INSANELY TWISTED RABBITS

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This document briefly describes the movies shown during the lecture I gave at CIRM. They aim to give a proof, purely based on pictures, of the main result of [1]. Although I have striven to include enough material in this document to make it self-contained, it should be read in conjuction with the article [1].

1. The problem

Let $f : S^2 \to S^2$ be the "rabbit" polynomial $z^2 + c$ with $c \cong (-0.1 + 0.8i)$ a root of $(c^2 + c)^c + c = 0$. Let $T : S^2 \to S^2$ be the Dehn about c and $c^2 + c$.

We are interested in such maps up to combinatorial equivalence, that is, letting P_f denote the critical values of f and their forward images, we declare two maps $f, g : S^2 \to S^2$ to be equivalent if there exist orientation-preserving homeomorphisms $\phi_0, \phi_1 : (S^2, P_f) \to (S^2, P_g)$ such that $f\phi_0 = \phi_1 g$ and ϕ_0, ϕ_1 are isotopic rel P_f.

By Thurston's theorem, there are, up to combinatorial equivalence, only three branched coverings of S^2 whose post-critical points follow the same orbit as f's: the "rabbit", its complex conjugate, and the "airplane" $\cong z^2 - 1.7$.

Consider the maps $T^m \circ f$. They are all branched coverings with same post-critical orbit as f, and each is therefore equivalent to the rabbit, the corabbit or the airplane. The question is, which?

This question was asked by Douady and Hubbard in [2].

Theorem 1 ([1, Theorem 4.7])**.** *Write m in base 4, that is, $m = \sum_{i \geq 0} m_i 4^i$ with $m_i \in \{0, 1, 2, 3\}$. Then:*

- *if $m_i \in \{1, 2\}$ for some i, then $T^m \circ f$ is an airplane;*
- *if $m_i \in \{0, 3\}$ for all i and m is non-negative, then $T^m \circ f$ is a rabbit;*
- *if $m_i \in \{0, 3\}$ for all i and m is negative, then $T^m \circ f$ is a corabbit.*

[1] Laurent Bartholdi and Volodymyr V. Nekrashevych, *Thurston equivalence of topological polynomials*, Acta Math. **197** (2006), no. 1, 1–51.

[2] Adrien Douady and John H. Hubbard, *A proof of Thurston's topological characterization of rational functions*, Acta Math. **171** (1993), no. 2, 263–297. MR **1251582 (94j:**58143)

[3] Volodymyr V. Nekrashevych, *Self-similar groups*, Mathematical Surveys and Monographs, vol. 117, Amer. Math. Soc., Providence, RI, 2005.

2. Generators of IMG($z^2 + c$)

The main notion behind the proof of the Theorem is that of "iterated monodromy groups", which is explained more thoroughly in [3].

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Consider a branched covering $f : S^2 \to S^2$, with post-critical set P. There is a Galois (monodromy) action of $\pi_1(S^2 \setminus P)$ on the fibres of f, and of its iterates. Fixing a basepoint $t \in S^2 \setminus P$, we get an action of $\pi_1(S^2 \setminus P)$ on $T = \bigsqcup_{n\geq 0} f^{-n}(t)$, and note that T naturally has the structure of a binary rooted tree (the root is t and there is an edge from z to $f(z)$).

The iterated monodromy group $G = \mathrm{IMG}(f)$ is the image of $\pi_1(S^2\setminus P)$ in $\mathrm{aut}(T)$. Furthermore, the action on T can be made quite explicit, by identifying T with two copies of itself linked by a caret at their roots; in this way, any automorphism of T is recursively described by two automorphisms of T and a permutation of the caret. The notation we use is

$$\tau = \langle\!\langle \tau_L,\ \tau_R \rangle\!\rangle \rho, \tag{1}$$

with $\sigma \in \{1, \sigma\}$ a permutation of $\{L, R\}$ and $\tau, \tau_L, \tau_R \in G$ the automorphism and its restriction to the left and right subtrees.

We choose as generating set for G the loops α, β, γ, based at ∞, and circling respectively $c, c^2 + c$ and 0. Their decomposition, following the notation in (1), is obtained as follows. Given $\tau \in G$, the permutation ρ is the monodromy action on $f^{-1}(t)$, which records whether the loop $\tau \in G$ lifts to two loops or to two paths between the preimages of t. The elements τ_L and τ_R are the loops at t obtained after the lifts of τ are completed into loops at t by pre- and post-pending little paths in the neighbourhood of t.

The formulas we obtain are:

$$\alpha = \left\langle\!\!\left\langle \alpha^{-1}\beta^{-1},\ \gamma\beta\alpha \right\rangle\!\!\right\rangle \sigma, \quad \beta) = \langle\!\langle \alpha,\ 1 \rangle\!\rangle, \quad \gamma = \langle\!\langle \beta,\ 1 \rangle\!\rangle.$$

These are justified by drawing the loops α, β, γ atop the Julia set of f, and seeing what they become along a deformation from the identity to f. The pictures are built as follows. For time t varying in $[0, 1]$, the black image is a convex combination (essentially $t+(1-t)f$) of the Julia set of f; in particular, it is the Julia set of f for $t = 0$ and $t = 1$. In colours, the paths are followed along the convex combination. Here are α in green (AVI, MOV), β in blue (AVI, MOV) and γ in red (AVI, MOV).

The Dehn twist T about c and c^2+c acts on G by conjugating α and β by $(\alpha\beta)^{-1}$. The iterated monodromy group of $T^m \circ f$ is therefore given by the recursions

$$\begin{aligned} \alpha = \Phi_{f_R}(\alpha)^{T^{-m}} &= \left\langle\!\!\left\langle \alpha^{-1}\beta^{-1},\ \gamma\beta\alpha \right\rangle\!\!\right\rangle\sigma, \\ \beta = \Phi_{f_R}(\beta)^{T^{-m}} &= \left\langle\!\!\left\langle \alpha^{\left(\alpha^{-1}\beta^{-1}\right)^m},\ 1 \right\rangle\!\!\right\rangle, \\ \gamma = \Phi_{f_R}(\gamma)^{T^{-m}} &= \left\langle\!\!\left\langle \beta^{\left(\alpha^{-1}\beta^{-1}\right)^m},\ 1 \right\rangle\!\!\right\rangle, \end{aligned} \tag{2}$$

and the problem of determining when $T^m \circ f$ and $T^n \circ f$ are equivalent is reduced to the problem of comparing these recursions.

3. Dehn Twists

This is a musical illustration of Dehn twists (AVI, MOV).

The Dehn twists can also be lifted through f, and written in the formalism of (1). The mapping class group of $S^2 \setminus P_f$ is free, generated by T and the twist S about $c^2 + c$ and 0. Here are their lifts: T (AVI, MOV), and S (AVI, MOV).

Their recursions are

$$T = \left\langle\!\!\left\langle 1,\ S^{-1}T^{-1} \right\rangle\!\!\right\rangle \sigma, \qquad S = \langle\!\langle T,\ 1 \rangle\!\rangle. \tag{3}$$

We easily check the following

Lemma 1. *Let U be a mapping class with decomposition $\langle\!\langle U_L,\, U_R\rangle\!\rangle\rho$. If $\rho = 1$ then $U \circ f$ is combinatorially equivalent to $U_L \circ f$, while if $\rho \neq 1$ then $U \circ f$ is combinatorially equivalent to $TU \circ f$.*

Now we compute $T^{4m} = \left\langle\!\!\left\langle (S^{-1}T^{-1})^{2m},\, (S^{-1}T^{-1})^{2m} \right\rangle\!\!\right\rangle$ and $(S^{-1}T^{-1})^{2m} = \langle\!\langle S^m,\, S^m\rangle\!\rangle$ and $S^m = \langle\!\langle T^m,\, T^m\rangle\!\rangle$, so $T^{4m} \circ f$ is equivalent to $T^m \circ f$. Similarly, $T^{4m+1} \circ f$ and $T^{4m+2} \circ f$ are equivalent to $T \circ f$, and $T^{4m+3} \circ f$ is equivalent to $T^m \circ f$.

To complete the proof of the Theorem, it suffices to show that $T \circ f$ is equivalent to the airplane, and that $T^{-1} \circ f$ is equivalent to the corabbit.

4. Action on Teichmller space

The mapping class group's recursion for the group $H = \langle S, T\rangle$ can be explained in terms of Teichmüller theory, as follows. Let $P \subset S^2$ be a finite subset of the sphere. Then the *Teichmüller space* $\mathcal{T}_P$ modelled on (S^2, P) is the space of homeomorphisms $\tau : S^2 \to \widehat{\mathbb{C}}$, where τ_1 and τ_2 are identified if there exists a biholomorphic isomorphism $\Theta : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ (i.e., an element of the Möbius group) such that $\Theta \circ \tau_1 = \tau_2$ on P and $\Theta \circ \tau_1$ is isotopic to τ_2 relative to P.

The *moduli space* $\mathcal{M}_P$ of (S^2, P) is the space of all injective maps $P \hookrightarrow \widehat{\mathbb{C}}$ modulo post-compositions with elements of the Möbius group. The Teichmüller space $\mathcal{T}_P$ is the universal cover of the moduli space $\mathcal{M}_P$, where the covering map is the restriction map of $\tau \in \mathcal{T}_P$ to P. Pulling-back complex structures via a branched covering f induces a map σ_f on Teichmüller space: we have a commutative diagram

$$\begin{array}{ccc} S^2 & \xrightarrow{\ f\ } & S^2 \\ {\scriptstyle \tau'}\downarrow & & \downarrow{\scriptstyle \tau} \\ \widehat{\mathbb{C}} & \xrightarrow[\ f_\tau\ ]{} & \widehat{\mathbb{C}} \end{array}$$

and $f_\tau = (\tau')^{-1} \cdot f \cdot \tau : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a rational function.

Let us return to the case when P is the post-critical set of the "rabbit" polynomial. Applying Möbius transformations, we may assume that three of the punctures are $0, 1, \infty$, so the moduli space of a fourfold punctured sphere is $\widehat{\mathbb{C}} \setminus \{0, 1, \infty\}$. The point $w \in \widehat{\mathbb{C}} \setminus \{0, 1, \infty\}$ corresponds to the element $\tau|_P$ such that $\tau|_P(\infty) = \infty, \tau|_P(0) = 0, \tau|_P(c) = 1$ and $\tau|_P(c^2 + c) = w$. The universal covering of modulie space is Teichmüller space, with H acting by deck transformations.

In our situation, the map σ_f on Teichmüller space descends to a map (in reverse direction) on moduli space, as

$$\begin{array}{ccc} \mathcal{T}_P & \xrightarrow{\ \sigma_f\ } & \mathcal{T}_P \\ \downarrow & & \downarrow \\ \mathcal{M}_P & \xleftarrow{\ F\ } & \mathcal{M}_P, \end{array}$$

and an explicit calculation gives $F(w) = 1 - w^{-2}$. The iterated monodromy group of this map is precisely given by the recursions (3).

Points in Teichmüller space correspond to branched coverings with precribed post-critical orbit. Mapping classes in H correspond to loops in moduli space, and therefore to paths in Teichmüller space. The "twisted rabbit" problem is solved, for an arbitrary $U \in H$, as follows. One starts at the point f_0 in Teichmüller space corresponding to the rabbit. There is a unique path in Teichmüller space representing U and starting at f_0; it ends at f_1. There is then a unique lift of that path starting at f_1, which ends at f_2; and so on. The points f_n accumulate, and their accumulation point is a fixed point of $w \mapsto 1 - w^{-2}$. There are three fixed points, corresponding to the rabbit, the corabbit and the airplane.

I have prepared movies of the loop in moduli space representing S (AVI, MOV) and T (AVI, MOV). The most complex movie lifts the loop S and shows that it converges to the airplane (AVI, MOV).

These movies show an amusing by-product of the solution: assembling the moduli and dynamical coördinaters gives a two-dimensional dynamical system

$$\begin{matrix} z \\ w \end{matrix} \mapsto \begin{matrix} 1 - z^2 w^{-2} \\ 1 - w^{-2} \end{matrix}$$

The Julia set of this dynamical system (as a fibered space over $\widehat{\mathbb{C}}$) is represented along the movie, as the basepoint w moves according to the figure in the upper right corner.

5. Matings

Xavier Buff has noted that this construction gives a simple visual description of the mating of two polynomials. Let $f, g : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be two polynomials of same degree d. They both act as $z \to z^d$ in the neighbourhood of ∞, so one may remove a small disk around ∞ from $\widehat{\mathbb{C}}$; glue together two copies of the resulting space, possibly after rotating one of them; and define a branched covering on it as f on the first copy and g on the second.

The moduli space of the mating is essentially a product of the moduli spaces of the original polynomials. There is again a fibered product Julia set. For points in moduli space that separate well the post-critical orbits of f and g, the fiber will be essentially the disjoint union of the Julia sets of f and g. As one moves in moduli space to bring both post-critical orbits together, the Julia sets "mate".

The following pictures illustrate this mating procedure:

Mating two rabbits	AVI	MOV
From $w = 0$ to $w =$ Rabbit	AVI	MOV
Spiral from $w = 0.3$ to $w =$ Rabbit	AVI	MOV
Circle from $w = 0.99$ to $w = 0.99$Rabbit	AVI	MOV

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