

On the distribution of small points on abelian and toric varieties

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Introduction

0.1 In 1972, J.-P. Serre proved the following remarkable result.

Theorem (Serre). *Let K be an algebraic number field and E be an elliptic curve over K without complex multiplication over \overline{K} . Then, for almost every prime number l the Galois group $\text{Gal}(\overline{K}/K)$ acts surjectively on the l -torsion points of E .*

For $x \in E(\overline{K})$ we denote by δ_x the Dirac measure associated to its Galois orbit, i.e. if $x \in E(F)$ for some number field $F \supseteq K$, then $\delta_x(f) := \frac{1}{\#\{\sigma: F \hookrightarrow \mathbb{C}\}} \sum_{\sigma} f(\sigma(x))$ for each $f \in C^0(E(\mathbb{C}))$. In this language, Serre's theorem immediately implies the following corollary.

Corollary. *Let K be an algebraic number field and E be an elliptic curve over K without complex multiplication over \overline{K} . If $\{x_i\}_{i \in \mathbb{N}}$ is a sequence of torsion points in E which does not contain a constant subsequence then the associated sequence $\{\delta_{x_i}\}_{i \in \mathbb{N}}$ of measures on $E(\mathbb{C})$ automatically converges weakly versus the Haar measure of volume one.*

In 1997, making use of the powerful methods of Arakelov geometry, L. Szpiro, E. Ullmo and S. Zhang [SUZ] showed a generalization of this corollary from torsion points, being of Néron-Tate height zero, to points of a small positive height. Their result is as follows.

Theorem (Szpiro, Ullmo, Zhang). *Let A be an abelian variety over some number field K and $\{x_i\}_{i \in \mathbb{N}}$ a sequence of closed points in A . Assume $\{x_i\}_{i \in \mathbb{N}}$ converges to the generic point of A in the sense of the Zariski topology and assume further that $\lim_{i \rightarrow \infty} h_{NT}(x_i) = 0$.*

Then the associated sequence $\{\delta_{x_i}\}_{i \in \mathbb{N}}$ of measures on $A(\mathbb{C})$ automatically converges weakly to the Haar measure of volume one.

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Surprisingly, there is a completely parallel theorem for the naive height on the projective space which was proven by Yuri Bilu [Bi97]. In fact, for \mathbf{P}^1 he proved this result already in 1988 [Bi88] and it would have been easy to deduce the general result from this.

Theorem (Bilu). *Let $\{x_i\}_{i \in \mathbb{N}}$ a sequence of closed points in $\mathbf{P}^n_{\mathbb{Q}}$. Assume $\{x_i\}_{i \in \mathbb{N}}$ converges to the generic point of \mathbf{P}^n in the sense of the Zariski topology and assume further that $\lim_{i \rightarrow \infty} h_{\text{naive}}(x_i) = 0$.*

Then, the associated sequence $\{\delta_{x_i}\}_{i \in \mathbb{N}}$ of measures on $\mathbf{P}^n(\mathbb{C})$ automatically converges weakly to the Haar measure of volume one on the product of unit circles $(S^1)^n \subset \mathbb{C}^n \subset \mathbf{P}^n(\mathbb{C})$.

Both the Néron-Tate height and the naive height are examples of *canonical* heights. This means we consider a situation like this.

Situation. We work on a projective variety P over \mathbb{Q} equipped with an ample invertible sheaf $\mathcal{L} \in \text{Pic}(P)$. These data fix a height up to a bounded summand in that way that one may put $h_{\mathcal{L}}(x) := h_{\text{naive}}(i_{\mathcal{L}}(x))$ where $i_{\mathcal{L}}: P \rightarrow \mathbf{P}^N$ is an embedding of P into a projective space defined by \mathcal{L} . (We assume \mathcal{L} to be very ample for the moment.)

Further, we assume to be given a self map $f: P \rightarrow P$ such that there is some isomorphism $\Phi: \mathcal{L}^{\otimes d} \xrightarrow{\cong} f^* \mathcal{L}$.

Then, the *canonical height* $h_{f, \mathcal{L}}$ is given by $h_{f, \mathcal{L}}(x) := \lim_{n \rightarrow \infty} \frac{1}{\deg f} h_{\mathcal{L}}(f^{(n)}(x))$ where $f^{(n)}(x)$ means the n -th iteration of f .

Note that the limit process defining $h_{f, \mathcal{L}}$ is a generalization of the classical one for the Néron-Tate height.

0.2 The goal of the present paper is to prove a common generalization of the two theorems above. We will give an Arakelovian approach to a situation which covers the case of the Néron-Tate height as well as the naive height. For a general self map f iteration of f will lead to fractals. Unfortunately, not much is known about the corresponding fractal heights.

Theorem. *Let P/\mathbb{Q} be a regular projective variety containing a group scheme $G \subseteq P$ as an open dense subset. Further, let a morphism $f: P \rightarrow P$ be given and assume that*

- a) *the unit e is a repelling fixed point, i.e. all eigenvalues of the tangent map $T_e f: T_e G \rightarrow T_e G$ have absolute value strictly bigger than 1,*
- b) *$f|_G$ is a group homomorphism $f|_G: G \rightarrow G$ and*
- c) *there is some compact and Zariski dense subgroup $K \subseteq G(\mathbb{C})$ which is both backward and forward invariant under $m_{\mathbb{C}}$.*

Finally, let $\mathcal{L} \in \text{Pic}(P)$ be ample and $\Phi: \mathcal{L}^{\otimes d} \xrightarrow{\cong} f^ \mathcal{L}$ be an isomorphism.*

In this situation, if $\{x_i\}_{i \geq 0}$ converges to the generic point of P and $h_{f, \mathcal{L}}(x_i) \rightarrow 0$, then the sequence of measures associated to the Galois orbits, $\{\delta_{x_i}\}_{i \geq 0}$ converges weakly to the measure τ on $P(\mathbb{C})$ being the zero measure on $P(\mathbb{C}) \setminus K$ and the Haar measure of volume 1 on K .

0.3 The condition that e is a repelling fixed point is fulfilled, in particular, if G is commutative and $m: g \mapsto g^l$ the homomorphism raising to the l -th power for $l \geq 2$. This includes the two standard cases, namely

a) $P = G = A$ is an abelian variety and $\mathcal{L} \in \text{Pic}(A)$ a symmetric, ample invertible sheaf. One has

$$f = [l]: A \rightarrow A$$

and puts $K = A(\mathbb{C})$. Then, independently on the choice of l , $h_{f, \mathcal{L}}$ is the Néron-Tate height corresponding to f .

b) $P = \mathbf{P}^n, G = \mathbb{G}_m^{n+1}/\mathbb{G}_m \cong \mathbb{G}_m^n$ and $\mathcal{L} = \mathcal{O}(1)$. We have

$$f = [l]: (x_0, \dots, x_n) \mapsto (x_0^l, \dots, x_n^l)$$

and put $K := U(1)^n$. Here $h_{f, \mathcal{L}}$ is the naive height.

For abelian varieties our proof coincides with the one of Szpiro, Ullmo, and Zhang. In the case of a projective space we provide a new, more geometric, proof for Bilu's theorem.

0.4 Further, there are the following

Examples. i) One may combine abelian varieties and projective spaces and consider a *split semi-abelian variety* $A \times \mathbf{P}^n$. Here, $G = A \times \mathbb{G}_m^n$, $K = A \times U(1)$, and $\mathcal{L} = \mathcal{S} \boxtimes \mathcal{O}(p)$ where \mathcal{S} is a symmetric, ample invertible sheaf on A . One may put $f := [l] \times [l^2]$.

The equidistribution result of Chambert-Loir [Ch] for quasi-split semi-abelian varieties can easily be deduced from this.

ii) Another special case is that of a regular, projective toric variety X .

For that consider a free abelian group N of finite rank and a finite rational polyhedral decomposition $\{\sigma_\alpha\}_\alpha$ of $N \otimes_{\mathbb{Z}} \mathbb{R}$. These data define a proper toric variety X (see e.g. [KKMS, Chapter I, §§ 2 and 3] for details on toric varieties). The multiplication map $N \rightarrow N, n \mapsto ln$ induces a self-morphism

$$f = [l]: X \rightarrow X.$$

Further, consider a function $g: N \otimes_{\mathbb{Z}} \mathbb{R} \rightarrow \mathbb{R}$ which is strictly convex in the sense of [KKMS, ???]. Then $\mathcal{L} := \mathcal{F}_g$ is a T -invariant ample invertible sheaf on X . One has $[l]^* \mathcal{F}_g \cong \mathcal{F}_{lg} = \mathcal{F}_g^{\otimes l}$.

In this situation $P = X$, $G = T \cong \mathbb{G}_m^n$ is the Zariski dense torus in X , and $K \subset T$ is its maximal compact subgroup.

This case clearly contains the case of the naive height on a projective space. Notice that it also contains the blow-up $\text{Bl}_{(0,0)}(\mathbf{P}^2)$ of the projective plane in the origin (and even the blow-up of \mathbf{P}^2 in up to three points) and the canonical height defined by the ample divisor $nH - mE$ for $n, m \in \mathbb{N}, n > m$.

0.5 Remark. Theorem 0.2 is the starting point for another equidistribution theorem (Theorem 4.6). It will state that, if there is a small and generic sequence $\{x_i\}_{i \geq 0}$ on a closed subvariety $X \subset P$ then the associated sequence of measures $\{\delta_{x_i}\}_{i \geq 0}$ converges weakly to some well-defined measure with support in $X \cap K$.

0.6 To understand the canonical heights better, one may use the language Arakelov geometry, initiated by Arakelov, Faltings, and Deligne and further developed by Gillet, Soulé, and Zhang. This theory gives us a concept of a height not only for points but for subvarieties [BGS].

For that extend \mathcal{L} arbitrarily to some projective model \mathcal{P}/\mathbb{Z} of P and equip it with any hermitian metric $\|\cdot\|$. One puts $h_{\mathcal{L},0}(X) := h_{\mathcal{L},\|\cdot\|}(X)$ and, inductively, $h_{\mathcal{L},i+1}(X) := \frac{1}{(\deg f)^{\dim X+1}} h_{\mathcal{L},i}(f(X))$, where $f(X)$ should be understood as a cycle, i.e. possibly with multiplicities. This sequence converges pointwise to a limit $h_{f,\mathcal{L}}$ which is independent of the metric chosen. Clearly,

$$h_{\mathcal{L},i}(X) = \frac{1}{(\deg f)^{i(\dim X+1)}} h_{f^{(i)*}(\mathcal{L},\|\cdot\|)}(X).$$

If the isomorphism $\Phi: \mathcal{L}^{\otimes d} \xrightarrow{\cong} f^* \mathcal{L}$ would extend to \mathcal{P} we could say

$$h_{\mathcal{L},i}(X) = h_{\mathcal{L},\Phi^{-1}f^* \dots \Phi^{-1}f^* \|\cdot\|^{1/d^i}}(X).$$

The sequence of metrics obtained on \mathcal{L} converges versus some continuous limit metric. In general, it will be of a fractal nature. Unfortunately, not much is known about the heights related to such metrized invertible sheaves.

1 The adelic Picard group

In this section we recall S. Zhang's adelic Picard group [Zh1] and fix notation.

1.1 The local case. Preparations.

1.1 Let K be an algebraically closed valuation field. The cases we have in mind are $K = \overline{\mathbb{Q}}_p$ for a prime number p and $K = \overline{\mathbb{Q}}_\infty = \mathbb{C}$ but, of course, there are many

other examples, all of which are non-archimedean. Further, let X/K be a projective scheme.

Definition. A *metric* on an invertible sheaf $\mathcal{L} \in \text{Pic}(X)$ is a system of K -norms on the K -vector spaces $\mathcal{L}(x)$ for all $x \in X(K)$.

1.2 Example. Let K be non-archimedean and \mathcal{O}_K the ring of integers in K . Then, a *model* of X , i.e. a flat projective scheme $\pi: \mathcal{X} \rightarrow \mathcal{O}_K$ such that $\mathcal{X}_K \cong X$, and an invertible sheaf $\widetilde{\mathcal{L}} \in \text{Pic}(\mathcal{X})$ with $\widetilde{\mathcal{L}}|_X \cong \mathcal{L}^{\otimes n}$ for some $n > 0$ induce a metric $\|\cdot\|$ on \mathcal{L} as follows:

$x \in X(K)$ has a unique extension $\bar{x}: \text{Spec } \mathcal{O}_K \rightarrow \mathcal{X}$. Then $\bar{x}^* \widetilde{\mathcal{L}}$ is a projective \mathcal{O}_K -module of rank 1. Each $l \in \mathcal{L}(x)$ induces $l^{\otimes n} \in \mathcal{L}^{\otimes n}(x)$ and a rational section of $\bar{x}^* \widetilde{\mathcal{L}}$. Put

$$\|l\|_x := \left[\inf \left\{ |a| \mid a \in K, l \in a \cdot \bar{x}^* \widetilde{\mathcal{L}} \right\} \right]^{\frac{1}{n}}.$$

That $\|\cdot\|$ is called the metric on \mathcal{L} being *induced* by the *model* $(\mathcal{X}, \widetilde{\mathcal{L}})$.

Note here that \mathcal{O}_K will in general be a *non-discrete* valuation ring. In particular, it will usually not be Noetherian.

Nevertheless, projectivity always includes being of finite type [EGA II, Définition (5.5.2)]. This means, for the description of \mathcal{X} only a finite number of elements from \mathcal{O}_K will be needed. In the special case of $\overline{\mathbb{Q}_p}$ the group of values of $\mathcal{O}_{\overline{\mathbb{Q}_p}}$ is isomorphic to $(\mathbb{Q}, +)$ and for any finite set $\{a_1, \dots, a_s\}$ of elements of $\mathcal{O}_{\overline{\mathbb{Q}_p}}$ there exists a *discrete* valuation ring $O \subseteq \mathcal{O}_{\overline{\mathbb{Q}_p}}$ containing a_1, \dots, a_s . By consequence, \mathcal{X} is the base change of a scheme being projective over a discrete valuation ring.

1.3 Definition. Let K be an algebraically closed valuation field.

a) Assume $K = \mathbb{C}$.

Then a metric $\|\cdot\|$ on $\mathcal{L} \in \text{Pic}(X)$ is called *continuous* resp. *smooth*, if for each $x \in X(\mathbb{C})$ there exist some open neighborhood U_x and a holomorphic section $s \in \Gamma(U_x, \mathcal{L})$ such that $\|s\|$ is continuous resp. smooth.

Here a function h on a complex space Z is said to be smooth, if there are an embedding of Z into some complex manifold \overline{Z} and a smooth continuation of h to the whole of \overline{Z} .) $\|\cdot\|$ is called *bounded*, if for a continuous metric the quotient $\|\cdot\|/\|\cdot\|'$ is a bounded function.

b) Assume K to be non-archimedean.

Then a metric $\|\cdot\|$ on $\mathcal{L} \in \text{Pic}(X)$ is called *continuous* resp. *bounded*, if $\|\cdot\| = f \cdot \|\cdot\|'$, where $\|\cdot\|'$ is induced by some model and f is a continuous resp. bounded function on $X(K)$.

1.2 The global case. The adelic Picard group.

Let now X/\mathbb{Q} be a projective variety and $\mathcal{L} \in \text{Pic}(X)$ be a invertible sheaf.

1.4 Definition. An *adelic metric* on \mathcal{L} is a system $\|\cdot\| = \{\|\cdot\|_p\}_{p \in \{2,3,5,\dots;\infty\}}$ of continuous and bounded metrics $\|\cdot\|_p$ on $\mathcal{L}_{\overline{\mathbb{Q}}_p} \in \text{Pic}(X_{\overline{\mathbb{Q}}_p})$ such that

- i) for each $p \in \{2,3,5,\dots;\infty\}$ the metric $\|\cdot\|_p$ is $\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ -invariant,
 - ii) for some $n \in \mathbb{N}$ there exist a projective model \mathcal{X} of X over $\text{Spec } \mathbb{Z}[\frac{1}{n}]$ and an invertible sheaf $\overline{\mathcal{L}} \in \text{Pic}(\mathcal{X})$ with $\overline{\mathcal{L}}|_X \cong \mathcal{L}$, which induces $\|\cdot\|_p$ for almost all p .
- \mathcal{L} being equipped with an adelic metric, is called an *adelicly metrized invertible sheaf*. The group of all adelicly metrized invertible sheaves on X will be denoted by $\text{Pic}^{\text{ad}}(X)$.

1.5 Remark. Let $\mathcal{X}/\text{Spec } \mathbb{Z}$ be a model of X . Then there are the following two natural homomorphisms given by the induced metrics.

$$\begin{aligned} i_{\mathcal{X}}: \widehat{\text{Pic}}(\mathcal{X}) &\rightarrow \text{Pic}^{\text{ad}}(X) \\ a_{\mathcal{X}}: \ker(\text{Pic}(\mathcal{X}) \rightarrow \text{Pic}(X)) \otimes_{\mathbb{Z}} \mathbb{Q} &\rightarrow \text{Pic}^{\text{ad}}(X) \end{aligned}$$

Here $\widehat{\text{Pic}}(\mathcal{X})$ denotes the arithmetic Picard group of Gillet and Soulé [GS]. Further one has the forgetful homomorphism

$$v: \text{Pic}^{\text{ad}}(X) \rightarrow \text{Pic}(X).$$

Note, if $\zeta \in \text{Im } a_{\mathcal{X}}$, then there exists some $n \in \mathbb{N}$ such that $n\zeta \in \text{Im } i_{\mathcal{X}}$.

1.6 Definition. a) The group

$$\text{Pic}^{\text{in}}(X) := \langle \text{Im } i_{\mathcal{X}}, \text{Im } a_{\mathcal{X}} \rangle_{\mathcal{X}/\text{Spec } \mathbb{Z} \text{ model of } X} \subseteq \text{Pic}^{\text{ad}}(X)$$

is called the group of the invertible sheaves on X *with induced metrics*.

b) $\zeta \in \text{Pic}^{\text{in}}(X)$ is called *semipositive*, $\zeta \in C^{\geq 0}$, if there are a natural number n , a model \mathcal{X} of X and $\overline{\mathcal{L}} \in \widehat{\text{Pic}}(\mathcal{X})$ with $n\zeta = i_{\mathcal{X}}(\overline{\mathcal{L}})$, where

- i) $\overline{\mathcal{L}}$ has (after resolution of the singularities of X) a non-negative curvature form $c_1(\overline{\mathcal{L}})$ and
- ii) for every prime number p the underlying bundle \mathcal{L} admits a non-negative degree on each curve in $\mathcal{X}_{\mathbb{F}_p}$.

1.7 Remarks. i) There is also a concept of ampleness in this context. It is given by the conditions that there exists some model $(\mathcal{X}, \overline{\mathcal{L}})$ inducing ζ , for which $\overline{\mathcal{L}}$ be ample and the curvature form be positive.

ii) The property of being semipositive can be tested on every model of X , where there exists a metrized invertible sheaf inducing ζ . Note that the same is not true for ampleness.

1.8 Definition (Metric on $v^{-1}(\mathcal{L}) \subseteq \text{Pic}^{\text{ad}}(X)$). Let $(\mathcal{L}, \|\cdot\|)$ and $(\mathcal{L}, \|\cdot\|')$ be two adelicly metrized invertible sheaves with the same underlying bundle. Then the *distance* between $(\mathcal{L}, \|\cdot\|)$ and $(\mathcal{L}, \|\cdot\|')$ is given by

$$\delta((\mathcal{L}, \|\cdot\|), (\mathcal{L}, \|\cdot\|')) := \sum_{p \in \{2, 3, 5, \dots; \infty\}} \sup_{x \in X(\overline{\mathbb{Q}}_p)} \left| \log \frac{\|\cdot\|'_p}{\|\cdot\|_p} \right|.$$

1.9 Remark. In order to prove that δ is indeed a metric, the only point that requires some consideration, is to show that the sum is always finite.

For this note first that each summand is finite. This is clear for $p = \infty$ and for finite p the metrics are bounded by definition, so assume they are induced. Going over to the pull-back on the fiber product and to tensor powers one may assume $\|\cdot\|$ and $\|\cdot\|'$ to come from two invertible sheaves $\overline{\mathcal{L}}, \overline{\mathcal{L}}' \in \text{Pic}(\mathcal{X})$ on the same model of X with $\overline{\mathcal{L}}|_X \cong \overline{\mathcal{L}}'|_X \cong \mathcal{L}^{\otimes n}$. Then $1 \in \Gamma(X, (\overline{\mathcal{L}}' \otimes \overline{\mathcal{L}}^{-1})|_X)$ can be extended to $\overline{\mathcal{L}}' \otimes \overline{\mathcal{L}}^{-1} \otimes \pi^*(p^k)$ for $k \gg 0$ and therefore $\log \frac{\|\cdot\|'_p}{\|\cdot\|_p}$ is bounded above. Analogously, it is bounded below.

Further almost all the summands vanish. Indeed, as above we may assume the two metrics to be induced by $\overline{\mathcal{L}}, \overline{\mathcal{L}}' \in \text{Pic}(\mathcal{X})$, where $\mathcal{X} / \text{Spec } \mathbb{Z}[\frac{1}{n}]$ is a model of X . But $\overline{\mathcal{L}}$ and $\overline{\mathcal{L}}'$ coincide necessarily over some open neighborhood of the generic fiber.

1.10 Lemma. Let $\mathcal{X} / \text{Spec } \mathbb{Z}$ be a flat, projective scheme of dimension $d + 1$. Denote its generic fiber by X .

Further, let $(\mathcal{L}', \|\cdot\|'), (\mathcal{L}'', \|\cdot\|''), \overline{\mathcal{L}}_1, \dots, \overline{\mathcal{L}}_d \in \widehat{\text{Pic}}(\mathcal{X})$, where \mathcal{L}' and \mathcal{L}'' are extensions of one and the same invertible sheaf $\mathcal{L} \in \text{Pic}(X)$. If $i_{\mathcal{X}}(\overline{\mathcal{L}}_1), \dots, i_{\mathcal{X}}(\overline{\mathcal{L}}_d)$ are semipositive, then

$$\begin{aligned} & \left| \widehat{c}_1(\mathcal{L}', \|\cdot\|') - \widehat{c}_1(\mathcal{L}'', \|\cdot\|'') \right| \cdot \widehat{c}_1(\overline{\mathcal{L}}_1) \cdot \dots \cdot \widehat{c}_1(\overline{\mathcal{L}}_d) \leq \\ & \leq \delta(i_{\mathcal{X}}(\mathcal{L}', \|\cdot\|'), i_{\mathcal{X}}(\mathcal{L}'', \|\cdot\|'')) \cdot c_1(\mathcal{L}_1|_X) \cdot \dots \cdot c_1(\mathcal{L}_d|_X). \end{aligned}$$

Proof. This result is implicitly contained in the work of Zhang [Zh1, Theorem 1.4]. The main ingredient is the fact, that on a variety over a field the intersection product of semipositive divisors is always non-negative. \square

1.11 Definition. Let $\overline{C}^{\geq 0}$ be the closure of the semipositive cone in $\text{Pic}^{\text{ad}}(X)$. The subgroup $\widetilde{\text{Pic}}(X) \subseteq \text{Pic}^{\text{ad}}(X)$ generated by $\overline{C}^{\geq 0}$ is called the *adelic Picard group* of X . If $(\mathcal{L}, \|\cdot\|) \in \widetilde{\text{Pic}}(X)$ then $\|\cdot\|$ is called an *integrable metric* on \mathcal{L} , while $(\mathcal{L}, \|\cdot\|)$ is said to be an *integrably metrized invertible sheaf*.

1.12 Theorem (Zhang). *Let X/\mathbb{Q} be projective and equidimensional. Then there is exactly one continuous map, the "arithmetic intersection product"*

$$\gamma: \underbrace{\widetilde{\text{Pic}}(X) \times \dots \times \widetilde{\text{Pic}}(X)}_{\dim X + 1 \text{ times}} \rightarrow \mathbb{R},$$

such that for every model $\mathcal{X}/\text{Spec } \mathbb{Z}$ of X the diagram

$$\begin{array}{ccc} \widetilde{\text{Pic}}(X) \times \dots \times \widetilde{\text{Pic}}(X) & \xrightarrow{\gamma} & \mathbb{R} \\ \uparrow i_{\mathcal{X}}^{\dim X + 1} & \nearrow \gamma_{\text{Gillet-Soulé}} & \\ \widehat{\text{Pic}}(\mathcal{X}) \times \dots \times \widehat{\text{Pic}}(\mathcal{X}) & & \end{array}$$

commutes. γ is multi-linear.

Proof. Note first that $i_{\mathcal{X}}: \widehat{\text{Pic}}(\mathcal{X}) \rightarrow \widetilde{\text{Pic}}(X)$ is well-defined since $\widehat{\text{Pic}}(X)$ is generated by its semipositive elements.

The intersection product of Gillet and Soulé is compatible with pull-backs under birational maps. All the models of X form an inverse system of schemes, where all the morphisms are birational. Thus all their Picard groups form a direct system and one gets a commutative diagram with $\varinjlim \widehat{\text{Pic}}(\mathcal{X}) \hookrightarrow \text{Pic}^{\text{in}}(X)$ instead of $\widetilde{\text{Pic}}(X)$. Since each element in $\text{Pic}^{\text{in}}(X)$ has a multiple belonging to the direct limit we get purely formally a uniquely determined multi-linear map γ' making the diagram

$$\begin{array}{ccc} \text{Pic}^{\text{in}}(X) \times \dots \times \text{Pic}^{\text{in}}(X) & \xrightarrow{\gamma'} & \mathbb{R} \\ \uparrow i_{\mathcal{X}}^{\dim X + 1} & \nearrow \gamma_{\text{Gillet-Soulé}} & \\ \widehat{\text{Pic}}(\mathcal{X}) \times \dots \times \widehat{\text{Pic}}(\mathcal{X}) & & \end{array}$$

commute. Restricting the left side of the diagram to the semipositive elements one finds a similar diagram of multi-additive maps. But Lemma 1.10 shows that γ' is continuous on $C^{\geq 0} \times \dots \times C^{\geq 0}$ such that one can go over to the closure $\overline{C^{\geq 0}}$ and to the group $\widehat{\text{Pic}}(X)$ spanned by it. \square

1.13 Remarks. i) Zhang's adelic Picard group [Zh1] is a priori a little smaller, because he considers a more restrictive notion of convergence. He takes under consideration only sequences of adelic metrics that are constant outside a finite number of places and where there is uniform convergence at the finitely many other places. ii) One could easily enlarge $\widetilde{\text{Pic}}(X)$ without losing any of the properties proven. Omit just condition ii) in the definition of an adelic metric. Then the distance δ will no longer be a metric, but only a separated écart, since infinite distances are possible, but that does not cause any difficulty.

1.14 Example. *Let K be a number field. Then $\widetilde{\text{Pic}}(\text{Spec } K) = \widehat{\text{Pic}}(\text{Spec } \mathcal{O}_K)$, i.e. there is an exact sequence*

$$0 \longrightarrow \text{Cl}_K \longrightarrow \widetilde{\text{Pic}}(\text{Spec } K) \xrightarrow{\text{deg}} \mathbb{R} \longrightarrow 0.$$

Proof. The models of $\text{Spec } K$ are the orders of K , but all induced metrics come from the maximal order. \square

1.15 Example. Let $C/\text{Spec } \mathbb{Q}$ be a semistable projective curve. Then there is an exact sequence

$$0 \rightarrow \mathbb{A} \rightarrow \widetilde{\text{Pic}}(C) \rightarrow \text{Pic}(C) \rightarrow 0,$$

where $\mathbb{A} = \bigoplus_{p \in \{2,3,5,\dots,\infty\}} C_0(C(\overline{\mathbb{Q}}_p))$. Here $C_0(C(\overline{\mathbb{Q}}_\infty)) = C(C(\overline{\mathbb{Q}}_\infty))^{F_\infty}$ is the space of all continuous F_∞ -invariant functions on $C(\mathbb{C})$ and $C_0(C(\overline{\mathbb{Q}}_p))$ for $p \neq \infty$ is given as follows.

Choose a minimal semistable regular model $\mathcal{C}/\text{Spec } \mathbb{Q}_p$ of C and embed it into the projective space. Let D_1, \dots, D_r be the irreducible divisors in the special fiber. The projective embedding induces on $C(\overline{\mathbb{Q}}_p)$ the structure of a metric space of diameter 1, where $d(x, y) < 1$, if and only if x and y have the same specialization. $C(\overline{\mathbb{Q}}_p)$ has a countable disjoint covering $\{U_{11}, U_{12}, \dots\}$ by open balls of radius $1, \dots$, a countable disjoint covering $\{U_{n1}, U_{n2}, \dots\}$ by open balls of radius $1/p^{n-\frac{1}{2}\min\{j,n\}}, \dots$. In this formula $j: C(\overline{\mathbb{Q}}_p) \rightarrow \mathbb{N}_0 \cup \{\infty\}$ denotes the function giving the minimal number of blow-ups being necessary to specialize some point of $C(\overline{\mathbb{Q}}_p)$ into a smooth point.

Then $C_0(C(\overline{\mathbb{Q}}_p))$ consists of all $\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ -invariant series of the type

$$\sum_{i=1}^r \alpha_i \|1\|_{\mathcal{C}, \sigma(D_i)} + \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \beta_{ij} \chi_{U_{ij}},$$

where $\|\cdot\|_{\mathcal{C}, \sigma(D_i)}$ is the induced metric, $\chi_{U_{ij}}$ denotes the characteristic function of the set U_{ij} , α_i and β_{ij} are real numbers, for each $i \in \mathbb{N}_0$ one has $\lim_{j \rightarrow \infty} \beta_{ij} = 0$ and the outer series converges uniformly.

Sketch of proof. All the invertible sheaves on models of C come from invertible sheaves on \mathcal{C} and its successive blow-ups. The invertible sheaves on \mathcal{C} , which are trivial on the generic fiber, give rise to sums of the type $\sum_{i=1}^r \alpha_i \|1\|_{\mathcal{C}, \sigma(D_i)}$, although only with rational coefficients. Choosing a primitive divisor, which is supported in the blow-up of a point of \mathcal{C} , yields $\chi_{U_{0j}}$, the general $\chi_{U_{ij}}$ appear by successive blow-ups.

The function j comes into play for the following reason. If one blows up a smooth point of the special fiber, in formal coordinates $\text{Spf } \mathbb{Z}_p[[X]]$, then X/p becomes the local coordinate of the exceptional curve, i.e. two points do still have the same specialization, if and only if their distance is less than $\frac{1}{p}$. But if one blows up a crossing point, in formal coordinates $\text{Spf } \mathbb{Z}_p[[X, Y]]/(XY - p)$, then $\frac{X}{Y} = \frac{X^2}{p}$ becomes the local coordinate and two points resist to have the same specialization, if and only if their distance is less than $\frac{1}{p^{1/2}}$. \square

1.16 Lemma. *Let $f: X \rightarrow Y$ be a morphism of projective varieties over \mathbb{Q} . Then there is a natural homomorphism*

$$f^*: \begin{array}{ccc} \widetilde{\text{Pic}}(Y) & \rightarrow & \widetilde{\text{Pic}}(X) \\ (\mathcal{L}, \|\cdot\|) & \mapsto & (f^*\mathcal{L}, f^*\|\cdot\|). \end{array}$$

Proof. Everything we have to show is that the pull-back of an induced metric is again induced. Thus for every model \mathcal{Y} of Y one needs a model \mathcal{X} of X and a morphism $\mathcal{F}: \mathcal{X} \rightarrow \mathcal{Y}$ extending f . Take for \mathcal{X} the closure of the graph of f in $\mathcal{X}' \times \mathcal{Y}$, where \mathcal{X}' is an arbitrary model of X . \square

1.17 Definition. Let M be a projective and equidimensional variety over \mathbb{C} and $\mathcal{G} \in \text{Pic}(M)$. We will call a metric $\|\cdot\|$ on \mathcal{G} *almost semiample*, if there are an embedding $j: M \rightarrow P$ into a compact complex manifold, some $\mathcal{P} \in \text{Pic}(P)$ with $j^*\mathcal{P} \cong \mathcal{G}$ and a sequence of smooth hermitian metrics $\{\|\cdot\|_i\}_{i \geq 0}$ with strictly positive curvature on \mathcal{P} such that $\{j^*\|\cdot\|_i\}_{i \geq 0}$ converges uniformly to $\|\cdot\|$.

1.18 Finally, we want to recall the theorem of successive minima. Let X be projective and equidimensional over \mathbb{Q} and $\overline{\mathcal{L}} \in \widetilde{\text{Pic}}(X)$. Then the *height* of X with respect to $\overline{\mathcal{L}}$ is given by the formula

$$h_{\overline{\mathcal{L}}}(X) := \frac{c_1(\overline{\mathcal{L}})^{\dim X + 1}}{(\dim X + 1) c_1(\mathcal{L}_{\mathbb{Q}})^{\dim X}}.$$

The *height* of a L -valued point $x \in X(L)$ for K a number field is given by

$$h_{\overline{\mathcal{L}}}(x) := \frac{1}{[L : \mathbb{Q}]} \deg \overline{\mathcal{L}}|_X.$$

For $i = 1, 2, \dots, \dim X + 1$ put

$$e_i(\overline{\mathcal{L}}) := \sup_{\text{cod } Y=i} \inf_{x \in X \setminus Y} h_{\overline{\mathcal{L}}}(x).$$

Theorem (Zhang). *Let $\overline{\mathcal{L}} \in \widetilde{\text{Pic}}(X)$, where \mathcal{L} is ample, $\|\cdot\|_p$ is semipositive for all $p \neq \infty$ and $\|\cdot\|_{\infty}$ is almost semiample. Then*

$$e_1(\overline{\mathcal{L}}) \geq h_{\overline{\mathcal{L}}}(X) \geq \frac{e_1(\overline{\mathcal{L}}) + \dots + e_{\dim X + 1}(\overline{\mathcal{L}})}{\dim X + 1}.$$

Proof. This is [Zh1, Theorem 1.10]. It is obtained by an easy limit argument from an analogous statement for arithmetic varieties [Zh2, (5.2)]. Note that the Arithmetic Riemann-Roch Theorem is the main ingredient here. \square

2 The dynamics of f

2.1 The heights $h_{f,\mathcal{L}}$ introduced in above fit into the context of the adelic Picard groups as follows. Equip \mathcal{L} with an adelic metric $\|\cdot\|_0$ such that $(\mathcal{L}, \|\cdot\|_0) \in \text{Pic}^{\text{in}}(P)$. For $p \in \{2, 3, 5, \dots; \infty\}$ consider the sequences $\{\|\cdot\|_{p,i}\}_{i \geq 0}$ such that $\|\cdot\|_{p,i+1} := \Phi^* f^* \|\cdot\|_{p,i}^{\frac{1}{d}}$. They converge uniformly to continuous and bounded metrics $\|\cdot\|_p$ which do not depend [Zh1, Theorem (2.2)] on $\|\cdot\|_{p,0}$ and the sequences are even constant for almost all p . In particular, $\overline{\mathcal{L}} := (\mathcal{L}, \|\cdot\|) \in \widetilde{\text{Pic}}(P)$, the $\|\cdot\|_p$ are semipositive for $p \neq \infty$ and $\|\cdot\|_\infty$ is the limit of a uniformly converging sequence of metrics with strictly positive curvature. In particular, its curvature $c_1(\mathcal{L}_\mathbb{C}, \|\cdot\|_\infty)$ is a non-negative $(1,1)$ -current. One has $h_{f,\mathcal{L}} = h_{\overline{\mathcal{L}}}$. As the construction depends on the iterated pull-backs under f , one should understand the dynamics of the system $\{f^i\}_{i \geq 1}$. As it will turn out to be sufficient for our purposes will pay attention only to the infinite place, i.e. to the dynamics of that system on $P(\mathbb{C})$.

2.2 Let us analyze the special situation of Theorem 0.2. Recall that P/\mathbb{Q} is a projective variety and $f: P \rightarrow P$ a morphism such that $\mathcal{L}^{\otimes d} \cong f^* \mathcal{L}$ for some ample $\mathcal{L} \in \text{Pic}(P)$. This implies automatically that f is finite of degree $d^{\dim P}$. $G \subseteq P$ is an open dense subset being a group scheme such that f has a restriction $m: G \rightarrow G$ being a group homomorphism. Therefore $\ker m$ is a group scheme of finite order $d^{\dim P}$. As f is surjective, $\text{im } g$ is a priori dense, as m is a homomorphism it is surjective, too. There is a compact subgroup $K \subseteq G(\mathbb{C})$ being both forward and backward invariant under $m_\mathbb{C}$. In particular, all the finite groups $\ker(m_\mathbb{C} \circ \dots \circ m_\mathbb{C})$ are contained in K .

2.3 All eigenvalues of the tangent map $T_e m$ are assumed to have absolute value strictly bigger than 1. Therefore, on $G(\mathbb{C})$ there is a left invariant Riemannian metric ν such that $q := \|(T_e m_\mathbb{C})^{-1}\| < 1$. Indeed, based on the Jordan normal form, one finds easily a hermitian scalar product on $T_e G(\mathbb{C})$ guaranteeing the inequality. Take its real part and transport this structure on all $G(\mathbb{C})$ by left multiplication. We will denote the metric on $G(\mathbb{C})$ induced by ν by δ .

Lemma. $m_\mathbb{C}$ induces an injective morphism $\overline{m}_\mathbb{C}: G(\mathbb{C})/K \rightarrow G(\mathbb{C})/K$ of homogeneous spaces. $\overline{m}_\mathbb{C}$ is expanding for the Riemannian metric $\overline{\nu}$ on $G(\mathbb{C})/K$ induced by ν . More precisely, $\overline{\delta}(\overline{m}_\mathbb{C}(Z_1), \overline{m}_\mathbb{C}(Z_2)) \geq \frac{1}{q} \overline{\delta}(Z_1, Z_2)$ for any two compact subsets $Z_1, Z_2 \subseteq G(\mathbb{C})/K$.

Proof. $\overline{m}_\mathbb{C}$ is injective as K is backward invariant under $m_\mathbb{C}$. For the inequality note that for every path p in $G(\mathbb{C})/K$ its preimage $\overline{m}^{-1}(p)$ is a path again. But all tangent maps are expanding by a factor $\geq \frac{1}{q}$. Therefore $l(\overline{m}^{-1}(p)) \leq q \cdot l(p)$, which implies the claim. \square

2.4 Corollary. Let $\overline{G(\mathbb{C})} = G(\mathbb{C}) \cup \{\infty\}$ be the one-point compactification of $G(\mathbb{C})$. Then, for each $x \in G(\mathbb{C}) \setminus K$ the sequence $\{x_i\}_{i \geq 0}$ with $x_0 = x$ and

$x_{i+1} = m_{\mathbb{C}}(x_i)$ converges to ∞ . More precisely, K is the Julia set of the dynamical system $\{m_{\mathbb{C}}^i\}_{i \geq 1}$ on $G(\mathbb{C})$.

Proof. Let $A_0 \subseteq \overline{G(\mathbb{C})} \setminus K$ be compact. Consider the sequence $\{A_i\}_{i \geq 0}$ with $A_{i+1} := m_{\mathbb{C}}(A_i)$. We have to show that for each compact $L \subseteq G(\mathbb{C})$ there are only finitely many i such that $A_i \cap L \neq \emptyset$. Replacing A by KA if necessary, we may assume A to be left K -invariant. But $\overline{m_{\mathbb{C}}}: G(\mathbb{C})/K \rightarrow G(\mathbb{C})/K$ is expanding. \square

2.5 Proposition. *Let μ_i be the measure induced by the smooth $(\dim P, \dim P)$ -form $c_1(\mathcal{L}, \|\cdot\|_{\infty, i}) \wedge \dots \wedge c_1(\mathcal{L}, \|\cdot\|_{\infty, i})$ on $P(\mathbb{C})$. The sequence $\{\mu_i\}_{i \geq 0}$ converges weakly to the measure μ which is the zero measure on $P(\mathbb{C}) \setminus K$ and the Haar measure ρ of volume $\deg_{\mathcal{L}} P$ on K .*

Proof. In order to simplify notations we will write f and m instead of $f_{\mathbb{C}}$ and $g_{\mathbb{C}}$ when there is no danger of confusion. First we note that all μ_i are of volume $\deg_{\mathcal{L}} P$ and observe that the sequence $\{\mu_i\}_{i \geq 0}$ obeys the recursive law $\mu_{i+1} = \frac{1}{d^{\dim P}} m^* \mu_i$.

Then let $g \in C_0(P(\mathbb{C}) \setminus K)$ be a continuous function with compact support. We consider the sequence $\{g_i\}_{i \geq 0}$ with $g_0 := g$ and $g_{i+1} := \frac{1}{d^{\dim P}} f_* g_i$ where f_* of a function is given by summation over the fibers of f . By definition $\mu_i(g) = \mu_0(g_i)$, so let us show $\mu_0(g_i) \rightarrow 0$ for $i \rightarrow \infty$. For that we note that $|g|$ is bounded by some constant C , which implies $|g_i| \leq C$ for all i . Further there is some $\varepsilon > 0$ such that $\text{supp}(g) \subseteq P(\mathbb{C}) \setminus U_{\varepsilon}(K)$. Consequently,

$$|\mu_0(g_i)| \leq C \int_{P(\mathbb{C})} \chi_{(P(\mathbb{C}) \setminus U_{\varepsilon/q^i})} d\mu_0,$$

where χ_A denotes the characteristic function of a measurable set A . The integrand converges monotonically to $\chi_{P(\mathbb{C}) \setminus G(\mathbb{C})}$, which has integral zero. The claim follows by the Theorem of Beppo Levi.

Next we remark that the union over all the groups $\ker(m_{\mathbb{C}} \circ \dots \circ m_{\mathbb{C}})$ is dense in K . Indeed, if not its closure would be a compact Lie group K' properly contained in K . But it is obvious that K' is forward and backward invariant under $m_{\mathbb{C}}$, therefore the map induced by m on the compact homogeneous space K/K' would be injective and expanding.

Let now $g \in C_0(G(\mathbb{C}))$ be a continuous function with compact support. Again we put $g_0 := g$ and $g_{i+1} := \frac{1}{d^{\dim P}} m_* g_i$ and have $\mu_i(g) = \mu_0(g_i)$. As the sequence $\{g_i\}_{i \geq 0}$ is uniformly bounded, it suffices to show that it converges pointwise to the constant $I := \frac{1}{\text{vol } K} \int_K g d\rho$. We claim that it converges even uniformly in every compact set $A \subseteq G(\mathbb{C})$. Note that

$$g_i(x) = \frac{1}{\#\ker m_{\mathbb{C}}^i} \sum_{k \in \ker m_{\mathbb{C}}^i} g(ky),$$

where $m_{\mathbb{C}}(y) = x$. We may assume without loss of generality A to be of type $A = \{x \in G \mid \delta(x, K) \leq N\}$. Then $(m^i)^{-1}(A)$ is of the same type and $N \rightarrow 0$ for

$i \rightarrow \infty$. But g is uniformly continuous on A and m is expanding. So for each $\varepsilon > 0$ there is some i such that for two arbitrary points $x, x' \in A$ one can find y, y' as in the formula above with $\delta(y, y') < \varepsilon$. Consequently, the total variation of g_i on A tends to zero for $i \rightarrow \infty$. Furthermore, I is the mean value of g_i on K . \square

2.6 Caution. It is not true in general that the curvature current $c_1(\mathcal{L}_{\mathbb{C}}, \|\cdot\|_{\infty})$ vanishes on $P(\mathbb{C}) \setminus K$. Nevertheless one has at least the following

2.7 Lemma. *The curvature current $c_1(\mathcal{L}_{\mathbb{C}}, \|\cdot\|_{\infty})$ has the properties below.*

- a) *Its restriction $c_1(\mathcal{L}_{\mathbb{C}}, \|\cdot\|_{\infty})|_{G(\mathbb{C})}$ is left and right K -invariant.*
- b) *It satisfies the equation $f^*c_1(\mathcal{L}_{\mathbb{C}}, \|\cdot\|_{\infty}) = dc_1(\mathcal{L}_{\mathbb{C}}, \|\cdot\|_{\infty})$.*

Proof. a) By construction $c_1(\mathcal{L}_{\mathbb{C}}, \|\cdot\|_{\infty, i})$ is invariant under $\ker(m_{\mathbb{C}}^i)$, so $c_1(\mathcal{L}_{\mathbb{C}}, \|\cdot\|_{\infty})$ being the weak limit of that sequence of currents, is invariant under $\bigcup_i \ker(m_{\mathbb{C}}^i)$. Invariance under any $k \in K$ follows as $k_i \rightarrow k$ implies that the test forms $k_i\omega$ converge to $k\omega$ in the Schwartz space. b) is clear. \square

3 Approximation by smooth metrics with non-negative curvature

3.1 Remark. One of the most natural ways to produce a new adelic metric from an old one is to replace $\|\cdot\|_{\infty}$ by $\|\cdot\|_{\infty} \cdot \exp(-g)$ for some continuous function $g \in C(P(\mathbb{C}))$. As we want to use Theorem 1.18 as a fundamental tool, it is necessary to understand for which g this metric will be the limit of a uniformly convergent sequence of smooth metrics with strictly positive curvature.

3.2 In this section we will continue to use the notations of Theorem 0.2. Further, denote by S the set of all continuous functions $g \in P(\mathbb{C})$ such that $\|\cdot\|_{\infty} \cdot \exp(-g)$ is almost semiample.

Lemma. a) *One has $C \in S$ for every constant C .*

b) *If $g_1, g_2 \in S$ and $0 < a < 1$ is a real number, then $ag_1 + (1 - a)g_2 \in S$.*

c) *If $g \in S$, then $\frac{1}{d}f^*g \in S$.*

d) *Let $g \in S$ be a function with $\text{supp } g \subseteq G(\mathbb{C})$. Then for each $k \in K$ one has $k \cdot g \in S$, where $(k \cdot g)(x) := g(k^{-1}x)$.*

e) *Let $\{g_i\}_{i \geq 0}$ be a sequence of functions $g_i \in S$ that converges uniformly to some $g \in C(P(\mathbb{C}))$. Then $g \in S$.*

Proof. e) is trivial.

a) $\|\cdot\|_{\infty}$ is defined by the limit procedure described in 2.1. Start with some $\|\cdot\|_{\infty, 0}$ having a strictly positive curvature form. Then all $\|\cdot\|_{\infty, i}$ have strictly positive

curvature, too. $\{\|\cdot\|_{\infty,i} \cdot \exp(-C)\}_{i \geq 0}$ will repeat their curvatures and converge uniformly to $\|\cdot\|_{\infty} \cdot \exp(-C)$.

b) Let $\{\|\cdot\|'_i\}_{i \geq 0}$ and $\{\|\cdot\|''_i\}_{i \geq 0}$ be sequences of smooth hermitian metrics with strictly positive curvature on $\mathcal{L}_{\mathbb{C}}$ that converge to $\|\cdot\|_{\infty} \cdot \exp(-g_1)$, resp. $\|\cdot\|_{\infty} \cdot \exp(-g_2)$. Put $\|\cdot\|_i := \|\cdot\|'^a_i \cdot \|\cdot\|''^{(1-a)}_i$. One sees easily that these metrics have strictly positive curvature and that uniformly $\|\cdot\|_i \rightarrow \|\cdot\|_{\infty} \cdot \exp(-ag_1 - (1-a)g_2)$.

c) Put $\|\cdot\|'_\infty := \|\cdot\|_{\infty} \cdot \exp(-g)$. Then

$$\Phi^* f^* \|\cdot\|'^{\frac{1}{d}}_\infty = \Phi^* f^* \|\cdot\|_{\infty} \cdot \exp(-\frac{1}{d} f^* g) = \|\cdot\|_{\infty} \cdot \exp(-\frac{1}{d} f^* g).$$

Therefore, if $\{\|\cdot\|'_{\infty,i}\}_{i \geq 0}$ is a sequence of metrics with positive curvature that converges uniformly to $\|\cdot\|'_\infty$, then we are allowed to consider $\{\Phi^* f^* \|\cdot\|'^{\frac{1}{d}}_{\infty,i}\}_{i \geq 0}$ for $\|\cdot\|_{\infty} \cdot \exp(-\frac{1}{d} f^* g)$.

d) By e) we may assume $k \in \ker(m^j)$ for some $j \in \mathbb{N}_+$. Denote by $e_k: G(\mathbb{C}) \rightarrow G(\mathbb{C})$ the left multiplication by k . To show $\|\cdot\|_{\infty} \cdot \exp(-k \cdot g)$ is almost semiample, it suffices to consider the hermitian metric $\|\cdot\|^{d^j}_{\infty} \cdot \exp(-k \cdot d^j g)$ on $\mathcal{L}_{\mathbb{C}}^{\otimes d^j}$. But there is a natural isomorphism $e_k^* \mathcal{L}_{\mathbb{C}}^{\otimes d^j}|_{G(\mathbb{C})} \cong e_k^*(f^j)^* \mathcal{L}_{\mathbb{C}}|_{G(\mathbb{C})} \cong (f^j)^* \mathcal{L}_{\mathbb{C}}|_{G(\mathbb{C})} \cong \mathcal{L}_{\mathbb{C}}^{\otimes d^j}|_{G(\mathbb{C})}$, where by construction $\|\cdot\|^{d^j}_{\infty,i}$ and $e_k^* \|\cdot\|^{d^j}_{\infty,i}$ coincide under the identification above as soon as $i \geq j$.

Consider any sequence $\{\|\cdot\|'_i\}_{i \geq 0}$ of smooth hermitian metrics with strictly positive curvature on $\mathcal{L}_{\mathbb{C}}^{\otimes d^j}|_{G(\mathbb{C})}$ that converges uniformly to $\|\cdot\|^{d^j}_{\infty} \cdot \exp(-d^j g)$. Then $\{e_k^* \|\cdot\|'_i\}_{i \geq j}$ will do what we want on $G(\mathbb{C})$, while $\{\|\cdot\|^{d^j}_{\infty,i}\}_{i \geq 0}$ does it on some neighborhood of $P(\mathbb{C}) \setminus G(\mathbb{C})$. Use some decomposition of the unit. \square

3.3 Proposition. *Let M be a projective complex manifold, $\mathcal{G} \in \text{Pic}(M)$ be ample and $\{U_1, \dots, U_n\}$ an open covering. Let $\|\cdot\|_1, \dots, \|\cdot\|_n$ be continuous hermitian metrics on $\mathcal{G}|_{U_1}, \dots, \mathcal{G}|_{U_n}$ such that for some $\delta > 0$ the sets*

$$D_{i,\delta} := \{x \in U_i \mid \|\cdot\|_{i,(x)} \leq (1 + \delta) \cdot \min\{\|\cdot\|_{1,(x)}, \dots, \|\cdot\|_{n,(x)}\}\}$$

are compact subsets of the U_i . Assume each $\|\cdot\|_i$ is the limit of a uniformly convergent sequence of smooth metrics with strictly positive curvature. Then $\|\cdot\| := \min_i \|\cdot\|_i$ is a continuous hermitian metric and almost semipositive.

Proof. Continuity is trivial. For almost semipositivity we may assume \mathcal{G} to be very ample. Let $\{\|\cdot\|_{ij}\}_{j \geq 0}$ be a sequence of smooth and strictly positively curved metrics on $\mathcal{G}|_{U_i}$ that converges uniformly to $\|\cdot\|_i$. Put $\|\cdot\|_{\tilde{j}} := \min\{\|\cdot\|_{1j}, \dots, \|\cdot\|_{nj}\}$ to be the pointwise minimum of the approximating metrics. It is a continuous hermitian metric on the whole of \mathcal{G} for $j \gg 0$ and the sequence $\{\|\cdot\|_{\tilde{j}}\}_{j \gg 0}$ converges uniformly to $\|\cdot\|$.

We choose an embedding $i: M \hookrightarrow \mathbf{P}^N$ such that $i^* \mathcal{O}(1) \cong \mathcal{G}$ and extend the $\|\cdot\|_{ij}$ to smooth metrics $\|\cdot\|'_{ij}$ on $\mathcal{O}(1)$ being defined on open subsets $W_i \supseteq U_i$ of \mathbf{P}^N . Under the tautological action $\mathrm{PGL}_n(\mathbb{C}) \times \mathbf{P}^N \rightarrow \mathbf{P}^N$ each $\gamma \in \mathrm{PGL}_n(\mathbb{C})$ defines an automorphism $e_\gamma: \mathbf{P}^N \rightarrow \mathbf{P}^N$ such that there is a natural identification $e_\gamma^* \mathcal{O}(1) \cong \mathcal{O}(1)$. We find an open neighborhood O of $e \in \mathrm{PGL}_n(\mathbb{C})$ such that for all $\gamma \in O$ and all $i \in \{1, \dots, n\}$ the pull-back $e_\gamma^* \|\cdot\|'_{ij}$ is well-defined on $D_{i,\delta}$ and $e_\gamma^* \|\cdot\|'_{ij}|_{U_i \cap e_\gamma^{-1}(W_i)}$ has strictly positive curvature in every point of $D_{i,\delta}$.

We claim that for each $\gamma \in O$ the perturbation

$$\|\cdot\|_j^{\sim,\gamma} := e_\gamma^*(\min\{\|\cdot\|'_{1j}, \dots, \|\cdot\|'_{nj}\})|_M$$

of $\|\cdot\|_j^{\sim}$ has a strictly positive curvature current in each holomorphic curve inside M . Indeed, this is a local statement, so for $x \in M$ fix a holomorphic section $s \in \Gamma(U, \mathcal{G})$ defined in some neighborhood U of x . Then

$$c_1(\mathcal{G}, \|\cdot\|_j^{\sim,\gamma})|_U = -dd^c \log (\|s\|_j^{\sim,\gamma})^2 = dd^c(\max\{-\log e_\gamma^* \|s\|_{1j}^{\prime 2}, \dots, -\log e_\gamma^* \|s\|_{nj}^{\prime 2}\}).$$

But the maximum of a finite system of plurisubharmonic functions is plurisubharmonic again.

Now copy Sobolev's averaging procedure. For every non-negative smooth function $\varphi \neq 0$ on $\mathrm{PGL}_n(\mathbb{C})$ with $\mathrm{supp} \varphi \subseteq O$ one gets the approximation

$$\|\cdot\|_j^{\sim,\varphi} := \int_{\mathrm{PGL}_n(\mathbb{C})} \varphi(\gamma) \cdot \|\cdot\|_j^{\sim,\gamma} d\rho(\gamma) \Big/ \int_{\mathrm{PGL}_n(\mathbb{C})} \varphi(\gamma) d\rho(\gamma)$$

of $\|\cdot\|_j^{\sim}$. Here ρ is the left Haar measure on $\mathrm{PGL}_n(\mathbb{C})$. Consider a sequence of such φ that converges weakly to δ_e . All $\|\cdot\|_j^{\sim,\varphi}$ are smooth, positively curved and they form a sequence that converges uniformly to $\|\cdot\|_j^{\sim}$. \square

3.4 Corollary. *If $g_1, g_2 \in S$, then $g := \max\{g_1, g_2\} \in S$.* \square

3.5 Corollary. *Let $l \in \mathbb{N}_+$, $0 < a < 1$ be a real number and $0 \neq s \in \Gamma(P(\mathbb{C}), \mathcal{L}_{\mathbb{C}}^{\otimes l})$ a global section. Then $g := \max\{a \log \|s\|^l, 0\} \in S$.*

Proof. Consider the function \bar{g} on $U := \{x \in P(\mathbb{C}) \mid \log \|s\|_{(x)}^l > -1\}$ with $\bar{g}(x) = a \log \|s\|_{(x)}^l$. We have to show that the hermitian metric

$$\|\cdot\|'_\infty := \|\cdot\|_\infty^l \cdot \exp(-\bar{g})$$

on $\mathcal{L}_{\mathbb{C}}^{\otimes l}|_U$ is the limit of a uniformly convergent sequence of smooth metrics with positive curvature. Put $\|\cdot\|'_{\infty,i} := \|\cdot\|_{\infty,i}^l / (\|s\|_{\infty,i}^l)^a$. It is clear that there is uniform

convergence $\|\cdot\|'_{\infty,i} \rightarrow \|\cdot\|'_\infty$. Further

$$\begin{aligned} & c_1(\mathcal{L}_{\mathbb{C}}|_U, \|\cdot\|'_{\infty,i}) \\ &= dd^c(-\log \|s\|_{\infty,i}^2) \\ &= dd^c(-\log(\|s\|_{\infty,i}^l)^{2(1-a)}) \\ &= (1-a)l \cdot c_1(\mathcal{L}_{\mathbb{C}}|_U, \|\cdot\|_{\infty,i}) > 0. \end{aligned} \quad \square$$

3.6 Lemma. *Let $g \in S$ with $\text{supp } g \subseteq G(\mathbb{C})$. Then also $\underline{g} \in S$ for the K -invariant function \underline{g} with*

$$\underline{g}(x) := \begin{cases} \max_{k \in K} g(kx) & \text{if } x \in G(\mathbb{C}), \\ g(x) & \text{otherwise.} \end{cases}$$

Proof. \underline{g} is the limit of the uniformly convergent sequence $\{g_i\}_{i \geq 0}$ with

$$g_i(x) := \max_{k \in \ker(m_{\mathbb{C}}^i)} g(kx). \quad \square$$

3.7 Proposition. *Let $\varepsilon > 0$ and $U \subseteq G(\mathbb{C})$ an open set containing K . Then there is some non-negative K -invariant function $0 \neq g \in \mathbb{R}_+ \cdot S$ such that $g(e) = 1$, $\max_{x \in P(\mathbb{C})} g(x) \leq 1 + \varepsilon$ and $\text{supp } g \subseteq U$.*

Proof. Let $D := P(\mathbb{C}) \setminus G(\mathbb{C})$. For some $j \in \mathbb{N}_+$ the coherent sheaf $\mathcal{L}^{\otimes j} \otimes \mathcal{I}_D$ has a section, that does not vanish in $e \in G$, i.e. $\mathcal{L}^{\otimes j}$ has a section s vanishing in D but not in e . Using Corollary 3.5 with $j = l$ we see $\tilde{g}_C := \max\{\frac{1}{2j} \log \|Cs\|^j, 0\} = \max\{\frac{1}{2j} \log \|s\|^j + \frac{\log C}{2j}, 0\} \in S$ for every $C > 0$. Obviously, $\text{supp } \tilde{g}_C \subseteq G(\mathbb{C})$ are compact sets.

With $A := \max_{x \in K} \frac{1}{2j} \log \|s\|_{(x)}^j$ and $B := \max_{x \in P(\mathbb{C})} \frac{1}{2j} \log \|s\|_{(x)}^j$ choose C such that $C+A > 0$ and $\frac{C+B}{C+A} \leq 1 + \varepsilon$. Then let g_0 be the associated K -invariant function to \tilde{g}_C as in Lemma 3.6 and put recursively $g_{i+1} := \frac{1}{d} f^* g_i$. Clearly, one has $g_i \in S$, $g_i(e) > 0$ and $\max_{x \in P(\mathbb{C})} g_i(x) / \max_{x \in K} g_i(x) \leq 1 + \varepsilon$ for every i . For $i \gg 0$ by Lemma 2.3 $\text{supp } g_i \subseteq U$. Therefore some multiple will have all the properties desired. \square

4 The equidistribution theorems

4.1 Let X/\mathbb{Q} be a projective variety and $\overline{\mathcal{L}} \in \widetilde{\text{Pic}}(X)$. One calls a sequence $\{x_i\}_{i \geq 0}$ of points in $X(\overline{\mathbb{Q}})$ *generic*, if no infinite subsequence is contained in a proper closed subvariety of X . In other words, $\{x_i\}_{i \geq 0}$ is generic, if the underlying sequence in X converges to the generic point.

Assume $h_{\overline{\mathcal{L}}}(X) = 0$ and $e_{\dim X+1}(\overline{\mathcal{L}}) = 0$. Then a sequence $\{x_i\}_{i \geq 0}$ in $X(\overline{\mathbb{Q}})$ is called *small*, if $h_{\overline{\mathcal{L}}}(x_i) \rightarrow 0$. Note, if X is a closed subvariety of a projective variety P/\mathbb{Q} and $\overline{\mathcal{L}} \in \overline{\text{Pic}}(P)$ as in Theorem 1.18, where $e_{\dim P+1}(\overline{\mathcal{L}}) \geq 0$, then $h_{\overline{\mathcal{L}}}(X) = 0$ implies the existence of a small and generic sequence in X .

For $x \in X(\overline{\mathbb{Q}})$ we denote by δ_x the distribution associated to its Galois orbit, i.e. if $x \in X(F)$ for some number field F , then

$$\delta_x(f) := \frac{1}{\#\{\sigma: K \hookrightarrow \mathbb{C}\}} \sum_{\sigma: F \hookrightarrow \mathbb{C}} f(\sigma(x))$$

for each $f \in C^\infty(X(\mathbb{C}))$.

4.2 Lemma. *Let L be a compact group and $T \subseteq C(L)$ a set of continuous real-valued functions such that*

- i) T contains all the constants,
- ii) T is closed under addition and multiplication with positive constants,
- iii) for $x \in L$ with $x \neq e$ there exists $g \in T$ such that $g(e) > g(x)$,
- iv) for each $l \in L$ and $g \in T$ one has the shift $l \cdot g \in T$,
- v) if $g \in T$, then $g_+ \in T$ where $g_+(x) = \max\{g(x), 0\}$,
- vi) T is closed under uniform convergence.

Then $T = C(L)$ contains all the continuous functions.

Proof. We fix a Haar measure ρ on L . The conditions iv) and vi) imply that, if $h \in T$ and g is non-negative, measurable and bounded, then their convolution $g * h \in T$. Restricting our considerations to the case g is continuous we easily see that it suffices to show that for each open U containing e there is some non-negative $g_U \in T$ with $g_U \neq 0$ and $\text{supp } g_U \subseteq U$. For this, by i), ii) and v) we need a function $g \in T$ having its maximum entirely in U .

Assume for some U_0 that would be impossible. Then for every $g \in T$ having its maximum in e we find a non-empty $A_g := \{y \in L \setminus U_0 \mid g(y) = g(e)\} \subseteq L \setminus U_0$, where that maximum is taken, too. If $\bigcap_{g \in T} A_g = \emptyset$, then by compactness already a finite intersection $A_{g_1} \cap \dots \cap A_{g_n} = \emptyset$ would be empty and $g_1 + \dots + g_n$ had its maximum in U_0 only.

Therefore $A_{U_0} := \bigcap_{g \in T} A_g$ is non-empty. Put $A := \{e\} \cup \bigcup_{U \subseteq U_0 \text{ open}} A_U$. Every function $g \in T$ having maximum in e has it in all A . But for $y \in A$ the shift $y^{-1} \cdot g$ has its maxima in $y^{-1}A$, therefore $A = y^{-1}A$ proving that A is a group. Similarly, one sees $yAy^{-1} = A$ for every $y \in L$, so A is even normal. Denote by $i: A \rightarrow L$ the natural inclusion.

Fix a Haar measure ρ_A on A . The construction of A , together with i), ii) and v), implies that there is a sequence of functions in T such that the associated distribu-

tions converge weakly to $i_*\rho_A$. The convolution argument above, i.e an application of iv) and vi) makes sure that all A -invariant continuous functions belong to T .

Now use iii). We have some $x_0 \in A$ and $g \in T$ such that g is non-positive on A , $g(e) = 0$ and $g(x_0) =: C < 0$. Define a continuous function M on L by $M(x) := -\sup_{a \in A} g(ax)$. Obviously, M is A -invariant, so $M \in T$. Consequently, $g + M \in T$, but $g(x) \leq 0$ for every $x \in L$, $g(e) = 0$ and $g(x_0) < 0$, which is a contradiction to the properties of A . \square

4.3 Now return to the situation of Theorem 0.2.

Proposition. *Let $\{x_i\}_{i \geq 0}$ be a generic and small sequence of points in $P(\overline{\mathbb{Q}})$. Assume some non-negative $f \in C(P(\mathbb{C}))$ fulfills either $\text{supp } f \subseteq P(\mathbb{C}) \setminus K$ or $f \in \mathbb{R}_+ \cdot S$. Then*

$$\liminf_{i \rightarrow \infty} \int_{P(\mathbb{C})} f d\delta_{x_i} \geq \int_{P(\mathbb{C})} f d\tau,$$

where τ is the zero measure on $P(\mathbb{C}) \setminus K$ and the Haar measure of volume 1 on K .

Proof. For f with $\text{supp } f \subseteq P(\mathbb{C}) \setminus K$ the right hand side is zero, so the statement is clear. If $f \in \mathbb{R}_+ \cdot S$, then almost the same argument works as in [SUZ] or [Zh3]. For any positive $\lambda \in \mathbb{R}$ let $\|\cdot\|_\lambda$ be the adelic metric on \mathcal{L} with $\|\cdot\|_{\lambda,p} = \|\cdot\|_p$ for the finite places and $\|\cdot\|_{\lambda,\infty} = \|\cdot\|_\infty \cdot \exp(-\lambda f)$. For $\lambda \rightarrow 0$ the adelic metric $\|\cdot\|_\lambda$ fulfills the assumptions of Theorem 1.18. Clearly, one has

$$h_{\mathcal{L}, \|\cdot\|_\lambda}(x_i) = h_{\overline{\mathcal{L}}}(x_i) + \lambda \int_{P(\mathbb{C})} f d\delta_{x_i}.$$

Further,

$$\begin{aligned} h_{\mathcal{L}, \|\cdot\|_\lambda}(P) &= \frac{1}{(\dim P + 1) c_1(\mathcal{L}_{\mathbb{Q}})^{\dim P}} \gamma((\mathcal{L}, \|\cdot\|_\lambda), \dots, (\mathcal{L}, \|\cdot\|_\lambda)) \\ &= h_{\overline{\mathcal{L}}}(P) + \frac{1}{\deg_{\mathcal{L}} P} \gamma((\mathcal{O}_P, |\cdot|_\infty \cdot \exp(-\lambda f)), (\mathcal{L}, \|\cdot\|), \dots, (\mathcal{L}, \|\cdot\|)) \\ &\quad + O(\lambda^2) \\ &= h_{\overline{\mathcal{L}}}(P) + \frac{1}{\deg_{\mathcal{L}} P} \lambda \int_{P(\mathbb{C})} f d\mu + O(\lambda^2), \end{aligned}$$

where μ is computed in Proposition 2.5. As $\frac{1}{\deg_{\mathcal{L}} P} \mu = \tau$, Theorem 1.18 yields the claim. \square

4.4 Theorem (Equidistribution in $P(\mathbb{C})$). *Let P/\mathbb{Q} and $\overline{\mathcal{L}} \in \widetilde{\text{Pic}}(P)$ be as described in Theorem 0.2. Then for each generic and small sequence $\{x_i\}_{i \geq 0}$ in P*

the associated sequence of measures $\{\delta_{x_i}\}_{i \geq 0}$ converges weakly to τ , being the zero measure on $P(\mathbb{C}) \setminus K$ and the Haar measure of volume one on K .

Proof. Let us show first $\delta_{x_i}|_{P(\mathbb{C}) \setminus K} \rightarrow 0$. We know, for every $\varepsilon_1, \varepsilon_2 > 0$ there is some non-negative $g_{\varepsilon_1, \varepsilon_2} \in S$ such that $\text{supp } g_{\varepsilon_1, \varepsilon_2} \subseteq U_{\varepsilon_1}(K)$, $\max_{x \in P(\mathbb{C})} g_{\varepsilon_1, \varepsilon_2} \leq 1 + \varepsilon_2$ and $\int_{P(\mathbb{C})} g_{\varepsilon_1, \varepsilon_2} d\tau = 1$. Therefore, by Proposition 4.3,

$$\liminf_{i \rightarrow \infty} \int_{P(\mathbb{C})} g_{\varepsilon_1, \varepsilon_2} d\delta_{x_i} \geq 1, \quad \text{i.e.} \quad \limsup_{i \rightarrow \infty} \int_{P(\mathbb{C})} (1 - g_{\varepsilon_1, \varepsilon_2}) d\delta_{x_i} \leq 0.$$

But

$$1 - g_{\varepsilon_1, \varepsilon_2}(x) \geq \begin{cases} 1 & \text{if } x \in P(\mathbb{C}) \setminus U_{\varepsilon_1}(K), \\ -\varepsilon_2 & \text{if } x \in U_{\varepsilon_1}(K), \end{cases}$$

therefore

$$\limsup_{i \rightarrow \infty} [\delta_{x_i}(P(\mathbb{C}) \setminus U_{\varepsilon_1}(K)) - \varepsilon_2 \delta_{x_i}(U_{\varepsilon_1}(K))] \leq 0,$$

i.e. $\limsup_{i \rightarrow \infty} \delta_{x_i}(P(\mathbb{C}) \setminus U_{\varepsilon_1}(K)) \leq \varepsilon_2$. As this is true for every $\varepsilon_2 > 0$ we have

$$\lim_{i \rightarrow \infty} \delta_{x_i}(P(\mathbb{C}) \setminus U_{\varepsilon_1}(K)) = 0.$$

But this is true for each $\varepsilon_1 > 0$, i.e. $\{\delta_{x_i}|_{P(\mathbb{C}) \setminus K}\}_{i \geq 0}$ converges weakly to the zero measure.

Now let $f \in C(P(\mathbb{C}))$ be an arbitrary continuous function. As we can interchange the roles of f and $-f$ it suffices to prove

$$\liminf_{i \rightarrow \infty} \int_{P(\mathbb{C})} f d\delta_{x_i} \geq \int_{P(\mathbb{C})} f d\tau.$$

Lemma 3.2 and Corollaries 3.4 and 3.5 make sure that $\mathbb{R}_+ \cdot S$ fulfills all the assumptions of Lemma 4.2, except closedness under uniform convergence. Therefore for every $\varepsilon > 0$ there is some $f_1 \in \mathbb{R}_+ \cdot S$ such that $|f(x) - f_1(x)| < \frac{\varepsilon}{4}$ for all $x \in K$. Then an easy application of the decomposition of the unit shows that there is some $f_2 \in C(P(\mathbb{C}))$ with $\text{supp } f_2 \subseteq P(\mathbb{C}) \setminus K$ and $|f(x) - f_1(x) - f_2(x)| < \frac{\varepsilon}{2}$ for all $x \in P(\mathbb{C})$. We showed above that

$$\lim_{i \rightarrow \infty} \int_{P(\mathbb{C})} f_2 d\delta_{x_i} = 0.$$

Proposition 4.3 shows

$$\liminf_{i \rightarrow \infty} \int_{P(\mathbb{C})} f_1 d\delta_{x_i} \geq \int_{P(\mathbb{C})} f_1 d\tau.$$

Consequently,

$$\liminf_{i \rightarrow \infty} \int_{P(\mathbb{C})} f d\delta_{x_i} \geq \int_{P(\mathbb{C})} f d\tau - \varepsilon.$$

But $\varepsilon > 0$ was arbitrary. □

4.5 Corollary. *Let $X \subset P$ be an irreducible closed subvariety and $\{x_i\}_{i \geq 0}$ a small and generic sequence in X . Then the sequence $\{\delta_{x_i}|_{X(\mathbb{C}) \setminus K}\}_{i \geq 0}$ converges weakly to the zero measure.*

Proof. As there are only finitely many i such that $x_i \notin G$ let us assume the whole sequence $\{x_i\}_{i \geq 0}$ lies in G . Then choose some K -invariant metric on $G(\mathbb{C})$ and assume that for some $\varepsilon > 0$ one would have

$$\liminf_{i \rightarrow \infty} \delta_{x_i}(X(\mathbb{C}) \setminus U_\varepsilon(K)) = \delta > 0.$$

We want to construct another sequence $\{y_j\}_{j \geq 0}$ being generic and small on the whole of P . Note first that we have $h_{f, \mathcal{L}}(x) = \frac{1}{\deg f} h_{f, \mathcal{L}}(f(x))$. Consequently, $h_{f, \mathcal{L}}$ is invariant under shift by a torsion point $t \in K_{\text{tor}} = \bigcup_{j \in \mathbb{N}} \ker(m_{\mathbb{Q}}^j)$. In particular, all the sequences $\{t \cdot x_i\}_{i \geq 0}$ fulfill

$$\liminf_{i \rightarrow \infty} \delta_{t \cdot x_i}(P(\mathbb{C}) \setminus U_\varepsilon(K)) = \delta$$

and for each i the union $\bigcup_{t \in K_{\text{tor}}} \{t \cdot x_i\}$ is a Zariski dense subset in P . Finally, observe that there are only countably many proper closed subvarieties $P_0, P_1, P_2, \dots \subset P$. Therefore we can choose a sequence $\{y_j\}_{j \geq 0}$ such that for every $i \in \mathbb{N}$ one has $y_i \in P \setminus P_0 \setminus \dots \setminus P_i$, $y_i \in \bigcup_{t \in K_{\text{tor}}, i \geq 0} \{t \cdot x_i\}$, but also $\delta_{y_i}(P(\mathbb{C}) \setminus U_\varepsilon(K)) > \delta/2$ and $h_{f, \mathcal{L}}(y_i) \leq \frac{1}{i}$. This is a contradiction to Theorem 4.4 above. □

4.6 Let $X \subset P$ be an irreducible closed subvariety. Then there is a linear functional I on $\mathbb{R} \cdot S \subseteq C(X(\mathbb{C}))$ defined by $\overline{\mathcal{L}} \in \widetilde{\text{Pic}}(P)$ that is compatible with uniform convergence. Indeed, put

$$I(f) := \gamma \left((\mathcal{O}_X, |\cdot|_\infty \cdot \exp(-f)), \underbrace{(\mathcal{L}, \|\cdot\|), \dots, (\mathcal{L}, \|\cdot\|)}_{\dim X \text{ times}} \right),$$

where γ is Zhang's intersection product (cp. Theorem 1.12). Note that $f \in \mathbb{R} \cdot S$ makes sure that $|\cdot|_\infty \cdot \exp(-f)$ is integrable, i.e. $(\mathcal{O}_X, |\cdot|_\infty \cdot \exp(-f)) \in \widetilde{\text{Pic}}(X)$.

Theorem (Equidistribution on $X(\mathbb{C})$). *Let P/\mathbb{Q} and $\overline{\mathcal{L}} \in \widetilde{\text{Pic}}(P)$ be as described in Theorem 0.2 and $X \subset P$ an irreducible closed subvariety. Assume on X there is a small and generic sequence of points $\{x_i^0\}_{i \geq 0}$.*

a) Then there is a unique extension of I to some measure σ on $X(\mathbb{C})$ such that $\sigma|_{X(\mathbb{C}) \setminus K} = 0$ is the zero measure.

b) The sequence $\{I_i\}_{i \geq 0}$ of the measures with

$$I_i(f) := \gamma \left((\mathcal{O}_X, |\cdot|_\infty \cdot \exp(-f)), (\mathcal{L}, \|\cdot\|_i), \dots, (\mathcal{L}, \|\cdot\|_i) \right)$$

converges weakly to σ .

c) For any small and generic sequence $\{x_i\}_{i \geq 0}$ in X the associated sequence of measures $\{\delta_{x_i}\}_{i \geq 0}$ converges weakly to $\frac{1}{\deg_{\mathcal{L}} X} \sigma$.

Proof. c) and a) Let us work with functions on the whole of $P(\mathbb{C})$. If $f \in \mathbb{R}_+ \cdot S \subseteq C(P(\mathbb{C}))$ we can repeat the argument from Proposition 4.3 almost word-by-word and obtain

$$\liminf_{i \rightarrow \infty} \int_{X(\mathbb{C})} f d\delta_{x_i} \geq \frac{1}{\deg_{\mathcal{L}} X} I(f).$$

Therefore one has as well

$$\limsup_{i \rightarrow \infty} \int_{X(\mathbb{C})} f d\delta_{x_i} \leq \frac{1}{\deg_{\mathcal{L}} X} I(f),$$

if $f \in \mathbb{R}_- \cdot S$. Further, by Corollary 4.5 we know

$$\lim_{i \rightarrow \infty} \int_{X(\mathbb{C})} f d\delta_{x_i} = 0,$$

if $\text{supp } f \subseteq P(\mathbb{C}) \setminus K$. But as we saw in the proof of Theorem 4.4, every $f \in C(P(\mathbb{C}))$ can be uniformly approximated by sums of a function from $\mathbb{R}_+ \cdot S$ and a continuous function whose support is contained in $P(\mathbb{C}) \setminus K$ and as well by sums of a function in $\mathbb{R}_- \cdot S$ and a continuous function whose support is contained in $P(\mathbb{C}) \setminus K$.

b) Assume for some f_0 with $\text{supp } f_0 \subseteq P(\mathbb{C}) \setminus K$ one would have $\liminf_{i \rightarrow \infty} I_i(f_0) > 0$. Clearly, we may assume f_0 even bigger and multiply with constants, so require without restriction that, for some $\varepsilon_1, \varepsilon_2 > 0$,

$$(1 - f_0)(x) = \begin{cases} 1 & \text{if } x \in U_{\varepsilon_1}(K), \\ 0 & \text{if } x \in P(\mathbb{C}) \setminus U_{\varepsilon_2}(K) \end{cases}$$

and $\limsup_{i \rightarrow \infty} I_i(1 - f_0) < \deg_{\mathcal{L}} X$. But by Proposition 3.7 for each $\varepsilon > 0$ there is some non-negative $f \in \mathbb{R}_+ \cdot S$ with $f \leq 1 - f_0$ and $\lim_{i \rightarrow \infty} I_i(f) = \sigma(f) > \deg_{\mathcal{L}} X \cdot (1 - \varepsilon)$. \square

4.7 Remark. This Theorem shows, in particular, that, if $X \cap K = \emptyset$, then $h_{f, \mathcal{L}}(X) > 0$, i.e. there are no small and generic sequences in X .

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