Funktionalanalysis
Lösungen für Woche 1

21. April 2004

1. Let $\{e_1, \ldots, e_n\}$ be a set of pairwise-orthogonal and non-zero vectors in some Hilbert space. Consider scalars $\alpha_1, \ldots, \alpha_n$ such that

   $$\alpha_1 e_1 + \cdots + \alpha_n e_n = 0$$

   Then for each k,

   $$0 = \langle \alpha_1 e_1 + \cdots + \alpha_n e_n, e_k \rangle = \alpha_1 \langle e_1, e_k \rangle + \cdots + \alpha_n \langle e_n, e_k \rangle = \alpha_k \langle e_k, e_k \rangle$$

   since the set $\{e_1, \ldots, e_n\}$ is pairwise-orthogonal. But $e_k \neq 0$, so $\langle e_k, e_k \rangle \neq 0$, and $\alpha_k = 0$. It follows that the set $\{e_1, \ldots, e_n\}$ is linearly independent.

2. Let H be a real Hilbert space, and let $x, y \in H$ be orthogonal. Then $\langle x, y \rangle = 0$. Hence for any real number $\alpha \in \mathbb{R}$, we know that

   $$\begin{aligned} \|x + \alpha y\|^2 &= \langle x + \alpha y, x + \alpha y \rangle \\ &= \langle x, x \rangle + \langle y, y \rangle \end{aligned}$$

   Similarly

   $$\begin{aligned} \|x - \alpha y\|^2 &= \langle x - \alpha y, x - \alpha y \rangle \\ &= \langle x, x \rangle + \langle y, y \rangle \end{aligned}$$

   so $\|x + \alpha y\| = \|x - \alpha y\|$.

   Conversely, suppose that $\|x + \alpha y\| = \|x - \alpha y\|$ for all real numbers $\alpha \in \mathbb{R}$. Then $\|x + \alpha y\|^2 = \|x - \alpha y\|^2$ for all α. Observe:

   $$\begin{aligned} \|x + \alpha y\|^2 &= \langle x + \alpha y, x + \alpha y \rangle \\ &= \langle x, x \rangle + 2\alpha \langle x, y \rangle + \langle y, y \rangle \end{aligned}$$

   and

   $$\begin{aligned} \|x + \alpha y\|^2 &= \langle x + \alpha y, x + \alpha y \rangle \\ &= \langle x, x \rangle - 2\alpha \langle x, y \rangle + \langle y, y \rangle \end{aligned}$$

   It follows that $\alpha \langle x, y \rangle = 0$ for all real numbers α. In particular, $\langle x, y \rangle = 0$, so the vectors x and y are orthogonal.

3. (a) Let X be the space of all continuous functions $f\colon [0,1] \to \mathbb{R}$, with the L^2-scalar product

$$\langle f, g\rangle = \int_{-1}^{1} fg$$

Write

$$A = \{f \in X \mid f(x) = f(-x) \text{ for all } x \in [-1,1]\}$$

and

$$B = \{f \in X \mid f(x) = -f(-x) \text{ for all } x \in [-1,1]\}$$

If $f \in A \cap B$, then $f(-x) = -f(-x)$ for all $x \in [-1,1]$. It follows that $f(-x) = 0$ for all $x \in [-1,1]$, and so $f = 0$. Hence $A \cap B = \{0\}$.
Given a continuous function $f\colon [-1,1] \to \mathbb{R}$, let us write

$$f_A(x) = \frac{1}{2}(f(x) + f(-x)) \qquad f_B(x) = \frac{1}{2}(f(x) - f(-x))$$

Then $f_A \in A$, $f_B \in B$, and $f = f_A + f_B$. Therefore $X = A \oplus B$.
More generally, let $f \in A$ and $g \in B$. Then $f(x)g(x) = -f(-x)g(-x)$. It follows that

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)\ dx = \int_{0}^{1} f(x)g(x)\ dx - \int_{0}^{1} f(x)g(x)\ dx = 0$$

Therefore $A \perp B$.

(b) Write

$$A = \{f \in X \mid f(x,y,z) = f(-x,y,z) \text{ for all } (x,y,z) \in \mathbb{R}^3\}$$

and

$$B = \{f \in X \mid f(x,y,z) = -f(-x,y,z) \text{ for all } (x,y,z) \in \mathbb{R}^3\}$$

If $f \in A \cap B$, then $f(-x,y,z) = -f(-x,y,z)$ for all $(x,y,z) \in \mathbb{R}^3$. It follows that $f(-x,y,z) = 0$ for all $(x,y,z) \in \mathbb{R}^3$, and so $f = 0$. Hence $A \cap B = \{0\}$.
Given a continuous function $f\colon \mathbb{R}^3 \to \mathbb{R}$, let us write

$$f_A(x,y,z) = \frac{1}{2}(f(x,y,z) + f(-x,y,z)) \qquad f_B(x,y,z) = \frac{1}{2}(f(x,y,z) - f(-x,y,z))$$

Then $f_A \in A$, $f_B \in B$, and $f = f_A + f_B$. Therefore $X = A \oplus B$.

4. Let H be a complex Hilbert space, and let $x, y \in H$ be orthogonal. Then $\langle x, y\rangle = 0$. Hence for any complex number $\alpha \in \mathbb{C}$, we know that

$$\begin{aligned} \|x + \alpha y\|^2 &= \|x\|^2 + |\alpha|^2\|y\|^2 \\ &\geq \|x\|^2 \end{aligned}$$

Hence $\|x + \alpha y\| \geq \|x\|$.

Conversely, suppose we have two vectors $x, y \in H$ such that $\|x + \alpha y\| \geq \|x\|$ for all $\alpha \in \mathbb{C}$. Then for any complex number $\alpha \in \mathbb{C}$, we know that $\|x + \alpha y\|^2 \geq \|x\|^2$. Observe:

$$\|x + \alpha y, x\|^2 = \|x\|^2 + \alpha\langle x, y\rangle + \overline{\alpha\langle x, y\rangle} + |\alpha|^2\|y\|^2$$

so

$$\alpha\langle x, y\rangle + \overline{\alpha\langle x, y\rangle} + |\alpha|^2\|y\|^2 =\geq 0$$

Suppose that $\|y\| \leq \|x\|$, and let

$$\alpha = -\lambda\overline{\langle x, y\rangle}$$

where $\lambda \in \mathbb{R}\backslash\{0\}$.

Then the above inequality tells us that

$$-2|\langle x, y\rangle|^2 + \lambda|\langle x, y\rangle|\|y\|^2 \geq 0$$

But this inequality can only hold for all $\lambda \in \mathbb{R}\backslash\{0\}$ if $\langle x, y\rangle = 0$, ie: x and y are orthogonal.

5. See the solution of problem 1, next week.

6. We will use the Gram-Schmidt orthonormalisation process. Let $g_1(x) = 1$, $g_2(x) = x$, and $g_3(x) = x^2$. Then we define:

$$f_1(x) = \frac{g_1}{\|g_1\|}$$

Here

$$\|g_1\|^2 = \int_{-1}^{1} 1\ dx = 2$$

so $f_1(x) = 1/\sqrt{2}$.

For the next term,

$$\tilde{f}_2 = g_2 - \langle f_1, g_2\rangle f_1 \qquad f_2 = \frac{\tilde{f}_2}{\|f_2\|}$$

Here

$$\tilde{f}_2(x) = x - \left(\int_{-1}^{1} \frac{x}{\sqrt{2}}\ dx\right)\frac{1}{\sqrt{2}} = x$$

and

$$\|\tilde{f}_2\|^2 = \int_{-1}^{1} x^2\ dx = \frac{2}{3}$$

so $f_2(x) = \sqrt{3/2}x$.

For the final term,

$$\tilde{f}_3 = g_3 - \langle f_1, g_3 \rangle f_1 - \langle f_2, g_3 \rangle f_2 quad f_2 = \frac{\tilde{f}_2}{\|f_2\|}$$

Here

$$\tilde{f}_3(x) = x^2 - \left(\int_{-1}^{1} \frac{x^2}{\sqrt{2}} \; dx \right) \frac{1}{\sqrt{2}} - \left(\int_{-1}^{1} \sqrt{3/2} x^3 \; dx \right) x = x^2 - \frac{1}{3}$$

and

$$\|\tilde{f}_3\|^2 = \int_{-1}^{1} (x^2 - 1/3)^2 \; dx = \frac{8}{45}$$

so $f_2(x) = \sqrt{\frac{5}{2}} \frac{3}{2} (x^2 - 1/3)$.