

1 Branner-Hubbard conjecture

Theorem 1.1. *(Qiu-Yin, Kozlovski-van Strien) Let g be a polynomial with a disconnected its filled Julia set K_g . Assume that every critical component of K_g is aperiodic under the iteration of g . Then K_g is a Cantor set.*

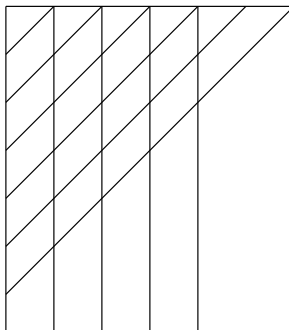
Set up. Let $f : \mathcal{E} \rightarrow \mathcal{L}$ be a proper holomorphic map such that $\mathcal{L} = \sqcup_{i \in I} L_i$ is the disjoint union of finitely many hyperbolic discs, $\mathcal{E} \subset \subset \mathcal{L}$, and $\mathcal{E} = \sqcup_{i,j \in I} \sqcup_{k \in \Lambda_{ij}} E_{ijk}$ is the union of finitely many discs with disjoint closures. Critical points are non-escaping and are in

distinct L_i 's.
$$\begin{array}{ccc} L_i & - & L_j \\ \cup & \nearrow_{f \text{ proper}} & \\ E_{ijk} & & \end{array}$$

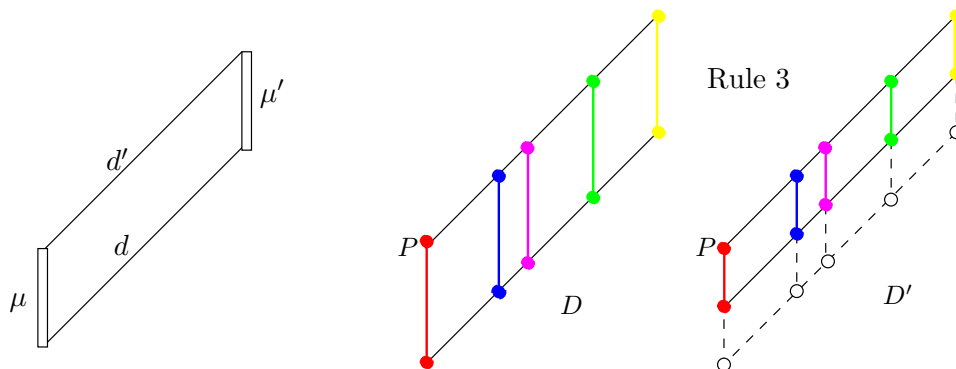
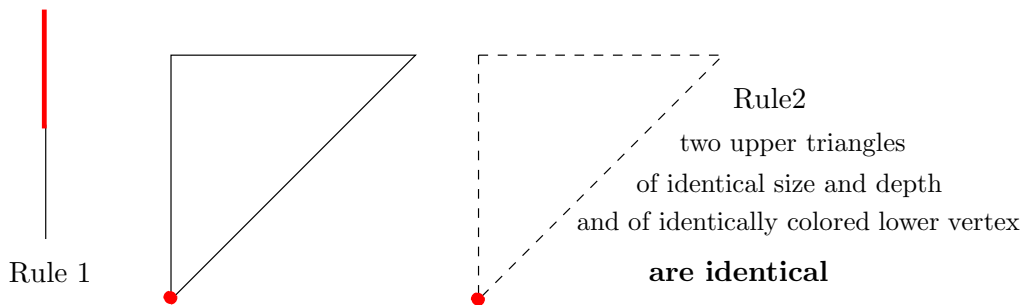
Set $b = \# \text{Crit}(f)$ and $\delta := \sup_{c \in \text{Crit}(f)} \deg_c(f)$.

Theorem 1.2. *Assume that f is persistently recurrent (see below). Then there is $C > 0$, such that for any $c \in \text{Crit}(f)$, there is a sequence of disjoint annuli A_n contained essentially in $P_0(c) \setminus \{c\}$ whose moduli $\mu(A_n)$ satisfy $\mu(A_n) \geq C > 0$. Moreover the constant $C = C(b, \delta)$ depends only on b and δ , and $A_n \cap \widehat{\mathcal{P}}_f = \emptyset$.*

Sketch of the proof. Starting from any critical piece K_{-1} , we define inductively a sequence $K_{n-1} \supset \supset K'_n \supset \supset K_n$ following Kozlovski-Shen-Strien and then a family of Kahn-Lyubich type maps with the same constants. We then prove that the moduli of $K'_n \setminus K_n$ are bounded from below, using the Kahn-Lyubich covering lemma. \square



$\mathcal{T}(x) \subset \mathbb{R}^- \times \mathbb{R}$ a graph with
 $\{\text{vertices}\} = -\mathbb{N} \times \mathbb{N}$, some are colored
 3 types of edges: $-$, $|$, $/$ (length=1)
 a modulus on each vertical segments (subadditive)
 a degree on each diagonal segment starting
 with a given colored vertex (multiplicative)
 $\text{degree}(/ \text{ edge})=1$ if the left vertex is in colored

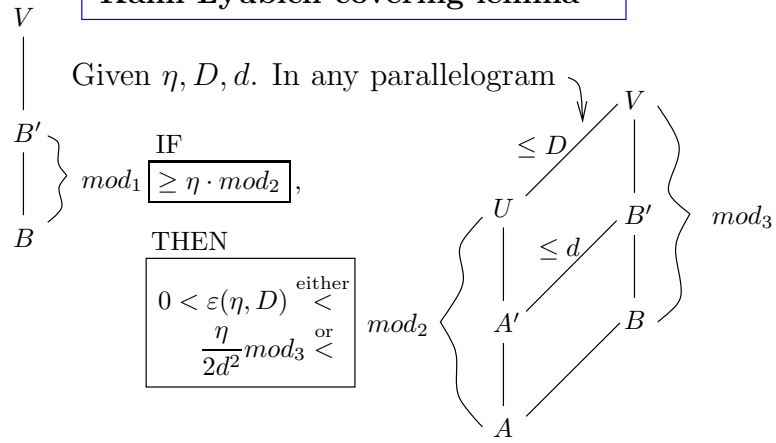


In a parallelogram, $d' \geq d$, $\mu \geq \frac{1}{d'}\mu$.

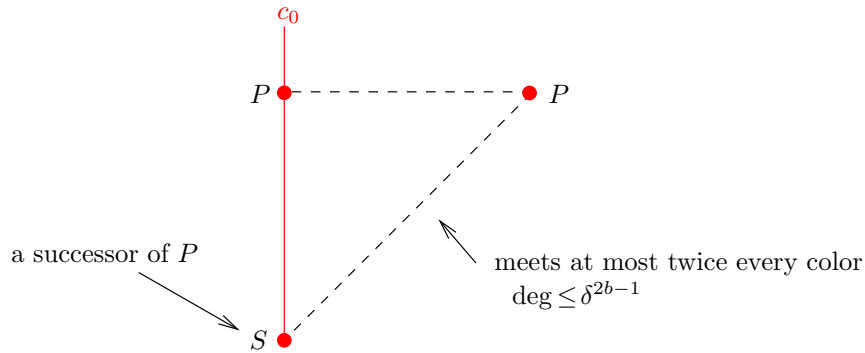
Some vertical segments are called **vacuous**, i.e. $\cap \widehat{\mathcal{P}}_f = \emptyset$

right side vacuous \implies left side vacuous, $d' = d$ and $\mu = \frac{1}{d'}\mu'$

Kahn-Lyubich covering lemma



1.1 Successors and persistent recurrence

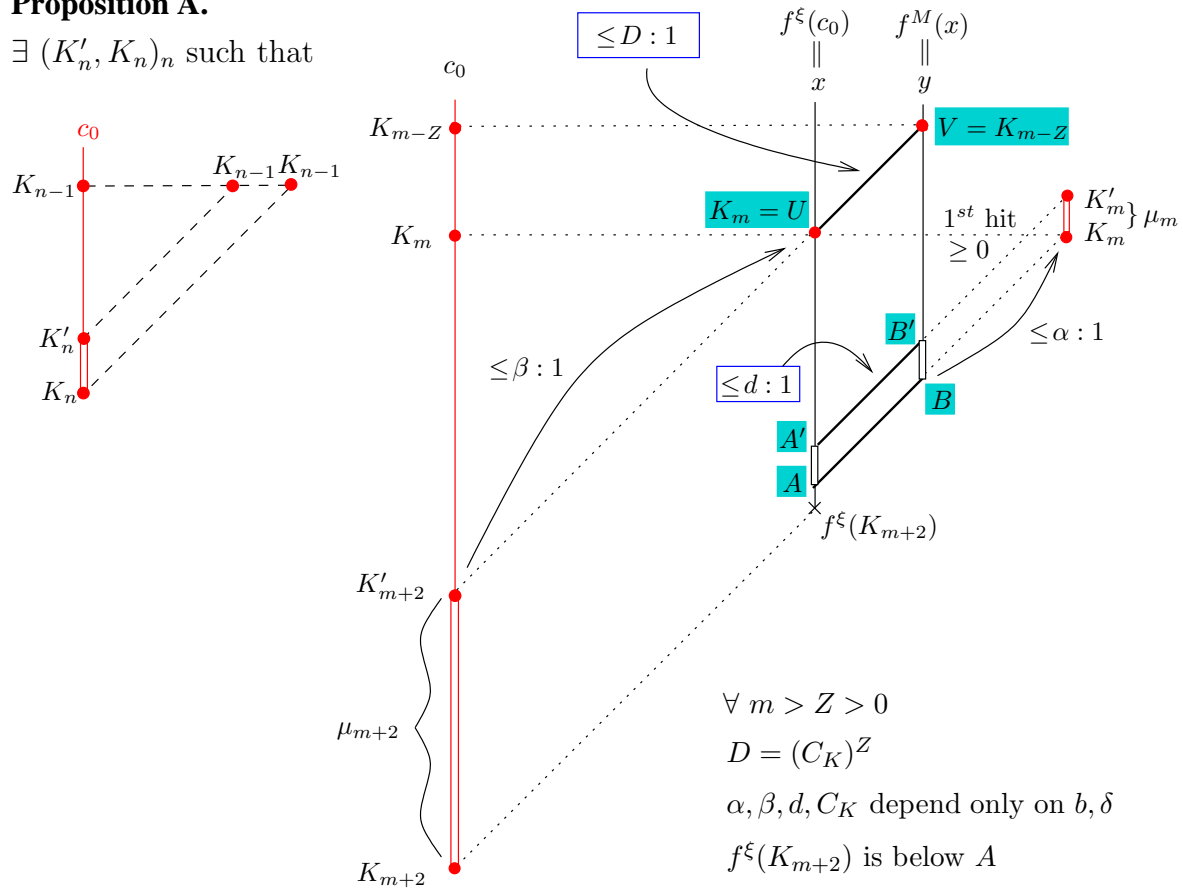


We say that f is **persistently recurrent** if:

- given any couple $(c, c') \in \text{Crit}(f)^2$, marching horizontally to the right of any position in $\mathcal{T}(c)$, one will meet a position that has the color of c' ; and $\mathcal{T}(c)$ contains no infinite long critical nests of c' (except of course on the 0th column);
- every critical piece of any color has $< +\infty$ successors.

Proposition A.

$\exists (K'_n, K_n)_n$ such that



Proof of Theorem 1.2.

Assume there is a sequence $k_n \rightarrow \infty$ such that $\mu_{m'} \stackrel{m' < k_n}{\geq} \mu_{k_n}$ (otherwise $\mu_n \geq \mu_{n_0} > 0$ for some n_0). Fix any n satisfying $k_n - 2 > Z$. Set $m = k_n - 2$. Then $m + 2 = k_n$ and $\mu_{m'} \geq \mu_{m+2}$ for any $m' \leq m + 2$. For this m , construct as above $(U, V, M, \xi, x, y, B', B, A', A)$. Then,

$$\mu_{m+2} \geq \frac{\text{mod}\left(\begin{array}{c} U \\ | \\ f^\xi(K_{m+2}) \end{array}\right)}{\deg\left(\begin{array}{c} K'_{m+2} \\ \setminus U \end{array}\right)} \stackrel{(e)}{\underset{(d)}{\geq}} \frac{\text{mod}\left(\begin{array}{c} U \\ | \\ A \end{array}\right)}{\beta}; \quad (1)$$

$$\text{mod}\left(\begin{array}{c} B' \\ | \\ B \end{array}\right) \stackrel{\text{no } \widehat{\mathcal{P}}_f}{=} \frac{\mu_m}{\deg\left(\begin{array}{c} B' \\ \setminus K'_m \end{array}\right)} \stackrel{\text{choice of } k_n}{\underset{(b)}{\geq}} \frac{\mu_{m+2}}{\alpha} \stackrel{(1)}{\geq} \frac{\text{mod}\left(\begin{array}{c} U \\ | \\ A \end{array}\right)}{\alpha\beta} =: \eta \cdot \text{mod}\left(\begin{array}{c} U \\ | \\ A \end{array}\right).$$

$$\beta \cdot \mu_{m+2} \stackrel{(1)}{\geq} \text{mod}\left(\begin{array}{c} U \\ | \\ A \end{array}\right) \stackrel{\text{either}}{>} \varepsilon(\eta, D) > 0; \quad \text{or} \quad \frac{\eta}{2d^2} \text{mod}\left(\begin{array}{c} V \\ | \\ B \end{array}\right) \stackrel{(*)}{\geq} \frac{\eta}{2d^2\alpha} (\mu_m + \cdots + \mu_{m-Z+1}) \stackrel{\text{choice of } k_n}{\geq} \frac{\eta Z}{2d^2\alpha} \mu_{m+2},$$

where the inequality $(*)$ is proved by using the pullback of the first ≥ 0 hit of y to K_j for $j = m, m - 1, \dots, m - Z + 1$, and the property (K') that $(K'_j \setminus K_j) \cap \widehat{\mathcal{P}}_f = \emptyset$.

Set $Z = 2d^2\alpha^2\beta^2 + 1$. Then $Z > 2d^2\alpha^2\beta^2 = \frac{2d^2\alpha\beta}{\eta}$. Hence the second line above is impossible. So $\mu_{k_n} = \mu_{m+2} > \frac{1}{\beta} \cdot \varepsilon(\eta, D) > 0$. Therefore

$$\forall l \in \mathbb{N}, \quad \mu_l \geq \lim_{n \rightarrow \infty} \mu_{k_n} \geq \frac{1}{\beta} \cdot \varepsilon(\eta, D) =: C(b, \delta) > 0.$$

□

2 Construction of the KSS nest $(K'_n, K_n)_n$

We will define four pullback operators $\mathcal{Y}, \mathcal{A}, \mathcal{B}, \mathcal{C}$ from a critical nest into itself, and then set inductively $K_n = \mathcal{B} \circ \mathcal{A} \circ \mathcal{Y}(K_{n-1})$ and $K'_n = \mathcal{C} \circ \mathcal{Y}(K_{n-1})$.

2.1 First hit has bounded degree

$$\deg \left(\begin{array}{c} B' \\ \diagup \\ K'_m \end{array} \right) = \deg \left(\begin{array}{c} B \\ \diagup \\ K_m \end{array} \right) \quad \text{and} \quad \frac{y}{\times \begin{array}{c} \text{1st hit} \\ \geq 0 \end{array} K_m \text{ (red)}} \quad (2)$$

$$\left| \begin{array}{c} \diagup \\ B \end{array} \right. \longleftarrow \deg \leq \delta^{b-1} = \alpha$$

Proof. $\text{diag}_{[B/\diagup K'_m]}$ does not meet red color, and meets every other color at most once. Otherwise we get two upper triangles of different sizes, both with K_m as the right vertex, and both have a left side of the same coloring. Now moving the smaller triangle to the left and applying Rule 2 would imply an earlier hit of red spot on the horizontal segment containing K_m . \square

2.2 Last successor operator Γ and the operator $\mathcal{Y} = \Gamma^{3b}$

For every critical puzzle piece P ,

$$\Gamma(P) = \text{last successor of } P,$$

$$\mathcal{Y}(P) := \Gamma^{3b}(P) = \text{the } 3b\text{-th generation of last successors.}$$

Lemma 2.1. *(a consequence of tableaux rules) Let P be a critical puzzle piece.*

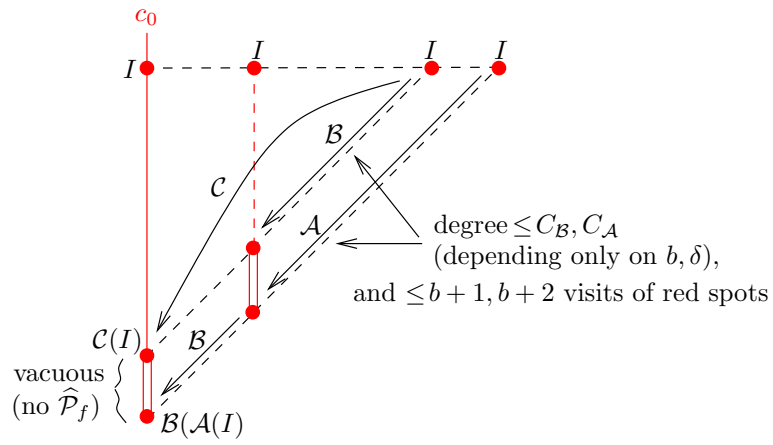
$$a) \quad 2 \leq \#\{\text{successors of } P\} < +\infty.$$

$$b) \quad \deg(\mathcal{Y}(P)/P) \leq C_{\mathcal{Y}} := \delta^{(2b-1)3b}.$$

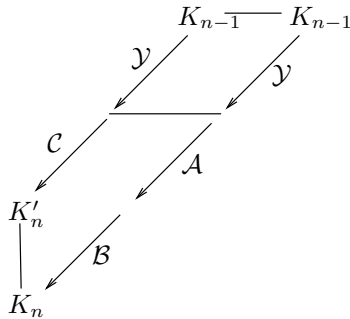
The only difficulty in this Lemma is the part that P has at least two successors.

2.3 The operators \mathcal{A} , \mathcal{B} and \mathcal{C}

Lemma 2.2. *For any critical puzzle piece I , there are $\mathcal{A}(I), \mathcal{B}(I), \mathcal{C}(I)$ such that*



2.4 The KSS nest



Fix $c_0 \in \text{Crit}(f)$. Let K_{-1} be a critical puzzle piece of c_0 . We define inductively

$$K_n = \mathcal{B}(\mathcal{A}(\mathcal{Y}(K_{n-1}))) \quad K'_n = \mathcal{C}(\mathcal{Y}(K_{n-1})).$$

They satisfy (following Lemmas 2.1, 2.2):

$$(\mathbf{K}) \begin{cases} \deg(K_n / K_{n-1}) \leq C_K := C_B \cdot C_A \cdot C_Y \quad (\text{depending only on } b, \delta); \\ 3b + 2 \leq \#(\{\text{red spots}\} \cap \text{diag}_{[K_n / K_{n-1}]}) \leq b + b + 1 + 3b = 5b + 1. \end{cases}$$

$$(\mathbf{K}') \quad (K'_n \setminus K_n) \cap \widehat{\mathcal{P}}_f = \emptyset, \quad \deg(K'_n / K_{n-1}) \leq C_B^2 \cdot C_Y.$$

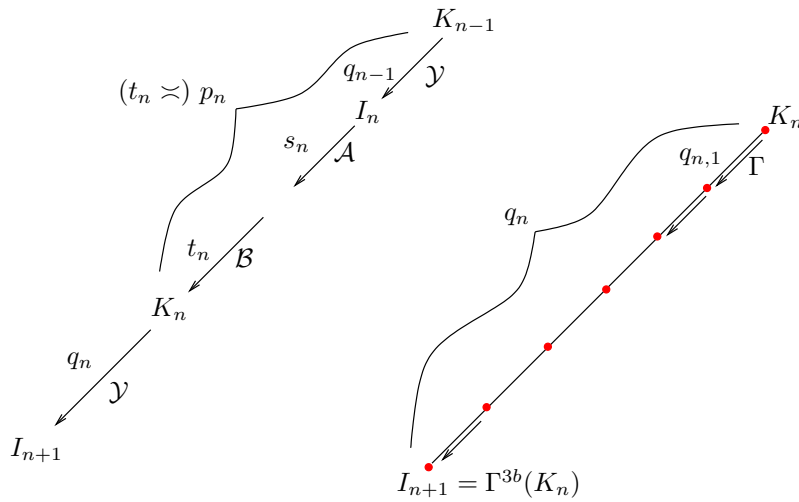
Therefore:

$$\deg(K_m / K_{m-z}) \leq (C_K)^z =: D; \quad (3)$$

$$\deg(K'_{m+2} / K_m) \leq \deg(K'_{m+2} / K_{m+1}) \cdot \deg(K_{m+1} / K_m) \leq C_B^2 \cdot C_Y \cdot C_K =: \beta. \quad (4)$$

2.5 Proof of Proposition A

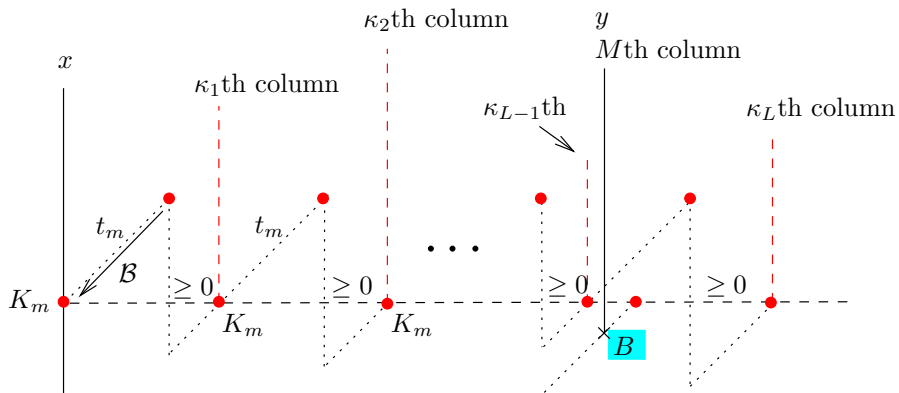
It remains to prove $\deg(A'/B') = \deg(A/B) \leq d$ and that $f^\xi(K_{m+2})$ is below A .



For J colored piece, denote by $r(J) = \min\{|J - J|, \text{ among all tableaux}\}$.

Lemma 2.3. *We have*

- (1) $2 \cdot r(K_n) \leq q_{n,1} \leq \frac{1}{2^{3b-1}} r(I_{n+1})$;
- (2) any two consecutive red spots on $\text{row}(K_n)$ of $\mathcal{T}(c_0)$ have a column difference $\leq q_{n,1}$;
- (3) $t_n \leq p_n \leq (b+4)t_n$, and $p_n \geq 2p_{n-1}$;
- (4) For any $m > m^-$, and for M such that $f^M(K_m) = K_{m^-}$, we have $M < 2p_m$.



L minimal s.t. $\kappa_L > M$

$$(L-1) \cdot t_m \leq \kappa_{L-1} \leq M \stackrel{\text{Lem.2.3.(4)}}{<} 2p_m$$

$$\stackrel{\text{Lem.2.3.(3)}}{\leq} 2(b+4) \cdot t_m = (2b+8) \cdot t_m.$$

So $L \leq 2b+8$.

$$\deg_A / B \leq (C_B \cdot \alpha)^{2b+8} =: d$$

Proof of that $f^\xi(K_{m+2})$ is below A .

For this we will estimate the number of red spots on the top edge of the two upper triangles T' and T , where T' has a left side $\begin{array}{c} K_m \\ | \\ f^\xi(K_{m+2}) \end{array}$ on $column(y)$ of $\mathcal{T}(c_0)$, and T

has a left side $\begin{array}{c} K_m \\ | \\ A \end{array}$ on the same column. Set $r = |T'|$. Then

$$q_{m,1} \cdot (\#\{\text{reds} \in \text{top}(T')\} - 1) \stackrel{\text{Lem.2.3.(2)}}{\geq} r \stackrel{*}{\geq} r(I_{m+2}) \stackrel{\text{Lem.2.3.(1)}}{\geq} \frac{r(I_{m+2})}{r(I_{m+1})} 2^{3b-1} q_{m,1} \stackrel{**}{\geq} 2^{6b-1} q_{m,1}. \quad (5)$$

where the $*$ inequality is due to the fact that r is also equal to the column difference of two red spots on $row(I_{m+2})$ (see Figure 2.2), and the $**$ inequality is due to $r(I_{n+1}) \geq 2^{3b} r(I_n) \geq 2^{3b} r(I_n)$ for all $n \geq 0$ (by Lemma 2.3.(1)).

On the other hand,

$$\#\{\text{reds} \in \text{top}(T)\} - 1 \stackrel{\text{Property of } \mathcal{B}}{\leq} \stackrel{\text{def. of } L}{L} \cdot (b+1) \leq (2b+8)(b+1) \stackrel{b \geq 1}{\leq} 2^{6b-1}. \quad (6)$$

Now (5) and (6) together imply that $f^\xi(K_{m+2})$ is below A . \square