Continuous Families of Rational Surface Automorphisms

Eric Bedford, Indiana U.
Kyounghee Kim, Florida State U.

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Question

What are the 2-dimensional complex surfaces with interesting automorphisms?

Dimension 1:

- \( \hat{\mathbb{C}} \) Riemann sphere \( \Rightarrow \text{Aut}(\hat{\mathbb{C}}) \) is the group of Möbius transformations.

- \( \mathbb{T}^1 = \mathbb{C}/\mathcal{L} \Rightarrow \text{Aut}(\mathbb{T}^1) \) is essentially the group of translations.

- \( M \) a compact Riemann surface \( \Rightarrow \text{Aut}(M) \) is finite.
Dimension 2:

- $\mathbb{C}^2$: Hénon maps
  - There is no complex compactification.
  - **Hubbard, Buzzard**: Constructed (non-complex) compactifications for Hénon maps.

**Theorem (Cantat)**

*Suppose $\mathcal{X}$ is compact algebraic surface. If $f \in \text{Aut}(\mathcal{X})$ with $\text{entropy}(f) > 0$ then $\mathcal{X}$ is either a torus, a $K3$, or a rational surface (or certain quotients).*

- $\mathbb{T}^2 = \mathbb{C}^2/\mathcal{L}$
- $K3$
- Rational surfaces.
A rational surface $\mathcal{X}$ is a surface birationally equivalent to the projective plane $\mathbb{P}^2$.

**Theorem (Nagata)**

Suppose $f$ is an automorphism on a rational surface $\mathcal{X}$ and $f_*$ has infinite order. Then there is a holomorphic, birational map $\pi : \mathcal{X} \to \mathbb{P}^2$ where the map $\pi$ is obtained by a finite blowing up process (possibly iterated blowups).

- Suppose $f$ is a linear on $\mathbb{P}^2 \Rightarrow f_* = id$.

**Theorem (Hirschowitz, Koitabashi)**

For a generic rational surface $\mathcal{X}$, $\text{Aut}(\mathcal{X}) = 1$. 
Theorem (Bedford-K, McMullen)

For each \( n \geq 3 \), there exist \( a, b \) which satisfy two polynomial equations \( P_n(a, b) = 0, Q_n(a, b) = 0 \) such that

\[
  f_{a,b} : (x, y) \mapsto \left( y, \frac{y + a}{x + b}\right)
\]

induces an automorphism of a surface \( \pi : \mathcal{X}_{a,b} \to \mathbb{P}^2 \) where \( \mathcal{X}_{a,b} \) is obtained by blowing up \( n \) points.

- This family gives a countable family of nontrivial rational surface automorphisms.
- Are there larger families?
\[ f(x, y) = \left( y, \frac{y + a}{x + b} \right) \] is birational.

For generic \((a, b)\), \(\delta(f_{a,b}) \sim 1.324 \cdots\) is the largest root of \(x^3 - x - 1\).

Blowdown/blowup behavior:
\[ \Sigma_\gamma := \{y + a = 0\} \to q = (-a, 0), \quad p = (-b, -a) \to \{x = -a\} \]

Set in parameter space:
\[ \mathcal{V}_n = \{(a, b) \in \mathbb{C}^2 : f^n_{a,b}(-a, 0) = (-b, -a)\} \]

**Theorem (B-Kim).** For \((a, b) \in \mathcal{V}_n\), we construct a new complex manifold \(X_{a,b}\) by blowing up the projective plane \(\mathbb{P}^2\) at the \(n + 3\) points: \(e_1, e_2, q, f q, \ldots, f^n q = p\). Then \(f_{a,b}\) induces an automorphism on the manifold \(X_{a,b}\). Further, \(\delta(f_{a,b})\) is the largest real root of the polynomial

\[ \chi_n(x) = -1 + x^2 + x^3 - x^{1+n} - x^{2+n} + x^{4+n}. \]

If \(n \geq 7\), then \(\delta(f_{a,b}) > 1\).
\[ \mathbb{P}^2 = \{ [t : x : y] \mid [t : x : y] = [\lambda t : \lambda x : \lambda y] \} \]

Imbed \( \mathbb{C}^2 \) into \( \mathbb{P}^2 \) via map \( \mathbb{P}^2 \ni [1 : x : y] \leftrightarrow (x, y) \in \mathbb{C}^2 \)

Blowups & Iterated Blowups

First blowup of the origin \([1 : 0 : 0]\)
\( \pi_1 : \mathcal{X}_1 \to \mathbb{P}^2 \)
\( (s, \xi) \mapsto [1 : s : s\xi] \) near \( \{ t \neq 0 \} \)
\( \mathcal{X}_1 \) consists of “all the lines through the origin made disjoint”.

Second blowup of the point \((0, 0) \in \mathcal{X}_1\)
\( \pi_2 : \mathcal{X}_2 \to \mathcal{X}_1 \)
\( \pi_1 \circ \pi_2 : (s, \xi) \mapsto [1 : s : s^2\xi] \)
In \( \mathcal{X}_2 \), curves \( y = cx^2, c \in \mathbb{C} \) are disjoint.
McMullen has a different construction of an automorphism, starting with an invariant cubic and a root $t$ of $\chi_n$. These maps are equivalent to $f_{a,b}$'s.

**Theorem (McMullen).** $V_n \neq \emptyset$ for every $n$.

**Theorem (Diller-Favre).** For every automorphism, the characteristic polynomial of $f_*$ is a Salem polynomial, i.e., there is a unique root $\lambda > 1 > 1/\lambda$, and all other roots are on the unit circle.

**Theorem (McMullen).** For each $n \geq 7$, there is a unique root $\lambda_n > 0$, and let $(a, b) \in V_n$ be the corresponding parameter values. Then $f_{a,b}$ has a unique repelling fixed point, and the basin has full measure.

Question: Is the previous picture “honest”? That is, is there actually a real lamination? complex lamination? Evidence:

**Theorem (B-Kim).** The entropy of restriction of this map to real points is the same as the entropy of the full map.
Theorem (McMullen, B-Kim). Let $n \geq 7$, and let $|t| = 1$ be a root of $\chi_n$ which is not a root of unity. Let $(a,b) = (a(t),b(t))$. Then $f_{a,b}$ has a rank 1 rotation domain. Further, if $n$ is divisible by 2 or 3, then there are roots $|t| = 1$ for which $f_{a,b}$ also has a rank 2 rotation domain.

Question: Are there only finitely many rotation domains? (The computer suggests that for $n = 9$ or $n = 12$ you can choose $t$ so that the map has 5 or more rotation domains.)

Theorem (B-Kim). If $f_{a,b}$, $(a,b) \in \mathcal{V}_n$ has an invariant curve, then the curve is a cubic, and thus $f_{a,b}$ is one of the maps corresponding to a root of $\chi_n$.

Question: $\# \mathcal{V}_n \sim (1.324)^n$ for large $n$? In particular, the majority of maps in $\mathcal{V}_n$ do not have invariant curves.

Question: Are the general maps in $\mathcal{V}_n$ as nice as the ones with invariant curves?
Theorem

Let $n \geq 2$ be given, and let $\delta$ be a root of a Salem polynomial $x^n - (x^{n-1} + \cdots + x^2 + x) + 1$, which is not a cube root of $-1$. For $c = 2\sqrt{\delta} \cos(j\pi/n)$ with $(j, n) = 1$

$$h(x, y) = \left(y, -\delta x + cy + \frac{1}{y}\right)$$

is an automorphism with entropy $\log \lambda_{n,1}$.

There are two possibilities:

- $\delta \in \mathbb{R}$, $\delta > 1 > \frac{1}{\delta}$

- $|\delta| = 1$, not a root of unity.

Points of the line $\Sigma_0$ are fixed by $h^n$, and the multiplier in the normal direction is $-\delta^{-n/2}$.

If we take $\delta > 1$, we have real maps whose area growth on $\mathbb{R}^2$ is the same as the exponential of the entropy.
Theorem

If $\delta > 1$, then $\Sigma_0$ is center of an attracting basin $\mathcal{B}$. Further, $\mathcal{B}$ has full volume in the sense that $\mathcal{X} - \mathcal{B}$ has zero volume.

$n = 4, \ j = 3, \ \delta = 1.72208$
Theorem

If $|\delta| = 1$ then there is a neighborhood $\mathcal{N}$ of $\Sigma_0$ inside $\mathcal{X}$ and $h^n|_{\mathcal{N}} \simeq$ rotation which fixes $\Sigma_0$.

In this case, $\mathcal{N}$ is a 1-complex parameter family of Siegel disks centered at points of $\Sigma_0$. 
Take \( k \geq 1 \) and a \( k-1 \)-tuple \( a = (a_1, a_2, \ldots, a_{k-1}) \in \mathbb{C}^{k-1} \).

Let \( 0 < j < n, (j, n) = 1 \) and \( c = 2 \cos(j\pi/n) \).

**Theorem**

With \( k, a, c \) above,

\[
f(x, y) = \begin{pmatrix}
y, -x + cy + \sum_{j=1}^{k-1} \frac{a_j}{y^{2j}} + \frac{1}{y^{2k}}
\end{pmatrix}
\]

defines an automorphism of a rational surface \( \mathcal{X}_a \).

- \( f^*(dx \wedge dy) = dx \wedge dy \).
- The compact manifold \( \mathcal{X}_a \) is constructed by iterated blowups.
\[ f(x, y) = \left( y, -x + \frac{a}{y^2} + \frac{1}{y^4} \right) \quad a = -2.64 \]
Remark 1. This construction is again a blowup procedure, but it involves a lot of blowups on top of blowups.

Remark 2. With a similar map, the jacobian can be any root of unity (constant).

Remark 3. (An iterate of) the map is tangent to the identity on most blowup fibers.

Remark 4. The number $k$ can be as large as you want, so these maps come in arbitrarily high-dimensional families.
\[ \mathcal{F}_0 \rightarrow \cdots \rightarrow \mathcal{F}_{n-1} \rightarrow \mathcal{F}_0, \quad \mathcal{F}_0^{k+1} \rightarrow \cdots \rightarrow \mathcal{F}_{n-1}^{k+1} \rightarrow \mathcal{F}_0^{k+1} \]

\[ \mathcal{F}_0^j \rightarrow \cdots \rightarrow \mathcal{F}_{n-1}^j \rightarrow \mathcal{F}_0^{2k+2-j} \rightarrow \cdots \mathcal{F}_{n-1}^{2k+2-j} \rightarrow \mathcal{F}_0^j, \quad 2 \leq j \leq k \]

\[ \{ y = 0 \} \rightarrow \mathcal{F}_0^{2k+1} \rightarrow \cdots \rightarrow \mathcal{F}_{n-1}^{2k+1} \rightarrow \{ x = 0 \} \]