Some trends in higher dimensional holomorphic dynamics

(a personal view)

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Setting

We virtually consider $f : X \dashrightarrow X$ a dominant meromorphic map on a (compact) complex manifold and want to study f as a dynamical system.

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To keep some intuition from 1-dimensional dynamics, we restrict to polynomial automorphisms of $\mathbb{C}^2$.

We have the following structure result:

THEOREM (Friedland-Milnor) Any polynomial automorphism of $\mathbb{C}^2$ with non trivial dynamics is conjugate to a product of Hénon mappings

$$(z, w) \longmapsto (aw + p(z), az), a) \in \mathbb{C}^* \text{ and } p \text{ a polynomial.}$$

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Notice that for $a \approx 0$, $(aw + p(z), az), a) \approx (p(z), 0)$ so a Hénon map with small a is a perturbation of a one dimensional map.

In the following by Hénon map we mean polynomial automorphism of $\mathbb{C}^2$ with nontrivial dynamics.

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THEOREM (Gromov, Yomdin, Smillie) The topological entropy of a Hénon map is $\log(\deg p)$

THEOREM(Bedford-Lyubich-Smillie) A Hénon map admits a unique measure of entropy $\log(\deg p)$, which is mixing, has non-zero Lyapunov exponents, and describes the asymptotic distribution of periodic orbits.

(in particular a Hénon map admits a canonical invariant measure)

Topological entropy and ergodic theory

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- Analogous results are conjectured to hold for "all" $f : X \dashrightarrow X$. Even the case $\dim X = 2$ is not complete yet.

A subset of this problem is to understand the degree growth of sequences of rational mappings (e.g. polynomial mappings in $\mathbb{C}^k$), which is already very difficult.

Fatou and Julia sets

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- Another basic open question is whether

$$J = \overline{\{\text{saddle periodic points}\}}.$$

(here J denotes the set of points such that iterates are not locally normal in both forward and backward time)

Fatou and Julia sets

- There is a characterization of Hénon maps with connected Julia set in terms of "critical points" not escaping (Bedford-Smillie).

COROLLARY A hyperbolic Hénon map with $|a| \leq 1$ and connected Julia set has an attracting orbit.

Fatou and Julia sets

▶ There is a characterization of Hénon maps with connected Julia set in terms of "critical points" not escaping (Bedford-Smillie).

COROLLARY A hyperbolic Hénon map with $|a| \leq 1$ and connected Julia set has an attracting orbit.

▶ On the other hand there is no criterion for J to be a Cantor set in terms of "all critical points escaping".

QUESTION Let a hyperbolic Hénon map with $|a| \leq 1$ and no attracting orbit. Is J a Cantor set?

Fatou and Julia sets

- There is no known example of Hénon maps with “smooth” Julia set –probably does not exist?

Some rational mappings with “anomalous smoothness properties” for Julia sets, invariant measures, etc. actually exist and it is an interesting problem to classify them.
It is also interesting to classify rational mappings possessing invariant objects, like foliations, etc.

Exploration of parameter space

- Hénon mappings with Julia set contained in $\mathbb{R}^2$ are well understood.

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- There exist hyperbolic Hénon maps which are not perturbations of 1D polynomials (Ishii).

Exploration of parameter space

- On the other hand, the stability vs. bifurcations scheme for Hénon maps is poorly understood.

In dimension 1, there is a codimension 1 dense phenomenon in the bifurcation locus: prerepelling critical points. Do tangencies play the same role for Hénon maps?

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THEOREM (Deserti, Cantat-Lamy) $SL_n(\mathbb{Z})$ with $n \geq 3$ admits no morphism to $\mathrm{Aut}(\mathbb{C}^2)$ with infinite image.
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- All this also makes sense in the *Cremona group* $\mathrm{Bir}(\mathbb{P}^2)$, and analogous results hold for this group.

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▶ This also raises the problem of finding a compactification of $\mathbb{C}^2$ "adapted" to the study of a given polynomial map (this is related e.g. to the question of understanding degree growth).