On Thurston's combinatorial characterization of rational functions

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October 2008

Def. A Thurston map $F : S^2 \to S^2$ is an orientation-preserving branched covering map for which the set

$$P_F = \bigcup_{n>0} F^n(\text{Crit(F)})$$

is finite.

Def. Thurston maps F, G are equivalent if $\exists h_0, h_1 : (S^2, P_F) \to (S^2, P_G)$ with $h_0 \circ F = G \circ h_1$ and $h_0 \simeq h_1$ rel P_F.

Remarks:

1. Equivalence is like homotopy.

2. Algebraic formulation of equivalence (Nekrashevych; Kameyama; P.)

3. Algebra useful since equivalence is difficult to check.

Thurston's Theorem (Douady-Hubbard).

$$\left\{\begin{matrix}\text{rational}\\ \text{maps}\end{matrix}\right\}/\text{Aut}(\widehat{\mathbb{C}}) \to \left\{\begin{matrix}\text{Thurston}\\ \text{maps}\end{matrix}\right\}/\simeq$$

Characterization: Which Thurston maps arise?

Rigidity: Fibers are singletons, or integral Lattès families.

Remarks:

1. Rigidity $\leftrightarrow$ transversality

Each prescription

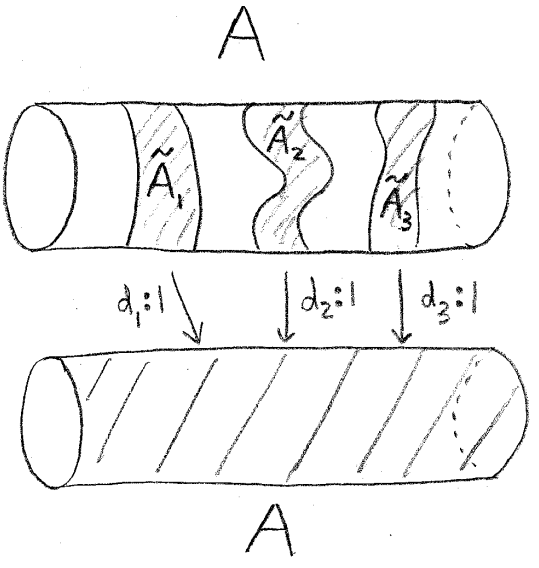
$$z_i \mapsto z_j, \ \ \deg(f, z_i) = k_i$$

gives k_i equations in the unknown coefficients of the map and unknown locations of the z_i's.

Riemann-Hurwitz $\Rightarrow$

equations = # unknowns.

2. Example of obstruction.



For a rational map, $\sum_i \frac{1}{d_i} > 1$ cannot occur. Equality can hold only for Lattès maps.

Significance.

1. Monotonicity of entropy in logistic family $\lambda x(1-x)$ (Milnor-Thurston)

2. Density of hyperbolicity equivalent to:

$$\forall c \ \infty - \text{renormalizable}, \ \ c = \lim c_n$$

where $z^2 + c_n$ are Thurston maps (Yoccoz, McMullen).

3. Combinations & surgeries: *matings* (Rees, Shishikura, Tan, et. al.)

4. Like rational points: via limiting processes, can build maps with desired combinatorial behavior, e.g.:
(Sorensen) J_{z^2+c} non-locally connected $\rightsquigarrow$
(Epstein-Yampolsky) intertwining $\rightsquigarrow$
(Henriksen) failure of combinatorial rigidity for cubic polynomials

5. *Iterated monodromy groups* of Thurston maps provide wealth of important examples (Bux-Perez; Grigorchuk-Zuk; Bartholdi-Kaimanovich-Nekrashevych-Virag)

6. *Galois theory* (Aitken-Hajir-Maire; Silverman; P.)

7. Simplest $>$ 1-dimensional noninvertible dynamical systems which are not coverings.

$$\left\{\begin{matrix}\text{expanding}\\ \text{maps}\end{matrix}\right\}/\text{top. conj.} \to \left\{\begin{matrix}\text{Thurston}\\ \text{maps}\end{matrix}\right\}/\simeq$$

Rigidity: Fibers are singletons

Characterization: (Cannon-Floyd-Parry-P.)

8. Dynamics in several variables (Koch)

Proof of Thurston's theorem: iteration on Teichmüller space.

Koch: for critically n-periodic $z^2 + c$, this iteration is essentially inverse of

$$g : \mathbb{CP}^{n-2} \to \mathbb{CP}^{n-2}$$

where g is critically finite & holomorphic.

9. Analogy with Kleinian ("Sullivan dictionary") and Gromov hyperbolic groups

Have

$$J_f \approx \partial_\infty X$$

in several different natural ways: (Nekrashevych, Cannon-Floyd-Parry, Haissinsky-P.)

Quasi-isometry type is canonical.