Funktionalanalysis
Lösungen für Woche 9

17. Juni 2004

1. (a) Let

$$A = \{a \in [0,1] \mid (f_n(a)) \text{ converges}\}$$

Let $a \in \overline{A}$. Let (a_k) be a sequence in A with limit a. Write

$$b_k = \lim_{n\to\infty} f_n(a_k)$$

Let $\varepsilon > 0$. Then we can find $\delta > 0$ such that $|f_n(a_k) - f_n(a_l)| < \varepsilon$ whenever $|a_k - a_l| < \delta$ and $n \in \mathbb{N}$. Since the sequence (a_k) is Cauchy, we can find $N \in \mathbb{N}$ such that $|a_k - a_l| < \delta$ whenever $k, l \geq N$. It follows that $|f_n(a_k) - f_n(a_l)| < \varepsilon$ whenever $k, l \geq N$ and $n \in \mathbb{N}$.

Taking limits, we see that $|b_k - b_l| \leq \varepsilon$ whenever $k, l \geq N$. Hence the sequence (b_k) is Cauchy.

Let

$$b = \lim_{k\to\infty} b_k$$

We claim that

$$b = \lim_{n\to\infty} f_n(a)$$

so $a \in A$, and the set $\overline{A}$ is closed.

Observe

$$|b - f_n(a)| \leq |b - b_k| + |b_k - f_n(a_k)| + |f_n(a_k) - f_n(a)|$$

We can find $\delta > 0$ such that $f_n(a) - f_n(a_k)| < \varepsilon/3$ whenever $|a - a_k| < \delta$. We can find $N, k \in \mathbb{N}$ such that $|b - k_k| < \varepsilon/3$, $|b_k - f_n(a_k)| < \varepsilon/3$, and $|a - a_k| < \delta$ whenever $n \geq N$. Hence

$$|b - f_n(a)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

whenever $n \geq N$, and we are done.

(b) Let

$$f_n(x) = 1 + x + x^2 + \cdots + x^n$$

Then each function $f_n\colon [0,1] \to \mathbb{C}$ is continuous. The sequence $(f_n(a))$ converges when $a \in [0,1)$, and diverges when $a = 1$. Therefore the set

$$A = \{a \in [0,1] \mid (f_n(a)) \text{ converges}\}$$

is not closed.

2. (a) The set $\{e_k \mid k \in \mathbb{Z}\}$ is an orthogonal basis for the sapce $L^2_p(\mathbb{R})$. Hence, for any function $f \in L^2_p(\mathbb{R})$, there are unique complex numbers $\alpha_k \in \mathbb{C}$ such that

$$f = \sum_{k \in Z} \alpha_k e_k$$

Here convergence is in the $L^2 - norm$. By definition, $f \in H$ if and only if $\alpha_k = 0$ whenever $k < 0$.

Observe that

$$e_m(x)e_n(x) = e_{m+n}(x)$$

Hence

$$M_{e_n} f = \sum_{k \in \mathbb{Z}} \alpha_k e_{k+n}$$

and

$$PM_{e_n} f = \sum_{k \geq -n} \alpha_k e_{k+n}$$

Similarly

$$PM_{e_n} f = \sum_{k \geq 0} \alpha_k e_{k+n}$$

so

$$(PM_{e_n} - M_{e_n} P)f = \sum_{k=-n}^{-1} \alpha_k e_{k+n}$$

Thus the operator $PM_{e_n} - M_{e_n} P$ has finite-dimensional range, and so is compact. Therefore $e_n \in S$.

(b) Let $Q(x) = \alpha_0 e_0(x) + \cdots + \alpha_n e_n(x)$, where $\alpha_i \in \mathbb{C}$. Observe that

$$M_Q = \alpha_0 M_{e_0} + \cdots + \alpha_n M_{e_n}$$

so

$$PM_Q - M_Q P = \alpha_0 (PM_{e_0} - M_{e_0} P) + \cdots + \alpha_n (PM_{e_n} - M_{e_n} P)$$

By the above exercise, the operator $PM_Q - M_Q P$ is thus a finite linear combination of compact operators, and so is itself compact. Therefore $Q \in \S$.

(c) Let $g \in S$ with $g(x) \neq 0$ for all $x \in \mathbb{R}$. Then the operator $S = PM_{1/g}$ is well-defined. We already know that the operator

$$K = PM_g - M_g P$$

is compact. Hence

$$T_g S = PM_g PM_{1/g} = P(PM_g - K)M_{1/g}P = P^2 M_g M_{1/g} P + L = I + L$$

where L is a compact operator, since P is the projection onto H. A similar calculation tells us that $ST_g = I + L'$, where L' is a compact operator. Hence, by exercise 5, the operator PM_g is Fredholm.

(d) Let $f \in H$. Write

$$f = \sum_{k \in Z} \alpha_k e_k$$

Let $n \geq 0$. Then

$$T_{e_n} f = \sum_{k=0}^{\infty} \alpha_k e_{k+n}$$

Thus

$$\dim(\ker T_{e_n}) = 0 \qquad \dim(coker\, T_{e_n}) = n$$

and

$$Index(T_{e_n}) = -n$$

Similarly, if $n < 0$, then

$$\dim(\ker T_{e_n}) = -n \qquad \dim(coker\, T_{e_n}) = 0$$

and

$$Index(T_{e_n}) = -n$$

Extra Exercise: Let $g \in C_p(\mathbb{R})$. Then the function g is a uniform limit of functions of the form

$$P_n = \sum_{k=-n}^{n} \alpha_k e_k$$

The above work tells us that $P_n \in S$ for all n. By the uniform convergence theorem, we see that the operator $PM_g - M_g P$ is the norm limit of the operators $PM_{P_n} - M_{P_n} P$. But the operators $PM_{P_n} - M_{M_n} P$ are all of finite rank. Therefore the operator $PM_g - M_g P$ is compact, and $g \in S$.

3. By exercise 5, there are operators G_1 and G_2, and compact operators K_1, K_1', K_2, K_2' such that

$$F_1 G_1 = 1 - K_1 \qquad G_1 F_1 = 1 - K_1'$$

and

$$F_2G_2 = 1 - K_2 \qquad G_2F_2 = 1 - K_2'$$

Hence

$$(F_1F_2)(G_2G_1) = F_1(1 - K_2)G_1 = 1 - K_1 - F_1K_2G_1$$

The operator $K_1 + F_1K_2G_1$ is compact. Similarly, $(G_2G_1)(F_1F_2) = 1 - L$ for some compact operator L. Hence, again applying exercise 5, the operator F_1F_2 is Fredholm.

Suppose (temporarily) that $Index(F_2) = 0$. Then $\dim(\ker F_2) = \dim(coker\, F_2) = \dim(im\, F_2^{\perp})$. The restricted operator $F_2|_{(\ker F_2)^{\perp}} \colon (\ker F_2)^{\perp} \to im\, F_2$ is bijective, and so by the open mapping theorem has a continuous inverse G.

Let $A \colon \ker F_2 \to (im\, F_2)^{\perp}$ be a bijection. We can define a finite rank (and so compact) operator $K \colon H \to H$ by writing

$$K(x + y) = A(x)$$

whenever $x \in \ker F_2$ and $y \in (\ker F_2)^{\perp}$.

The operator $F_2 + K$ is invertible, with inverse defined by the formula

$$(F_2 + K)^{-1}(x + y) = G(x) + A^{-1}(y)$$

whenever $x \in im\, F_2$ and $y \in im\, F_2^{\perp}$. It follows that

$$Index(F_1) = Index(F_1(F_2 + K)) = Index(F_1F_2 + F_1K) = Index(F_1F_2)$$

since the composition F_1K is compact.

Now, suppose that $Index(F_2) = -n < 0$. Consider the shift operator $S \colon l^2(\mathbb{N}) \to l^2(\mathbb{N})$ defined by the usual formula

$$S(\xi_1, \xi_2, \xi_3, \dots) = (\xi_2, \xi_3, \dots)$$

Then the operator S is clearly Fredholm, and $Index(S^n) = n$. We have Fredholm operators

$$F_1 \oplus 1 \colon H \oplus l^2(\mathbb{N}) \to H \oplus l^2(\mathbb{N}) \qquad F_2 \oplus S^n \colon H \oplus l^2(\mathbb{N}) \to H \oplus l^2(\mathbb{N})$$

where $Index(F_1 \oplus 1) = Index(F_1)$ and $Index(F_2 \oplus S^n) = -n + n = 0$.

Hence, by the above calculation

$$Index(F_1F_2) + n = Index(F_1F_2 \oplus S^n) = Index((F_1 \oplus 1)(F_2 \oplus S^n)) = Index(F_1)$$

It follows that

$$Index(F_1F_2) = Index(F_1) + Index(F_2)$$

as required.

If $Index(F_2) > 0$, we can use the above trick, but with the right shift operator

$$T(\xi_1, \xi_2, \xi_3, \ldots) = (0, \xi_1, \xi_2, \ldots)$$

instead of S.

Observe that

$$(F_2 + K)(x + y) = A(x) + F_2(y)$$

whenever $x \in \ker F_2$ and $y \in (\ker F_2)^{\perp}$. The operator $F_2 + K$

4. Observe that the operator T is compact. Hence the operator $I - T$ is Fredholm, with index zero. If we can show that $\ker(I - T) = \{0\}$, then $coker(I - T) = \{0\}$ and the operator $I - T$ is bijective. We will have proven that the equation

$$F - Tf = g$$

has a unique solution in the space $L^2[0, 1]$.

It remains to prove that $\ker(I - T) = \{0\}$. Let $f \in \ker(I - T)$. Then $f = Tf$. But

$$\|Tf\|_2^2 = \int_0^1 \left| \int_0^s (s-t)f(t)\ dt \right|^2\ ds \leq \int_0^1 \int_0^s (s-t)^2 |f(t)|\ dt\ ds$$

By the Cauchy-Schwarz inequality:

$$\left| \int_0^s (s-t)f(t)\ dt \right|^2 \leq \left(int_0^s (s-t)^2\ dt \right) \|f\|_2^2 < \frac{1}{3} \|f\|_2^2$$

Hence $\|Tf\|_2 < \|f\|_2/\sqrt{3}$. It follows that $f = 0$, and we are done.

5. Let $F\colon H \to H$ be a Fredholm operator. Then we have a bounded linear bijection

$$F|_{(\ker F)^{\perp}} \colon (\ker F)^{\perp} \to \Im F$$

We know that the image $\Im F$ is closed. By the open mapping theorem, the above map has a bounded linear inverse $G\colon \Im F \to (\ker F)^{\perp}$. We can define a map $\overline{G}\colon H \to H$ by writing

$$\overline{G}(x + y) = G(x)$$

whenever $x \in \Im F$ and $y \in (\Im F)^{\perp}$.

Observe:

$$F\overline{G}(x + y) = x \qquad (I - F\overline{G})(x + y) = y$$

Since the operator F is Fredholm, the space $(\Im F)^{\perp}$ is finite-dimensional. Therefore the operator $I - F\overline{G}$ has finite rank, and so is compact.

A similar calculation tells us that the operator $I - \overline{G}F$ has finite rank.

Covnersely, suppose we have operators $F, G\colon H \to H$ and compact operators K and K' such that $FG = I - K$ and $GF = I - K'$.

Suppose that the space $\ker F$ is not finite-dimensional. Then we can find an orthonormal sequence (x_n) in $\ker F$. We know that

$$K' x_n = (I - GF)(x_n) = x_n$$

for all n, since $F(x_n) = 0$.

If B is the unit ball in the space $\ker F$, then the above calculation tells us that the image $K'[B]$ contains an infinite-dimensional unit ball. Hence the image $K'[B]$ cannot have compact closure, and the operator K' is not compact.

This statement is a contradiction. Thus the space $\ker F$ is finite-dimensional.

Now observe that

$$\dim(\mathit{coker}\, F) = \dim((\mathit{im}\, F)^{\perp}) = \dim(\ker F^{\star})$$

We know that

$$G^{\star} F^{\star} = I - K^{\star}$$

and the operator $K^{\star}$ is compact. An argument similar to the above tells us that the sapce $\ker F^{\star}$ is finite-dimensional. Therefore the operator F is Fredholm, as required.