1 Branner-Hubbard conjecture

Theorem 1.1. (Qiu-Yin, Kozlovski-van Strien) Let $g$ be a polynomial with a disconnected its filled Julia set $K_g$. Assume that every critical component of $K_g$ is aperiodic under the iteration of $g$. Then $K_g$ is a Cantor set.
Set up. Let \( f : \mathcal{E} \to \mathcal{L} \) be a proper holomorphic map such that \( \mathcal{L} = \bigcup_{i \in I} L_i \) is the disjoint union of finitely many hyperbolic discs, \( \mathcal{E} \subset\subset \mathcal{L} \), and \( \mathcal{E} = \bigcup_{i,j \in I} \bigcup_{k \in \Lambda_{ij}} E_{ijk} \) is the union of finitely many discs with disjoint closures. Critical points are non-escaping and are in \( L_i \)'s. \( L_i \cup_{/ \text{proper}} E_{ijk} \)

Set \( b = \# \text{Crit}(f) \) and \( \delta := \sup_{c \in \text{Crit}(f)} \deg_c(f) \).

**Theorem 1.2.** Assume that \( f \) is persistently recurrent (see below). Then there is \( C > 0 \), such that for any \( c \in \text{Crit}(f) \), there is a sequence of disjoint annuli \( A_n \) contained essentially in \( P_0(c) \setminus \{c\} \) whose moduli \( \mu(A_n) \) satisfy \( \mu(A_n) \geq C > 0 \). Moreover the constant \( C = C(b, \delta) \) depends only on \( b \) and \( \delta \), and \( A_n \cap \hat{P}_f = \emptyset \).

**Sketch of the proof.** Starting from any critical piece \( K_{-1} \), we define inductively a sequence \( K_{-1} \supset\supset K_n' \supset\supset K_n \) following Kozlovski-Shen-Strien and then a family of Kahn-Lyubich type maps with the same constants. We then prove that the moduli of \( K'_n \setminus K_n \) are bounded from below, using the Kahn-Lyubich covering lemma. \( \square \)
A graph with $T(x) \subset \mathbb{R}^2 \times \mathbb{R}$, some vertices are colored.

3 types of edges: $-,$,$|,$,$/$ (length=1)

A modulus on each vertical segment (subadditive)

A degree on each diagonal segment starting with a given colored vertex (multiplicative)

Degree($/$ edge) = 1 if the left vertex is incolored.

Rule 1

Rule 2

two upper triangles of identical size and depth

of identically colored lower vertex

are identical

Rule 3

In a parallelogram, $d' \geq d$, $\mu \geq \frac{1}{d'} \mu$.

Some vertical segments are called vacuous, i.e., $\cap D_f = \emptyset$

right side vacuous $\implies$ left side vacuous, $d' = d$ and $\mu = \frac{1}{d'} \mu'$
Kahn-Lyubich covering lemma

Given $\eta, D, d$. In any parallelogram $V$ either

\[
\begin{align*}
&\text{IF } \mod_3 \geq \eta \cdot \mod_2, \\
&\text{THEN } 0 < \varepsilon(\eta, D) < \frac{\eta}{2d^2} \mod_3 < \end{align*}
\]

or
1.1 Successors and persistent recurrence

We say that $f$ is **persistently recurrent** if:

- given any couple $(c, c') \in \text{Crit}(f)^2$, marching horizontally to the right of any position in $T(c)$, one will meet a position that has the color of $c'$; and $T(c)$ contains no infinite long critical nests of $c'$ (except of course on the 0th column);
- every critical piece of any color has $< +\infty$ successors.
Proposition A.

\[ \exists (K'_n, K_n)_n \text{ such that} \]

\[ \forall m > Z > 0 \]

\[ D = (C_K)^Z \]

\[ \alpha, \beta, d, C_K \text{ depend only on } b, \delta \]

\[ f^\xi(K_{m+2}) \text{ is below } A \]
Proof of Theorem 1.2.

Assume there is a sequence $k_n \to \infty$ such that $\mu_{m'} \geq \mu_{k_n}$ (otherwise $\mu_n \geq \mu_{n_0} > 0$ for some $n_0$). Fix any $n$ satisfying $k_n - 2 > Z$. Set $m = k_n - 2$. Then $m + 2 = k_n$ and $\mu_{m'} \geq \mu_{m + 2}$ for any $m' \leq m + 2$. For this $m$, construct as above $(U, V, M, x, y, B', B, A, A')$. Then,

\[
\begin{align*}
\mu_{m+2} & \geq \text{mod} \left( \frac{U}{f^\xi(K_{m+2})} \right) \text{mod} \left( \frac{U}{A} \right) \\
\text{deg} \left( K_{m+2} \setminus U \right) & \geq \beta ;
\end{align*}
\]

(1)

\[
\begin{align*}
\text{choice of } k_n & \quad \mu_{m+2} \geq \frac{\alpha}{\alpha \beta} ; \\
\text{choice of } k_n & \quad \mu_{m+2} \geq \frac{\alpha}{\alpha \beta} =: \eta \cdot \text{mod} \left( \frac{U}{A} \right) .
\end{align*}
\]

\[
\begin{align*}
\text{either } U & \quad \text{mod} \left( \frac{B'}{B} \right) ; \\
\beta \cdot \mu_{m+2} & \quad \geq \frac{\alpha}{\alpha \beta} \cdot \text{mod} \left( \frac{V}{B} \right) \cdot \frac{\alpha}{\alpha \beta} \cdot \frac{\eta Z}{2d^2 \alpha} \mu_{m+2} ,
\end{align*}
\]

where the inequality (*) is proved by using the pullback of the first $\geq 0$ hit of $y$ to $K_j$ for $j = m, m - 1, \ldots, m - Z + 1$, and the property (K') that $(K'_j \setminus K_j) \cap \hat{P}_j = \emptyset$.

Set $Z = 2d^2 \alpha^2 \beta^2 + 1$. Then $Z > 2d^2 \alpha^2 \beta^2 = \frac{2d^2 \alpha \beta}{\eta}$. Hence the second line above is impossible. So $\mu_{k_n} = \mu_{m + 2} > \frac{1}{\beta} \cdot \epsilon(\eta, D) > 0$. Therefore

\[
\forall l \in \mathbb{N}, \quad \mu_l \geq \lim_{n \to \infty} \mu_{k_n} \geq \frac{1}{\beta} \cdot \epsilon(\eta, D) =: C(b, \delta) > 0.
\]

\[\square\]
2 Construction of the KSS nest \((K'_n, K_n)_n\)

We will define four pullback operators \(\mathcal{Y}, A, B, C\) from a critical nest into itself, and then set inductively \(K_n = B \circ A \circ \mathcal{Y}(K_{n-1})\) and \(K'_n = C \circ \mathcal{Y}(K_{n-1})\).

2.1 First hit has bounded degree

\[
\deg \left( \frac{B'}{K'_m} \right) = \deg \left( \frac{B}{K_m} \right) \quad \text{and} \quad \begin{array}{c}
\text{\text{y}} \\
\times \text{first hit} \\
\geq 0 \\
\left\langle \frac{K_m \text{ (red)}}{B} \right\rangle \\
\text{\text{deg} \leq \delta^{b-1} = \alpha}
\end{array}
\] (2)

Proof. \(\text{diag}_{B/K_m}\) does not meet red color, and meets every other color at most once. Otherwise we get two upper triangles of different sizes, both with \(K_m\) as the right vertex, and both have a left side of the same coloring. Now moving the smaller triangle to the left and applying Rule 2 would imply an earlier hit of red spot on the horizontal segment containing \(K_m\). \(\square\)
2.2 Last successor operator $\Gamma$ and the operator $\mathcal{Y} = \Gamma^{3b}$

For every critical puzzle piece $P$,

$\Gamma(P)$ = last successor of $P$, 

$\mathcal{Y}(P) := \Gamma^{3b}(P)$ = the 3b-th generation of last successors.

Lemma 2.1. (a consequence of tableaux rules) Let $P$ be a critical puzzle piece.

\begin{enumerate}
  \item $2 \leq \#\{\text{successors of } P\} < +\infty$.
  \item $\deg\left(\mathcal{Y}(P) / P\right) \leq C_Y := \delta^{(2b-1)3b}$.
\end{enumerate}

The only difficulty in this Lemma is the part that $P$ has at least two successors.
2.3 The operators $A$, $B$ and $C$

**Lemma 2.2.** For any critical puzzle piece $I$, there are $A(I), B(I), C(I)$ such that

- $A(\hat{P_f})$ is vacuous
- $B(I) \leq b + 1, b + 2$ visits of red spots
- Degree $\leq C_B, C_A$ (depending only on $b, \delta$), and
- $C(I) \leq b + 1, b + 2$ visits of red spots
2.4 The KSS nest

Fix $c_0 \in Crit(f)$. Let $K_{-1}$ be a critical puzzle piece of $c_0$. We define inductively $K_n = \mathcal{B}(\mathcal{A}(\mathcal{Y}(K_{n-1})))$ and $K'_n = \mathcal{C}(\mathcal{Y}(K_{n-1}))$.

They satisfy (following Lemmas 2.1, 2.2):

(K) \[
\deg \left( \frac{K_n}{K_{n-1}} \right) \leq C_{K} := C_B \cdot C_A \cdot C_Y \quad (\text{depending only on } b, \delta);
\]

\[
3b + 2 \leq \#(\text{red spots}) \cap \text{diag}_{\frac{K_n}{K_{n-1}}} \leq b + b + 1 + 3b = 5b + 1.
\]

(K') \[
(K'_n \setminus K_n) \cap \hat{\mathcal{P}}_f = \emptyset, \quad \deg \left( \frac{K'_n}{K_{n-1}} \right) \leq C_{B}^2 \cdot C_Y.
\]

Therefore:

\[
\deg \left( \frac{K_m}{K_{m-2}} \right) \leq (C_K)^2 =: D;
\]

\[
\deg \left( \frac{K_{m+2}}{K_m} \right) \leq \deg \left( \frac{K_{m+2}}{K_{m+1}} \right) \cdot \deg \left( \frac{K_{m+1}}{K_m} \right) \leq C_B^2 \cdot C_Y \cdot C_K =: \beta.
\]
2.5 Proof of Proposition A

It remains to prove $\deg (A' / B') = \deg (A / B) \leq d$ and that $f^\xi(K_{m+2})$ is below $A$.

For $J$ colored piece, denote by $r(J) = \min \{|J - J|, \text{ among all tableaux}\}$.

**Lemma 2.3.** We have

1. $2 \cdot r(K_n) \leq q_{n,1} \leq \frac{1}{2m-1} r(I_{n+1})$;
2. any two consecutive red spots on row($K_n$) of $T(c_0)$ have a column difference $\leq q_{n,1}$;
3. $t_n \leq p_n \leq (b + 4)t_n$, and $p_n \geq 2p_{n-1}$;
4. For any $m > m^-$, and for $M$ such that $f^M(K_m) = K_{m^-}$, we have $M < 2p_m$. 
\( L \) minimal s.t. \( \kappa_L > M \)

\[(L - 1) \cdot t_m \leq \kappa_{L-1} \leq M < 2p_m \]

\[\text{Lem.2.3.}(4)\]

\[\leq 2(b + 4) \cdot t_m = (2b + 8) \cdot t_m.\]

So \( L \leq 2b + 8. \)

\[\text{deg}_A / B \leq (C_B \cdot \alpha)^{2b+8} =: d\]
**Proof of that** $f^\xi(K_{m+2})$ **is below** $A$.

For this we will estimate the number of red spots on the top edge of the two upper triangles $T'$ and $T$, where $T'$ has a left side $K_m$ on column $(y)$ of $T(c_0)$, and $T$ has a left side $A$ on the same column. Set $r = |T'|$. Then

$$q_{m,1} \cdot (\# \{ \text{reds} \in \text{top}(T') \} - 1) \geq r \geq r(I_{m+2}) \geq \frac{r(I_{m+2})}{r(I_{m+1})} 2^{3b-1} q_{m,1} \geq 2^{6b-1} q_{m,1}. \tag{5}$$

where the * inequality is due to the fact that $r$ is also equal to the column difference of two red spots on row $(I_{m+2})$ (see Figure 2.2), and the ** inequality is due to $r(I_{n+1}) \geq 2^{3b} r(K_n) \geq 2^{3b} r(I_n)$ for all $n \geq 0$ (by Lemma 2.3.(1)).

On the other hand,

$$\# \{ \text{reds} \in \text{top}(T) \} - 1 \leq L \cdot (b + 1) \leq (2b + 8)(b + 1) \leq 2^{6b-1}. \tag{6}$$

Now (5) and (6) together imply that $f^\xi(K_{m+2})$ is below $A$. \qed