Funktionalanalysis
Lösungen für Woche 12

28. Juni 2004

1. Let $(a_n) \in D$. Then $\sum_{n=1}^{\infty} |a_n| < \infty$. Thus there exists N such that $|a_n| \leq 1$ when $n \geq N$. Therefore $|a_n|^2 \leq |a_n$ whenever $n \geq N$, and the series $\sum_{n=1}^{\infty} |a_n|^2 < \infty$. Hence $D \subseteq l^2(\mathbb{N})$.

   We claim that D is dense in $l^2(\mathbb{N})$. To see this, let $\varepsilon > 0$, and $(a_n) \in l^2(\mathbb{N})$. Then there exists N such that
   $$\sum_{n=N+1}^{\infty} |a_n|^2 < \varepsilon^2$$
   Let
   $$b_n = \begin{cases} a_n & n \leq N \\ 0 & n \geq N+1 \end{cases}$$
   Then $b_n \in D$, and $|(b_n) - (a_n)|_2 < \varepsilon$. Thus D is dense in $l^2(\mathbb{N})$, and we are done.

   Now, for our operator T, we know that the domain $D(T^\star)$ consists of all points $z \in l^2(\mathbb{N})$ such that there exists $z \in l^2(\mathbb{N})$ with
   $$\langle Tx, z \rangle = \langle x, z' \rangle$$
   for all $x \in D$.

   Write $z = (z_n)$ and $z' = (z'_n)$. Then
   $$\langle Tx, z \rangle = \sum_{m,n=1}^{\infty} \overline{x_n y_m} z_m$$
   and
   $$\langle x, z' \rangle = \sum_{n=1}^{\infty} \overline{x_n} z'_n$$
   Thus
   $$z'_n = \sum_{m=1}^{\infty} \overline{y_m} z_m$$

for all n.

But $(z'_n) \in l^2(\mathbb{N})$, so $\sum_{n=1}^{\infty} |z'_n|^2 < \infty$. Therefore

$$\sum_{m=1}^{\infty} \overline{y_m} z_m = 0$$

that is $\langle y, z \rangle = 0$. We have proved that $y \perp D(T^\star)$.

It follows that $D(T^\star)^\perp \neq \{0\}$, and so the space $D(T^\star)$ is not dense in $l^2(\mathbb{N})$.

2. Let $T = T^\star$, and $\ker T = \{0\}$. Then certainly $\ker T^\star = \{0\}$.

   Let $\in (im\, T)^\perp$. Then $\langle u, Tv \rangle = \langle Tu, v \rangle = 0$ for all $v \in D$. Since D is a dense subset of H, it follows that $Tu = 0$, and so $u = 0$.

   We have proved that $(im\, T)^\perp = \{0\}$, and therefore $im\, T$ is dense in H.

   It is true in general that $(T^{-1})^\star = (T^\star)^{-1}$ when T is injective. Therefore T^{-1} is self-adjoint when T is self-adjoint.

   Let us define $T \colon l^2(\mathbb{N}) \to l^2(\mathbb{N})$ be the formula

$$x = (x_n) \qquad Tx = (x_n/n)$$

   Then it is easy to see that T is bounded, self-adjoint, and injective.

   However, given $N \in \mathbb{N}$, we can find $x \in l^2(\mathbb{N})$ such that $\|x\| = 1$, and $\|Tx\| = 1/n$. Hence the inverse T^{-1} is not bounded.

3. Let T be symmetric. Then

$$\langle Tx, y \rangle = \langle x, Ty \rangle$$

   for all $x, y \in D$. In particular

$$\langle Tx, x \rangle = \langle x, Tx \rangle = \overline{\langle Tx, x \rangle}$$

   for all $x \in D$. Therefore $\langle Tx, x \rangle \in \mathbb{R}$ for all $x \in D$.

   The converse follows by polarisation, as in problem 1 last week.

4. Let $x \in D(ST)$ and $y \in D(T^\star S^\star)$. Then $y \in D(S^\star)$, $S^\star y \in D(T^\star)$, and $x \in D(T)$, $Tx \in D(S)$. Hence the formula

$$\langle S(Tx), y \rangle = \langle Tx, S^\star y \rangle = \langle x, T^\star S^\star \rangle$$

   makes sense, and $T^\star S^\star \subseteq (ST)^\star$.

   Now, let $S \colon H \to H$ be bounded. We need to show that $D((ST)^\star \subseteq D(T^\star S^\star)$.

   Let $y \in (ST)^\star$. Then there is an element $y' \in H$ such that

$$\langle STx, y \rangle = \langle x, y' \rangle$$

for all $x \in D(ST)$.

Since S is bounded and $D(S) = H$:

$$\langle STx, y\rangle = \langle Tx, S^\star y\rangle = \langle x, y'\rangle$$

Hence $S^\star y \in D(T^\star)$. Since $y \in H = D(S^\star)$, it follows that $y \in D(T^\star S^\star)$ and we are done.

5. We can certainly find a function $c \in C_c^\infty(\mathbb{R})$ such that $|c|_\infty = 1$, and $|c'|_{L^2} = \alpha \neq 0$.

   Define $c_n(x) = c(nx)$. Then

   $$|c_n|_\infty = 1 \qquad |c_n'(x)|^2 = n^2|c'(nx)|$$

   so

   $$|c_n|_2^2 = \int_{\mathbb{R}} n^2|c'(nx)|^2\ dx = n\alpha^2$$

   and

   $$|c_n|_2 = \alpha\sqrt{n}$$

   It follows that the given function $T\colon C_c^\infty(\mathbb{R}) \to L^2(\mathbb{R})$ is unbounded.

6. Since $T \in \mathcal{B}(H)$ is normal, we can apply the continuous functional calculus. If $f\colon \sigma(T) \to \sigma(T)$ is defined by the formula $f(\lambda) = \overline{\lambda}$, then $f(T) = T^\star$.

   (a) Let $\sigma(T) \in \mathbb{R}$. Then $f(\lambda) = \lambda$ for all $\lambda \in \sigma(T)$, and so $T = T^\star$.

   (b) Let $|\lambda| \in \sigma(T)$. Then $|\lambda|^2 = 1$, so $f(\lambda)\lambda = \lambda f(\lambda) = |\lambda|^2 = 1$. It follows that

   $$T^\star T = TT^\star = I$$