Funktionalanalysis
Lösungen für Woche 11

2. Juli 2004

1. Sei H komplexer Hilbertraum.

   (a) The first four statements are trivial consequences of the fact that the formula
   $$\langle A^\star u, v\rangle = \langle u, Av\rangle$$
   holds for all $u, v \in H$.
   To prove the last statement, suppose for now that A has finite-dimensional image U. Then U is closed, so $H = U \oplus U^\perp$.
   Let $u, v \in H$. Write $u = u_1 + u_2$ where $v_1 \in U$ and $v_2 \in U^\perp$. Then
   $$\langle A^\star u, v\rangle = \langle u_1, Av\rangle + \langle u_2, Av\rangle = \langle u_1, Av\rangle = \langle A^\star u_1, v\rangle$$
   It follows that $im\, A^\star \subseteq A^\star[U]$, so the operator $A^\star$ has finite-dimensional image.
   More generally, let A be a compact operator. Then we can find a sequence of finite-rank operators (A_N) such that
   $$\lim_{N\to\infty} \|A - A_N\| = 0$$
   Hence
   $$\lim_{N\to\infty} \|A^\star - A_N^s tar\| = 0$$
   and the operator $A^\star$ is also compact.

   (b) Let $u, v \in H$. Then
   $$\phi(A^\star u)v = \langle A^\star u, v\rangle = \langle u, Av\rangle$$
   and
   $$A'(\phi u)v = (\phi u)(Av) = \langle u, Av\rangle$$
   Hence $\phi \circ A^\star = A' \circ \phi$.

2. Let $c(x) = 1$ for all $x \in [0, 1]$. Then

$$Tc(x) = i \int_0^x 1\ dt = ix$$

We know from last week's problems that

$$T^\star f(x) = -i \int_x^1 f(t)\ dt$$

so in our case

$$T^\star c(x) = -i \int_x^1 t\ dt = i(x-1)$$

Clearly $T^\star c \neq Tc$, so T is not self-adjoint.

Suppose that $f \in H$, so

$$\int_0^1 f(t)\ dt = 0$$

Then

$$\int_0^x f(t)\ dt + \int_x^1 f(t)\ dt = 0$$

and

$$i \int_0^x f(t)\ dt - (-i) \int_x^1 f(t)\ dt = 0$$

that is

$$Tf(x) - T^\star f(x) = 0$$

[There is a problem with the following calculation, since $T|_H[H] \not\subseteq H$. We will pretend that this problem does not exist, in order to see how the work should go].

Thus $T|_H$ is self-adjoint. The operator $T|_H$ is certainly compact, so the spectrum consists of the point zero, and some real eigenvalues.

Suppose that $Tf = \lambda f$ where $f \in H$, and $\lambda \in \mathbb{R}\backslash\{0\}$. Then

$$i \int_0^x f(t)\ dt = \lambda f(x) \qquad \int_0^1 f(t)\ dt = 0$$

Let us assume that the function f is differentiable. Then

$$f'(x) = \frac{i}{\lambda} f(x)$$

We have solution $f(x) = Ae^{(i/\lambda)x}$, where A is constant. The condition $\int_0^1 f(t)\ dt = 0$ means that

$$\frac{1}{\lambda} = 2\pi k \qquad k \in \mathbb{Z}$$

Hence

$$\sigma(A) = \{0\} \cup \{1/2\pi k \mid k \in \mathbb{Z}\}$$

[We have ignored here the obvious condition $f(0) = 0$. This condition would make our whole calculation nonsense...]

For each point of the spectrum, $\lambda \neq 0$, the λ-eigenspace is 1-dimensional. Further, $\lambda \in [1/16, 1]$ precisely when $1/\lambda \in [1, 16]$. Let $\lambda = 2\pi k$. Then, for $\lambda \in [1/16, 1]$, we know that $k = 1$ or $k = 2$.

Each eigenspace has dimension one.

The map $\chi_{[1/16,1]}(T|_H)$ is projection onto the sum of all eigenspaces, with eigenvectors in the interval $[1/16, 1]$, and so has dimension 2.

3. (a) Supose that there is a sequence (x_n) in H such that $\|x_n\| = 1$ for all n, and

$$\lim_{n\to\infty} \|Ax_n - \lambda x_n\| = 0$$

Suppose that the map $A - \lambda$ has an inverse S. Then

$$\lim_{n\to\infty} \left(\frac{\|S(A-\lambda)x_n\|}{\|Ax_n - \lambda x_n\|} \right) = \lim_{n\to\infty} \frac{1}{\|Ax_n - \lambda x_n\|} = \infty$$

Therefore the operator S is not continuous., It follows that the map $A - \lambda$ has no continuous inverse, so $\lambda \in \sigma(A)$.
Conversely, suppose that there is no sequence (x_n) such that $\|x_n\| = 1$ for all n and

$$\lim_{n\to\infty} \|Ax_n - \lambda x_n\| = 0$$

Then there is a constant $C > 0$ such that

$$\|Ax - \lambda x\| \geq C\|x\|$$

for all $x \in H$. It follows that the map $A - \lambda$ is injective, and the image $im(A - \lambda) \subseteq H$ is closed.
But

$$im(A-\lambda)^{\perp}) = \ker(A^\star - \overline{\lambda}) = \ker(A - \lambda) = \{0\}$$

so $im(T) = H$. Hence $\lambda \notin \sigma(T)$, and we are done.

(b) Let $A \in \mathcal{B}(H)$, $A \geq 0$. We have already shown (last week's questions) that A is self-adjoint.
Now, let $\lambda = \alpha + i\beta$, where $\beta \neq 0$. Let $x \in H$, with $\|x\| = 1$. Then

$$\|(A-\lambda)x\|^2 = \|(A-\alpha)x\|^2 + \beta^2\|x\|^2 \geq \beta^1 2\|x\|^2$$

Hence, by part (a), $\alpha + i\beta \notin \sigma(A)$, and $\sigma(A) \subseteq \mathbb{R}$.
Now, let $\lambda > 0$. Let $x \in H$. Then $\langle Tx, x\rangle \geq 0$, so

$$\lambda\|x\|^2 \leq \langle \lambda x, x\rangle + \langle Tx, x\rangle = \langle (T+\lambda)x, x\rangle \leq \|(T+\lambda)x\|\|x\|$$

by the Cauchy-Schwarz inequality. Hence

$$\|(T+\lambda)x\| \geq \lambda\|x\|$$

By part (a), $-\lambda \notin \sigma(A)$, and $\sigma(A) \subseteq [0, \infty)$.

4. By the spectral theorem, it is clear that

$$(g(T)f)(x) = g(x)f(x)$$

for all $x \in [0,1]$. Let $g = \chi_{[a,b]}$. Then

$$(g(T)f)(x) = \begin{cases} f(x) & x \in [a,b] \\ 0 & \text{otherwise} \end{cases}$$

It is clear that $g(T)$ is the orthogonal projection onto the space

$$im\, g(T) = \{f \in L^2[0,1] \mid \; Supp(f) \subseteq [a,b]\}$$

and

$$\ker g(T) = \{f \in L^2[0,1] \mid \; Supp(f) \subseteq [0,1] \backslash (a,b)\}$$

5. Let $U \in \mathcal{B}(H)$ be unitary, and let $\lambda \in \sigma(U)$. Then $\|U\| = 1$, so $|\lambda| \leq 1$.

Let $|\lambda| < 1$. Then $\|U^\star\| = 1$, so $\|\lambda U^\star\| < 1$. Hence the operator $I - \lambda U^\star$ is invertible. But

$$U - \lambda I = U(I - \lambda U^\star)$$

so the operator $U - \lambda I$ is also invertible, and $\lambda \notin \sigma(U)$.

To summarise, $\sigma(U) \subseteq \{z \in \mathbb{C} \mid |z| = 1\}$.