cubic surface. The group of all permutations of the 27 lines which respect the intersection matrix is isomorphic to the Weyl group $W(E_6)$.

For a smooth cubic surface $S \subset \mathbf{P}^3$ over $\mathbb{Q}$, the 27 lines are, in general, not defined over $\mathbb{Q}$ but over an algebraic field extension L. The Galois group $\mathrm{Gal}(L/\mathbb{Q})$ is a subgroup of $W(E_6)$. It is known that equality holds for general cubic surfaces while for diagonal cubic surfaces the Galois group is significantly smaller. It may be of order 54 at most.

1.2. In this article, we describe our search for explicit examples of cubic surfaces over $\mathbb{Q}$ such that the Galois group $\mathrm{Gal}(L/\mathbb{Q})$ is exactly the index two subgroup $D^1W(E_6) \subset W(E_6)$. This is the simple group of order 25 920.

Our approach is as follows. We consider cubic surfaces in pentahedral normal form with rational coefficients. For these, we study the discriminant Δ. We show that $\mathrm{Gal}(L/\mathbb{Q})$ is contained in the index two subgroup if and only if $(-3)\Delta$ is a perfect square. This leads to a point search on the double covering of $\mathbf{P}^4$ ramified at the degree 32 discriminantal variety.

A generalized Cremona transform reduces the degree to eight. We discuss the asymptotic of the $\mathbb{Q}$-rational points of bounded height on the resulting double covering and construct an accumulating subvariety. A final section is devoted to the problem to which extent this subvariety is unique.

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2. The discriminant and the index two subgroup

2.1. One way to write down a cubic surface in explicit form is the so-called *pentahedral normal form.* Denote by $S^{(a_0,\ldots,a_4)}$ the cubic surface given in $\mathbf{P}^4$ by the system of equations

$$\begin{aligned} a_0X_0^3 + a_1X_1^3 + a_2X_2^3 + a_3X_3^3 + a_4X_4^3 &= 0\,, \\ X_0 + X_1 + X_2 + X_3 + X_4 &= 0\,. \end{aligned}$$

Remarks 2.2. a) A general cubic surface over an algebraically closed field may be brought into pentahedral normal form over that field. Further, the coefficients are unique up to permutation and scaling. This is a classical result which was first observed by J. J. Sylvester [11]. A proof is given in [2].

Cubic surfaces in pentahedral normal form with rational coefficients are, however, special to a certain extent.

b) One should keep in mind that $S^{(0,a_1,\ldots,a_4)}$ is simply the diagonal cubic surface with coefficients $a_1, \ldots, a_4$.

Definition 2.3. The expression

$$\begin{aligned} &\Delta(S^{(a_0,\ldots,a_4)}) := \\ &a_0^8 \cdot \ldots \cdot a_4^8 \cdot \\ &\prod_{i_1,i_2,i_3,i_4 \in [0,1]} \Big(\frac{1}{\sqrt{a_0}} + (-1)^{i_1} \frac{1}{\sqrt{a_1}} + (-1)^{i_2} \frac{1}{\sqrt{a_2}} + (-1)^{i_3} \frac{1}{\sqrt{a_3}} + (-1)^{i_4} \frac{1}{\sqrt{a_4}} \Big) \end{aligned}$$

is called the *discriminant* of the cubic surface $S^{(a_0,\ldots,a_4)}$. Instead of $\Delta(S^{(a_0,\ldots,a_4)})$, we will usually write $\Delta(a_0, \ldots, a_4)$.

Remark 2.4. One has

$$\begin{aligned} &\Delta(a_0, \ldots, a_4) := \\ &\prod_{i_1,i_2,i_3,i_4 \in [0,1]} (\sqrt{a_1a_2a_3a_4} + (-1)^{i_1}\sqrt{a_0a_2a_3a_4} + (-1)^{i_2}\sqrt{a_0a_1a_3a_4} + \ldots \\ &\qquad\qquad\qquad \ldots + (-1)^{i_3}\sqrt{a_0a_1a_2a_4} + (-1)^{i_4}\sqrt{a_0a_1a_2a_3})\,. \end{aligned}$$

Lemma 2.5. $\Delta \in \mathbb{Q}[a_0, \ldots, a_4]$ *is a symmetric polynomial, homogeneous of degree* 32*, and absolutely irreducible.*

Proof. The remark shows $\Delta \in \mathbb{Q}[\sqrt{a_0}, \ldots, \sqrt{a_4}]$. Further, the expression is obviously invariant under the action of $G := \mathrm{Gal}(\mathbb{Q}(\sqrt{a_0}, \ldots, \sqrt{a_4})/\mathbb{Q}(a_0, \ldots, a_4))$. This yields $\Delta \in \mathbb{Q}[a_0, \ldots, a_4]$. Symmetry and homogeneity are obvious.

Definition 2.3 provides us with the decomposition of Δ into irreducible factors in the unique factorization domain $\overline{\mathbb{Q}}[\sqrt{a_0}, \ldots, \sqrt{a_4}, \frac{1}{a_0}, \ldots, \frac{1}{a_4}]$. Since G operates transitively on the sixteen factors, we see that Δ is irreducible in $\overline{\mathbb{Q}}[a_0, \ldots, a_4, \frac{1}{a_0}, \ldots, \frac{1}{a_4}]$.

It remains to exclude the possibility that Δ might be divisible by a polynomial which is a unit in $\overline{\mathbb{Q}}[a_0, \ldots, a_4, \frac{1}{a_0}, \ldots, \frac{1}{a_4}]$. I.e., by a non-trivial monomial. For this, note that the formula given in Remark 2.4 immediately shows $\Delta(0, a_1, a_2, a_3, a_4) = (a_1a_2a_3a_4)^8$. Δ is not divisible by a_0. □

Lemma 2.6. *Writing* σ_i*, for the elementary symmetric function of degree* i *in* $a_0, \ldots, a_4$*, one may express the discriminant as follows,*

$$\Delta = (A^2 - 64B)^2 - 2^{11}(8D + AC)\,.$$

Here,

$$A := \sigma_4^2 - 4\sigma_3\sigma_5, \quad B := \sigma_1\sigma_5^3, \quad C := \sigma_4\sigma_5^4, \quad D := \sigma_2\sigma_5^6 \, .$$

Proof. This formula may easily be established, for example, using `maple`. □

Remarks 2.7. i) Together with $E := \sigma_5^8$, the expressions A, B, C, and D are called the *fundamental invariants* of the cubic surface $S^{(a_0,\ldots,a_4)}$. This notion is due to A. Clebsch [2].

ii) Lemma 2.6 is originally due to G. Salmon [10]. Note that there is a misprint in Salmon's original work which has been repeatedly copied by several people throughout the 20th century. The correct formula may be found in [4].

Fact 2.8. *Assume that* $a_0 \cdot \ldots \cdot a_4 \neq 0$. *Then, the singular points on* $S^{(a_0,\ldots,a_4)}$ *are exactly those of the form*

$$\Big(\frac{1}{\sqrt{a_0}} : (-1)^{i_1}\frac{1}{\sqrt{a_1}} : (-1)^{i_2}\frac{1}{\sqrt{a_2}} : (-1)^{i_3}\frac{1}{\sqrt{a_3}} : (-1)^{i_4}\frac{1}{\sqrt{a_4}}\Big)$$

which lie on the hyperplane given by $X_0 + X_1 + X_2 + X_3 + X_4 = 0$.

Proof. A $\overline{\mathbb{Q}}$-valued point $(x_0 : \ldots : x_4)$ on $S^{(a_0,\ldots,a_4)}$ is singular if and only if the Jacobian matrix

$$\begin{pmatrix} 3a_0x_0^2 & 3a_1x_1^2 & 3a_2x_2^2 & 3a_3x_3^2 & 3a_4x_4^2 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

is not of maximal rank. This yields the form of the point claimed. Observe that $(\frac{1}{\sqrt{a_0}} : (-1)^{i_1}\frac{1}{\sqrt{a_1}} : (-1)^{i_2}\frac{1}{\sqrt{a_2}} : (-1)^{i_3}\frac{1}{\sqrt{a_3}} : (-1)^{i_4}\frac{1}{\sqrt{a_4}}) \in S(\overline{\mathbb{Q}})$ if and only if the sum of the coordinates is zero. □

Examples 2.9. i) The cubic surface $S^{(1,1,1,1,\frac{1}{4})}$ has exactly four singular points. These are $(1 : -1 : -1 : -1 : 2)$ and permutations of the first four coordinates. This is the famous Cayley cubic.

ii) The cubic surface $S^{(1,1,1,\frac{1}{9},\frac{1}{16})}$ has exactly three singular points, namely $(1 : -1 : -1 : -3 : 4)$ and permutations of the first three coordinates.

iii) The cubic surface $S^{(1,1,\frac{1}{4},\frac{1}{9},\frac{1}{25})}$ has exactly two singular points. These are $(1 : -1 : -2 : -3 : 5)$ and permutations of the first two coordinates.

iv) $(-1 : -1 : -1 : -1 : 4)$ is the only singular point of the cubic surface $S^{(1,1,1,1,\frac{1}{16})}$.

Corollary 2.10. *The cubic surface* $S^{(a_0,\ldots,a_4)}$ *is non-singular if and only if* $\Delta(a_0, \ldots, a_4) \neq 0$.

Proof. We have that $\Delta(0, a_1, a_2, a_3, a_4) = (a_1a_2a_3a_4)^8$. Correspondingly, a diagonal cubic surface is singular if and only if one of its four coefficients vanishes. In the case that $a_0 \cdot \ldots \cdot a_4 \neq 0$, the assertion follows from Fact 2.8. □

Remark 2.11. The same is true over any ground field of characteristic $\neq 3$. Therefore, with the possible exception of the prime 3, for $a_0, \ldots, a_4 \in \mathbb{Z}$ such that $\gcd(a_0, \ldots, a_4) = 1$, the prime divisors of $\Delta(a_0, \ldots, a_4)$ are exactly the primes where $S^{(a_0,\ldots,a_4)}$ has bad reduction.

One might want to renormalize Δ in order to overcome the defect at the prime 3. For this, observe that $S^{(\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3})}$ has only integral coefficients after the substitution $x_4 := -x_0 - \ldots - x_3$. It turns out that this surface has good reduction at 3. Since $\Delta(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}) = -5 \cdot 3^{-27}$, actually $\pm 3^{27}\Delta(a_0, \ldots, a_4)$ could have the property desired. Theorem 2.12 below indicates that the minus sign should be correct.

Theorem 2.12. *Let $a_0, \dots, a_4 \in \mathbb{Q}$ such that $\Delta(a_0, \dots, a_4) \neq 0$. Then, the Galois group operating on the 27 lines on $S^{(a_0,\dots,a_4)}$ is contained in the index two subgroup $D^1W(E_6) \subset W(E_6)$ if and only if $(-3)\Delta(a_0, \dots, a_4) \in \mathbb{Q}$ is a perfect square.*

Proof. *First step.* Construction of a ramified covering of degree two of $\mathbf{P}^4$.

Define $\mathscr{C} \subset \mathbf{P}^4_{(X)} \times \mathbf{P}^4_{(x)}$ by the system of equations

$$\begin{aligned} x_0X_0^3 + x_1X_1^3 + x_2X_2^3 + x_3X_3^3 + x_4X_4^3 &= 0\,, \\ X_0 + \quad X_1 + \quad X_2 + \quad X_3 + \quad X_4 &= 0\,. \end{aligned}$$

The projection $\pi\colon \mathscr{C} \to \mathbf{P}^4 (= \mathbf{P}^4_{(x)})$ is the family of the cubic surfaces in pentahedral normal form. The fiber of π over $(x_0 : \ldots : x_4)$ is the cubic surface $S^{(x_0,\dots,x_4)}$.

The fiber $\mathscr{C}_\eta$ over the generic point $\eta \in \mathbf{P}^4$ is a smooth cubic surface over $\mathbb{Q}(\eta) = \mathbb{Q}(x_1/x_0, x_2/x_0, x_3/x_0, x_4/x_0)$. Its 27 lines are defined over a finite extension L of $\mathbb{Q}(\eta)$. We claim that $\mathrm{Gal}(L/\mathbb{Q}(\eta)) = W(E_6)$.

Indeed, this is the maximal possible group. The inclusion "$\subseteq$" is, therefore, trivially fulfilled. On the other hand, according to a result of B. L. van der Waerden, the generic Galois group $\mathrm{Gal}(L/\mathbb{Q}(\eta))$ can not be smaller than that for a particular fiber. Specializing, for example, to $(x_0 : x_1 : x_2 : x_3 : x_4) = (1 : 2 : 3 : 7 : 17)$, [5, Algorithm 10] shows that the Galois group is equal to $W(E_6)$.

Consequently, there exists a unique intermediate field K of $L/\mathbb{Q}(\eta)$ which is quadratic over $\mathbb{Q}(\eta)$. This induces a scheme V together with a finite morphism $p\colon V \to \mathbf{P}^4$ of degree two.

In fact, this is a standard construction. For each affine open set $\mathrm{Spec}\, A = U \subseteq \mathbf{P}^4$, take the spectrum of the integral closure of A in the extension K. Note that A is integrally closed in $\mathbb{Q}(\eta)$ since $\mathbf{P}^4$ is a normal scheme. The morphism $p\colon V \to \mathbf{P}^4$ is finite according to the finiteness of the integral closure.

Second step. $p\colon V \to \mathbf{P}^4$ is unramified outside the divisor R given by "$\Delta = 0$".

For this, let us describe the double covering V more precisely. We have $\mathbf{P}^3_{\mathbb{Q}(\eta)} \subset \mathbf{P}^4_{\mathbb{Q}(\eta)}$ given by the equation $X_0 + \ldots + X_4 = 0$ and a smooth cubic surface

$$\mathscr{C}_\eta \subset \mathbf{P}^3_{\mathbb{Q}(\eta)}\,.$$

On $\mathscr{C}_\eta$, there are the 45 tritangent planes. These give rise to a subscheme of $(\mathbf{P}^3)^\vee_{\mathbb{Q}(\eta)}$ which is finite of length 45 and étale over $\mathbb{Q}(\eta)$.

This, according to Galois theory, induces a set $M = \{e_1, \dots, e_{45}\}$ of 45 elements together with an operation of $\mathrm{Gal}(\overline{Q(\eta)}/Q(\eta))$. Actually, only a finite quotient isomorphic to $W(E_6)$ is operating. The set M, in turn, gives rise to the two element set $\{\pm e_1 \wedge \ldots \wedge e_{45}\}$ which is again acted upon by $\mathrm{Gal}(\overline{Q(\eta)}/Q(\eta))$. The fixgroup of this operation corresponds to the quadratic field extension $K/\mathbb{Q}(\eta)$.

The same may be done in the relative situation over $\mathbf{P}^4 \backslash R$. The 45 tritangent planes yield a closed subscheme of $(\mathbf{P}^3)^\vee \times \mathbf{P}^4$ which is finite and étale of degree 45 over $\mathbf{P}^4 \backslash R$. According to A. Grothendieck's theory of the étale fundamental group [7], this induces a set $M = \{e_1, \dots, e_{45}\}$ of 45 elements together with an operation of $\pi_1^{\text{ét}}(\mathbf{P}^4 \backslash R, *)$. This group is canonically a quotient of $\mathrm{Gal}(\overline{Q(\eta)}/Q(\eta))$. Again, we get a canonical operation on the two element set $\{\pm e_1 \wedge \ldots \wedge e_{45}\}$. Corresponding to this, there is an étale covering $p'\colon V' \to \mathbf{P}^4 \backslash R$ of degree two.

V' is, by construction, a normal scheme with function field K. In particular, over an affine open set $\operatorname{Spec} A = U \subseteq \mathbf{P}^4 \backslash R$, we have the spectrum of the integral closure of A in the extension K. This shows that V and V' coincide over $\mathbf{P}^4 \backslash R$.

Third step. The equation.

As R is irreducible, the ramification locus of $p\colon V \to \mathbf{P}^4$ might be either empty or equal to R. If the ramification locus were empty then, as $\pi_1^{\text{ét}}(\mathbf{P}^4, *) = 0$, we had a trivial covering by a non-connected scheme. However, V is connected by construction. The generic fiber of p is a scheme consisting of a single point.

Hence, p is ramified exactly at R. This implies that V is given by the equation $w^2 = \lambda\Delta$ for a suitable constant λ.

Fourth step. Specialization.

Let $(a_0 : \ldots : a_4) \in \mathbf{P}^4(\mathbb{Q})$ such that $\Delta(a_0, \ldots, a_4) \neq 0$. Then, by virtue of the construction above, we have the following statement.

Denote by l the field of definition of the 27 lines on $S^{(a_0,\ldots,a_4)}$. Then, the smallest intermediate field k of $l/\mathbb{Q}$ such that $\operatorname{Gal}(l/k)$ acts on the 45 tritangent planes on $S^{(a_0,\ldots,a_4)}$ only via even permutations is exactly $k = \mathbb{Q}(\sqrt{\lambda\Delta(a_0, \ldots, a_4)})$.

This extension splits if and only if $\lambda\Delta(a_0, \ldots, a_4)$ is a perfect square in $\mathbb{Q}$. Except for the determination of the constant λ, this proves the assertion.

Fifth step. The constant λ.

We consider the particular cubic surface $S^{(0,1,1,1,1)}$. I.e., the diagonal cubic surface given by $x_1^3 + x_2^3 + x_3^3 + x_4^3 = 0$.

Here, the 27 lines are defined over the field $\mathbb{Q}(\zeta_3) = \mathbb{Q}(\sqrt{-3})$. They may be given explicitly in the form

$$x_i + \zeta_3^m x_j = 0, \qquad x_k + \zeta_3^n x_l = 0$$

for $\{i, j, k, l\} = \{1, 2, 3, 4\}$ and $m, n \in \{0, 1, 2\}$. Exactly three of these lines are defined over $\mathbb{Q}$.

They form a triangle which is cut out by the equation $x_1 + x_2 + x_3 + x_4 = 0$. There are six more tritangent planes which consist of a rational line and two lines conjugate to each other. These are given by $x_i + x_j = 0$ for $\{i, j\} \subset \{1, 2, 3, 4\}$ any subset of size two. To summarize, $\operatorname{Gal}(\mathbb{Q}(\sqrt{-3})/\mathbb{Q})$ operates on the 45 tritangent planes as a product of 19 two-cycles while seven tritangent planes are fixed. This is an odd permutation.

Consequently, in this case, $k = \mathbb{Q}(\sqrt{-3})$ is the smallest field such that $\operatorname{Gal}(l/k)$ acts on the 45 tritangent planes only by even permutations. As $\Delta(0, 1, 1, 1, 1) = 1$, this shows $\lambda = -3$ up to a factor which is a perfect square. The proof is complete. □

Remark 2.13. This result was essentially known to H. Burkhardt [1, p. 341] in 1893. Burkhardt gives credit to C. Jordan [8] who was the first to study the automorphism group of the configuration of the 27 lines on a cubic surface.

3. Rational points on the discriminantal covering

Definition 3.1. We will call the twofold covering of $\mathbf{P}^4_{\mathbb{Q}}$ given by the equation

$$w^2 = -3\Delta(a_0, \ldots, a_4) \tag{3.1}$$

the *discriminantal covering.*

3.2. There are two surprising constraints which equation (3.1) imposes on the coefficients $a_0, \dots, a_4$.

Proposition 3.3 (The two constraints)**.** — *Suppose* $a_0, \dots, a_4 \in \mathbb{Z}$ *are such that* $\gcd(a_0, \dots, a_4) = 1$ *and* $(-3)\Delta(a_0, \dots, a_4) \neq 0$ *is a perfect square in* $\mathbb{Q}$.

a) *Then,* $a_0, \dots, a_4$ *all have the same sign.*

b) *Further, for every prime number* $p \equiv 2 \pmod 3$*, all the* p*-adic valuations* $\nu_p(a_0), \dots, \nu_p(a_4)$ *are even.*

Proof. Observe first that the assumption ensures $a_0, \dots, a_4 \neq 0$. Indeed, $\Delta(0, a_1, \dots, a_4) = (a_1a_2a_3a_4)^8 \geq 0$.

a) Assume the contrary. Then, there is a product of four, say $a_1 \cdot \ldots \cdot a_4$, which is negative. The formula given in Remark 2.4 implies that $\Delta(a_0, \dots, a_4)$ is the norm of an element of $\mathbb{Q}(\sqrt{a_1 \cdot \ldots \cdot a_4})$. As this is an imaginary quadratic field, we see that $\Delta(a_0, \dots, a_4) \geq 0$. Contradiction!

b) Again, assume the contrary. Then, there is a product of four, say $a_1 \cdot \ldots \cdot a_4$, the p-adic valuation of which is odd. We have the fact that $\Delta(a_0, \dots, a_4)$ is the norm of an element of $\mathbb{Q}(\sqrt{a_1 \cdot \ldots \cdot a_4})$. On the other hand, $(-3)\Delta(a_0, \dots, a_4)$, being perfect square by assumption, is a norm, too. Consequently, (-3) is the norm of an element of $\mathbb{Q}(\sqrt{a_1 \cdot \ldots \cdot a_4})$.

Since $\nu_p(a_1 \cdot \ldots \cdot a_4)$ is odd, the norm equation $(-3) = x^2 - a_1 \cdot \ldots \cdot a_4 \cdot y^2$ ensures that $\nu_p(x) = 0$ and $\nu_p(a_1 \cdot \ldots \cdot a_4 \cdot y^2) > 0$. Therefore, (-3) is a quadratic residue modulo p. This is a contradiction. □

3.4. We are interested in smooth cubic surfaces $S^{(a_0,\dots,a_4)}$ such that the Galois group operating on the 27 lines is exactly equal to $D^1W(E_6)$.

By Theorem 2.12, this implies that $(a_0 : \ldots : a_4) \in \mathbf{P}^4(\mathbb{Q})$ gives rise to a $\mathbb{Q}$-rational point on the discriminantal covering. Further, according to Corollary 2.10, $(a_0 : \ldots : a_4)$ is supposed not to lie on the ramification locus.

Finally, if two of the coefficients were the same, say $a_0 = a_1$, then $S^{(a_0,\dots,a_4)}$ allowed the tritangent plane "$x_0 + x_1 = 0$" which was defined over $\mathbb{Q}$. Consequently, the order of the group acting on the lines could not be higher than 1152.

3.5. A naive search. For these reasons, we searched for $\mathbb{Q}$-rational points $(w; a_0 : \ldots : a_4)$ satisfying equation (3.1) and the extra conditions below,

i) $w \neq 0$.

ii) No two of the five coordinates $a_0, \dots, a_4$ are the same.

A rather simple computation led to the $\mathbb{Q}$-rational points $(3 : 4 : 21 : 36 : 63)$, $(4 : 7 : 12 : 28 : 84)$, and $(12 : 28 : 36 : 63 : 84)$. Up to symmetry, these are the only solutions of height ≤ 100.

Remark 3.6. The three rational points given above really lead to cubic surfaces such that the 27 lines are acted upon by the simple group $D^1W(E_6)$. To prove this, we ran the algorithm below which is an obvious modification of [5, Algorithm 10].

Algorithm 3.7 (Verifying $G \supseteq D^1W(E_6)$)**.** — Given the equation $f = 0$ of a smooth cubic surface, this algorithm verifies that $G \subseteq W(E_6)$ is of index at most two.

i) Compute a univariate polynomial $0 \neq g \in \mathbb{Z}[d]$ of minimal degree such that

$$g \in (f(\ell(0)), f(\ell(\infty)), f(\ell(1)), f(\ell(-1))) \subset \mathbb{Q}[a, b, c, d]$$

where $\ell\colon t \mapsto (1 : t : (a+bt) : (c+dt))$.

If g is not of degree 27 then terminate with an error message. In this case, the coordinate system is not sufficiently general.

ii) Factor g modulo all primes below a given limit. Ignore the primes dividing the leading coefficient of g.

iii) If one of the factors is multiple then go to the next prime immediately. Otherwise, check whether the decomposition type is $(1,1,5,5,5,5,5)$ or $(9,9,9)$.

iv) If each of the two cases occurred at least once then output the message "The Galois group contains $D^1W(E_6)$." and terminate.

Otherwise, output "Can not prove that the Galois group contains $D^1W(E_6)$."

4. The generalized Cremona transform

4.1. Δ is a homogeneous form of degree 32. Naively, one would expect that there are not many solutions of the equation

$$w^2 = -3\Delta(a_0, a_1, a_2, a_3, a_4)\,.$$

The constraints proven above reduce expectations even more. Nevertheless, three rational points of height ≤ 100 have been found. The reason for this is the following observation.

Fact 4.2. *There is form Δ' homogeneous of degree 8 such that*

$$\Delta(a_0, \ldots, a_4) = (a_0 \cdot \ldots \cdot a_4)^8 \cdot \Delta'(1/a_0, \ldots, 1/a_4)\,.$$

Proof. The octic Δ' is given by the formula

$$\Delta'(x_0, \ldots, x_4) := \prod_{i_1,i_2,i_3,i_4 \in [0,1]} \left(\sqrt{x_0} + (-1)^{i_1}\sqrt{x_1} + (-1)^{i_2}\sqrt{x_2} + (-1)^{i_3}\sqrt{x_3} + (-1)^{i_4}\sqrt{x_4}\right). \qquad \square$$

Definition 4.3. We will call the birational automorphism ι of $\mathbf{P}^4$ given by

$$(a_0 : \ldots : a_4) \mapsto (1/a_0 : \ldots : 1/a_4)$$

a *generalized Cremona transform.* Note that the standard Cremona transform of $\mathbf{P}^2$ is given by $(a_0 : a_1 : a_2) \mapsto (1/a_0 : 1/a_1 : 1/a_2)$.

The generalized Cremona transform ι provides a bijection of

$$\{(x_0 : \ldots : x_4) \in \mathbf{P}^4(\mathbb{Q}) \mid x_0 \cdot \ldots \cdot x_4 \neq 0\}$$

to itself.

Corollary 4.4. $(x_0 : \ldots : x_4) \in \mathbf{P}^4(\mathbb{Q})$, $x_0 \cdot \ldots \cdot x_4 \neq 0$, *gives rise to a solution of*

$$w^2 = (-3)\Delta'(x_0, \ldots, x_4)$$

if and only if $\iota\big((x_0 : \ldots : x_4)\big)$ *yields a rational point on the discriminantal covering.*

$\square$

Lemma 4.5. a) $\Delta' \in \mathbb{Q}[x_0, \ldots, x_4]$ *is a symmetric polynomial, homogeneous of degree eight and absolutely irreducible.*

b) *One has* $\Delta'(0, x_1, \ldots, x_4) = D^2$ *for a symmetric, homogeneous quartic form* $D \in \mathbb{Q}[x_1, \ldots, x_4]$.

Proof. a) By definition, $\Delta' \in \mathbb{Q}[\sqrt{x_0}, \ldots, \sqrt{x_4}]$. Further, the expression for Δ' is obviously invariant under the action of $G := \mathrm{Gal}(\mathbb{Q}(\sqrt{x_0}, \ldots, \sqrt{x_4})/\mathbb{Q}(x_0, \ldots, x_4))$. This yields $\Delta \in \mathbb{Q}[x_0, \ldots, x_4]$. Symmetry and homogeneity are obvious.

Finally, we have a decomposition of Δ' into irreducible factors in the unique factorization domain $\overline{\mathbb{Q}}[\sqrt{x_0}, \ldots, \sqrt{x_4}]$. Since G operates transitively on the sixteen factors, Δ is absolutely irreducible.

b) $\Delta'(0, x_1, \ldots, x_4)$ is the square of

$$D(x_1, \ldots, x_4) := \prod_{i_2, i_3, i_4 \in [0,1]} \left(\sqrt{x_1} + (-1)^{i_2}\sqrt{x_2} + (-1)^{i_3}\sqrt{x_3} + (-1)^{i_4}\sqrt{x_4}\right). \qquad \square$$

Remarks 4.6. i) The ramification locus $R :=$ "$\Delta' = 0$" is a rational threefold. The parametrization $\iota\colon \mathbf{P}^3 \to R$ given by

$$\iota\colon (t_0 : \ldots : t_3) \mapsto \left(t_0^2 : t_1^2 : t_2^2 : t_3^2 : (t_0 + \ldots + t_3)^2\right)$$

is a finite birational morphism.

ii) The equation $D = 0$ defines the Roman surface of J. Steiner.

5. More rational points on the discriminantal covering

5.1. A point search.

5.1.1. On the double covering $\pi\colon O \to \mathbf{P}^4_{\mathbb{Q}}$, given by

$$w^2 = (-3)\Delta'(x_0, \ldots, x_4),$$

we searched for rational points such that

i) $w \neq 0$,

ii) the five coordinates $x_0, \ldots, x_4$ are pairwise different from each other.

5.1.2. Surprisingly many solutions have been found. It turned out that there are 4 900 907 essentially different solutions up to a height limit of 3000. Under symmetry, they give rise to 120 solutions each. The smallest ones are $(1 : 3 : 7 : 9 : 12)$, $(1 : 3 : 4 : 7 : 13)$, $(1 : 3 : 7 : 12 : 13)$, and $(3 : 7 : 9 : 12 : 13)$. For a few height limits, we indicate the number of solutions up to that limit in the table below.

Table 1. Numbers of solutions up to various height limits

limit	#	limit	#	limit	#	limit	#
25	20	200	10 039	500	93 680	1500	1 111 303
50	209	300	25 778	750	236 403	2000	2 088 752
100	1 481	400	54 331	1000	460 330	3000	4 900 907

Remark 5.1.3. We used the constraints shown above to optimize the searching algorithm. On one hand, it is sufficient to search for solutions such that $0 < x_0 < x_1 < x_2 < x_3 < x_4$. On the other hand, only 751 of the positive integers up to 3000 fulfill the condition that all prime divisors $p \equiv 2 \pmod 3$ have an even exponent.

5.2. The conjecture of Manin.

5.2.1. Let X be a non-singular (weak) Fano variety over $\mathbb{Q}$. Assume that $X(\mathbb{Q}) \neq \emptyset$. Then, the conjecture of Manin [6] makes the following prediction for the number of $\mathbb{Q}$-rational points on X of bounded anticanonical height.

There exists some $\tau > 0$ such that, for every Zariski open set $X^\circ \subseteq X$ which is sufficiently small but non-empty,

$$\#\{x \in X^\circ(\mathbb{Q}) \mid h_{-K}(x) < B\} \sim \tau B \log^r B$$

for $r := \operatorname{rk} \operatorname{Pic}(X) - 1$ and $B \gg 0$.

Unfortunately, O is singular. In this situation, one has to consider a resolution $\widetilde{O}$ of singularities and compare heights.

Proposition 5.2.2. *The singular locus of O is reducible into ten components. The component $S_{(x_0,x_1)}$ is given by*

$$x_0 - x_1 = 0\,, \quad x_2^2 + x_3^2 + x_4^2 - 2x_2x_3 - 2x_2x_4 - 2x_3x_4 = 0\,.$$

The others are obtained by permuting coordinates.

Proof. *First case.* $x_0 \cdot \ldots \cdot x_4 \neq 0$.

Then, the morphism $p\colon \mathbf{P}^4_{\mathbb{Q}} \to \mathbf{P}^4_{\mathbb{Q}}$ given by

$$(t_0 : \ldots : t_4) \mapsto (t_0^2 : \ldots : t_4^2)$$

is étale over $(x_0 : \ldots : x_4)$. We may therefore test the fiber product $O \times_{\pi,\mathbf{P}^4_{\mathbb{Q}},p} \mathbf{P}^4_{\mathbb{Q}}$ for smoothness. It is given explicitly by

$$w^2 = (-3) \prod_{i_1,i_2,i_3,i_4 \in [0,1]} \big(t_0 + (-1)^{i_1} t_1 + (-1)^{i_2} t_2 + (-1)^{i_3} t_3 + (-1)^{i_4} t_4\big)\,.$$

Here, the singular points are exactly the singular points of the ramification locus. That, in turn, consists of 16 hyperplanes such that precisely the intersection points are singular. Going back to O, we see that the singular points are those where at least two of the expressions

$$\sqrt{x_0} + (-1)^{i_1}\sqrt{x_1} + (-1)^{i_2}\sqrt{x_2} + (-1)^{i_3}\sqrt{x_3} + (-1)^{i_4}\sqrt{x_4}$$

vanish.

If these expressions coincide in one or four signs then this enforces one coordinate to be zero. The cases that there are two or three signs in common are essentially equivalent to each other. Without restriction,

$$\sqrt{x_0} - \sqrt{x_1} + \sqrt{x_2} + \sqrt{x_3} + \sqrt{x_4} = \sqrt{x_0} - \sqrt{x_1} - \sqrt{x_2} - \sqrt{x_3} - \sqrt{x_4} = 0\,.$$

Then, $\sqrt{x_0} = \sqrt{x_1}$ and $\sqrt{x_2} + \sqrt{x_3} + \sqrt{x_4} = 0$. The first equation yields $x_0 = x_1$. The quadratic relation given is equivalent to $\sqrt{x_2} \pm \sqrt{x_3} \pm \sqrt{x_4} = 0$.

Second case. $x_0 \cdot \ldots \cdot x_4 = 0$.

The singular locus is a Zariski closed subset. Therefore, the points satisfying the equations given above are clearly singular. It remains to prove that the others are non-singular.

Without restriction, we may assume that $x_0 = 0$ and that exactly one of the expressions $\sqrt{x_1}+(-1)^{i_2}\sqrt{x_2}+(-1)^{i_3}\sqrt{x_3}+(-1)^{i_4}\sqrt{x_4}$, say $\sqrt{x_1}+\sqrt{x_2}+\sqrt{x_3}+\sqrt{x_4}$, is equal to zero. Then, the partial derivative of

$$(\sqrt{x_0}+\sqrt{x_1}+\sqrt{x_2}+\sqrt{x_3}+\sqrt{x_4})(\sqrt{x_0}-\sqrt{x_1}-\sqrt{x_2}-\sqrt{x_3}-\sqrt{x_4}) = \\ = x_0-(\sqrt{x_1}+\sqrt{x_2}+\sqrt{x_3}+\sqrt{x_4})^2$$

by x_0 is non-zero. As the other factors do not vanish, the product over all the 16 factors has non-zero derivative at this point. The assertion follows. $\square$

Theorem 5.2.3. *Let* $\mathrm{pr}\colon \widetilde{O} \to O$ *be the proper and birational morphism obtained by blowing up the ten singular components.*

a) *Then,* $\widetilde{O}$ *is non-singular. I.e.,* pr *is a resolution of singularities.*

b) *Further,* $\operatorname{rk}\operatorname{Pic}(\widetilde{O}) = 11$.

c) *The canonical divisor of* $\widetilde{O}$ *is* $K = \mathrm{pr}^*K_O$ *for* $K_O = -\pi^*H$ *and* H *a hyperplane section of* $\mathbf{P}^4$.

Proof. a) This may be tested locally. Let $(w; x_0 : \ldots : x_4)$ be a point in the singular locus of O.

First case. $x_0 \cdot \ldots \cdot x_4 \neq 0$.

Near $(x_0 : \ldots : x_4)$, the morphism

$$p\colon \mathbf{P}^4_{\mathbb{Q}} \to \mathbf{P}^4_{\mathbb{Q}}, \quad (t_0 : \ldots : t_4) \mapsto (t_0^2 : \ldots : t_4^2)$$

is étale. We may take square roots $t_0^{(0)}, \ldots, t_4^{(0)}$ of $x_0, \ldots, x_4$ and consider

$$w^2 = (-3) \prod_{i_1,i_2,i_3,i_4 \in [0,1]} \big(t_0 + (-1)^{i_1}t_1 + (-1)^{i_2}t_2 + (-1)^{i_3}t_3 + (-1)^{i_4}t_4\big)\,.$$

Actually, only the linear factors vanishing at $(t_0^{(0)} : \ldots : t_4^{(0)})$ need to be taken into consideration.

Without restriction, suppose that $(x_0 : \ldots : x_4) \in S_{(x_0,x_1)}$. Then, again without restriction,

$$t_0^{(0)} - t_1^{(0)} + t_2^{(0)} + t_3^{(0)} + t_4^{(0)} = t_0^{(0)} - t_1^{(0)} - t_2^{(0)} - t_3^{(0)} - t_4^{(0)} = 0\,.$$

The corresponding linear forms X, Y are linearly independent which means that we blow up a scheme, locally given by the equation $W^2 = XY$, at the ideal (X, Y). The result is clearly non-singular.

Now suppose that $(x_0 : \ldots : x_4)$ is a point of intersection of at least two singular components. Without loss of generality, the second singular component might be either $S_{(x_0,x_2)}$ or $S_{(x_2,x_3)}$. The latter variant enforces that $(x_0 : \ldots : x_4) = (1 : 1 : 1 : 1 : 4)$ is the point corresponding to the Cayley cubic. This is actually a special case of the first variant.

Thus, assume that $(x_0 : \ldots : x_4) \in S_{(x_0,x_1)} \cap S_{(x_0,x_2)}$. Then, without restriction, $t_0^{(0)} = t_1^{(0)} = t_2^{(0)}$ and $t_0^{(0)} + t_3^{(0)} + t_4^{(0)} = 0$. We have the three vanishing linear forms $t_0 + t_1 - t_2 + t_3 + t_4$, $t_0 - t_1 + t_2 + t_3 + t_4$, and $t_0 - t_1 - t_2 - t_3 - t_4$. Only when $x_3 = x_0$ (or $x_4 = x_0$), another linear form vanishes.

Altogether, there are four linearly independent linear forms X, Y, Z, and U. We blow up $W^2 = XYZU$ or $W^2 = XYZ$ at (X, Y), (X, Z), and (Y, Z), (as well as (X, U), (Y, U), and (Z, U)). The resulting scheme is non-singular.

Second case. Exactly one of the coordinates $x_0, \ldots, x_4$ vanishes.

Then, without loss of generality, $(x_0 : \ldots : x_4) = (a : a : b : b : 0)$. We may take square roots $t_0, \ldots, t_3$ of $x_0, \ldots, x_3$ such that t_0 and t_1 as well as t_2 and t_3 are of the same sign. Then, the right hand side goes over into the product over all $(t_0 \pm t_1 \pm t_2 \pm t_3)^2 - x_4$. Among these, $(t_0 - t_1 + t_2 - t_3)^2 - x_4$ and $(t_0 - t_1 - t_2 + t_3)^2 - x_4$ do vanish.

Hence, for two linearly independent linear forms X and Y, we consider the scheme given by $W^2 = (X^2 - x_4)(Y^2 - x_4)$. The singular components $S_{(x_0,x_1)}$ and $S_{(x_2,x_3)}$ correspond to the ideals $(X^2 - x_4, X + Y)$ and $(X^2 - x_4, X - Y)$, respectively. Blowing up the first ideal amounts to the substitutions $x_4 = X^2 + v(X + Y)$ and, for the other affine chart, $x_4 = X^2 + \frac{1}{v}(X + Y)$. The first substitution leads to $(W')^2 = v(X - Y + v)$ becoming smooth after blowing up $(v, X - Y)$ which is the next step. On the other hand, the second substitution yields $(W')^2 = v(X - Y) + 1$ which is clearly non-singular near $v = 0$.

There is the exceptional case that $a = b$. Then, $(t_0 + t_1 - t_2 - t_3)^2 - x_4$ is a third factor vanishing. We have to consider a scheme locally given by $W^2 = (X^2 - x_4)(Y^2 - x_4)(Z^2 - x_4)$. Here, the substitution $x_4 = X^2 + v(X + Y)$ yields $(W')^2 = v(X - Y + v)[Z^2 - X^2 - v(X + Y)]$. The next step, to blow up $(v, X - Y)$, leads to $(W'')^2 = v_1(1 + v_1)[Z^2 - X^2 - v_1(X^2 - Y^2)]$. Here, for the other affine chart, we find a formula of the same structure. Further, it is sufficient to consider the singularity at $v_1 = 0$. That at $v_1 = -1$ is analogous.

Actually, to blow up $S_{(x_1,x_2)} \cup S_{(x_0,x_3)}$ suffices to resolve this singularity. Indeed, the substitution $Z^2 - X^2 = v_2 v_1$ yields $(W''')^2 = v_2 - X^2 + Y^2$ which is clearly non-singular. On the other hand, putting $Z^2 - X^2 = \frac{1}{v_2} v_1$ leads to $(W''')^2 = v_2(1 - v_2(X^2 - Y^2))$ which is obviously smooth near $v_2 = 0$.

Third case. Exactly two of the coordinates $x_0, \ldots, x_4$ vanish.

Here, without restriction, $(x_0 : \ldots : x_4) = \big(0 : 0 : \big(t_2^{(0)}\big)^2 : \big(t_3^{(0)}\big)^2 : \big(t_4^{(0)}\big)^2\big)$ for $t_2^{(0)} + t_3^{(0)} + t_4^{(0)} = 0$. Therefore, precisely four of the sixteen factors of the right hand side vanish. These are $\sqrt{x_0} \pm \sqrt{x_1} \pm (t_2 + t_3 + t_4)$. We find $W^2 = X^2 - 2YT^2 + T^4$ for the new coordinate functions $X := x_0 - x_1$, $Y := x_0 + x_1$, and $T := t_2 + t_3 + t_4$.

When blowing up (X, T), the substitution $X := uT$ leads to $(W')^2 = u^2 - 2Y + T^2$ which is non-singular. On the other hand, $T := uX$ yields $(W')^2 = 1 - 2u^2 Y + u^4 X^2$ being clearly smooth near $u = 0$.

Fourth case. Three of the coordinates $x_0, \ldots, x_4$ vanish.

Without restriction, $(x_0 : \ldots : x_4) = (0 : 0 : 0 : 1 : 1)$. Take square roots t_3, t_4 of x_3 and x_4 which are of the same sign. The eight factors $\sqrt{x_0} \pm \sqrt{x_1} \pm \sqrt{x_2} \pm (t_3 - t_4)$ vanish at $(0 : 0 : 0 : 1 : 1)$. We find the local equation $W^2 = D(x_0, x_1, x_2, t^2)$ for $t := t_3 - t_4$ and D the symmetric, homogeneous quartic from Lemma 4.5.b).

Blowing up $(x_0 - x_1, x_2 - t^2)$ amounts to substituting $x_2 := t^2 + u(x_0 - x_1)$ and, for the other affine chart, $x_2 := t^2 + \frac{1}{u}(x_0 - x_1)$. Then, the ideals $(x_0 - x_2, x_1 - t^2)$ and $(x_1 - x_2, x_0 - t^2)$ to be blown up subsequently go over to $(u - 1, x_1 - t^2)$ and $(u + 1, x_0 - t^2)$. The substitutions

$$\begin{aligned} x_2 &:= t^2 + u(x_0 - x_1)\,, \\ x_1 &:= t^2 + u_1(u - 1)\,, \\ x_0 &:= t^2 + u_2(u + 1)\,, \end{aligned}$$

yield

$$W^2 = (Y - uZ)^2 + 8Zt^2$$

for the new functions $Y := u_1 + u_2$ and $Z := u_1 - u_2$.

For the other seven affine charts of this triple blow-up, the equations are completely analogous. The differences are that the definitions of Y and Z may be replaced by $Y, Z := 1 \pm u_1 u_2$. Further, instead of $Y - uZ$, we may have $uY - Z$.

The last step is to blow up the ideal $(Y - uZ, t)$ corresponding to the component $S_{(x_3,x_4)}$. The substitution $Y - uZ = vt$ yields $(W')^2 = v^2 + 8Z$ which is non-singular. Indeed, otherwise we must have $v = 0$ and $W' = 0$ which implies $Z = 0$. But, in this situation, Z is a local parameter. On the other hand, $Y - uZ = \frac{1}{v}t$ leads to $(W')^2 = 1 + 8v^2 Z$ which is clearly smooth at $v = 0$.

b) We claim that O is normal. To see this, note first that O is a hypersurface in weighted projective space $\mathbf{P} = \mathbf{P}(4, 1, 1, 1, 1, 1)$. This is a scheme equipped with a canonical rational map $\iota\colon \mathbf{P} \dashrightarrow \mathbf{P}(1, 1, 1, 1, 1) = \mathbf{P}^4$. ι is undefined at exactly one point which is the only singularity of $\mathbf{P}$.

By construction, the double covering O does not meet the singular point. Consequently, O is Gorenstein and, in particular, Cohen-Macaulay. Further, the singularities of O are in codimension 2. Serre's criterion [9, Theorem 23.8] shows that O is normal.

We assert that, after each step of blowing up, the resulting scheme is still normal. In fact, the centre of the blowing up is a codimension two complete intersection. The blow-up $\mathrm{Bl}_{S_{(x_0,x_1)}}(O)$ is, therefore, locally given by a single equation in a $\mathbf{P}^1$-bundle over O. This ensures $\mathrm{Bl}_{S_{(x_0,x_1)}}(O)$ is Cohen-Macaulay. Further, the smooth part of O is untouched under blowing up. Thus, regularity in codimension two could be destroyed only if the whole exceptional set were singular. As this is a $\mathbf{P}^1$-bundle over $S_{(x_0,x_1)}$, that is clearly not the case. The same argument works for each of the subsequent steps.

By Lemma 5.2.4, it suffices to show that the Picard rank grows by one in each step. Again, let us explain this for the first step in order to simplify notation. We have $\mathrm{Bl}_{S_{(x_0,x_1)}}(O) = \mathbf{Proj}(\mathscr{O} \oplus \mathscr{I} \oplus \mathscr{I}^2 \oplus \ldots)$ for $\mathscr{I} := \mathscr{I}_{S_{(x_0,x_1)},O}$. We assert that the twisting sheaf $\mathscr{O}(1)$ is linearly independent of the pull-backs of $\mathrm{Pic}(O)$ in $\mathrm{Pic}(\mathrm{Bl}_{S_{(x_0,x_1)}}(O))$. Indeed, $\mathscr{O}(n)$ for $n \neq 0$ is non-trivial when restricted to one of the exceptional fibers which is just a $\mathbf{P}^1$.

c) As O is a Gorenstein scheme, its dualizing sheaf ω_O is invertible [3, Theorem 3.5.1]. To describe ω_O completely, we may restrict it to O^{reg} since O is normal. Here, $\omega_O|_{O^{\mathrm{reg}}} \cong \Omega^4_{O^{\mathrm{reg}}}$. A 4-form with a simple pole at "$x_0 = 0$" is given by $(x_0^4/w) \cdot d(x_1/x_0) \wedge \ldots \wedge d(x_4/x_0)$. Hence, $\omega_O = \pi^*\mathscr{O}(-1)$.

Further, pr is an isomorphism outside the exceptional fibers. This implies that K and pr^*K_O coincide up to a sum of exceptional divisors. Due to symmetry, the coefficients at $E_1, \ldots, E_{10}$ are equal to each other. To determine the actual number, consider a general point $P \in S_{(x_0,x_1)}$. Near P, we blow up a double covering of the type $w^2 = XY$. This is a quadric cone times a neighbourhood of $(0, 0) \in \mathbf{A}^2$. Its blow-up is the Hirzebruch surface Σ_2 times that neighbourhood. The exceptional curve $E \subset \Sigma_2$ is a (-2)-curve, hence $\omega_{\Sigma_2}|_E$ is trivial. The coefficients desired are equal to zero. $\square$

Lemma 5.2.4. *Let $p\colon X \to Y$ be a surjective and birational morphism of Noetherian, normal, integral schemes. Then, the pull-back homomorphism $p^*\colon \operatorname{Pic}(Y) \to \operatorname{Pic}(X)$ is injective.*

Proof. Suppose, for $\mathscr{L} \in \operatorname{Pic}(Y)$, the pull-back $p^*\mathscr{L} \in \operatorname{Pic}(X)$ would be trivial. This means, we have a section $s \in \Gamma(X, p^*\mathscr{L})$ without zeroes or poles. Corresponding to each codimension one point $\xi \in Y$, there is a discrete valuation ring $\mathscr{O}_\xi$. Further, there is a codimension one point $\zeta \in X$ mapping to ξ. As $\mathscr{O}_\xi$ is integrally closed, we see that $\mathscr{O}_\xi \cong \mathscr{O}_\zeta$.

Consequently, s gives rise to a section $t \in \Gamma(Y^\circ, \mathscr{L}|_{Y^\circ})$ without zeroes or poles for $Y^\circ \subseteq Y$ the complement of a closed subset of codimension ≥ 2. [9, Theorem 12.4.i)] implies that t may be extended to a global section. Hence, $\mathscr{L} \cong \mathscr{O}_Y$ is trivial. □

Remark 5.2.5 (The prediction—Manin's conjecture for the double covering O)**.** Theorem 5.2.3.c) implies

$$h_{-K}(y) = h_{-\mathrm{pr}^* K_O}(y) = h_{-K_O}(\mathrm{pr}(y)) = h_{\mathrm{naive},\mathbf{P}^4}(\pi(\mathrm{pr}(y)))$$

for every $y \in \widetilde{O}(\mathbb{Q})$. Manin's conjecture therefore predicts that, for every sufficiently small, non-empty, Zariski open subset $O^\circ \subseteq O$,

$$\#\{x \in O^\circ(\mathbb{Q}) \mid h_{\mathrm{naive},\mathbf{P}^4}(\pi(x)) < B\} \sim \tau B \log^{10} B\,.$$

The reader might want to compare Table 2 below where the actual numbers are given for a reasonably chosen Zariski open subset.

Remark 5.2.6. We actually found that $\operatorname{Pic}(\widetilde{O}) \cong \mathbb{Z}^{11}$ is a trivial $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-module. This implies that there is no Brauer-Manin obstruction present on $\widetilde{O}$.

5.3. **Infinitely many solutions.**

Proposition 5.3.1. *There are infinitely many $\mathbb{Q}$-rational points on O. In fact, over the quadric surface Q in $\mathbf{P}^4$, given by $l = q = 0$ for*

$$\begin{aligned} l &:= x_0 + x_1 + x_2 - 3x_3 - 3x_4\,, \\ q &:= x_0^2 + x_1^2 + x_2^2 + 9x_3^2 - x_0x_1 - x_0x_2 - 3x_0x_3 - x_1x_2 - 3x_1x_3 - 3x_2x_3\,, \end{aligned}$$

the double covering $\pi\colon O \to \mathbf{P}^4_\mathbb{Q}$ splits. In particular, there are one or two $\mathbb{Q}$-rational points above each $\mathbb{Q}$-rational point of Q.

Proof. Modulo $\mathscr{I}_Q$, one has actually

$$(-3)\Delta(x_0, \ldots, x_4) = [\tfrac{64}{3}(x_0 - x_1)(x_0 - x_2)(x_1 - x_2)(x_3 - x_4)]^2\,. \tag{5.1}$$

□

Remarks 5.3.2. i) The difference of the two octic forms in equation (5.1) consists of 495 monomials. To verify the assertion, one may first use the linear equation to eliminate x_4 and then check that the remaining octic form in $x_0, \ldots, x_3$ is divisible by the quadratic form q.

Actually, a simple Gröbner base calculation quarries the fact that equation (5.1) is true even modulo $\mathscr{I}_Q^2$.

ii) There is another proof for Lemma 5.3.1 which is somehow easier from the computational point of view but less canonical. In fact, Q is parametrized by the birational map $\iota\colon\ \mathbf{P}^2 \dashrightarrow Q$,

$$\begin{aligned}&(t_0 : t_1 : t_2) \mapsto \\ &\qquad \big((t_0^2 + t_1^2 + t_2^2 - t_0t_1 - t_0t_2 - t_1t_2) : (t_0^2 + t_1^2 + t_2^2 - t_0t_1 + 2t_0t_2 - t_1t_2) : \\ &\qquad\qquad : (t_0^2 + t_1^2 + t_2^2 - t_0t_1 - t_0t_2 + 2t_1t_2) : t_2^2 : (t_0^2 + t_1^2 - t_0t_1)\big)\end{aligned}$$

which is defined over $\mathbb{Q}$. The locus where ι is undefined does not contain any $\mathbb{Q}$-rational point since the quadratic form $t_0^2 + t_1^2 - t_0t_1$ does not represent zero over $\mathbb{Q}$. A direct calculation shows

$$(-3)\Delta'\big(\iota(t_0, t_1, t_2)\big) = [576t_0t_1(t_0 - t_1)t_2^3(t_0^2 + t_1^2 - t_0t_1 - t_2^2)]^2\,.$$

Here, the factor t_0 corresponds to $(x_0 - x_1)$, t_1 to $(x_0 - x_2)$, $(t_0 - t_1)$ to $(x_1 - x_2)$, and $(t_0^2 + t_1^2 - t_0t_1 - t_2^2)$ to $(x_3 - x_4)$. The factor t_2^3 is somehow artificial. For $t_2 = 0$, the parametrization is constant to $(1 : 1 : 1 : 0 : 1)$.

The parametrization ι is actually constructed in a very naive manner. Start with the point $(1 : 1 : 1 : 0 : 1)$ and determine for which value of $\tau \neq 0$ the point

$$(1 : (1 + \tau t_0) : (1 + \tau t_1) : (\tau t_2/3) : (1 + \tau(t_0 + t_1 - t_2)/3))$$

is contained in the quadric surface Q. Many other parametrizations would serve the same purpose.

Remarks 5.3.3. i) The surface Q is obviously symmetric under permutations of $\{x_0, x_1, x_2\}$. It is symmetric under switch of x_3 and x_4, too. All in all, there are ten mutually different copies of Q.

ii) Q is a smooth quadric surface. The two pencils of lines on Q are defined over $\mathbb{Q}(\sqrt{-3})$ and conjugate to each other.

iii) This implies that $\mathrm{Pic}(Q) = \mathbb{Z}$. The Picard group has two generators as soon as the ground field contains $\mathbb{Q}(\sqrt{-3})$.

For quadrics such as Q, Manin's conjecture is proven. The number of points of height $\leq B$ is asymptotically $\tau_Q B^2$ for some constant τ_Q. This means that $\pi^{-1}(Q) \subset O$ is an example of a so-called *accumulating subvariety.* The growth of the number of rational points on $\pi^{-1}(Q)$ is faster than predicted for a sufficiently small Zariski open subset of O.

Remark 5.3.4. The quadric surface Q was detected by a statistical investigation of the rational points found on O. Nevertheless, as the height limit of 3000 is too low, most of these points are actually not contained in $\pi^{-1}(Q)$ or one of its copies. Cf. Table 2 below for the numbers of points on O with those over the copies of Q excluded.

TABLE 2. Numbers of solutions, accumulating subvarieties excluded

limit	#	limit	#	limit	#	limit	#
25	12	200	8 989	500	86 897	1500	1 049 502
50	156	300	23 496	750	221 187	2000	1 977 863
100	1 248	400	50 070	1000	432 737	3000	4 651 857

Remarks 5.3.5. i) The smallest $\mathbb{Q}$-rational points on Q with no two coordinates equal are $(3 : 9 : 12 : 1 : 7)$ and $(1 : 7 : 13 : 3 : 4)$. Algorithm 3.7 shows that, indeed, these two points yield cubic surfaces such that the 27 lines are acted upon by the

simple group $D^1W(E_6)$. According to B. L. van der Waerden, this is the generic behaviour on Q.

ii) When testing the cubic surface corresponding to $(3:9:12:1:7)$, Algorithm 3.7 works with the primes 19 and 73. Therefore, we have an explicit infinite set of $\mathbb{Q}$-rational points which lead to the group $D^1W(E_6)$. It is given by those points on Q reducing to $(3:9:12:1:7)$ modulo both 19 and 73.

5.3.6. Some of the surprising properties of Q are described by the following two facts.

Fact 5.3.7. *Q meets the octic R only within its singular locus. Actually,*

$$Q \cap R \subset S_{(x_0,x_1)} \cup S_{(x_0,x_2)} \cup S_{(x_1,x_2)} \cup S_{(x_3,x_4)} \,.$$

Proof. Suppose $(x_0 : \ldots : x_4) \in Q \cap R$. Then, formula (5.1) implies that $x_0 = x_1$, $x_0 = x_2$, $x_1 = x_1$, or $x_3 = x_4$. The equation $x_0 = x_1$ yields $x_0 = (-x_2+3x_3+3x_4)/2$. Substituting this into the quadratic relation $q(x_0, \ldots, x_4) = 0$ from the definition of Q shows

$$x_2^2 + x_3^2 + x_4^2 - 2x_2x_3 - 2x_2x_4 - 2x_3x_4 = 0\,.$$

For the relations $x_0 = x_2$, $x_1 = x_2$, and $x_3 = x_4$, the situation is analogous. $\square$

Fact 5.3.8. *Q is tangent to all five coordinate hyperplanes.*
The points of tangency are $(0:3:3:1:1)$*,* $(3:0:3:1:1)$*,* $(3:3:0:1:1)$*,* $(1:1:1:0:1)$*, and* $(1:1:1:1:0)$*.* $\square$

5.3.9. The quadric surface Q determines the linear form l uniquely. On the other hand, the quadratic form q is unique only up to a multiple of l. One might have the idea to fix a canonical representative $\underline{q}$ by the requirement that the quadric threefold "$\underline{q} = 0$" contain some of the singular components entirely. This is possible to a certain extent.

Fact. a) *There is no quadric threefold in* $\mathbf{P}^4$ *containing the singular components* $S_{(x_0,x_1)}$ *and* $S_{(x_3,x_4)}$*.*

b) *There is, however, a one-dimensional family of quadric threefolds in* $\mathbf{P}^4$ *containing* $S_{(x_0,x_1)}$ *and* $S_{(x_0,x_2)}$*. It is given by* $f_t = 0$ *for a parameter* t *and*

$$\begin{aligned} f_t := {} & -tx_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 - \\ & - (1-t)x_0x_1 - (1-t)x_0x_2 + 2x_0x_3 + 2x_0x_4 + \\ & + (1-t)x_1x_2 - 2x_1x_3 - 2x_1x_4 - 2x_2x_3 - 2x_2x_4 - 2x_3x_4 = 0\,. \end{aligned}$$

Proof. The statement that a quadric threefold contains $S_{(x_0,x_1)}$ is equivalent to saying it is given by an equation of the form $\underline{q} = 0$ for

$$\underline{q} := a(x_2^2 + x_3^2 + x_4^2 - 2x_2x_3 - 2x_2x_4 - 2x_3x_4) + (a_0x_0 + \ldots + a_4x_4) \cdot (x_0 - x_1)\,.$$

The assumptions of a) yield a linear system of equations which is only trivially solvable. On the other hand, the system of equations for b) leads to a two-dimensional vector space. $\square$

Remark 5.3.10. This family is attached to the rational map $f\colon \mathbf{P}^4 \dashrightarrow \mathbf{P}^1$,

$$\begin{aligned} (x_0 : \ldots : x_4) \mapsto (& x_1^2 + x_2^2 + x_3^2 + x_4^2 - x_0x_1 - x_0x_2 + 2x_0x_3 + 2x_0x_4 + \\ & + x_1x_2 - 2x_1x_3 - 2x_1x_4 - 2x_2x_3 - 2x_2x_4 - 2x_3x_4) \\ & : (x_0^2 - x_0x_1 - x_0x_2 + x_1x_2)\,. \end{aligned}$$

The map f enjoys the following remarkable properties.

i) Its locus of indeterminacy is equal to $S_{(x_0,x_1)} \cup S_{(x_0,x_2)}$.

ii) The fiber at $t = -1$ is a singular quadric of rank three. The fiber at infinity is reducible into the two hyperplanes "$x_0 = x_1$" and "$x_0 = x_2$". All other special fibers are smooth.

iii) The special fiber at $t = \frac{1}{3}$ may also be written as

$$4q + (-7x_0 + 5x_1 + 5x_2 + 9x_3 - 3x_4)l = 0\,.$$

In particular, the accumulating subvariety Q is contained within this fiber.

iv) The fiber at $t = \frac{1}{3}$ contains more of the rational points known than any other, even after deleting the accumulating subvarieties. The singular fiber at $t = -1$ follows next.

6. Accumulating subvarieties

6.1. The goal of this section is to prove that there are no other accumulating subvarieties which are, in a certain sense, similar to Q. Similarity shall include to be a non-degenerate quadric surface over which the double covering $\pi\colon O \to \mathbf{P}^4_{\mathbb{Q}}$ splits.

In view of the first constraint established above, this implies that the real points on such a quadric surface S are contained in the 16-ant

$$\{\, (x_0 : \ldots : x_4) \in \mathbf{P}^4(\mathbb{R}) \mid x_0, \ldots, x_4 \geq 0 \text{ or } x_0, \ldots, x_4 \leq 0 \,\}\,.$$

Further, there are strong restrictions for the behaviour at the boundary. By Lemma 4.5.b), we know that Δ' is a perfect square on the coordinate hyperplane H_0 given by "$x_0 = 0$". On the other hand, we require $(-3)\Delta'$ to be a perfect square on S.

A way to realize both of these, seemingly contradictory, requirements is to make $S \cap H_0$ a curve of degree two on which (-3) is the square of a rational function. The only such examples are two lines over $\mathbb{Q}(\sqrt{-3})$ which are conjugate to each other. This implies that S must necessarily be tangent to H_0 and the point of tangency is a $\mathbb{Q}$-rational point on the ramification locus R.

Theorem 6.2. *Suppose $S \subset \mathbf{P}^4_{\mathbb{Q}}$ is a smooth quadric surface such that the double covering $\pi\colon O \to \mathbf{P}^4_{\mathbb{Q}}$ splits over S. Assume further that S is tangent to the five coordinate hyperplanes $H_0, \ldots, H_4$ and that, for each i, the point of tangency is actually contained in one of the three lines on $H_i \cap R$.*

Then, S is equal to Q or one of its copies under permutation of coordinates.

Remark 6.3. On the Steiner surface $H_0 \cap R$, there are two types of $\mathbb{Q}$-rational points. There are the three lines given by $(0 : r : r : s : s)$ and permutations of the four coordinates to the right. The other $\mathbb{Q}$-rational points are of the form $(0 : t_1^2 : \ldots : t_4^2)$ for $t_1, \ldots, t_4 \in \mathbb{Q}$ such that $t_1 + \ldots + t_4 = 0$.

Lemma 6.4. *Assume S is as in Theorem 6.2. Further, write*

$$P^{(0)} := (0 : x_1^{(0)} : x_2^{(0)} : x_3^{(0)} : x_4^{(0)})$$

for the point of tangency of S with the coordinate hyperplane H_0.

Then, $x_1^{(0)}, x_2^{(0)}, x_3^{(0)}, x_4^{(0)} \neq 0$.

Proof. Assume, to the contrary, that $x_1^{(0)} = 0$. The assumption on the type of the points of tangency made in Theorem 6.2 implies that one more coordinate must vanish. Without restriction, we may assume $P^{(0)} = (0:0:0:1:1)$. The tangent plane at $P^{(0)}$ is given by $x_0 = 0$ and another linear relation $C_1x_1 + \ldots + C_4x_4 = 0$. Whatever the coefficients are, there is a tangent vector $(v_0, \ldots, v_4)$ such that $v_1 < 0$ or $v_2 < 0$. The implicit function theorem yields a real point $(x_0 : \ldots : x_3 : 1) \in S(\mathbb{R})$ satisfying $x_1 < 0$ or $x_2 < 0$. This is a contradiction. $\square$

Lemma 6.5. *Assume that the quadric surface S is tangent to the coordinate hyperplanes H_0, H_1, and H_2 in $(0 : x_1^{(0)} : x_2^{(0)} : x_3^{(0)} : x_4^{(0)})$, $(x_0^{(1)} : 0 : x_2^{(1)} : x_3^{(1)} : x_4^{(1)})$, and $(x_0^{(2)} : x_1^{(2)} : 0 : x_3^{(2)} : x_4^{(2)})$, respectively.*

Then,

$$x_1^{(0)}x_2^{(1)}x_0^{(2)} - x_2^{(0)}x_0^{(1)}x_1^{(2)} = 0$$

or

$$\begin{aligned}
x_1^{(0)}x_2^{(1)}x_0^{(2)} + x_2^{(0)}x_0^{(1)}x_1^{(2)} &= 0\,, \\
x_1^{(0)}x_0^{(1)}x_3^{(2)} - x_1^{(0)}x_0^{(2)}x_3^{(1)} - x_0^{(1)}x_1^{(2)}x_3^{(0)} &= 0\,, \\
x_1^{(0)}x_2^{(1)}x_3^{(2)} + x_2^{(0)}x_1^{(2)}x_3^{(1)} - x_2^{(1)}x_1^{(2)}x_3^{(0)} &= 0\,, \\
x_2^{(0)}x_0^{(1)}x_3^{(2)} - x_2^{(0)}x_0^{(2)}x_3^{(1)} + x_2^{(1)}x_0^{(2)}x_3^{(0)} &= 0\,.
\end{aligned}$$

Proof. The linear equation by which S is defined may be written

$$L_0x_0 + L_1x_1 + L_2x_2 + L_3x_3 + L_4x_4 = 0\,. \tag{6.1}$$

We distinguish three cases.

First case. $L_4 \neq 0$.

Then, we may use the linear equation (6.1) to eliminate x_4 from the quadratic equation. Write

$$\begin{aligned}
&Q_0x_0^2 + Q_1x_1^2 + Q_2x_2^2 + Q_3x_3^2 + \\
&\qquad + Q_4x_0x_1 + Q_5x_0x_2 + Q_6x_0x_3 + Q_7x_1x_2 + Q_8x_1x_3 + Q_9x_2x_3 \;=\; 0\,.
\end{aligned}$$

Tangency of H_0 at $(0 : x_1^{(0)} : x_2^{(0)} : x_3^{(0)} : x_4^{(0)})$ means that the two linear forms

$$\begin{aligned}
&(Q_4x_1^{(0)} + Q_5x_2^{(0)} + Q_6x_3^{(0)})x_0 + (2Q_1x_1^{(0)} + Q_7x_2^{(0)} + Q_8x_3^{(0)})x_1 + \\
&\qquad + (2Q_2x_2^{(0)} + Q_7x_1^{(0)} + Q_9x_3^{(0)})x_2 + (2Q_3x_3^{(0)} + Q_8x_1^{(0)} + Q_9x_2^{(0)})x_3\,, \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad L_0x_0 + L_1x_1 + L_2x_2 + L_3x_3 + x_4\,,
\end{aligned}$$

together generate x_0. This enforces the linear relations

$$\begin{aligned}
2x_1^{(0)}Q_1 + x_2^{(0)}Q_7 + x_3^{(0)}Q_8 &= 0\,, \\
2x_2^{(0)}Q_2 + x_1^{(0)}Q_7 + x_3^{(0)}Q_9 &= 0\,, \\
2x_3^{(0)}Q_3 + x_1^{(0)}Q_8 + x_1^{(0)}Q_9 &= 0\,.
\end{aligned} \tag{6.2}$$

The two other points of tangency yield relations which are completely analogous. Altogether, we find the homogeneous linear system of equations associated with the 9×10-matrix

$$\begin{pmatrix}
0 & 2x_1^{(0)} & 0 & 0 & 0 & 0 & 0 & x_2^{(0)} & x_3^{(0)} & 0 \\
0 & 0 & 2x_2^{(0)} & 0 & 0 & 0 & 0 & x_1^{(0)} & 0 & x_3^{(0)} \\
0 & 0 & 0 & 2x_3^{(0)} & 0 & 0 & 0 & 0 & x_1^{(0)} & x_2^{(0)} \\
2x_0^{(1)} & 0 & 0 & 0 & 0 & x_2^{(1)} & x_3^{(1)} & 0 & 0 & 0 \\
0 & 0 & 2x_2^{(1)} & 0 & 0 & x_0^{(1)} & 0 & 0 & 0 & x_3^{(1)} \\
0 & 0 & 0 & 2x_3^{(1)} & 0 & 0 & x_0^{(1)} & 0 & 0 & x_2^{(1)} \\
2x_0^{(2)} & 0 & 0 & 0 & x_1^{(2)} & 0 & x_3^{(2)} & 0 & 0 & 0 \\
0 & 2x_1^{(2)} & 0 & 0 & x_0^{(2)} & 0 & 0 & 0 & x_3^{(2)} & 0 \\
0 & 0 & 0 & 2x_3^{(2)} & 0 & 0 & x_0^{(2)} & 0 & x_1^{(2)} & 0
\end{pmatrix}.$$

If this matrix is of rank 9 then the quadratic equation defining S is, up to scaling, determined uniquely. In fact, this case is degenerate. There is a linear form in $x_0, \ldots, x_3$ only, vanishing on the three points given. The unique solution of the system corresponds to the square of this linear form.

Consequently, the rank is at most 8. The ten 9×9-minors must all vanish. These minors are polynomials in $x_0^{(0)}, \ldots, x_3^{(2)}$ having

$$(x_1^{(0)} x_2^{(1)} x_0^{(2)} - x_2^{(0)} x_0^{(1)} x_1^{(2)})$$

as their greatest common divisor. After division by this, we are left with ten sextics. It turns out that they are precisely the squares and pairwise products of the four cubics $x_1^{(0)} x_2^{(1)} x_0^{(2)} + x_2^{(0)} x_0^{(1)} x_1^{(2)}$, $x_1^{(0)} x_0^{(1)} x_3^{(2)} - x_1^{(0)} x_0^{(2)} x_3^{(1)} - x_0^{(1)} x_1^{(2)} x_3^{(0)}$, $x_1^{(0)} x_2^{(1)} x_3^{(2)} + x_2^{(0)} x_1^{(2)} x_3^{(1)} - x_2^{(1)} x_1^{(2)} x_3^{(0)}$, and $x_2^{(0)} x_0^{(1)} x_3^{(2)} - x_2^{(0)} x_0^{(2)} x_3^{(1)} + x_2^{(1)} x_0^{(2)} x_3^{(0)}$.

Second case. $L_4 = 0$ and $L_3 \neq 0$.

As the roles of the third and fourth coordinates may be interchanged, we have, as in the first case, $x_1^{(0)} x_2^{(1)} x_0^{(2)} - x_2^{(0)} x_0^{(1)} x_1^{(2)} = 0$ or

$$x_1^{(0)} x_2^{(1)} x_0^{(2)} + x_2^{(0)} x_0^{(1)} x_1^{(2)} = 0\,.$$

Suppose that the second variant is present. Then, the linear equation (6.1) implies that the vector $(x_3^{(0)}, x_3^{(1)}, x_3^{(2)})^t$ is linearly dependent of $(0, x_0^{(1)}, x_0^{(2)})^t$, $(x_1^{(0)}, 0, x_1^{(2)})^t$, and $(x_2^{(0)}, x_2^{(1)}, 0)^t$. For these vectors instead of $(x_3^{(0)}, x_3^{(1)}, x_3^{(2)})^t$, the three more relations asserted are clearly true.

Third case. $L_3 = L_4 = 0$.

In this situation, we may write the three points of tangency in the form $(0 : L_2 : (-L_1) : x_3^{(0)} : x_4^{(0)})$, $(L_2 : 0 : (-L_0) : x_3^{(1)} : x_4^{(1)})$, and $(L_1 : (-L_0) : 0 : x_3^{(2)} : x_4^{(2)})$. It turns out that the relation

$$x_1^{(0)} x_2^{(1)} x_0^{(2)} + x_2^{(0)} x_0^{(1)} x_1^{(2)} = 0$$

is automatically fulfilled. Further, $L_0, L_1, L_2 \neq 0$. Each of the three equations still to be proven reduces to $L_0 x_3^{(0)} - L_1 x_3^{(1)} + L_2 x_3^{(2)} = 0$.

We may use the linear equation (6.1) to eliminate x_0 from the quadratic equation. Write

$$\begin{aligned} &Q_0 x_1^2 + Q_1 x_2^2 + Q_2 x_3^2 + Q_3 x_4^2 + \\ &\qquad + Q_4 x_1 x_2 + Q_5 x_1 x_3 + Q_6 x_1 x_4 + Q_7 x_2 x_3 + Q_8 x_2 x_4 + Q_9 x_3 x_4 \;=\; 0\,. \end{aligned}$$

Tangency of H_0 at $(0 : L_2 : (-L_1) : x_3^{(0)} : x_4^{(0)})$ yields the linear relations

$$\begin{aligned}
L_2(2L_2Q_0 - L_1Q_4 + x_3^{(0)}Q_5 + x_4^{(0)}Q_6) - \\
- L_1(-2L_1Q_1 + L_2Q_4 + x_3^{(0)}Q_7 + x_4^{(0)}Q_8) &= 0\,, \\
2x_3^{(0)}Q_2 + x_1^{(0)}Q_5 + x_2^{(0)}Q_7 + x_4^{(0)}Q_9 &= 0\,, \\
2x_4^{(0)}Q_3 + x_1^{(0)}Q_6 + x_2^{(0)}Q_8 + x_3^{(0)}Q_9 &= 0\,.
\end{aligned}$$

Tangency of H_1 and H_2 leads to linear relations completely analogous to those given in (6.2). Altogether, we find the homogeneous linear system of equations associated with the 9×10-matrix

$$\begin{pmatrix}
2L_2^2 & 2L_1^2 & 0 & 0 & -2L_1L_2 & x_3^{(0)}L_2 & x_4^{(0)}L_2 & -x_3^{(0)}L_1 & -x_4^{(0)}L_1 & 0 \\
0 & 0 & 2x_3^{(0)} & 0 & 0 & x_1^{(0)} & 0 & x_2^{(0)} & 0 & x_4^{(0)} \\
0 & 0 & 0 & 2x_4^{(0)} & 0 & 0 & x_1^{(0)} & 0 & x_2^{(0)} & x_3^{(0)} \\
0 & -2L_0 & 0 & 0 & 0 & 0 & 0 & x_3^{(1)} & x_4^{(1)} & 0 \\
0 & 0 & 2x_3^{(1)} & 0 & 0 & 0 & 0 & -L_0 & 0 & x_4^{(1)} \\
0 & 0 & 0 & 2x_4^{(1)} & 0 & 0 & 0 & 0 & -L_0 & x_3^{(1)} \\
-2L_0 & 0 & 0 & 0 & 0 & x_3^{(2)} & x_4^{(2)} & 0 & 0 & 0 \\
0 & 0 & 2x_3^{(2)} & 0 & 0 & -L_0 & 0 & 0 & 0 & x_4^{(2)} \\
0 & 0 & 0 & 2x_4^{(2)} & 0 & 0 & -L_0 & 0 & 0 & x_3^{(2)}
\end{pmatrix}.$$

If this matrix is of rank 9 then, again, we have a degenerate case. There is a linear form in $x_1, \ldots, x_4$ only, vanishing on the three points given. The unique solution of the system corresponds to the square of this linear form.

Consequently, all the ten 9×9-minors must vanish. Actually, when deleting the fourth column, the corresponding minor is

$$-16L_0^4L_1L_2(L_0x_3^{(0)} - L_1x_3^{(1)} + L_2x_3^{(2)})^2\,. \qquad \square$$

Remark 6.6 (Interpretation)**.** The relations established in Lemma 6.5 may be interpreted as follows. The coordinates of three points of tangency form a 3×5-matrix

$$\begin{pmatrix}
0 & x_1^{(0)} & x_2^{(0)} & x_3^{(0)} & x_4^{(0)} \\
x_0^{(1)} & 0 & x_2^{(1)} & x_3^{(1)} & x_4^{(1)} \\
x_0^{(2)} & x_1^{(2)} & 0 & x_3^{(2)} & x_4^{(2)}
\end{pmatrix}.$$

We may scale such that $x_0^{(1)} = x_1^{(0)}$ and $x_0^{(2)} = x_2^{(0)}$.

i) Then, the leftmost 3×3-block is either symmetric, i.e., $x_1^{(2)} = x_2^{(1)}$, or symmetric up to sign. Then, $x_1^{(2)} = -x_2^{(1)}$.

ii) In the latter case, the column vector $(x_3^{(0)}, x_3^{(1)}, x_3^{(2)})^t$ is a linear combination of the column vectors $(0, x_0^{(1)}, x_0^{(2)})^t$, $(x_1^{(0)}, 0, x_1^{(2)})^t$, and $(x_2^{(0)}, x_2^{(1)}, 0)^t$.

Remarks 6.7. i) In the non-symmetric variant, $(x_4^{(0)}, x_4^{(1)}, x_4^{(2)})^t$ is a linear combination of the column vectors $(0, x_0^{(1)}, x_0^{(2)})^t$, $(x_1^{(0)}, 0, x_1^{(2)})^t$, and $(x_2^{(0)}, x_2^{(1)}, 0)^t$, too. The roles of the third and fourth coordinates may be interchanged.

ii) Actually, in this variant, linear dependence of the three vectors $(0, x_0^{(1)}, x_0^{(2)})^t$, $(x_1^{(0)}, 0, x_1^{(2)})^t$, and $(x_2^{(0)}, x_2^{(1)}, 0)^t$ is a non-trivial condition. Observe, they do not form a base of $\mathbb{R}^3$. In the symmetric variant, an analogous condition would be empty.

Remark 6.8. For each triple consisting of points of tangency of S with a coordinate hyperplane, relations of the same kind must be fulfilled.

Proof of Theorem 6.2. For each of the five points of tangency, we have at least two pairs $\{i, j\} \subset \{0, \ldots, 4\}$ such that $x_i = x_j$. There are two cases.

First case. Each of the ten pairs of $\{0, \ldots, 4\}$ appears exactly once.

Without restriction, the point of tangency to H_0 is $(0 : 1 : 1 : t : t)$. Again without loss of generality, $(1 : 0 : s : 1 : s)$ is the point of tangency to H_1. The structure of the remaining three points of tangency is then fixed. The five points form a matrix as follows,

$$\begin{pmatrix} 0 & 1 & 1 & t & t \\ 1 & 0 & s & 1 & s \\ 1 & r & 0 & r & 1 \\ t & q & t & 0 & q \\ t & t & p & p & 0 \end{pmatrix}.$$

Lemma 6.5 implies that $r = s$. Indeed, $r = -s$ would enforce that both $(t, 1, -s)^t$ and $(t, s, 1)^t$ are linearly dependent of $(0, 1, 1)^t$, $(1, 0, -s)^t$, and $(1, s, 0)^t$. This is a contradiction since $(0, s - 1, s + 1)^t$ is not in the span of these three.

For the same reason, $p = q$. Further, we have $q = \pm 1$ and $s = \pm t$ such that we end up with four one-parameter families,

$$\begin{pmatrix} 0 & 1 & 1 & t & t \\ 1 & 0 & t & 1 & t \\ 1 & t & 0 & t & 1 \\ t & 1 & t & 0 & 1 \\ t & t & 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 & t & t \\ 1 & 0 & -t & 1 & -t \\ 1 & -t & 0 & -t & 1 \\ t & 1 & t & 0 & 1 \\ t & t & 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 & t & t \\ 1 & 0 & t & 1 & t \\ 1 & t & 0 & t & 1 \\ t & -1 & t & 0 & -1 \\ t & t & -1 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 & t & t \\ 1 & 0 & -t & 1 & -t \\ 1 & -t & 0 & -t & 1 \\ t & -1 & t & 0 & -1 \\ t & t & -1 & -1 & 0 \end{pmatrix}.$$

The linear equation of S requires that the matrices considered are of rank at most 4. However, in the second and third families, the determinants $(t^2 - t - 1)(t^3 + 2t - 1)$ and $(t^2 + t - 1)(t^3 - 2t^2 - 1)$ have no rational zeroes. For the fourth family, we find $(t + 1)(t^2 - t + 1)(t^2 + 3t + 1)$ for the determinant. But, for $t = -1$, we had four equal coordinates in several of the points of tangency. Finally, for the first family, the determinant is $(t + 1)(t^2 - 3t + 1)^2$ and the value $t = -1$ could be possible.

The corresponding data lead to systems of equations which are uniquely solvable up to scaling. The resulting quadric surface is given by $l = q = 0$ for

$$\begin{aligned} l &:= x_0 + x_1 + x_2 + x_3 + x_4\,, \\ q &:= x_0^2 + x_1^2 - x_2^2 - x_3^2 + 3x_0x_1 + x_0x_2 - x_0x_3 - x_1x_2 + x_1x_3 - 3x_2x_3\,. \end{aligned}$$

This surface is indeed smooth and tangent to all five coordinate hyperplanes but the double covering $\pi\colon O \to \mathbf{P}^4_{\mathbb{Q}}$ does not split over it.

Second case. One of the ten pairs of $\{0, \ldots, 4\}$ appears at least twice.

Without loss of generality, the points of tangency to H_0 and H_1, respectively, are $(0 : 1 : 1 : t : t)$ and $(1 : 0 : 1 : s : s)$. If the point of tangency to H_2 were $(1 : (-1) : 0 : 1 : (-1))$ then, by Lemma 6.5, both $(t, s, 1)^t$ and $(t, s, -1)^t$ had to be linear combinations of $(0, 1, 1)^t$, $(1, 0, -1)^t$, and $(1, 1, 0)^t$. This is a contradiction since $(0, 0, 2)^t$ is not in the span of these three.

Consequently, the five points of tangency form a matrix as follows,

$$\begin{pmatrix} 0 & 1 & 1 & t & t \\ 1 & 0 & 1 & s & s \\ 1 & 1 & 0 & r & r \\ t & \pm s & \pm r & 0 & q \\ t & \pm s & \pm r & \pm q & 0 \end{pmatrix}.$$

Assume that one of the "r" or "s" actually carries a minus sign. Without restriction, there is "$-r$" in the fourth line. Then, Lemma 6.5 yields the contradiction that $(t,r,q)^t$ must be a linear combination of $(0,1,t)^t$, $(1,0,-r)^t$, and $(t,r,0)^t$. Further, if there were a "$-q$" in the fifth line then $(1,r,r)^t$ had to be a linear combination of $(0,t,t)^t$, $(t,0,-q)^t$, and $(t,q,0)^t$ which is not the case, either.

Finally, in the fourth line, we must have two pairs of equal entries. Without restriction, suppose that $q = t$ and $r = s$. All in all, we find a matrix of the form

$$\begin{pmatrix} 0 & 1 & 1 & t & t \\ 1 & 0 & 1 & s & s \\ 1 & 1 & 0 & s & s \\ t & s & s & 0 & t \\ t & s & s & t & 0 \end{pmatrix}.$$

For the determinant, one calculates $2t^2(4s-t-1)$. We may conclude that $s = \frac{t+1}{4}$.

For every $t \neq 0$, these data lead to systems of equations which are uniquely solvable up to scaling. The result is the one-parameter family S_t of quadric surfaces given by $l_t = q_t = 0$ for

$$\begin{aligned} l_t &:= (t-1)x_0 - 2tx_1 - 2tx_2 + 2x_3 + 2x_4\,, \\ q_t &:= (t+1)^2x_0^2 + 4(t+1)tx_1^2 + 4(t+1)tx_2^2 + 16x_3^2 - \\ &\qquad - 4(t+1)tx_0x_1 - 4(t+1)tx_0x_2 + 8(t-1)x_0x_3 + \\ &\qquad\qquad + 8(t-1)tx_1x_2 - 16tx_1x_3 - 16tx_2x_3\,. \end{aligned}$$

For each $t \neq 0$, the quadric surface S_t is indeed smooth and tangent to all five coordinate hyperplanes.

In order to check for which values of t the double covering $\pi\colon O \to \mathbf{P}^4_{\mathbb{Q}}$ splits over S_t, we first restrict to the intersection $C_t := S_t \cap$ "$x_1 = x_0 + x_2$". This is a smooth conic for each $t \neq 0$. A parametrization $\iota_t\colon \mathbf{P}^1 \to C_t$ is given by

$$\begin{aligned} &(u:v) \mapsto (16tu^2 : ((t^2+18t+1)u^2 + 8(t+1)tuv + 16t^2v^2) : \\ &\quad : ((t^2+2t+1)u^2 + 8(t+1)tuv + 16t^2v^2) : ((t^2+2t+1)tu^2 + 8(t-1)t^2uv + 16t^3v^2) : \\ &\qquad\qquad : ((t^2+10t+9)tu^2 + 8(t+3)t^2uv + 16t^3v^2))\,. \end{aligned}$$

The binary form $(-3)\Delta'(\iota_t(u,v))$ of degree 16 factors into $u^6((t+1)u+4tv)^4$ and a form of degree six which is irreducible for general t. We ask for the values of t for which this sextic is a perfect square. According to `magma`, its discriminant is equal to

$$C(t-3)(t-1)^6(3t-1)^6t^{83}(t^2+8t-1)^4(19t^3-82t^2+59t-16)^2$$

for C a 103-digit integer. Over S_1, the double covering $\pi\colon O \to \mathbf{P}^4_{\mathbb{Q}}$ does not split. The cases $t = 3$ and $t = \frac{1}{3}$ both yield the accumulating subvariety Q studied in subsection 5.3. They are equivalent to each other under the permutation $(0)(13)(24)$ of coordinates. $\square$

REFERENCES

[1] Burkhardt, H.: *Untersuchungen aus dem Gebiete der hyperelliptischen Modulfunctionen III,* Math. Ann. **41** (1893), 313–343

[2] Clebsch, A.: *Ueber eine Transformation der homogenen Functionen dritter Ordnung mit vier Veränderlichen,* J. für die Reine und Angew. Math. **58** (1861), 109–126

[3] Conrad, B.: *Grothendieck duality and base change,* Lecture Notes in Mathematics 1750, Springer, Berlin 2000

[4] Edge, W. L.: *The discriminant of a cubic surface,* Proc. Roy. Irish Acad. **80** (1980), 75–78

[5] Elsenhans, A.-S. and Jahnel, J.: *Experiments with general cubic surfaces,* to appear in: The Manin Festschrift

[6] Franke, J., Manin, Yu. I., and Tschinkel, Yu.: *Rational points of bounded height on Fano varieties,* Invent. Math. **95** (1989), 421–435

[7] Grothendieck, A.: *Revêtements étales et groupe fondamental* (SGA1), Lecture Notes Math. 224, Springer, Berlin 1971

[8] Jordan, C.: *Sur la trisection des fonctions abéliennes et sur les 27 droites des surfaces du troisième ordre,* C. R. Acad. Sci. Paris **68** (1869), 865–869

[9] Matsumura, H.: *Commutative ring theory,* Cambridge Studies in Advanced Mathematics 8, Cambridge University Press, Cambridge 1989

[10] Salmon, G.: *A treatise on the analytic geometry of three dimensions,* Hodges/Smith, Dublin 1865

[11] Sylvester, J. J.: *Sketch of a memoir on elimination, transformation, and canonical forms,* Cambridge and Dublin Mathematical Journal **6** (1851), 186–200

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