KK-theory spectra for C^*-categories and discrete groupoid C^*-algebras

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Contents

Abstract

In this paper we refine a version of bivariant K-theory developed by Cuntz to define symmetric spectra representing the KK-theory of C^*-categories and discrete groupoid C^*-algebras. In both cases, the Kasparov product can be expressed as a smash product of spectra.

1 Introduction

In [2], J.Cuntz developed KK-theory for locally convex algebras in order to look at versions of the Chern character for bivariant theories. This approach, using the thesis of A.B.Thom, was simplified in [4].

The purpose of this article is to go in a slightly different direction with the KK-theory machinery by looking at the KK-theory of C^*-categories and of discrete groupoid C^*-algebras. In both of these cases, the theory can naturally be expressed in termes of symmetric spectra, and the Kasparov product can be realised at the level of spectra.

Thus, for C^*-categories $\mathcal{A}$ and $\mathcal{B}$ (or as a special case C^*-algebras), we have a symmetric spectrum $\mathbb{KK}(\mathcal{A}, \mathcal{B})$ representing KK-theory. In fact, if we are working over the complex numbers, this spectrum is a symmetric $\mathbb{KK}(\mathbb{C}, \mathbb{C})$-module spectrum. Over the real numbers, we have a $\mathbb{KK}(\mathbb{R}, \mathbb{R})$-module spectrum. In the special case that $\mathcal{A} = \mathcal{B}$, the spectrum $\mathbb{KK}(\mathcal{A}, \mathcal{A})$ is a symmetric ring spectrum.

Similar results hold in the equivariant case. To be precise, if $\mathcal{G}$ is a discrete groupoid (or as a special case, a discrete group), and A and B are $\mathcal{G}$-C^*-algebras, then we have a symmetric spectrum $\mathbb{KK}_{\mathcal{G}}(A, B)$ representing equivariant KK-theory. This spectrum is a symmetric $\mathbb{KK}_{\mathcal{G}}(\mathbb{C}, \mathbb{C})$-module spectrum in the complex case, and a symmetric $\mathbb{KK}_{\mathcal{G}}(\mathbb{R}, \mathbb{R})$-module spectrum in the real case. The spectrum $\mathbb{KK}_{\mathcal{G}}(A, A)$ is a symmetric ring spectrum.

There are a several potential applications of the new machinery. The constructions in this article are both simpler and have more structure than the KK-theory spectra constructed in [13, 14], where KK-theory spectra are developed in order to examine analytic assembly maps. The extra structure present should be useful when homotopy-theoretic arguments involving KK-theory are applied, for example (see [15]) in the proof that the Baum-Connes conjecture implies the stable Gromov-Lawson-Rosenberg conjecture.

2 C^*-categories

Let $\mathbb{F}$ denote either the field of real numbers or the field of complex numbers. Recall (see for example [10]) that a small category $\mathcal{A}$ is called an *unital algebroid* (over the fireld $\mathbb{F}$) if each morphism set $Hom(a, b)_{\mathcal{A}}$ is a vector space over the field $\mathbb{F}$ and composition of morphisms is bilinear.

An *involution* on a unital algebroid $\mathcal{A}$ is a collection of maps

$$Hom(a, b)_{\mathcal{A}} \to Hom(b, a)_{\mathcal{A}}$$

written $x \mapsto x^*$ such that:

- $(\alpha x + \beta y)^* = \overline{\alpha} x^* + \overline{\beta} y^*$ for all scalars $\alpha, \beta \in \mathbb{F}$ and morphisms $x, y \in Hom(a, b)_{\mathcal{A}}$.
- $(xy)^* = y^* x^*$ for all composable morphisms x and y.
- $(x^*)^* = x$ for every morphism x.

A *non-unital* algebroid with involution is a collection of objects, morphisms, and maps similar to a unital algebroid with involution, except that there need not exist identity morphisms $1 \in Hom(a, a)_{\mathcal{A}}$. Thus, a non-unital algebroid with involution is no longer a category, but rather a slightly more general object which could be termed a *non-unital category.*

Given unital algebroids with involution, $\mathcal{A}$ and $\mathcal{B}$, we call a functor $F \colon \mathcal{A} \to \mathcal{B}$ a *C^*-functor* if each map $F\colon\ Hom(a, b)_{\mathcal{A}} \to Hom(F(a), F(b))_{\mathcal{B}}$ is linear, and $F(x^*) = F(x)^*$ for each morphism x in the category $\mathcal{A}$. We can similarly define C^*-functors between non-unital categories with involution.

Definition 2.1 Let $\mathcal{A}$ be a category with involution. Then we call $\mathcal{A}$ a *pre-C^*-category* if each morphism set is a normed vector space and the following three axioms hold:

- Let x and y be composable morphisms in $\mathcal{A}$. Then $\|xy\| \leq \|x\|\|y\|$.
- Let $x \in Hom(a,b)_{\mathcal{A}}$. Then the product x^*x is a positive element of the algebra $Hom(a,a)_{\mathcal{A}}$.[1]
- The *C^*-identity* $\|x^*x\| = \|x\|^2$ holds for any morphism of the category $\mathcal{A}$.

A collection of norms on the morphism sets of an algebroid with involution that turns it into a pre-C^*-category is called a *C^*-norm.* A pre-C^*-category is called a *C^*-category* if every morphism set is complete.

A C^*-algebra can be regarded as a C^*-category with only one object. Conversely, C^*-categories and C^*-functors have a number of useful properties similar to those of C^*-algebras and $*$-homomorphisms. For example, any C^*-functor between C^*-categories is norm-decreasing (and thus continuous) and has a closed image on morphism sets. Any faithful C^*-functor is an isometry. Perhaps the most important elementary result is the following; see [6] or [12] for further details.

Theorem 2.2 *Let $\mathcal{A}$ be a C^*-category. Let $\mathcal{L}$ be the C^*-category of all Hilbert spaces and bounded linear maps.*[2] *Then there exists a faithful C^*-functor $\rho \colon \mathcal{A} \to \mathcal{L}$.* □

A C^*-functor $\rho \colon \mathcal{A} \to \mathcal{L}$ is called a *representation* of $\mathcal{A}$.

Definition 2.3 A sequence

$$\mathcal{I} \xrightarrow{i} \mathcal{E} \xrightarrow{j} \mathcal{B}$$

of C^*-categories and C^*-functors is termed *exact* if:

- The categories $\mathcal{I}$, $\mathcal{E}$, and $\mathcal{B}$ have the same set of objects, and the functors i and j act as the identity map on the set of objects.
- Each sequence of vector spaces

  $$0 \to Hom(a,b)_{\mathcal{I}} \xrightarrow{i} Hom(a,b)_{\mathcal{E}} \xrightarrow{j} Hom(a,b)_{\mathcal{B}} \to 0$$

  is exact.

We will generally write

$$0 \to \mathcal{I} \xrightarrow{i} \mathcal{E} \xrightarrow{j} \mathcal{B} \to 0$$

when we have an exact seqence. Such an exact sequence is termed *split exact* if it comes equipped with a C^*-functor $s \colon \mathcal{B} \to \mathcal{E}$ such that $j \circ s = 1_{\mathcal{B}}$. Such a C^*-functor s is called a *splitting.*

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[1] That is to say the spectrum is a subset of the positive real numbers.

[2] Strictly speaking, the category $\mathcal{L}$ is not a C^*-category since it is not small. This problem does not matter to us since we will not be doing constructions directly involving the category $\mathcal{L}$; we can always pick a small full subcategory.

Definition 2.4 Let $\mathcal{A}$ be a C^*-category. Then we define the category $\mathcal{A}[0,1]$ to be the C^*-category with the same set of objects as $\mathcal{A}$, where the morphism set $Hom(a,b)_{\mathcal{A}[0,1]}$ consists of all continuous functions $f\colon [0,1] \to Hom(a,b)_{\mathcal{A}}$. The norm on the space $Hom(a,b)_{\mathcal{A}[0,1]}$ is the supremum norm.

We define the *cone*, $C\mathcal{A}$ to be the subcategory with the same set of objects as $\mathcal{A}$, and morphism sets

$$Hom(a,b)_{C\mathcal{A}} = \{f \in Hom(a,b)_{\mathcal{A}[0,1]} \mid f(0) = 0\}$$

The *suspension*, $\Sigma\mathcal{A}$ is the subcategory with the same set of objects as $\mathcal{A}$, and morphism sets

$$Hom(a,b)_{\Sigma\mathcal{A}} = \{f \in Hom(a,b)_{C\mathcal{A}} \mid f(1) = 0\}$$

We have a canonical inclusion C^*-functor $i\colon \Sigma\mathcal{A} \to C\mathcal{A}$. There is also a C^*-functor $j\colon C\mathcal{A} \to \mathcal{A}$, defined to be the identity on the set of objects, and by the formula $j(f) = f(1)$ for each morphism $f \in Hom(a,b)_{C\mathcal{A}}$. It is easy to check that we have an exact sequence

$$0 \to \Sigma\mathcal{A} \to C\mathcal{A} \to \mathcal{A} \to 0$$

The above exact sequence has a splitting $s\colon \mathcal{A} \to C\mathcal{A}$, defined to be the identity the set of objects, and by the formula

$$s(x)(t) = tx \qquad t \in [0,1],\ x \in Hom(a,b)_{\mathcal{A}}$$

Further, the above exact sequence and splitting are natural in the sense that the assignments $\mathcal{A} \mapsto C\mathcal{A}$ and $\mathcal{A} \mapsto \Sigma\mathcal{A}$ are C^*-functors depending functorially on the C^*-category $\mathcal{A}$, and the maps i, j, and s are natural transformations. Given a C^*-functor $\alpha\colon \mathcal{A} \to \mathcal{B}$, we write $\Sigma\alpha\colon \Sigma\mathcal{A} \to \mathcal{B}$ to denote the induced C^*-functor.

Definition 2.5 Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-categories. Then we define the *algebraic tensor product* $\mathcal{A} \odot \mathcal{B}$ to be the category with the set of objects

$$Ob(\mathcal{A} \otimes \mathcal{B}) = \{A \otimes B \mid A \in Ob(\mathcal{A}, B \in Ob(\mathcal{B})\}$$

where the morphism set $Hom(a \otimes b, a' \otimes b')_{\mathcal{A}\otimes\mathcal{B}}$ is a the algebraic tensor product of vector spaces $Hom(a,b)_{\mathcal{A}} \odot Hom(a',b')_{\mathcal{B}}$.

The following result is easy to check.

Proposition 2.6 *The algebraic tensor product $\mathcal{A} \odot \mathcal{B}$ can be equipped with a C^*-norm defined by the formula*

$$\|u\| = \inf \left\{ \|\sum_{j=1}^{n} x_j x_j^*\|^{\frac{1}{2}} \|\sum_{j=1}^{n} y_j^a s t y_j\|^{\frac{1}{2}} \mid u = \sum_{j=1}^{n} x_j \otimes y_j,\ x_j \in Hom(a,b)_{\mathcal{A}}, y_j \in Hom(a,b)_{\mathcal{B}} \right\}$$

□

Definition 2.7 We define the *tensor product*, $\mathcal{A} \otimes \mathcal{B}$, of the C^*-categories $\mathcal{A}$ and $\mathcal{B}$ to be the completion of the algebraic tensor product with respect top the above norm.

In the special case of C^*-algebras, the above tensor product is the Haagerup tensor product, as defined in [5]. In particular, the above tensor product is different to the spatial tensor product of C^*-categories defined through representations in [12].

For any C^*-category $\mathcal{A}$, the tensor product $C[0,1]\otimes\mathcal{A}$ is naturally isomorphic to the category $\mathcal{A}[0,1]$. The proof is the same as that of the corresponding result for C^*-algebras[3]; see for example appendix T of [16]. The following result for cones and suspensions follows.

Proposition 2.8 *Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-categories. Then we have natural isomorphisms*

$$C(\mathcal{A}\otimes\mathcal{B}) \cong \mathcal{A}\otimes C\mathcal{B} \cong (C\mathcal{A})\otimes\mathcal{B}$$

and

$$\Sigma(\mathcal{A}\otimes\mathcal{B}) \cong \mathcal{A}\otimes \Sigma\mathcal{B} \cong (\Sigma\mathcal{A})\otimes\mathcal{B}$$

□

Definition 2.9 Let $\mathcal{A}$ be a C^*-category. Given objects $a,b\in Ob(\mathcal{A})$, let us define

$$Hom(a,b)_{\mathcal{A}}^{\otimes(k+1)} = \bigoplus_{c_i\in Ob(\mathcal{A})} Hom(a,c_1)\otimes Hom(c_1,c_2)\otimes\cdots\otimes Hom(c_k,b)$$

We equip this morphism set with the norm defined by taking the tensor product of the C^*-category $\mathcal{A}$ with itself k times.

The *tensor category*, $T\mathcal{A}$, is the C^*-category with the same set of objects as the category $\mathcal{A}$ where the morphism set $Hom(a,b)_{T_{\mathrm{alg}}\mathcal{A}}$ is the completion of the vector space

$$\bigoplus_{k=1}^{\infty} Hom(a,b)_{\mathcal{A}}^{\otimes k}$$

with respect to the given norm.

Here the the space $Hom(a,b)_{\mathcal{A}}^{\otimes 1}$ is simply the morphism set $Hom(a,b)_{\mathcal{A}}$. Composition of morphisms in the tensor category is defined by concatenation of tensors.

There is a canonical C^*-functor $\sigma\colon\mathcal{A}\to T\mathcal{A}$ defined by mapping each morphism set of the category $\mathcal{A}$ onto the first summand.

Formation of the tensor category defines a functor from the category of C^*-categories and C^*-functors to itself. Further, the tensor category has a universal property.

Proposition 2.10 *Let $\alpha\colon\mathcal{A}\to\mathcal{B}$ be a C^*-functor between C^*-categories. Then there is a unique C^*-functor $\varphi\colon T\mathcal{A}\to\mathcal{B}$ such that $\alpha=\varphi\circ\sigma$.*

Proof: We can define an appropriate C^*-functor $\varphi\colon T\mathcal{A}\to\mathcal{B}$ by writing $\varphi(a)=\alpha(a)$ for each object $a\in Ob(\mathcal{A})$, and

$$\varphi(x_1\otimes\cdots\otimes x_n) = \alpha(x_n)\ldots\alpha(x_1)$$

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[3] The proof works for any sensible tensor product of C^*-algebras since the C^*-algebra $C[0,1]$ is commutative and therefore nuclear.

for morphisms $x_i \in Hom(C_i, C_{i+1})$. □

We have a natural C^*-functor $\pi\colon T\mathcal{A} \to \mathcal{A}$ defined to be the identity on the set of objects, and by the formula

$$\varphi(x_1 \otimes \cdots \otimes x_n) = x_n \ldots x_1$$

for morphisms $x_i \in Hom(c_i, c_{i+1})$. It follows that there is a C^*-category $J\mathcal{A}$ with the same objects as the category $\mathcal{A}$, and morphism sets

$$Hom(a,b)_{J\mathcal{A}} = \ker \pi\colon\ Hom(a,b)_{T\mathcal{A}} \to Hom(a,b)_{\mathcal{A}}$$

This category fits into a natural exact sequence

$$0 \to J\mathcal{A} \hookrightarrow T\mathcal{A} \xrightarrow{\pi} \mathcal{A} \to 0$$

with natural splitting $\sigma\colon \mathcal{A} \to T\mathcal{A}$. Given a C^*-functor $\alpha\colon \mathcal{A} \to \mathcal{B}$, we write $J\alpha\colon \mathcal{A} \to \mathcal{B}$ to denote the induced C^*-functor.

Proposition 2.11 *Let*

$$0 \to \mathcal{I} \to \mathcal{E} \to \mathcal{B} \to 0$$

be a split exact sequence of C^*-categories. Let $\alpha\colon \mathcal{A} \to \mathcal{B}$ be a C^*-functor. Then there are C^*-functors $\tau\colon T\mathcal{A} \to \mathcal{E}$ and $\gamma\colon J\mathcal{A} \to \mathcal{I}$ such that we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & J\mathcal{A} \arrow[r] \arrow[d] & T\mathcal{A} \arrow[r] \arrow[d] & \mathcal{A} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{I} \arrow[r] & \mathcal{E} \arrow[r] & \mathcal{B} \arrow[r] & 0
\end{tikzcd}$$

Proof: Let $s\colon \mathcal{B} \to \mathcal{E}$ be a splitting. Then by the universal property of the tensor category. we have a homomorphism $\tau\colon T\mathcal{A} \to \mathcal{E}$ such that $s \circ \alpha = \tau \circ \sigma$. It follows that the homomorphism τ fits into the above diagram. The homomorphism γ is defined by restriction of τ. □

Actually, for C^*-algebras, the above classifying map exists (for reasons connected with properties of the Haagerup tensor product) whenever we have an exact sequence with a completely positive splitting. However, we will not use this fact here.

Definition 2.12 The homomorphism γ is called the *classifying map* of the diagram

$$\begin{tikzcd}
 & & & \mathcal{A} \arrow[d] & \\
0 \arrow[r] & \mathcal{I} \arrow[r] & \mathcal{E} \arrow[r] & \mathcal{B} \arrow[r] & 0
\end{tikzcd}$$

3 C^*-algebras associated to C^*-categories

In [8], M. Joachim defined the K-theory spectrum of a C^*-category by associating a certain C^*-algebra to a C^*-category and then defining the K-theory of the C^*-category to be the K-theory of the associated C^*-algebra. In this section, we will make a similar construction in order to look at KK-theory spectra. However, our C^*-algebra will be based on constructions of K-theory spectra in [11] rather than M. Joachim's C^*-algebra.

Definition 3.1 Let $\mathcal{A}$ be a C^*-category. Then we define the *additive completion*, $\mathcal{A}_\oplus$, to be the algebroid in which the objects are formal sums $a_1 \oplus \cdots a_m$, where $a_i \in Ob(\mathcal{A})$. For natural numbers $m, n \in \mathbb{N}$, the morphism set $Hom(a_1 \oplus \cdots \oplus a_m, b_1 \oplus \cdots \oplus b_n)$ is the set of matrices

$$\left\{ \begin{pmatrix} x_{1,1} & \cdots & x_{1,m} \\ \vdots & \ddots & \vdots \\ x_{n,1} & \cdots & x_{n,m} \end{pmatrix} \mid x_{i,j} \in Hom(a_j, b_i)_{\mathcal{A}} \right\}$$

Composition of matrices is defined by matrix multiplication. The involution is defined by the formula

$$\begin{pmatrix} x_{1,1} & \cdots & x_{1,m} \\ \vdots & \ddots & \vdots \\ x_{n,1} & \cdots & x_{n,m} \end{pmatrix}^* = \begin{pmatrix} x_{1,1}^* & \cdots & x_{n,1}^* \\ \vdots & \ddots & \vdots \\ x_{1,m}^* & \cdots & x_{n,m}^* \end{pmatrix}$$

Given objects $a = a_1 \oplus \cdots \oplus a_n$ and $b = b_1 \oplus \cdots \oplus b_n$, we define

$$a \oplus b = a_1 \oplus \cdots \oplus a_n \oplus b_1 \oplus \cdots \oplus b_n$$

As a special case, we define 0 to be the formal sum of no objects. We have morphism sets $Hom(0, a)_{\mathcal{A}} = \{0\}$ and $Hom(a, 0)_{\mathcal{A}} = \{0\}$ for each object $a \in Ob(\mathcal{A}_\oplus)$.

It is clear that the additive completion of a C^*-category is an additive category. Given a C^*-functor $\alpha \colon \mathcal{A} \to \mathcal{B}$ where the C^*-category $\mathcal{B}$ is additive, there is an obvious induced additive functor $\alpha \colon \mathcal{A}_\oplus \to \mathcal{B}$. In particular, given a faithful representation $\rho \colon \mathcal{A} \to \mathcal{L}$, there is an induced faithful representation $\rho_\oplus \colon \mathcal{A}_\oplus \to \mathcal{L}$.

We can define a C^*-norm on the category $\mathcal{A}_\oplus$ by deeming the induced representation $\rho_\oplus$ to be an isometry. The category $\mathcal{A}$ is a C^*-category with respect to this norm. Further, the norm does not depend on the representation ρ. Thus the additive completion is a functor from the category of C^*-categories and C^*-functors to the category of additive C^*-categories and additive C^*-functors.

Further details of this construction and proofs of the above statements can by found in [11]. The following result is easy to check.

Proposition 3.2 *Let $\mathcal{A}$ be a C^*-category. Then we have natural isomorphisms*

$$(\Sigma\mathcal{A})_\oplus \cong \Sigma(\mathcal{A}_\oplus) \qquad (J\mathcal{A})_\oplus \cong J(\mathcal{A}_\oplus) \qquad (C\mathcal{A})_\oplus \cong C(\mathcal{A}_\oplus)$$

□

Given a C^*-category $\mathcal{A}$, we can define a category $\mathcal{O}_{\mathcal{A}}$. The set of objects consists of all compositions of inclusions $Hom(a \oplus c, a \oplus c)_{\mathcal{A}_\oplus} \to Hom(a \oplus b \oplus c, a \oplus b \oplus c)_{\mathcal{A}_\oplus}$ of the form

$$\begin{pmatrix} w & x \\ y & z \end{pmatrix} \mapsto \begin{pmatrix} w & 0 & x \\ 0 & 0 & 0 \\ y & 0 & z \end{pmatrix}$$

A morphism set between two inclusions has precisely one element if the inclusions are composable; otherwise, it is empty.

We can define a functor, $H_{\mathcal{A}}$, from the category $\mathcal{O}_{\mathcal{A}}$ to the category of C^*-algebras by associating the C^*-algebra $Hom(a \oplus c, a \oplus c)_{\mathcal{A}_\oplus}$ to the inclusion $Hom(a \oplus c, a \oplus c)_{\mathcal{A}_\oplus} \to Hom(a \oplus b \oplus c, a \oplus b \oplus c)_{\mathcal{A}_\oplus}$. If i and j are composable inclusions, then the one morphism in the set $Hom(i,j)_{\mathcal{O}_{\mathcal{A}}}$ is mapped to the inclusion i itself.

It is a well-known fact (see for example appendix L of [16]) that the category of C^*-algebras is closed under the formation of direct limits. The following definition therefore makes sense.

Definition 3.3 Let $\mathcal{A}$ be a C^*-category. Then we define the C^*-algebra $\mathcal{A}^H$ to be the colimit of the functor $H_{\mathcal{A}}$.

By construction, the assignment $\mathcal{A} \mapsto \mathcal{A}^H$ is a functor from the category of C^*-categories and C^*-functors to the category of C^*-algebras. There is an obvious natural faithful C^*-functor $H \colon \mathcal{A}_\oplus \to \mathcal{A}^H$.

Proposition 3.4 *Let A be a C^*-algebra. Then the C^*-algebra A^H is naturally isomorphic to the tensor product $A \otimes \mathcal{K}$, where $\mathcal{K}$ is the C^*-algebra of compact operators on a seperable Hilbert space.*

Proof: Let $\mathcal{O}'_A$ be the full subcategory of the category $\mathcal{O}_A$ in which the set of objects consists of all inclusions of the form

$$x \mapsto \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}$$

Then the restriction of the functor H_A to the category $\mathcal{O}'_A$ has colimit $A \otimes \mathcal{K}$ (see for example [16], appendix L).

But the categories $\mathcal{O}_A$ and $\mathcal{O}'_A$ are directed systems, and the category $\mathcal{O}'_A$ is cofinal in the category $\mathcal{O}_A$. The result now follows. $\square$

The following result is easy to check.

Proposition 3.5 *Let $\mathcal{A}$ be a C^*-category. Then we have natural isomorphisms*

$$(\Sigma\mathcal{A})^H \cong \Sigma(\mathcal{A}^H) \qquad (J\mathcal{A})^H \cong J(\mathcal{A}^H) \qquad (C\mathcal{A})^H \cong C(\mathcal{A}^H)$$

$\square$

4 The KK-theory spectrum

Recall that at the most basic level, a *spectrum*, $\mathbb{E}$, is a sequence of topological spaces, E_n, each of which is equipped with a basepoint, together with continuous maps $\epsilon \colon E_n \to \Omega E_n$. The book [1] can be consulted for further details on the theory of spectra at this elementary level.

Definition 4.1 Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-categories. Then we define $F(\mathcal{A}, \mathcal{B})$ to be the space of all C^*-functors $\mathcal{A} \to \mathcal{B}^H$. The topology is the compact open topology.

Let $\mathcal{A}$ be a fixed C^*-category. Then the assignment $\mathcal{B} \mapsto F(\mathcal{A}, \mathcal{B})$ is a covariant functor from the category of C^*-categories and C^*-functors to the category of topological spaces with basepoint. Given a C^*-category $\mathcal{B}$, the assignment $\mathcal{A} \mapsto F(\mathcal{A}, \mathcal{B})$ is a contravariant functor from the category of C^*-categories to the category of topological spaces with basepoint.

The following result follows by the definition of the construction $\mathcal{B} \mapsto \mathcal{B}^H$.

Lemma 4.2 *Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-categories. Let $F^0(\mathcal{A}, \mathcal{B})$ to be the set of all C^*-functors $\mathcal{A} \to \mathcal{B}_\oplus$. Then the image of the natural map $F^0(\mathcal{A}, \mathcal{B}) \to F(\mathcal{A}, \mathcal{B})$ is dense.* $\square$

Now, let $J^k\mathcal{A}$ be defined by iterating the construction of the category $J\mathcal{A}$ defined in the previous section. Consider a homomorphism $\alpha \in F(J^k\mathcal{A}, \mathcal{B})$. Then by functoriality of the construction $\mathcal{B} \mapsto \mathcal{B}^H$ and proposition 3.5 we have a split exact sequence

$$0 \to \Sigma\mathcal{B}^H \to C\mathcal{B}^H \to \mathcal{B}^H \to 0$$

so, by proposition 2.11, a classifying map

$$\eta(\alpha) \colon J^{k+1}\mathcal{A} \to \Sigma\mathcal{B}^H$$

Given a C^*-functor $\alpha \in F^0(J^k\mathcal{A}, \mathcal{B})$, our classifying map $\eta(\alpha)$ can be considered to lie in the space $F^0(J^{k+1}\mathcal{A}, \Sigma\mathcal{B})$, that is to say we have a C^*-functor

$$\eta(\alpha) \colon J^{k+1}\mathcal{A} \to \Sigma\mathcal{B}_\oplus$$

Definition 4.3 We define $\mathbb{KK}(\mathcal{A}, \mathcal{B})$ to be the spectrum with sequence of spaces $(F(J^{2n}\mathcal{A}, \Sigma^n\mathcal{B}))$. The structure map

$$\epsilon \colon F(J^{2n}\mathcal{A}, \Sigma^n\mathcal{B}) \to \Omega F(J^{2n+2}\mathcal{A}, \Sigma^{n+1}\mathcal{B}) \cong F(J^{2n+2}\mathcal{A}, \Sigma^{n+2}\mathcal{B})$$

is defined by applying the above classifying map construction twice, that is to say writing $\epsilon(\alpha) = \eta(\eta(\alpha))$ whenever $\alpha \in F(J^{2n}\mathcal{A}, \Sigma^n\mathcal{B})$.

We would like to be able to define certain products at the level of spectra. In order to do this, we need to have some extra structure. Symmetric spectra, as defined in [7] are suitable for our purposes.

Proposition 4.4 *The spectrum $\mathbb{KK}(\mathcal{A}, \mathcal{B})$ is a symmetric spectrum.*

Proof: The C^*-category $\Sigma^n\mathcal{B}$ can be viewed as the tensor product $C_0(0,1) \otimes \cdots \otimes C_0(0,1) \otimes \mathcal{B}$, where there are n copies of the C^*-algebra $C_0(0,1)$. There is therefore a canonical action of the permutation group S_n on the space $(F(J^{2n}\mathcal{A}, \Sigma^n\mathcal{B}))$ defined by permuting the copies of the C^*-algebra $C_0(0,1)$.

By naturality of the classifying map construction, the iterated structure map $\epsilon^k \colon F(J^{2n}\mathcal{A}, \Sigma^n\mathcal{B}) \to \Omega F(J^{2n+2k}\mathcal{A}, \Sigma^{n+k}\mathcal{B})$ is $S_n \times S_k$-equivariant, and so we have a symmetric spectrum as required. $\square$

Let $\mathbb{E}$, $\mathbb{F}$, and $\mathbb{G}$ be symmetric spectra, with spaces E_n, F_n, and G_n respectively. Then there is a notion of a smash product of symmetric spectra $\mathbb{E} \wedge \mathbb{F}$. A collection of continuous basepoint-preserving $S_m \times S_n$-equivariant maps $E_m \wedge F_n \to G_{m+n}$ which commute with the structure maps of the spectra define a map of spectra $\mathbb{E} \wedge \mathbb{F} \to \mathbb{G}$.

Proposition 4.5 *Let $\mathcal{A}$ be a C^*-category, and let k and l be natural numbers. Then there is a natural C^*-functor $s\colon J^k\Sigma^l\mathcal{A} \to \Sigma^k J^l\mathcal{A}$.*

Proof: The classifying map of the diagram

$$\begin{tikzcd}
 & & & \Sigma\mathcal{A} \arrow[d, equal] & \\
0 \arrow[r] & \Sigma J\mathcal{A} \arrow[r] & \Sigma T\mathcal{A} \arrow[r] & \Sigma\mathcal{A} \arrow[r] & 0
\end{tikzcd}$$

is a natural C^*-functor $J\Sigma\mathcal{A} \to \Sigma J\mathcal{A}$. The C^*-functor s is defined by iterating the above construction. □

Definition 4.6 Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be C^*-categories. Let $\alpha \in F^0(J^{2m}\mathcal{A}, \Sigma^m\mathcal{B})$ and $\beta \in F^0(J^{2n}\mathcal{B}, \Sigma^n\mathcal{C})$. Then we define the product $\alpha\sharp\beta$ to be the composition

$$J^{2m+2n}\mathcal{A} \xrightarrow{J^{2n}\alpha} J^{2n}\Sigma^m\mathcal{B}_\oplus \xrightarrow{s} \Sigma^m J^{2n}\mathcal{B}_\oplus \xrightarrow{\Sigma^m\beta_\oplus} \Sigma^{m+n}\mathcal{C}_\oplus$$

Theorem 4.7 *Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be C^*-categories. Then there is a natural map of spectra*

$$\mathbb{KK}(\mathcal{A},\mathcal{B}) \wedge \mathbb{KK}(\mathcal{B},\mathcal{C}) \to \mathbb{KK}(\mathcal{A},\mathcal{C})$$

defined by the formula

$$\alpha \wedge \beta \mapsto \alpha\sharp\beta \qquad \alpha \in F^0(J^m\mathcal{A},\mathcal{B}),\ \beta \in F^0(J^n\mathcal{B},\mathcal{C})$$

Further, the above product is associative. To be precise, let $\alpha \in F(J^m\mathcal{A},\mathcal{B})$, $\beta \in F(J^n\mathcal{B},\mathcal{C})$, and $\gamma \in F(J^p\mathcal{C},\mathcal{D})$. Then $(\alpha\sharp\beta)\sharp\gamma = \alpha\sharp(\beta\sharp\gamma)$.

Proof: Our construction gives us a natural continuous $S_m \times S_n$-equivariant map $F^0(J^m\mathcal{A},\mathcal{B}) \wedge F^0(J^n\mathcal{B},\mathcal{C}) \to F^0(J^{m+n}\mathcal{A},\mathcal{C})$. By continuity, this map extends to a map $F(J^m\mathcal{A},\mathcal{B}) \wedge F(J^n\mathcal{B},\mathcal{C}) \to F(J^{m+n}\mathcal{A},\mathcal{C})$.

Compatibility with the structure maps follows since naturality of the classifying map construction gives us a commutative diagram

$$\begin{tikzcd}
J^{2m+2n+2}\mathcal{A} \arrow[r, "J^{2n+2}(\alpha)"] \arrow[d, equal] & J^{2n+2}\Sigma^m\mathcal{B}_\oplus \arrow[r, "s"] \arrow[d] & \Sigma^m J^{2n+2}\mathcal{B}_\oplus \arrow[r, "\Sigma^m\eta^2(\beta_\oplus)"] \arrow[d] & \Sigma^{m+n+2}\mathcal{C}_\oplus \arrow[d, equal] \\
J^{2m+2n+2}\mathcal{A} \arrow[r, "J^{2n}\eta^2(\alpha)"] & J^{2n}\Sigma^{m+2}\mathcal{B}_\oplus \arrow[r, "s"] & \Sigma^{m+2}J^{2n}\mathcal{B}_\oplus \arrow[r, "\Sigma^{m+2}\beta_\oplus"] & \Sigma^{m+n+2}\mathcal{C}_\oplus
\end{tikzcd}$$

where the non-trivial vertical maps come from iterating the classifying map of the diagram

$$\begin{tikzcd}
 & & & \mathcal{B}_\oplus \arrow[d, equal] & \\
0 \arrow[r] & \Sigma\mathcal{B}_\oplus \arrow[r] & C\mathcal{B}_\oplus \arrow[r] & \mathcal{B}_\oplus \arrow[r] & 0
\end{tikzcd}$$

We now need to check the statement concerning associativity. Consider C^*-functors

$$\alpha\colon J^{2m}\mathcal{A} \to \Sigma^m\mathcal{B}_\oplus \qquad \beta\colon J^{2n}\mathcal{B} \to \Sigma^n\mathcal{C}_\oplus \qquad \gamma\colon J^{2p}\mathcal{C} \to \Sigma^p\mathcal{D}_\oplus$$

Then we have a commutative diagram

$$\begin{tikzcd}
J^{2m+2n+2p}\mathcal{A} \arrow[r, equal] \arrow[d] & J^{2m+2n+2p}\mathcal{A} \arrow[d] \\
J^{2n+2l}\Sigma^{m}\mathcal{B}_{\oplus} \arrow[r, equal] \arrow[d] & J^{2n+2p}\Sigma^{m}\mathcal{B}_{\oplus} \arrow[d] \\
J^{2p}\Sigma^{m}J^{2n}\mathcal{B}_{\oplus} \arrow[r, "s"] \arrow[d] & \Sigma^{m}J^{2n+2p}\mathcal{B}_{\oplus} \arrow[d] \\
J^{2p}\Sigma^{m+n}\mathcal{C}_{\oplus} \arrow[r, "s"] \arrow[d] & \Sigma^{m}J^{2p}\Sigma^{n}\mathcal{C}_{\oplus} \arrow[d] \\
\Sigma^{m+n}J^{2p}\mathcal{D}_{\oplus} \arrow[r, equal] \arrow[d] & \Sigma^{m+n}J^{2p}\mathcal{D}_{\oplus} \arrow[d] \\
\Sigma^{m+n+p}\mathcal{D}_{\oplus} \arrow[r, equal] & \Sigma^{m+n+p}\mathcal{D}_{\oplus}
\end{tikzcd}$$

But the column on the left is the product $(\alpha\sharp\beta)\sharp\gamma$ and the column on the right is the product $\alpha\sharp(\beta\sharp\gamma)$ so associativity of the product follows. □

Given homomorphisms $\alpha \in F(\mathcal{A},\mathcal{B})$ and $\beta \in F(\mathcal{B},\mathcal{C})$, we can form the smash product $\alpha\sharp\beta$. By definition of our prodect, the following result holds.

Proposition 4.8 *Let $\alpha\colon \mathcal{A} \to \mathcal{B}$ and $\beta\colon \mathcal{B} \to \mathcal{C}$ be homomorphisms. Then $\alpha\sharp\beta = \beta \circ \alpha$.* □

There is one last construction involving KK-theory spectra to consider before moving on.

Proposition 4.9 *Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be C^*-categories. Then there is a map $\Delta\colon \mathbb{KK}(\mathcal{A},\mathcal{B}) \to \mathbb{KK}(\mathcal{A}\otimes\mathcal{C},\mathcal{B}\otimes\mathcal{C})$. This map is compatible with the product in the sense that we have a commutative diagram*

$$\begin{tikzcd}
\mathbb{KK}(\mathcal{A},\mathcal{B}) \wedge \mathbb{KK}(\mathcal{B},\mathcal{C}) \arrow[r] \arrow[d] & \mathbb{KK}(\mathcal{A},\mathcal{C}) \arrow[d] \\
\mathbb{KK}(\mathcal{A}\otimes\mathcal{D},\mathcal{B}\otimes\mathcal{D}) \wedge \mathbb{KK}(\mathcal{B}\otimes\mathcal{D},\mathcal{C}\otimes\mathcal{D}) \arrow[r] & \mathbb{KK}(\mathcal{A}\otimes\mathcal{D},\mathcal{C}\otimes\mathcal{D})
\end{tikzcd}$$

where the horizontal maps are defined by the product, and the vertical maps are copies of the map Δ.

Proof: Let $\alpha\colon J^{2n}\mathcal{A} \to \Sigma^{n}\mathcal{B}^{H}$ be a C^*-functor. Then, since $\Sigma\mathcal{B}^{H} = C_0(0,1)\otimes\mathcal{B}^{H}$, we have a naturally induced C^*-functor $\alpha\otimes 1\colon (J^{2n}\mathcal{A})\otimes\mathcal{C} \to \Sigma^{n}(\mathcal{B}^{H}\otimes\mathcal{C})$.

There is an obvious natural C^*-functor $\mathcal{B}_{\oplus}\otimes\mathcal{C} \to (\mathcal{B}\otimes\mathcal{C})_{\oplus}$, and therefore a natural C^*-functor $\beta\colon \mathcal{B}^{H}\otimes\mathcal{C} \to (\mathcal{B}\otimes\mathcal{C})^{H}$. There is a natural C^*-functor $\gamma\colon J(\mathcal{A}\otimes\mathcal{C}) \to (J\mathcal{A})\otimes\mathcal{C}$ defined as the classifying map of the diagram

$$\begin{tikzcd}
 & & & \mathcal{A}\otimes\mathcal{C} \arrow[d, equal] & \\
0 \arrow[r] & (J\mathcal{A})\otimes\mathcal{C} \arrow[r] & (T\mathcal{A})\otimes\mathcal{C} \arrow[r] & \mathcal{A}\otimes\mathcal{C} \arrow[r] & 0
\end{tikzcd}$$

We define the map Δ by writing $\Delta(\alpha) = \beta\circ(\alpha\otimes 1)\circ\gamma^{n}$. The relevant naturality properties are easy to check. □

5 Ring and Module Structure

A *symmetric monoidal category* is a category with a sensible idea of a product of objects, $\wedge$, along with a unit object e equipped with isomorphisms $e \wedge X \to X$ and $X \wedge e \to X$ for any object X in the category. Any standard book on category, for example [9], can be consulted for further details. It is proven in [7] that the category of symmetric spectra is a symmetric monoidal category. The product is the smash product of spectra. The unit is the sequence of spaces

$$e = (S^0, \star, \star, \ldots)$$

where $\star$ is the one point topological space, and S^0 is the disjoint union of two points. By definition of the smash product in the category of symmetric spectra, there is a unique natural isomorphism between the objects $e \wedge \mathbb{E}$, $\mathbb{E} \wedge e$, and $\mathbb{E}$ for any spectrum $\mathbb{E}$.

Definition 5.1 A *symmetric ring spectrum* is a monoid in the category of symmetric spectra.

To be more precise, a symmetric spectrum R is called a *symmetric ring spectrum* if it is equipped with an associative product $\mu\colon R \wedge R \to R$ and a unit map $\eta\colon e \to R$ such that we have a commutative diagram

$$\begin{tikzcd}
e \wedge R \arrow[r, "\eta\wedge 1"] \arrow[d] & R \wedge R \arrow[d] & R \wedge e \arrow[l, "1\wedge\eta"'] \arrow[d] \\
R \arrow[r, equal] & R \arrow[r, equal] & R
\end{tikzcd}$$

Here the central verticlal map is the product μ. The vertical maps on the left and right are the isomorphisms associated with the unit e.

Theorem 5.2 *Let $\mathcal{A}$ be a C^*-category. Then the spectrum $\mathbb{KK}(\mathcal{A}, \mathcal{A})$ is a symmetric ring spectrum.*

Proof: By theorem 4.7 we have an associative product

$$\mu\colon \mathbb{KK}(\mathcal{A}, \mathcal{A}) \wedge \mathbb{KK}(\mathcal{A}, \mathcal{A}) \to \mathbb{KK}(\mathcal{A}, \mathcal{A})$$

Recall that the unit, e, has the sequence of spaces $(S^0, \star, \star, \ldots)$. Thus there is only one point in the 0-th space that is not a basepoint. We can define a unit map $\eta\colon e \to \mathbb{KK}(\mathcal{A}, \mathcal{A})$ by mapping the base point of the n-th space of the spectrum e to the base point of the n-th space of the spectrum $\mathbb{KK}(\mathcal{A}, \mathcal{A})$, and mapping the point in S^0 which is not a basepoint to the point in the space $F(\mathcal{A}, \mathcal{A})$ coming from the identity C^*-functor $1\colon \mathcal{A} \to \mathcal{A}$.

Commutativity of the diagram involving the unit follows from proposition 4.8. □

The following definition comes from applying a defintion for symmetric monoidal categories to the category of symmetric specta.

Definition 5.3 Let R be a symmetric ring spectrum equipped with multiplication μ. Then a symmetric spectrum M is called a *symmetric* (*left*) *R-module* *spectrum* if it comes equipped with a multiplication $\mu'\colon R \wedge M \to M$ such that we have a commutative diagram

$$\begin{tikzcd}
R \wedge R \wedge M \arrow[r, "\mu\wedge 1"] \arrow[d] & R \wedge M \arrow[d] \\
R \wedge M \arrow[r, "\mu'"] & M
\end{tikzcd}$$

Here the vertical map on the left is the product $1 \wedge \mu'$ and the vertical map on the right is the product μ.

Theorem 5.4 *Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-categories. Then the spectrum $\mathbb{KK}(\mathcal{A},\mathcal{B})$ is a symmetric $\mathbb{KK}(\mathbb{F},\mathbb{F})$-module spectrum.*

Proof: By theorem 4.7 and proposition 4.9 we can define a suitable product

$$\mathbb{KK}(\mathbb{F},\mathbb{F}) \wedge \mathbb{KK}(\mathcal{A},\mathcal{B}) \xrightarrow{\Delta\wedge 1} \mathbb{KK}(\mathcal{A},\mathcal{A}) \wedge \mathbb{KK}(\mathcal{A},\mathcal{B}) \xrightarrow{\mu} \mathbb{KK}(\mathcal{A},\mathcal{B})$$

□

6 The Equivariant Case

Let $\mathcal{G}$ be a discrete groupoid. We will regard $\mathcal{G}$ as a small category in which every morphism is invertible. Taking this point of view, we define (see [14]) a $\mathcal{G}$-C^*-algebra to be a functor from the category $\mathcal{G}$ to the category of C^*-algebras.

Thus, if A is a $\mathcal{G}$-C^*-algebra, then for each object $a \in Ob(\mathcal{G})$ we have a C^*-algebra $A(a)$. A morphism $g \in Hom(a,b)_{\mathcal{G}}$ induces a $\star$-homomorphism $g\colon A(a) \to A(b)$.

We can regard an ordinary C^*-algebra C as a $\mathcal{G}$-C^*-algebra by writing $C(a) = C$ for each object $a \in Ob(\mathcal{G})$ and saying that each morphism in the groupoid $\mathcal{G}$ acts as the identity map.

A *$\mathcal{G}$-equivariant map* between $\mathcal{G}$-C^*-algebras A and B is a natural transformation from the functor A to the functor B. A *short exact sequence* of $\mathcal{G}$-C^*-algebras is a sequence of $\mathcal{G}$-C^*-algebras and equivariant maps

$$A \xrightarrow{f} B \xrightarrow{g} C$$

such that the sequence

$$0 \to A(a) \xrightarrow{f} B(a) \xrightarrow{g} C(a) \to 0$$

is exact for each object $a \in Ob(\mathcal{G})$. A splitting of a short exact sequence is defined in the obvious way.

We would like a version of the KK-theory spectrum for $\mathcal{G}$-C^*-algberas. To define such a spectrum, we need variations of the various constructions defined in section 2.

Definition 6.1 Let $\mathcal{G}$ be a discrete groupoid, and let A be a $\mathcal{G}$-C^*-algebra. Then we define $A[0,1]$ to be the $\mathcal{G}$-C^*-algebra where the algebra $A[0,1](a)$ consists of all continuous functions $f\colon [0,1] \to Hom\, A(a)$. The $\mathcal{G}$-action is defined by the formula

$$g(f)(t) = g(f(t)) \qquad g \in Hom(a,b)_{\mathcal{G}},\ f\colon [0,1] \to A(a),\ t \in [0,1]$$

We define the *cone*, CA to be the $\mathcal{G}$-C^*-algebra where

$$CA(a) = \{f \in A[0,1](a) \mid f(0) = 0\} \qquad a \in Ob(\mathcal{G})$$

and the $\mathcal{G}$-action is as defined above. The *suspension*, $\Sigma\mathcal{A}$ is defined similarly by writing

$$\Sigma A(a) = \{f \in CA(a) \mid f(1) = 0\} \qquad a \in Ob(\mathcal{G})$$

There is an obvious natural equivariant map $i\colon \Sigma A \to CA$ defined by inclusion. There is also an equivariant map $j\colon CA \to A$, defined by the formula $j(f) = f(1)$, where $f \in CA(a)$. It is easy to check that we have a short exact sequence

$$0 \to \Sigma A \to CA \to A \to 0$$

The above exact sequence has a natural splitting $s\colon A \to CA$ defined by the formula

$$s(x)(t) = tx \qquad t \in [0,1],\ x \in A(a)$$

Definition 6.2 Let A and B be $\mathcal{G}$-C^*-algebras. Then we define the *tensor product* $A \otimes B$ to be the $\mathcal{G}$-C^*-algebra where the algebra $(A \otimes B)(a)$ is the Haagerup tensor product[4] $A(a) \otimes B(a)$ and the $\mathcal{G}$-action is defined writing $g(x \otimes y) = g(x) \otimes g(y)$ whenever $g \in Hom(a,b)_{\mathcal{G}}$, $x \in A(a)$, and $y \in B(a)$.

We define the *direct sum* $A \oplus B$ to be the $\mathcal{G}$-C^*-algebra where $(A \oplus B)(a)$ is the spatial tensor product $A(a) \oplus B(a)$ for each object $a \in Ob(\mathcal{G})$ and the $\mathcal{G}$-action is defined by writing $g(x \oplus y) = g(x) \oplus g(y)$ whenever $g \in Hom(a,b)_{\mathcal{G}}$, $x \in A(a)$, and $y \in B(a)$.

Definition 6.3 Let A be a $\mathcal{G}$-C^*-algebra. Then we define $A^{\otimes k}$ to be the tensor product of A with itself k times. We define the *equivariant tensor algebra*, TA, to be the completion of the iterated direct sum

$$TA = \oplus_{k=1}^{\infty} A^{\otimes k}$$

There is a canonical equivariant map $\sigma\colon A \to TA$ defined by mapping each morphism set of the category $\mathcal{A}$ onto the first summand.

Formation of the equivariant tensor algebra defines a functor from the category of $\mathcal{G}$-C^*-algebras and equivariant maps to itself. Further, just as we showed for C^*-categories, the equivariant tensor algebra has a universal property.

Proposition 6.4 *Let A and B be $\mathcal{G}$-C^*-algebras. Let $\alpha\colon A \to B$ be an equivariant map. Then there is a unique equivariant map $\varphi\colon TA \to B$ such that $\alpha = \varphi \circ \sigma$.* □

There is a natural equivariant map $\pi\colon TA \to A$ defined by the formula

$$\varphi(x_1 \otimes \cdots \otimes x_n) = x_n \dots x_1 \qquad x_i \in A(a),\ a \in Ob(\mathcal{G})$$

We can thus define a $\mathcal{G}$-C^*-algebra JA by writing

$$J(a) = \ker \pi\colon TA(a) \to A(a)$$

---

[4] see [5] for relevant definitions.

for each object $a \in Ob(\mathcal{G})$. The $\mathcal{G}$-action is inherited from the equivariant tensor algebra. There is a natural short exact sequence

$$0 \rightarrow JA \hookrightarrow TA \xrightarrow{\pi} A \rightarrow 0$$

with natural splitting $\sigma\colon A \rightarrow TA$. The following result is proved in the same way as proposition 2.11.

Proposition 6.5 *Let*

$$0 \rightarrow I \rightarrow E \rightarrow B \rightarrow 0$$

be a split exact sequence of $\mathcal{G}$-C^*-algebras. Let $\alpha\colon A \rightarrow B$ be an equivariant map. Then there are equivariant maps $\tau\colon TA \rightarrow E$ and $\gamma\colon JA \rightarrow I$ such that we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & JA \arrow[r] \arrow[d] & TA \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I \arrow[r] & E \arrow[r] & B \arrow[r] & 0
\end{tikzcd}$$

□

As before, the homomorphism γ is called the *classifying map* of the diagram

$$\begin{tikzcd}
 & & & A \arrow[d] & \\
0 \arrow[r] & I \arrow[r] & E \arrow[r] & B \arrow[r] & 0
\end{tikzcd}$$

Definition 6.6 Let A and B be $\mathcal{G}$-C^*-algebras. Then we define $F_{\mathcal{G}}(A,B)$ to be the space of all equivariant maps $A \rightarrow B \otimes \mathcal{K}$. The topology is the compact open topology.

Lemma 6.7 *Let A and B be $\mathcal{G}$-C^*-algebras. Let $F^0(A,B)$ to be the set of all equivariant maps $A \rightarrow M_n(B)$, where $n \in \mathbb{N}$. Then the image of the natural map $F^0_{\mathcal{G}}(A,B) \rightarrow F_{\mathcal{G}}(A,B)$ is dense.*

Proof: The result follows from the fact that the union $\bigcup_{n=1}^{\infty} M_n(B)$ is a dense subset of the tensor product $B \otimes \mathcal{K}$. □

Consider an equivariant map $\alpha \in F_{\mathcal{G}}(J^kA, B)$. Then we can check (by looking at matrices and then taking direct limits) that we have a split exact sequence

$$0 \rightarrow \Sigma B \otimes \mathcal{K} \rightarrow CB \otimes \mathcal{K} \rightarrow B \otimes \mathcal{K} \rightarrow 0$$

so, by proposition 6.5, a classifying map

$$\eta(\alpha)\colon J^{k+1}A \rightarrow \Sigma B \otimes \mathcal{K}$$

Definition 6.8 We define $\mathbb{KK}_{\mathcal{G}}(A,B)$ to be the symmetric spectrum with sequence of spaces $(F_{\mathcal{G}}(J^{2n}A, \Sigma^n B))$ with S_n-action defined by permuting the order in which the suspensions are made. The classifying map

$$\epsilon\colon F_{\mathcal{G}}(J^{2n}A, \Sigma^n B) \rightarrow \Omega F_{\mathcal{G}}(J^{2n+2}A, \Sigma^{n+1}B) \cong F_{\mathcal{G}}(J^{2n+2}A, \Sigma^{n+2}B)$$

is defined by applying the above classifying map construction twice, that is to say writing $\epsilon(\alpha) = \eta(\eta(\alpha))$ whenever $\alpha \in F_{\mathcal{G}}(J^{2n}A, \Sigma^n B)$.

Definition 6.9 Let A, B, and C be $\mathcal{G}$-C^*-algebras. Let $\alpha \in F^0_{\mathcal{G}}(J^m A, B)$ and $\beta \in F^0_{\mathcal{G}}(J^n B, C)$. Then we define the product $\alpha \sharp \beta$ to be the composition

$$J^{m+n} A \overset{J^n \alpha}{\to} J^n M_\star(\mathcal{B}) \overset{M_\star(\beta)}{\to} M_\star(C)$$

The following result is proved in exactly the same way as theorem 4.7.

Theorem 6.10 *Let A, B, and C be $\mathcal{G}$-C^*-algebras. Then there is a natural map of spectra*

$$\mathbb{KK}(A,B) \wedge \mathbb{KK}(B,C) \to \mathbb{KK}(A,C)$$

defined by the formula

$$\alpha \wedge \beta \mapsto \alpha \sharp \beta \qquad \alpha \in F^0(J^m A, B),\ \beta \in F^0(J^n B, C)$$

Further, the above product is associative. To be precise, let $\alpha \in F_{\mathcal{G}}(J^m A, B)$, $\beta \in F_{\mathcal{G}}(J^n B, C)$, and $\gamma \in F_{\mathcal{G}}(J^p C, D)$. Then $(\alpha \sharp \beta) \sharp \gamma = \alpha \sharp (\beta \sharp \gamma)$. □

As before, the following result is obvious.

Proposition 6.11 *Let $\alpha \colon A \to B$ and $\beta \colon B \to C$ be equivariant maps. Then $\alpha \sharp \beta = \beta \circ \alpha$.* □

The following result is proved in exactly the same way as theorems 5.2 and 5.4.

Theorem 6.12 *Let $\mathcal{G}$ be a discrete groupoid, and let A be a $\mathcal{G}$-C^*-algebra. Then the spectrum $\mathbb{KK}_{\mathcal{G}}(A,A)$ is a symmetric ring spectrum.*

Let B be another $\mathcal{G}$-C^*-algebra. Then the spectrum $\mathbb{KK}_{\mathcal{G}}(A,B)$ is a symmetric $\mathbb{KK}_{\mathcal{G}}(\mathbb{F},\mathbb{F})$-module spectrum. □

Let $\theta \colon \mathcal{G} \to \mathcal{H}$ be a functor between groupoids, and let A be a $\mathcal{H}$-C^*-algebra. Abusing notation, we can also regard A as a $\mathcal{G}$-C^*-algebra; we write $A(a) = A(theta(a))$ for each object $a \in Ob(\mathcal{G})$, and define a $*$-homomorphism $g = \theta(g) \colon A(\theta(a)) \to A(\theta(b))$ for each morphism $g \in Hom(a,b)_{\mathcal{G}}$.

If A and B are $\mathcal{H}$-C^*-algebras, then we have an induced map $\theta^* \colon F_{\mathcal{H}}(A,B) \to F_{\mathcal{G}}(A,B)$ defined by the observation that any $\mathcal{H}$-equivariant map between A and $B \otimes \mathcal{K}$ is also $\mathcal{G}$-equivariant. This induced map is natural in the variables A and B. Going slightly further, we have the following easy to check result.

Proposition 6.13 *Let $theta \colon \mathcal{G} \to \mathcal{H}$ be a functor between groupoids, and let A and B be $\mathcal{H}$-C^*-algebras. Then there is an induced map of spectra $\theta^* \colon \mathbb{KK}_{\mathcal{H}}(A,B) \to \mathbb{KK}_{\mathcal{G}}(A,B)$. This induced map is compatible with the product in the sense that we have a commutative diagram*

$$\begin{tikzcd}
\mathbb{KK}_{\mathcal{H}}(A,B) \wedge \mathbb{KK}_{\mathcal{H}}(B,C) \arrow[r] \arrow[d] & \mathbb{KK}_{\mathcal{H}}(A,C) \arrow[d] \\
\mathbb{KK}_{\mathcal{G}}(A,B) \wedge \mathbb{KK}_{\mathcal{G}}(B,C) \arrow[r] & \mathbb{KK}_{\mathcal{G}}(A,C)
\end{tikzcd}$$

where the horizontal map is defined by the product and the vertical maps are restriction maps. □

The above map f^* is called the *restriction map.*

7 Descent

Apart from the last theorem, most of the definitions and results in this section come from [14].

Let $\mathcal{G}$ be a discrete groupoid, and let A be a $\mathcal{G}$-C^*-algebra. Then we define the *convolution algebroid* $\mathcal{G}A$ to be the algebroid with the same set of objects as the groupoid $\mathcal{G}$, and morphism sets

$$Hom(a,b)_{\mathcal{G}A} = \{\sum_{i=1}^m x_i g_i \mid x_i \in A(b), g_i \in Hom(a,b)_{\mathcal{G}},\ m \in \mathbb{N}\}$$

Composition of morphisms is defined by the formula

$$\left(\sum_{i=1}^m x_i g_i\right)\left(\sum_{j=1}^n y_j h_i\right) = \sum_{i,j=1}^{m,n} x_i g_i(y_j) g_i h_j$$

Further, we have an involution

$$\left(\sum_{i=1}^m x_i g_i\right)^* = \sum_{i=1}^m g_i^{-1}(x_i^*) g_i^{-1}$$

Definition 7.1 Let $\mathcal{G}$ be a discrete groupoid. Then a *unitary representation* of $\mathcal{G}$ is a functor $\rho\colon \mathcal{G} \to \mathcal{L}$ such that $\rho(g^{-1}) = \rho(g)^*$ for each morphism g in the groupoid $\mathcal{G}$.

Let A be a $\mathcal{G}$-C^*-algebra. Then a *covariant representation of* A is a pair (ρ,π), where ρ is a unitary representation of the groupoid $\mathcal{G}$, and π is a set of representations $\pi\colon A(a) \to \rho(a)$, where $a \in Ob(\mathcal{G})$, such that

$$\rho(g)\pi(x) = \pi(gx)\rho(g)$$

for each element $x \in A(a)$ and morphism $g \in Hom(a,b)_{\mathcal{G}}$

Given a covariant representation (ρ,π), we have a C^*-functor $(\rho,\pi)_*\colon A\mathcal{G} \to \mathcal{L}$ defined by mapping the object $a \in Ob(\mathcal{G})$ to the Hilbert space $\rho(a)$, and the morphism $\sum_{i=1}^m x_i g_i \in Hom(a,b)_{A\mathcal{G}}$ to the bounded linear map $\sum_{i=1}^m \pi(x_i)\rho(g_i)\colon \rho(a) \to \rho(b)$. The following result comes from [14].

Proposition 7.2 *Let A be a $\mathcal{G}$-C^*-algebra. Then any C^*-functor $A\mathcal{G} \to \mathcal{L}$ takes the form $(\rho,\pi)_*$ for some covariant representation (ρ,π).* □

Let A be a $\mathcal{G}$-C^*-algebra. Fix an object $a \in Ob(\mathcal{G})$, and choose a representation $\alpha\colon A(a) \to \mathcal{L}(H)$ on some Hilbert space H. For each object $b \in Ob(\mathcal{G})$, let $l^2(a,b)$ be the Hilbert space consisting of all sequences $(\eta_g)_{g\in Hom(a,b)_{\mathcal{G}}}$ in the space H such that the series $\sum_{g\in Hom(a,b)_{\mathcal{G}}} \|\eta_g\|^2$ converges.

We can define a unitary representation of the groupoid $\mathcal{G}$ by mapping the object $b \in Ob(\mathcal{G})$ to the Hilbert space $l^2(a,b)$, and the morphism $g \in Hom(b,c)_{\mathcal{G}}$ to the opertator $\rho(g)\colon l^2(a,b) \to l^2(a,c)$ defined by translation.

There are corresponding representations $\pi\colon A(b) \to \mathcal{L}(l^2(a,b))$ defined by writing

$$\pi(x)((\eta_g)_{g\in Hom(a,b)_{\mathcal{G}}}) = (\alpha(g^{-1}(x))\eta_g)_{g\in Hom(a,b)_{\mathcal{G}}}$$

It is straightforward to verify that the pair (ρ,π) is a covariant representation of A.

Definition 7.3 A covariant representation of the type described above is called a *regular representation.*

It is shown in [14] that we can define C^*-norms on the algebroid $A\mathcal{G}$ by writing

$$\|\mu\|_r = \sup\{(\rho,\pi)_a st(\mu) \mid (\rho,\pi) \text{ is a regular representation of } A\ \}$$

and

$$\|\mu\|_{\max} = \sup\{(\rho,\pi)_a st(\mu) \mid (\rho,\pi) \text{ is a covariant representation of } A\ \}$$

for any morphism μ in the algebroid $A\mathcal{G}$.

Definition 7.4 The *reduced crossed product,* $A \rtimes_r \mathcal{G}$ is the C^*-category defined by completion of the algebroid $A\mathcal{G}$ with respect to the norm $\| - \|_r$.

The *full crossed product,* $A \rtimes \mathcal{G}$ is the C^*-category defined by completion of the algebroid $A\mathcal{G}$ with respect to the norm $\| - \|_{\max}$.

Let $\mathcal{G}$ be a groupoid, and let $\alpha\colon A \to B$ be an equivariant map between $\mathcal{G}$-C^*-algebras. Then we have an induced C^*-functor $\alpha_*\colon A\mathcal{G} \to B\mathcal{G}$ defined to be the identity on the set of objects, and by the formula

$$f_*\left(\sum_{i=1}^m x_i g_i\right) = \sum_{i=1}^m \alpha(x_i) g_i \qquad x_i \in A(b),\ g_i \in Hom(a,b)_{\mathcal{G}}$$

on morphism sets. This functor is continuous with respect to either norm.

Let $f\colon \mathcal{G} \to \mathcal{H}$ be a functor between groupoids, and let A be a $\mathcal{H}$-C^*-algebra. Then we have an induced C^*-functor $f_*\colon A\mathcal{G} \to A\mathcal{H}$ defined to be the functor f on the set of objects, and by the formula

$$f_*\left(\sum_{i=1}^m x_i g_i\right) = \sum_{i=1}^m x_i f(g_i) \qquad x_i \in A(b),\ g_i \in Hom(a,b)_{\mathcal{G}}$$

on morphism sets.

This functor is continuous with respect to the norm $\|-\|_{\max}$, and continuous with respect to the norm $\| - \|_r$ if the functor is faithful. We thus have the following result.

Proposition 7.5 *Let $\mathcal{G}$ be a groupoid. Then the assignments $A \mapsto A \rtimes_r \mathcal{G}$ and $A \mapsto A \rtimes \mathcal{G}$ are functors from the category of $\mathcal{G}$-C^*-algebras and equivariant maps to the category of C^*-categories and C^*-functors.*

Let $f\colon \mathcal{G} \to \mathcal{H}$ be a functor between groupoids, and let A be an $\mathcal{H}$-C^*-algebra. Then we have a functorially induced C^*-functor $f_*\colon A \rtimes \mathcal{G} \to A \rtimes \mathcal{H}$. If the functor f is faithful, then we also have a functorially induced C^*-functor $f_*\colon A \rtimes_r \mathcal{G} \to A \rtimes_r \mathcal{H}$. □

Theorem 7.6 *Let $\mathcal{G}$ be a groupoid, and let A and B be $\mathcal{G}$-C^*-algebras. Then there are maps*

$$D\colon \mathbb{KK}_{\mathcal{G}}(A,B) \to \mathbb{KK}(A \rtimes \mathcal{G}, B \rtimes \mathcal{G}) \qquad D_r\colon \mathbb{KK}_{\mathcal{G}}(A,B) \to \mathbb{KK}(A \rtimes_r \mathcal{G}, B \rtimes_r \mathcal{G})$$

which compatible with the product in the sense that we have commutative diagrams

$$\begin{tikzcd}
\mathbb{KK}_{\mathcal{G}}(A,B) \wedge \mathbb{KK}_{\mathcal{G}}(B,C) \arrow[r] \arrow[d] & \mathbb{KK}(A,C) \arrow[d] \\
\mathbb{KK}(A \rtimes \mathcal{G}, B \rtimes \mathcal{G}) \wedge \mathbb{KK}(B \rtimes \mathcal{G}, C \rtimes \mathcal{G}) \arrow[r] & \mathbb{KK}(A \rtimes \mathcal{G}, C \rtimes \mathcal{G})
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathbb{KK}_{\mathcal{G}}(A,B) \wedge \mathbb{KK}_{\mathcal{G}}(B,C) \arrow[r] \arrow[d] & \mathbb{KK}(A,C) \arrow[d] \\
\mathbb{KK}(A \rtimes_r \mathcal{G}, B \rtimes_r \mathcal{G}) \wedge \mathbb{KK}(B \rtimes_r \mathcal{G}, C \rtimes_r \mathcal{G}) \arrow[r] & \mathbb{KK}(A \rtimes_r \mathcal{G}, C \rtimes_r \mathcal{G})
\end{tikzcd}$$

where the horizontal maps are defined by the product, and the vertical maps are copies of the map D or D_r respectively.

Proof: We have a natural C^*-functor $\gamma\colon J(A \rtimes \mathcal{G}) \to (JA) \rtimes \mathcal{G}$ defined as the classifying map of the diagram

$$\begin{tikzcd}
 & & & A \rtimes \mathcal{G} \arrow[d, equal] & \\
0 \arrow[r] & (JA) \rtimes \mathcal{G} \arrow[r] & (TA) \rtimes \mathcal{G} \arrow[r] & A \rtimes \mathcal{G} \arrow[r] & 0
\end{tikzcd}$$

Viewing the suspension of a $\mathcal{G}$-C^*-algebra or C^*-category as the tensor product with the C^*-algebra $C_0(0,1)$, there is an obvious C^*-functor $\beta\colon (\Sigma^n B \otimes \mathcal{K}) \rtimes \mathcal{G} \to \Sigma^n(B \rtimes \mathcal{G}) \otimes \mathcal{K}$.

Let $\mathcal{C}$ be any C^*-category. Then the tensor product $\mathcal{C} \otimes \mathcal{K}$ is a direct limit of C^*-categories of matrices with elements the morphisms of the category $\mathcal{C}$. It follows that we have a natural C^*-functor $\delta\colon \mathcal{C} \otimes \mathcal{K} \to \mathcal{C}^H$.

Let $\alpha\colon J^{2n}A \to \Sigma^n B \otimes \mathcal{K}$ be an equivariant map. Then by the above proposition we have a functorially induced C^*-functor $\alpha_*\colon (J^{2n}A) \rtimes_{\mathcal{G}} \to (\Sigma^n B \otimes \mathcal{K}) \rtimes \mathcal{G}$. We can define a map $D\colon \mathbb{KK}_{\mathcal{G}}(A,B) \to \mathbb{KK}(A \rtimes \mathcal{G}, B \rtimes \mathcal{G})$ by writing $D(\alpha) = \delta \circ \beta \circ \alpha_* \circ \gamma^n$. The relevant naturality properties are easy to check. The construction is the same if we consider reduced rather than full crossed products. □

Corollary 7.7 *Let $\mathcal{G}$ be a discrete groupoid, and let A and B be $\mathcal{G}$-C^*-algebras. Then the spectra $\mathbb{KK}(A \rtimes_r \mathcal{G}, B \rtimes_r \mathcal{G})$ and $\mathbb{KK}(A \rtimes \mathcal{G}, B \rtimes \mathcal{G})$ are symmetric $\mathbb{KK}_{\mathcal{G}}(\mathbb{F},\mathbb{F})$-module spectra.* □

8 Comparison with C^*-algebra K-theory

The spectra defined in this article are based on the spaces used to construct KK-theory groups in the articles [2, 4, 3]. The fact that our spectra can be used to define the usual Kasparov KK-theory for C^*-algebras is therefore no surprise. To be specific, the following result holds.

Theorem 8.1 *Let A and B be C^*-algebras. Then the stable homotopy group $\pi_n \mathbb{KK}(A,B)$ is naturally isomorphic to the group $KK^{-n}(A,B)$. If C is another C^*-algebra, the smash product of spectra*

$$\mathbb{KK}(A,B) \wedge \mathbb{KK}(B,C) \to \mathbb{KK}(A,C)$$

induces the Kasparov product.

Proof: By proposition 3.4, the k-th space of the spectrum $\mathbb{KK}(A,B)$ is the spaces of all $*$-homomorphisms $J^{2k}A \to \Sigma^k B \otimes \mathcal{K}$. The product is defined by composition of these $*$-homomorphisms.

The stable homotopy group $\pi_n \mathbb{KK}(A,B)$ is thus the direct limit

$$\lim_{k\to\infty} [J^{2n+2k}A, \Sigma^{n+k}B \otimes \mathcal{K}]$$

where the square brackets indicate homotopy-classes of $\star$-homomorphisms.

The result now follows by proposition 3.1 in [3]. $\square$

A similar result also holds in the equivariant case, and also follows from the arguments of [3]. To be precise, we have the following.

Theorem 8.2 *Let G be a discrete group, and let A and B be G-C^*-algebras. Then the stable homotopy group $\pi_n \mathbb{KK}_G(A,B)$ is naturally isomorphic to the group $KK_G^{-n}(A,B)$. If C is another G-C^*-algebra, the smash product of spectra*

$$\mathbb{KK}_G(A,B) \wedge \mathbb{KK}_G(B,C) \to \mathbb{KK}_G(A,C)$$

induces the Kasparov product. $\square$

In order to apply the above two theorems to spectra for C^*-categories and groupoid C^*-algebras, we need some more comparison results.

Definition 8.3 Let $\mathcal{A}$ and $\mathcal{B}$ be unital C^*-categories. Two C^*-functors $\alpha, \beta \colon \mathcal{A} \to \mathcal{B}$ are termed *equivalent* if there are elements $u_a \in Hom(\alpha(a), \beta(a))_{\mathcal{B}}$ for each object $a \in Ob(\mathcal{A})$ such that:

- $u_a^* u_a = 1_{\alpha(a)}$ and $u_a u_a^* = 1_{\beta(a)}$ for all objects $a \in Ob(\mathcal{A})$.
- Let $x \in Hom(a,b)_{\mathcal{A}}$. Then $\beta(x)u_a = u_b \alpha(x)$.

Two unital C^*-categories $\mathcal{A}$ and $\mathcal{B}$ are called equivalent if there is are C^*-functors $\alpha \colon \mathcal{A} \to \mathcal{B}$ and $\beta \colon \mathcal{A} \to \mathcal{B}$ such that the compositions $\alpha \circ \beta$ and $\beta \circ \alpha$ are equivalent to identity C^*-functors.

Lemma 8.4 *Let $\alpha, \beta \colon \mathcal{A} \to \mathcal{B}$ be equivalent C^*-functors. Then α and β lie in the same path-component of the space $F(\mathcal{A}, \mathcal{B})$.*

Proof: In the space $F(\mathcal{A},\mathcal{B})$, the C^*-functors α and β are the same as the C^*-functors $\alpha' \colon \mathcal{A} \to \mathcal{B}_\oplus$ and $\beta' \colon \mathcal{A} \to \mathcal{B}_\oplus$ defined by writing

$$\alpha'(a) = \alpha(a) \oplus \beta(a),\ \beta'(a) = \alpha(a) \oplus \beta(a) \qquad a \in Ob(\mathcal{A})$$

and

$$\alpha'(x) = \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \qquad \beta'(x) = \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix}$$

where $x \in Hom(a,b)_{\mathcal{A}}$.

Since the C^*-functors α and β are equivalent, we can find morphisms $u_a \in Hom(\alpha(a), \beta(a))_{\mathcal{B}}$ for each object $a \in Ob(\mathcal{A})$ such that $u_a^* u_a = 1_{\alpha(a)}$, $u_a u_a^* = 1_{\beta(a)}$, and $\beta(x)u_a = u_b\alpha(x)$ for all $x \in Hom(a,b)_{\mathcal{A}}$.

Let $t \in [0, \pi/2]$. Define

$$r_a(t) = \left(\begin{array}{cc} \cos\theta & u_a^* \sin\theta \\ -u_a \sin\theta & \cos\theta \end{array} \right) \in Hom(\alpha(a) \oplus \beta(a))_{\mathcal{B}} \qquad t \in [0,\ frac\pi 2]$$

Then we have a path of C^*-functors, $F_t \colon \mathcal{A} \to \mathcal{B}_\oplus$, from α' to β', defined by the formula

$$F_t(a) = \alpha(a) \oplus \beta(a) \qquad F_t(x) = r_a(t) \left(\begin{array}{cc} \alpha(x) & 0 \\ 0 & 1 \end{array} \right) r_b(t)^*$$

where $x \in Hom(a,b)_{\mathcal{A}}$. □

Theorem 8.5 *Let $\mathcal{A}$ and $\mathcal{A}'$ be equivalent C^*-categories. Let $\mathcal{B}$ be another C^*-category. Then the spectra $\mathbb{KK}(\mathcal{A},\mathcal{B})$ and $\mathbb{KK}(\mathcal{A}',\mathcal{B})$ are homotopy-equivalent, and the spectra $\mathbb{KK}(\mathcal{B},\mathcal{A})$ and $\mathbb{KK}(\mathcal{B},\mathcal{A}')$ are homotopy-equivalent.*

Proof: Since the C^*-categories $\mathcal{A}$ and $\mathcal{A}'$ are equivalent, by the above lemma we can find elements $\alpha \in F(\mathcal{A},\mathcal{A}')$ and $\beta \in F(\mathcal{A}',\mathcal{A})$ along with a path $\gamma_t \in F(\mathcal{A},\mathcal{A})$ such that $\gamma_0 = \beta \circ \alpha$ and γ_1 is the identity.

There are thus induced maps

$$\alpha\sharp \colon \mathbb{KK}(\mathcal{A},\mathcal{B}) \to \mathbb{KK}(\mathcal{A}',\mathcal{B}) \qquad \beta\sharp \colon \mathbb{KK}(\mathcal{A}',\mathcal{B}) \to \mathbb{KK}(\mathcal{A},\mathcal{B})$$

defined by the product with the elements α and β respectively such that the map $\gamma_t\sharp \colon \mathbb{KK}(\mathcal{A},\mathcal{B}) \to \mathbb{KK}(\mathcal{A},\mathcal{B})$ is a homotopy between the composite $\alpha' \circ \alpha$ and the identity map.

Similarly, the composite $\alpha' \circ \alpha$ is homotopic to the identity map. It follows that the spectra $\mathbb{KK}(\mathcal{A},\mathcal{B})$ and $\mathbb{KK}(\mathcal{A}',\mathcal{B})$ are homotopy-equivalent, and we have proved the first of the statements in the theorem.

The proof of the second statement in the theorem is almost identical. □

The above result along with theorem 8.2 can be used to prove certain formal properties involving the KK-theory of C^*-categories that are equivalent to C^*-algebras, which covers most examples found in geometric applications.

Theorem 8.6 *Let $\theta \colon \mathcal{G} \to \mathcal{H}$ be an equivalence of discrete groupoids. Let A and B be $\mathcal{H}$-C^*-algebras. Then the restriction map $\theta^* \colon \mathbb{KK}_{\mathcal{H}}(A,B) \to \mathbb{KK}_{\mathcal{G}}(A,B)$ is a homeomorphism of spectra.*

Proof: Since the functor θ is an equivalence, there is a functor $\phi \colon \mathcal{H} \to \mathcal{G}$ along with natural isomorphisms $G \colon \phi \circ \theta \to 1_{\mathcal{G}}$ and $H \colon \theta \circ \phi \to 1_{\mathcal{H}}$.

Thus, for each object $a \in Ob(\mathcal{G})$, we have a morphism $G_a \in Hom(\phi\theta(a), a)$. Let $\alpha \colon A \to B \otimes \mathcal{K}$ be an $\mathcal{H}$-equivariant map. Then the map α can be defined in terms of the restriction $\phi^*\theta^*\alpha \colon A \to B \otimes \mathcal{K}$ by the formula

$$\alpha(x) = \phi^*\theta^*\alpha(H(a)^{-1} x H(a)) \qquad a \in Ob(\mathcal{A})$$

Thus the equivariant map α is determined by the restriction $\phi^*\theta^*$. The natural isomorphism H therefore induces a homeomorphism of spectra $H_* \colon \mathbb{KK}_{\mathcal{H}}(A,B) \to \mathbb{KK}_{\mathcal{H}}(A,B)$ such that $H_* \circ \phi^* \circ \theta^* = 1_{\mathbb{KK}_{\mathcal{H}}(A,B)}$. There is similarly a homeomorphism $G_* \colon \mathbb{KK}_{\mathcal{G}}(A,B) \to \mathbb{KK}_{\mathcal{G}}(A,B)$ such that $H_* \circ \theta^* \circ \phi^* = 1_{\mathbb{KK}_{\mathcal{H}}(A,B)}$.

It follows that the map θ^* is a homeomorphism, and we are done. □

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