Measure Theory and Integration

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Contents

1 σ-Algebras

Definition 1.1 Let Ω be a set. A collection, $\mathcal{M}$, of subsets of Ω is termed a *σ-algebra* if:

- $\Omega \in \mathcal{M}$.
- If $A \in \mathcal{M}$, then $\Omega \backslash A \in \mathcal{M}$.
- If $A = \cup_{n=1}^{i} nftyA_n$, where $A_n \in \mathcal{M}$ for all n, then $A \in \mathcal{M}$.

A set Ω equipped with a σ-algebra, $\mathcal{M}$, is called a *measurable space*. The elements of $\mathcal{M}$ are termed the *measurable subsets* of Ω.

If X is a measurable space, the empty set is measurable, and a countable intersection of measurable sets are measurable. These facts follow immediately from the axioms.

Example 1.2

- Let Ω be any set. Then the set of *all* subsets of $\mathcal{M}$ is a σ-algebra. The set $\{\emptyset, \Omega\}$ is also a σ-algebra.
- Let Ω be any set. Let $\mathcal{M}$ be the set of all sets $A \subseteq \Omega$ such that either A or $\Omega \backslash A$ is a countable or finite set. Then $\mathcal{M}$ is a σ-algebra.

The following result is extremely useful when constructing examples.

Proposition 1.3 *Let $\mathcal{F}$ be a collection of subsets of a set Ω. Then there is a unique smallest σ-algebra, $\mathcal{M}_{\mathcal{F}}$, on Ω containing $\mathcal{F}$.*

Proof: Let M be the family of all σ-algebras which contain $\mathcal{F}$. The set of all subsets of Ω is certainly a σ-algebra, so $M \neq \emptyset$. Let $\mathcal{M}_{\mathcal{F}}$ be the intersection of all σ-algebras in the set M. We need to show that $\mathcal{M}_{\mathcal{F}}$ is a σ-algebra.

If $A_n \in \mathcal{M}_{\mathcal{F}}$ for all $n \in \mathbb{N}$, and $\mathcal{M} \in \Omega$, then $A_n \in \mathcal{M}$ for all n, and so $\cup_{n \in \mathbb{N}} A_n \in \mathcal{M}$. Therefore $\cup_{n \in \mathbb{N}} A_n \in \mathcal{M}_{\mathcal{F}}$. The other required properties of a σ-algebra are verified similarly. □

The σ-algebra $\mathcal{M}_{\mathcal{F}}$ is called the σ-algebra generated by $\mathcal{F}$.

Example 1.4 Let X be a topological space. The σ-algebra, $\mathcal{B}$, generated by all open subsets of X is called the collection of *Borel measurable sets* or simple *Borel sets* on X.

Obviously all open and all closed sets are Borel measurable, as are all countable unions of closed sets and countable intersections of open sets.

Example 1.5

- The half-open interval $[a, b)$ is a Borel subset of $\mathbb{R}$.
- If X is a topological space with the discrete topology, then every subset of X is Borel measurable.
- If X is a topological space carrying the topology $\{X, \emptyset\}$, then the σ-algebra of Borel sets is the set $\mathcal{B} = \{X, \emptyset\}$.

2 Measurable Functions

Definition 2.1 Let Ω be a measurable space, and Y a topological space. A map $f\colon \Omega \to Y$ is termed *measurable* if the set $f^{-1}[U]$ is measurable for every open subset $U \subseteq Y$.

Proposition 2.2 *Let $f\colon \Omega \to Y$ be a measurable function. Then the inverse image $f^{-1}[B]$ is measurable whenever $B \subseteq Y$ is a Borel set.*

Proof: Let $\mathcal{M}$ be the collection of all subsets $E \subseteq Y$ such that the inverse image $f^{-1}[E] \subseteq \Omega$ is measurable. It is easy to check the axioms required to show that $\mathcal{M}$ is a σ-algebra.

Since the function f is measurable, the σ-algebra $\mathcal{M}$ contains all open sets of Y, and therefore all Borel sets. Thus, by definition of the σ-algebra $\mathcal{M}$, the set $f^{-1}[B]$ is measurable whenever B is a Borel set. □

Definition 2.3 Let $f: X \to Y$ be a mapping between topological spaces. If f is measurable with respect to the σ-algebra of all Borel sets in X, then we call f a Borel function.

Thus a function is a Borel function if the inverse image of any open set is a Borel set. In particular, any continuous function is a Borel function.

Proposition 2.4 *Let $f: \Omega \to X$ be a measurable function, and let $g: X \to Y$ be a Borel function. Then the composite $g \circ f: \Omega \to Y$ is measurable.*

Proof: Let $U \subseteq Y$ be an open set. Then the inverse image $g^{-1}[U]$ is a Borel set, so by proposition 2.2 the inverse image $(g \circ f)^{-1}[U]$ is measurable. □

As a corollary, the composite of a measurable and a continuous function is measurable.

Example 2.5 Let X be a measure space, and let $E \subseteq X$ be a measurable set. Then the function $\chi_E: X \to \mathbb{C}$ given by the formula

$$\chi_E(x) = \begin{cases} 1 & x \in E \\ 0 & x \notin E \end{cases}$$

is measurable.

The function χ_E is called the *characteristic function* of E.

Proposition 2.6 *Let $u, v: X \to \mathbb{R}$ be measurable functions, and let $\Phi: \mathbb{R}^2 \to Y$ be continuous. Define a function $h: X \to Y$ by the formula*

$$h(x) = \Phi(u(x), v(x))$$

Then the function h is measurable.

Proof: Define $f: X \to \mathbb{R}^2$ by the formula $f(x) = (u(x), v(x))$. In view of proposition 2.4 it suffices to prove that the function f is measurable. Observe that:

$$f^{-1}((a,b) \times (c,d)) = u^{-1}(a,b) \cap v^{-1}(c,d)$$

so the inverse image $f^{-1}((a,b) \times (c,d))$ is measurable since u and v are measurable functions.

But every open set $U \subseteq \mathbb{R}^2$ is a countable union of rectangles of the form $(a,b) \times (c,d)$. The σ-algebra axioms thus ensure that the inverse image $f^{-1}[U]$ is measurable whenever $U \subseteq \mathbb{R}^2$ is an open set. □

The above proposition and proof still function if the function Φ is a Borel function rather than a continuous function.

Corollary 2.7 *Let $f: X \to \mathbb{C}$ be a function on a measurable space X. Then the function f is measurable if and only if the functions $\Re(f)$ and $\Im(f)$ are measurable.*

Proof: Let $u, v: X \to \mathbb{R}$ be measurable functions. Define $\Phi: \mathbb{R}^2 \to \mathbb{C}$ by the formula $\Phi(x, y) = x + iy$. Then the function $u + iv$ is measurable by proposition 2.6.

The converse follows immediately from proposition 2.4 since the functions $\Re$ and $\Im$ are continuous. □

Corollary 2.8 *Let $f, g: X \to \mathbb{C}$ be measurable functions. Then the functions $f + g$ and fg are measurable.*

Proof: In view of corollary 2.7 it suffices to prove this result for real-valued measurable functions. If we define continuous functions $\Phi_1, \Phi_2: \mathbb{R}^2 \to \mathbb{R}$ by the formulae $\Phi_1(s, t) = s + t$ and $\Phi_2(s, t) = st$, then the result follows immediately from proposition 2.6 □

Proposition 2.9 *Let $f: X \to \mathbb{C}$ be a measurable function. Then the function $|f|$ is measurable, and there is a measurable function $\alpha: X \to \mathbb{C}$ such that $|\alpha(x)| = 1$ for all $x \in X$, and $f = \alpha|f|$.*

Proof: Let $E = \{x \in X \mid f(x) = 0\}$. Then the set E is the inverse image of a closed subset, and so measurable. We can define a continuous function $\varphi: \mathbb{C}\backslash\{0\} \to \mathbb{C}$ by the formula $\varphi(z) = z/|z|$. It follows from example 2.5, corollary 2.8, and proposition 2.4 that the function $\alpha: X \to \mathbb{C}$ defined by the formula

$$\alpha(x) = \varphi(f(x) + \chi_E(x))$$

is measurable. The formulae $|\alpha(x)| = 1$ and $f = \alpha|f|$ are easy to check. □

3 Lim inf and Lim sup

Definition 3.1 Let (a_n) be a sequence of real numbers. Then we define

$$\limsup_{n\to\infty} a_n = \limsup_{n\to\infty} \{a_n, a_{n+1}, a_{n+2}, \ldots\}$$

and

$$\liminf_{n\to\infty} a_n = \liminf_{n\to\infty} \{a_n, a_{n+1}, a_{n+2}, \ldots\}$$

We can pass from results about lim sup to results about lim inf, or conversely, by the observation

$$\limsup_{n\to\infty} a_n = -\liminf_{n\to\infty} (-a_n)$$

It will occasionally be convenient to us to allow ∞ and $-\infty$ as values of limits and functions. This is a safe enough option provided we do not attempt to do arithmetic with these symbols; for example, expressions such as '$\infty - \infty$' are completely meaningless.

However, we can form 'intervals'

$$[a, \infty] = [a, \infty) \cup \{\infty\} \qquad [\infty, b] = (\infty, b] \cup \{\infty\}$$

and so on. These intervals are topological spaces. We can also allow ourselves the inequality

$$-\infty < a < \infty$$

for all $a \in \mathbb{R}$. The standard result about lim sup and lim inf can now be expressed quite simply; although a number of special cases need to be examined in the proof.

Theorem 3.2 *Let (a_n) be a real-valued sequence. Then the limits*

$$\liminf_{n\to\infty} a_n \in [-\infty, \infty) \qquad \limsup_{n\to\infty} a_n \in (-\infty, \infty]$$

exist and satisfy the inequality

$$\liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n$$

Further, the equality

$$\liminf_{n\to\infty} a_n = a = \limsup_{n\to\infty} a_n$$

holds precisely when the sequence (a_n) converges to the real number a. □

Note that the number a in the above result must be finite.

Proposition 3.3 *Let Ω be a measurable space, and let $f\colon \Omega \to [\infty, \infty]$ be any map. Suppose that the inverse image $f^{-1}((\alpha, \infty])$ is measurable for every point $\alpha \in \mathbb{R}$. Then the function f is measurable.*

Proof: Let

$$\mathcal{M} = \{E \subseteq [-\infty, \infty] \mid f^{-1}[E] \text{ is measurable } \}$$

By proposition 2.2 the set $\mathcal{M}$ is a σ-algebra. Choose points $\alpha \in \mathbb{R}$ and $\alpha_n < \alpha$ such that $\lim_{n\to\infty} \alpha_n = \alpha$. Since the set $(\alpha_n, \infty]$ is measurable by hypothesis, and

$$[-\infty, \alpha) = \bigcup_{n=1}^{\infty} [-\infty, \alpha_n] = \bigcup_{n=1}^{\infty} [-\infty, \infty] \backslash (\alpha_n, \infty]$$

it follows that $[-\infty, \alpha) \in \Omega$. Hence

$$(\alpha, \beta) = [-\infty, \beta) \cap (\alpha, \infty] \in \Omega$$

for every point $\alpha, \beta \in \mathbb{R}$. Since every open set in $[-\infty, \infty]$ is a countable union of such open intervals, the collection $\mathcal{M}$ contains every open set. Thus the map f is measurable. □

Corollary 3.4 *Let $f_n: X \to [-\infty, \infty]$ be measurable functions for $n \in \mathbb{N}$. Then the functions*

$$\sup\{f_n\} \quad \limsup_{n\to\infty} f_n \quad \inf\{f_n\} \quad \liminf_{n\to\infty} f_n$$

are measurable.

Proof: Let $a \in \mathbb{R}$. Observe that the set

$$(\sup\{f_n\})^{-1}(a, \infty] = \bigcup_{n=1}^{\infty} f_n^{-1}(a, \infty]$$

is measurable. Hence by the above proposition, the function $\sup\{f_n\}$ is measurable. The formula $\inf\{f_n\} = -\sup\{-f_n\}$ tells us that the function $\inf\{f_n\}$ is also measurable.

Now, for each point $x \in \Omega$, the sequence of numbers

$$g_n(x) = \sup\{f_n(x), f_{n+1}(x), f_{n+2}(x), \ldots\}$$

is monotonic increasing. It follows that

$$\limsup_{n\to\infty} f_n(x) = \inf\{g_n(x)\}$$

We know that each function f_n is measurable. The above argument tells us that each function g_n is measurable, and that the function $\limsup_{n\to\infty} f_n$ is measurable. A similar argument tells us that the function $\liminf_{n\to\infty} g_n$ is measurable. □

Corollary 3.5 *If $f, g: X \to [-\infty, \infty]$ are measurable functions, then so are the functions* $\max\{f, g\}$ *and* $\min\{f, g\}$. □

Corollary 3.6 *The limit of a pointwise-convergent sequence of meaurable functions is measurable.* □

4 Measure Spaces

Definition 4.1 Let Ω be a measurable space, equipped with a σ-algebra $\mathcal{M}$. A *measure* on Ω is a function $\mu: \mathcal{M} \to [0, \infty]$ such that:

- The function μ is *σ-additive*, ie:

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$$

  whenever (A_n) is a sequence of disjoint mesaurable sets.

- There is a measurable set A such that $\mu(A) < \infty$.

The number $\mu(A)$ is called the *measure* of a set A. A measurable space equipped with some measure is called a *measure space.*

For the above definition to make sense, we need to make a convention concerning our 'number' ∞, namely that $a + \infty = \infty$ whenever $a \in [0, \infty]$.

Example 4.2 Let Ω be a measurable space. For any measurable set $E \subseteq \Omega$, let us define $\mu(E) = |E|$, where $|E|$ denotes the number of elements of E. Then μ is a measure on Ω, called the *counting measure*.

Example 4.3 Let Ω be a measurable space, and let $x_0 \in \Omega$. For any measurable set $E \subseteq \Omega$, let us define

$$\mu(E) = \begin{cases} 1 & x_0 \in E \\ 0 & x_0 \notin E \end{cases}$$

Then μ is a measure on Ω, called the *Dirac measure*.

Proposition 4.4 *Let Ω be a measure space, with measure μ. Then $\mu(\emptyset) = 0$.*

Proof: Choose a measurable set A such that $\mu(A) < \infty$. Then

$$\mu(A) = \mu(A) + \mu(\emptyset) + \mu(\emptyset) + \cdots$$

Hence $\mu(\emptyset) = 0$. □

Corollary 4.5 *Let $A_1, \ldots, A_n$ be disjoint measurable sets. Then*

$$\mu(A_1 \cup \cdots \cup A_n) = \mu(A_1) + \cdots + \mu(A_n)$$

□

Corollary 4.6 *Let A and B be measurable set where $A \subseteq B$. Then $\mu(A) \leq \mu(B)$.*

Proof: The set $B \backslash A = B \cap (\Omega \backslash A)$ is measurable, the sets A and $B \backslash A$ are disjoint, and $B = A \cup B \backslash A$. By the above corollary

$$\mu(B) = \mu(A) + \mu(B \backslash A)$$

The inequality $\mu(A) \leq \mu(B)$ follows since $\mu(B \backslash A) \geq 0$. □

Proposition 4.7 *Let (A_n) be a sequence of measurable sets such that $A_n \subseteq A_{n+1}$ for all n. Let $A = \bigcup_{n=1}^{\infty} A_n$. Then $\lim_{n \to \infty} \mu(A_n) = \mu(A)$.*

Proof: Let $B_1 = A_1$, and $B_n = A_n \backslash A_{n-1}$ when $n \geq 2$. Then the sets B_n are measurable and disjoint. Further

$$A_n = B_1 \cup \cdots \cup A_n \qquad A = \bigcup_{n=1}^{\infty} B_n$$

Hence

$$\mu(A) = \sum_{n=1}^{\infty} \mu(B_n) = \lim_{N \to} \sum_{n=1}^{N} \mu(B_n) = \lim_{N \to \infty} \mu(A_N)$$

□

Corollary 4.8 *Let (A_n) be a sequence of measurable sets such that $\mu(A_1) < \infty$ and $A_{n+1} \subseteq A_n$ for all n. Let $A = \bigcap_{n=1}^{\infty} A_n$. Then $\lim_{n\to\infty} \mu(A_n) = \mu(A)$.*

Proof: Let $C_n = A_1 \backslash A_n$. Then the set C_n is measurable, $C_n \subseteq C_{n+1}$ for all n, and $\bigcup_{n=1}^{\infty} C_n = A_1 \backslash A$. Hence, by the above proposition

$$\lim_{n\to\infty} \mu(C_n) = \mu(A_1 \backslash A)$$

We know that the measure $\mu(A_1)$ is finite, and that we have disjoint unions

$$A_1 = A_n \cup C_n \qquad A_1 = A_1 \backslash A \cup A$$

Hence

$$\mu(A_1) - \lim_{n\to\infty} \mu(A_n) = \mu(A_1) - \mu(A)$$

and

$$\lim_{n\to\infty} \mu(A_n) = \mu(A)$$

□

The above corollary is false if we omit the assumption that $\mu(A_1) < \infty$.

5 Simple Functions

Definition 5.1 A function $s: \Omega \to \mathbb{C}$ on a measurable space Ω is called *simple* if the range of s is a finite set of points.

Let $s: \Omega \to \mathbb{C}$ be a simple function, with image $s[X] = \{0\} \cup \{\alpha_1, \ldots, \alpha_n\}$. Write $A_i = s^{-1}(\alpha_i)$. Then clearly

$$s = \sum_{i=1}^{n} \alpha_i \chi_{A_i}$$

and the function s is measurable if and only if each set A_i is measurable.

Proposition 5.2 *Let $f: \Omega \to [0, \infty]$ be a measurable function. Then there are simple measurable functions $s_n: X \to [0, \infty)$ such that the sequence $(s_n(x))$ is monotonically increasing, with limit $f(x)$ for each point $x \in X$.*

Proof: Let $n \in \mathbb{N}$, and $t \in [0, \infty]$. Then there is a unique integer $k_n(t)$ such that

$$k_n(T)2^{-n} \leq t \leq (k_n(t) + 1)2^{-n}$$

Define

$$\varphi_n(t) = \begin{cases} k_n(t)2^{-n} & 0 \leq t < n \\ n & n \leq t \leq \infty \end{cases}$$

The function $\varphi_n: [0, \infty] \to [0, \infty]$ is a Borel function, and

$$t - 2^{-n} \leq \varphi_n(t) \leq t$$

if $0 \leq t \leq n$. Thus we have a monotonically increasing sequence $(\varphi_n(t))$ with limit t. If we write $s_n = \varphi_n \circ f$, then (s_n) is a monotonically increasing sequence of simple measurable functions, with pointwise limit f as required. □

We now come to the first of our definitions of the integral.

Definition 5.3 Let Ω be a measure space, with measure μ. Let $s\colon \Omega \to \mathbb{C}$ be a measurable simple function, with set of non-zero values $\{\alpha_1, \ldots, \alpha_n\}$. Write

$$s = \sum_{k=1}^{n} \alpha_k \chi_{A_k}$$

Let $E \subseteq \Omega$ be a measurable subset of Ω. Then we define the *integral* of s over E to be the complex number

$$\int_E s \; d\mu = \sum_{k=1}^{n} \alpha_k \mu(A_k \cap E)$$

There are several simple computations we can do immediately with integrals. For example, with s as above:

$$\int_\Omega s\chi_E \; d\mu = \sum_{k=1}^{\infty} \alpha_k \mu(A_k \cap E) = \int_E s \; d\mu$$

Lemma 5.4 *Let Ω be a measure space, with measure μ. Let $s\colon \Omega \to [0, \infty)$ be a measurable simple function. Then we can define a new measure φ on Ω by the formula*

$$\varphi(E) = \int_E s \; d\mu$$

Proof: To begin with, observe that $\varphi(E) \geq 0$ for every measurable set E, and that if $\mu(E) < \infty$, then $\varphi(E) < \infty$, so there is at least one measurable set with finite measure. We need to test σ-additivity.

Let (E_n) be a sequence of disjoint measurable sets. We know that

$$\mu(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \mu(E_i)$$

Let $\{\alpha_1, \ldots, \alpha_k\}$ be the set of non-zero values of the simple function s. Then

$$\varphi(\bigcup_{i=1}^{\infty} E_i) = \sum_{k=1}^{n} \sum_{i=1}^{\infty} \alpha_k \mu(A_k \cap E_i)$$

Exchanging the summation signs is possible since all of the numbers involved in the above equation are positive. Therefore

$$\varphi(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \sum_{k=1}^{n} \alpha_k \mu(A_k \cap E_i) = \sum_{i=1}^{\infty} \varphi(E_i)$$

and we are done. □

Proposition 5.5 *Let $s, t\colon \Omega \to [0, \infty]$ be simple functions. Then*

$$\int_\Omega s + t \; d\mu = \int_\Omega s \; d\mu + \int_\Omega t \; d\mu$$

Proof: Write as usual

$$s = \sum_{i=1}^{m} \alpha_k \chi_{A_i} \qquad t = \sum_{j=1}^{n} \beta_j \chi_{B_j}$$

Let $E_{ij} = A_i \cap B_j$. Then certainly

$$int_{E_{ij}}(s+t)\ d\mu = (\alpha_i + \beta_j)\mu(E_{ij}) = \int_{E_{ij}} s\ d\mu + \int_{E_{ij}} t\ d\mu$$

Now the sets $\{0, \alpha_1, \ldots, \alpha_m\}$ and $\{0, \beta_1, \ldots, \beta_n\}$ are the ranges of the functions s and t respectively. Let $A_0 = s^{-1}[0]$ and $B_0 = t^{-1}[0]$. Then

$$\Omega = \bigcup_{i=0}^{m} A_i = \bigcup_{j=0}^{n} B_j$$

Hence

$$\Omega = \bigcup_{i,j=0}^{m,n} E_{ij}$$

The sets E_{ij} are certainly disjoint. Hence by the above lemma, we know that

$$\int_{\Omega} s + t\ d\mu = int_{\Omega} s\ d\mu + \int_{\Omega} t\ d\mu$$

and we are done. □

If s is a step function, and $\alpha \in \mathbb{C}$, then clearly

$$\int_{\Omega} \alpha s\ d\mu = \alpha \int_{\Omega} s\ d\mu$$

Hence we have proven linearity for integrals of positive-valued step functions.

6 Integration of Positive-Valued Functions

Definition 6.1 Let Ω be a measure space, with measure μ. Let $f\colon \Omega \to [0, \infty]$ be a measurable function, and let $E \subseteq \Omega$ be a measurable set. Let S be the set of simple functions, $s\colon \Omega \to [0, \infty)$, such that $s(x) \leq f(x)$ for all $x \in \Omega$. Then we define the *integral* of f over E:

$$\int_E f\ d\mu = \sup\{\int_E s\ d\mu \mid s \in S\}$$

A few properties of the integral are easy to prove. For example:

- Let $f\colon \Omega \to [0, \infty]$ and $E \subseteq \Omega$ be measurable. Then

$$\int_{\Omega} f\ d\mu = \int_{\Omega} f\chi_E\ d\mu$$

- Let $f, g\colon \Omega \to [0, \infty]$ be measurable functions such that $f \leq g$, that is to say $f(x) \leq g(x)$ for all $x \in \Omega$. Then

$$\int_E f \leq \int_E g$$

whenever $E \subseteq \Omega$ is a measurable subset.

Theorem 6.2 (The Monotone Convergence Theorem) *Let $f_n\colon \Omega \to [0, \infty]$ be a sequence of measurable functions, such that for each point $x \in \Omega$ the sequence $(f_n(x))$ is monotonically increasing, with limit $f(x)$. Then the function $f\colon \Omega \to [0, \infty]$ is measurable, and*

$$\int_\Omega f \; d\mu = \lim_{n\to\infty} \int_\Omega f_n \; d\mu$$

Proof: As the limit of a sequence of measurable functions, the function f is measurable. Since the seqyebce $(f_n(x))$ is monotonic increasing, with limit $f(x)$, we know that $F_n \leq f_{n+1} \leq f$ for all n. Therefore the sequence of integrals $\left(\int_\Omega f_n\right)$ is monotonic increasing, and

$$\int_\Omega f_n \; d\mu \leq \int_\Omega f \; d\mu$$

for all n.

Choose a simple function s such that $0 \leq s \leq f$. Let $0 < \alpha < 1$, and write

$$E_n = \{x \in \Omega \mid f_n(x) \geq \alpha s(x)\}$$

Each set E_n is measurable, and $E_n \subseteq E_{n+1}$ for all n since the sequence (f_n) is monotonic increasing. Since the sequence (f_n) has pointwise limit f, we see that

$$\Omega = \bigcup_{n=1}^{\infty} E_n$$

Further

$$\int_\Omega f_n \; d\mu \geq \int_{E_n} f_n \; d\mu \geq \alpha \int_{E_n} s \; d\mu \qquad (*)$$

By lemma 5.4 we can define a measure on the set Ω by the formula

$$\varphi(E) = \int_E s \; d\mu$$

Hence

$$\int_\Omega s \; d\mu = \varphi(\Omega) = \lim_{n\to\infty} \varphi(E_n) = \lim_{n\to\infty} \int_{E_n} s \; d\mu$$

by proposition 4.7.

Taking limits in inequality $(*)$, we see that

$$\lim_{n\to\infty} \int_\Omega f_n \; d\mu \geq \alpha \int_\Omega s \; d\mu$$

In particular, this inequality holds whenever $0 < \alpha < 1$ and $s \leq f$. By the definition of the integral, it follows that

$$\lim_{n\to\infty} \int_\Omega f_n \; d\mu \geq \int_\Omega f \; d\mu$$

and we are done. □

Let $f\colon \Omega \to [0,\infty]$ be a measurable function. By proposition 5.2, there is a monotonically increasing sequence of simple functions $s\colon \Omega \to [0,\infty)$ with pointwise limit f.

The monotone convergence theorem tells us that

$$\int_\Omega f = \lim_{n\to\infty} \int_\Omega s_n$$

and so gives us a new way of viewing the definition of the integral. Using this viewpoint, the following result follows immediately from proposition 5.5

Corollary 6.3 *Let $f, g\colon \Omega \to [0,\infty]$ be measurable functions, and let $\alpha, \beta \in [0,\infty)$. Then*

$$\int_\Omega (\alpha f + \beta g) \; d\mu = \alpha \int_\Omega f \; d\mu + \beta \int_\Omega g \; d\mu$$

□

We can also immediately deduce the following result from the monotone convergence theorem.

Corollary 6.4 *Consider a sequence of measurable functions $f_n\colon \Omega \to [0,\infty]$. Then for any measurable subset $E \subseteq \Omega$ we have the formula*

$$\sum_{n=1}^{\infty} \int_E f_n \; d\mu = \int_E \left(\sum_{n=1}^{\infty} f_n \right) \; d\mu$$

□

Theorem 6.5 (Fatou's lemma) *Let $f_n\colon \Omega \to [0,\infty]$ be a sequence of measurable functions. Then*

$$\int_\Omega \liminf{}_{n\to\infty} f_n \leq \liminf{}_{n\to\infty} \int_\Omega f_n$$

Proof: Let

$$g_n(x) = \inf\{f_n(x), f_{n+1}(x), f_{n+2}(x), \ldots\}$$

Then the function g_n is measurable, the sequence (g_n) is monotonic increasing, and the inequality $g_n \leq f_n$ holds for all n.

We know that

$$\lim_{n\to\infty} g_n(x) = \liminf{}_{n\to\infty} f_n(x)$$

Hence, by the monotone convergence theorem

$$\int_\Omega \liminf{}_{n\to\infty} f_n = \lim_{n\to\infty} \int_\Omega g_n \leq \liminf{}_{n\to\infty} \int_\Omega f_n$$

and we are done. □

The inequality

$$\int_\Omega \limsup_{n\to\infty} f_n \geq \limsup_{n\to\infty} \int_\Omega f_n$$

is easily deduced from Fatou's lemma.

7 Integration of Complex-Valued Functions

Definition 7.1 Let Ω be a measure space, with measure μ. We call a measurable function $f\colon \Omega \to \mathbb{C}$ *integrable* if

$$\int_\Omega |f| \, d\mu < \infty$$

We write $L^1(\Omega)$ to denote the set of all integrable functions.

Suppose we have a measurable function f and a positive-valued integrable function g such that $|f| \leq g$. Then it follows by the above definition that the function f is integrable. This integrability criterion is often used.

Definition 7.2 Let $f\colon \Omega \to \mathbb{R}$ be any real-valued function. Then we define functions $f^+, f^-\colon \Omega \to [0, \infty)$ by the formulae

$$f^+(x) = \max(f(x), 0)) \qquad f^-(x) = \max(-f(x), 0)$$

respectively.

Observe that $f = f^+ - f^-$. If the function f is measurable, then so are the functions f^+ and f^-.

Proposition 7.3 *Let $f\colon \Omega \to \mathbb{R}$ be an integrable function. Then the functions f^+ and f^- are also integrable.*

Proof: The functions f^+ and $|f|$ are positive-valued, and $f^+ \leq |f|$. We know that $\int_\Omega |f| < \infty$, so $\int_\Omega f^+ < \infty$.

The proof that the function f^- is integrable is identical to the above. □

Definition 7.4 Let $f\colon \Omega \to \mathbb{R}$ be an integrable function. Then we define we define the integral

$$\int_\Omega f \, d\mu := \int_\Omega f^+ \, d\mu - \int_\Omega f^- \, d\mu$$

It is easy to see that definition agrees with the previous definition when the function f is positive-valued. Further, the equation

$$\int_\Omega (\alpha f + \beta g) \, d\mu = \alpha \int_\Omega f \, d\mu + \beta \int_\Omega g \, d\mu$$

holds for all real numbers $\alpha, \beta \in \mathbb{R}$ and integrable functions $f, g\colon \Omega \to \mathbb{R}$.

Definition 7.5 Let $f, g\colon \Omega \to \mathbb{C}$ be integrable functions. Then we define the integral

$$\int_\Omega f\ d\mu := \int_\Omega \Re(f)\ d\mu + i \int_\Omega \Im(f)\ d\mu$$

An argument similar to that made above tells us that this integral is well-defined, agrees with the previous definition for real-valued functions, and is linear.

Proposition 7.6 *Let $f\colon \Omega \to \mathbb{C}$ be an integrable function. Then*

$$\left|\inf_\Omega f\ d\mu\right| \leq \int_\Omega |f|\ d\mu$$

Proof: Choose $\alpha \in \mathbb{C}$ such that $|\alpha| = 1$ and

$$\left|\inf_\Omega f\ d\mu\right| = \alpha \int_\Omega f\ d\mu = \int_\Omega \alpha f\ d\mu$$

Let $g = \Re(\alpha f)$ and $h = \Im(\alpha f)$. Then

$$\left|\inf_\Omega f\ d\mu\right| = \int_\Omega g\ d\mu + i \int_\Omega h\ d\mu$$

Certainly, $|\inf_\Omega f\ d\mu| \in \mathbb{R}$, so

$$\int_\Omega h\ d\mu$$

and

$$\left|\inf_\Omega f\ d\mu\right| = \int_\Omega g\ d\mu$$

However

$$g \leq |g| \leq |\alpha f| = |f|$$

It follows that

$$\left|\inf_\Omega f\ d\mu\right| \leq \int_\Omega |f|\ d\mu$$

and we are done. □

Observe that the proof of the above result uses only positivity and linearity of the integral.

Theorem 7.7 (The Dominated Convergence Theorem) *Let (f_n) be a sequence of measurable functions $f_n\colon \Omega \to \mathbb{C}$ such that:*

- *The limit*

$$f(x) = \lim_{n\to\infty} f_n(x)$$

  exists for all $x \in \Omega$.

- *There is an integrable function $g \in L^1(\Omega)$ such that $|f_n(x)| \leq g(x)$ for all $x \in \Omega$ and $n \in \mathbb{N}$.*

Then $f \in L^1(\Omega)$, *and*

$$\lim_{n\to\infty} \int_\Omega |f_n - f| \; d\mu = 0$$

Proof: Since each fucntion f_n is measurable, the limit function f is also measurable. We know that $|f_n| \le g$ for all n. Therefore $|f| \le g$. It follows that $f \in L^1(\Omega)$.

Now, let

$$h_n = 2g - |f_n - f|$$

Observe that $h_n \ge 0$ for all n. Hence by Fatou's lemma

$$\int_\Omega \liminf_{n\to\infty} h_n \le \liminf_{n\to\infty} \int_\Omega h_n$$

that is

$$\int_\Omega 2g \; d\mu \le \int_\Omega 2g \; d\mu - \liminf_{n\to\infty} \int_\Omega |f_n - f| \; d\mu$$

and so

$$\liminf_{n\to\infty} \int_\Omega |f_n - f| \; d\mu \le 0$$

Since $|f_n - f| \ge 0$ for all n, we deduce that

$$\lim_{n\to\infty} \int_\Omega |f_n - f| \; d\mu = 0$$

as required. □

Combining the dominated convergence theorem with proposition 7.6 we obtain the following corollary, also referred to as the dominated convergence theorem.

Corollary 7.8 (The Dominated Convergence Theorem) *Let* (f_n) *be a sequence of measurable functions* $f_n\colon \Omega \to \mathbb{C}$ *such that:*

- *The limit*

  $$f(x) = \lim_{n\to\infty} f_n(x)$$

  exists for all $x \in \Omega$.

- *There is an integrable function* $g \in L^1(\Omega)$ *such that* $|f_n(x)| \le g(x)$ *for all* $x \in \Omega$ *and* $n \in \mathbb{N}$.

Then $f \in L^1(\Omega)$, *and*

$$\lim_{n\to\infty} \int_\Omega f_n \; d\mu = \int_\Omega f \; d\mu$$

□

8 Null Sets

Definition 8.1 Let Ω be a measure space, with measure μ. Then a set $E \subseteq \Omega$ is called a *null set* if E is measurable, and $\mu(E) = 0$.

The measure space Ω is called *complete* if every subspace of a null set is measurable.

The usual manipulations of the axioms tell us that every measure space is contained in a unique smallest complete measure space. To be more precise, we have the following result.

Proposition 8.2 *Let Ω be a measure space, equipped with σ-algbebra $\mathcal{M}$, and measure μ. Let us define*

$$\mathcal{M}^\star := \{E \subseteq \Omega \mid A \subseteq E \subseteq B,\ A, B \in \Omega,\ \mu(B \backslash A) = 0\}$$

Then the set $\mathcal{M}^\star$ is a σ-algebra. We can define a measure $\mu^\star$ on the set $\mathcal{M}^\star$ by writing

$$\mu^\star(E) = \mu(A) \qquad A \subseteq E \subseteq B,\ A, B \in \Omega,\ \mu(B \backslash A) = 0$$

□

As we might expect from the terminology, null sets are irrelevant from the point of view of integration theory.

Theorem 8.3 *Let $f : \Omega \to [0, \infty]$ be a measurable function. Then the integral of f is zero if and only if the function f is equal to zero except on a null set.*

Proof: Suppose that the set

$$N = \{x \in \Omega \mid f(x) \neq 0\}$$

is a null set. Let $s : \Omega \to [0, \infty]$ be a simple function such that $s \leq f$. Then $s(x) = 0$ when $x \notin N$. The definition of the integral of a simple function tells us that

$$\int_\Omega s\ d\mu = 0$$

The definition of the integral of a non-negative function now implies that

$$\int_\Omega f\ d\mu = 0$$

Conversely, suppose that the integral of the function f is zero. Let

$$A_n = \{x \in \Omega \mid f(x) > 1/n\}$$

Then clearly

$$\frac{1}{n}\mu(A_n) \leq \int_{A_n} f\ d\mu \leq \int_\Omega f d\mu = 0$$

so $\mu(A_n) = 0$. But

$$\{x \in \Omega \mid f(x) > 0\} = \bigcup_{n=1}^\infty A_n$$

Thus σ-additivity implies that the set of all points $x \in \Omega$ such that $f(x) \neq 0$ has measure zero. □

Given two functions $f, g\colon \Omega \to \mathbb{C}$, let us say that f and g are equal *almost everywhere* if they are equal outside of some set of measure zero.

Corollary 8.4 *Let $f, g\colon \Omega \to \mathbb{C}$ be integrable functions that are equal almost everywhere. Then*

$$\int_\Omega f = \int_\Omega g$$

□

Corollary 8.5 *Let $f\colon \Omega \to \mathbb{C}$ be an integrable function. Suppose that*

$$\int_E f = 0$$

whenever the subset $E \subseteq \Omega$ is measurable. Then the function f is equal to zero almost everywhere.

Proof: Let us write

$$f(x) = u(x) + iv(x) = (u^+(x) - u^-(x)) + i(v^+(x) - v^-(x))$$

where the functions u and v are real and integrable, and the functions $u^\pm$ and $v^\pm$ are integrable and non-negative.

Let

$$E = \{x \in \Omega \mid u(x) \geq 0\}$$

Then

$$\Re\left(\int_E f\right) = \int_E u^+ = 0$$

By the above theorem, it follows that $u^+ = 0$ except on a null set. Similarly, it follows that $u^- = 0$ except on a null set. Since the union of two null sets is also a null set, we have shown that $u = 0$ almost everywhere.

A similar argument tells us that $v = 0$ almost everywhere. We conclude that $f = 0$ almost everywhere. □

9 The Riesz Representation Theorem

Before we are ready to state the Riesz representation theorem, we need some terminology from point-set topology.

Definition 9.1 Let X be a topological space. Then we define the *support* of a continuous function $f\colon X \to \mathbb{C}$ to be the closure

$$Supp(f) := \overline{\{x \in X \mid f(x) \neq 0\}}$$

We write $C_c(X)$ to denote the set of all continuous compactly supported functions $f\colon X \to \mathbb{C}$. The set $C_c(X)$ is a vector space under the operations of pointwise addition and scalar multiplication.

Definition 9.2 A linear map $\Lambda\colon C_c(X) \to \mathbb{C}$ is said to be a *positive functional* if $\Lambda(f) \geq 0$ whenever $f \geq 0$.

Let X be a topological space equipped with a Borel measure μ such that $\mu(K) < \infty$ whenever $K \subseteq X$ is a compact subspace. Then the integration map

$$f \mapsto \int_X f$$

defines a positive linear functional.

The Riesz representation theorem is essentially a converse of the above observation.

Theorem 9.3 *Let X be a locally compact Hausdorff space, and let $\Lambda\colon C_c(X) \to \mathbb{C}$ be a positive linear functional.*

Then the set X has a σ-algebra Ω containing all Borel sets, and a unique measure μ on Ω such that

$$\Lambda(f) = \int_X f \, d\mu$$

whenever $f \in C_c(X)$.

The proof of this theorem is in a series of lemmas; the proof is quite long. Before we begin the proof, let us note a theorem from general topology which we shall need.

Theorem 9.4 *Let X be a locally compact Hausdorff space, and let $\mathcal{U} = \{U_\alpha \mid \alpha \in A\}$ be an open cover of the space X. Then there is a* partition of unity *subordinate to the cover $\mathcal{U}$, that is to say a set of continuous functions $u_\alpha\colon X \to [0,1]$ such that $\mathrm{Supp}\, u_\alpha \subseteq U_\alpha$ and*

$$\sum_{\alpha \in A} u_\alpha(x) = 1$$

whenever $x \in X$. □

The following corollary is known as *Urysohn's lemma.*

Corollary 9.5 *Let X be a locally compact Hausdorff space, and let $K \subseteq X$ be a compact set, and let U be an open set containing K. Then there is a continuous function $f\colon X \to [0,1]$ such that*

$$\chi_K(x) \leq f(x) \leq \chi_U(x)$$

Proof: The collection $\{U, X \backslash K\}$ is an open cover of the space X. There is therefore a partition of unity $\{f, g\}$ subordinate to this open cover.

The definition of a partition of unity gives us the required inequality for the function f. □

We now begin our proof of the Riesz representation theorem with the definition of the measure we are looking for.

Definition 9.6 Let $\Lambda\colon C_c(X) \to \mathbb{C}$ be a positive linear functional. Let $U \subseteq X$ be open. Then we define

$$\mu(U) := \sup\{\Lambda f \mid f \leq \chi_U\}$$

In general, for a subset, $E \subset X$, we define

$$\mu(E) = \inf\{\{\mu(U) \mid U \text{ open }, E \subseteq U\}$$

Proposition 9.7 *Let $f, g \in C_c(X)$, and let $f \leq g$. Then $\Lambda f \leq \Lambda g$.*

Proof: Observe $g - f \geq 0$. The result follows from positivity and linearity of the function Λ. □

Corollary 9.8 *Let A and B be subsets of the space X where $A \subseteq B$. Then $\mu(A) \leq \mu(B)$.* □

Although we have defined a function μ for every subset of E, the definition is only sensible for a certain σ-algebra.

Definition 9.9 We define Ω_F to be the sollection of all subsets $E \subseteq X$ such that $\mu(E) < \infty$ and

$$\mu(E) = \sup\{\mu(K) \mid K \subseteq E, K \text{ compact}\}$$

We define Ω to be the collection of all subsets $E \subseteq X$ such that $E \cap K \in \Omega_F$ whenever K is compact.

We need to prove that the set Ω is a σ-algebra which contains all Borel sets; this statement is not obvious.

Proposition 9.10 *Let $V \subseteq X$ be an open subset such that $\mu(V) < \infty$. Then $V \in \Omega_F$.*

Proof: Choose a number $a < \mu(V)$. By the definition of μ, there is a function $f \in C_c(X)$ such that $f \leq \chi_V$ and $a < \Lambda f$. Write $K = Supp(f)$, and let W be an open set that contains K. Then $\Lambda f \leq \mu(W)$, so $\Lambda f \leq \mu(K)$, using the above proposition and corollary, and the definition of the function μ.

Thus $K \subseteq V$ and $\mu(K) > a$. It follows that

$$\mu(V) = \sup\{\mu(K) \mid K \subseteq E, K \text{ compact}\}$$

and we are done. □

Proposition 9.11 *Let $U_1, \ldots, U_N \subseteq X$ be open sets. Then $\mu(U_1 \cup \cdots \cup U_N) \leq \mu(U_1) + \cdots + \mu(U_N)$.*

Proof: Let $N = 2$. Choose a function $g \in C_c(X)$ such that $g \leq \chi_{U_1 \cup U_2}$. By theorem 9.4 there are functions $u_1, u_2 \in C_c(X)$ such that $u_1 \leq \chi_{U_1}$, $u_2 \leq \chi_{u_2}$, and $u_1(x) + u_2(x) = 1$ whenever $x \in U_1 \cup U_2$. It follows that

$$u_1 g \leq \chi_{U_1},\ u_2 g \leq \chi_{U_2} \qquad g = u_1 g + u_2 g$$

and therefore

$$\Lambda g = \Lambda(u_1 g) + \Lambda(u_2 g) \leq \mu(U_1) + \mu(U_2)$$

Since the above inequality holds for every function $g \in C_c(X)$ such that $g \leq \chi_{U_1 \cup U_2}$, the result follows from the definition of μ when $N = 2$. The general result follows by induction. □

Lemma 9.12 *Let $E_1, E_2, E_3, \ldots$ be subsets of the space X. Write*

$$E = \bigcup_{n=1}^{\infty} E_n$$

Then

$$\mu(E) \leq \sum_{n=1}^{\infty} \mu(E_n)$$

Proof: If $\mu(E_n) = \infty$ for some n, then the result is obviously true. Thus, let us suppose that $\mu(E_n) < \infty$ for all n. Choose $\varepsilon > 0$. By definition of the function μ, there are open sets $U_n \supseteq V_n$ such that

$$\mu(V_n) < \mu(E_n) + 2^{-n}\varepsilon$$

for all n.

Let $U = \bigcup_{n=1}^{\infty} U$, and choose $f \in C_c(X)$ such that $f \leq \chi_U$. The support of the function f is covered by the collection of sets $\{U_n \mid n = 1, 2, 3, \ldots\}$. Since the function f has compact support, it follows that it has a finite subcovering, and so

$$f \leq \chi_{U_1 \cup \cdots \cup U_N}$$

for some N. By the above proposition, we see that

$$\Lambda f \leq \mu(U_1 \cup \cdots \cup U_N) \leq \mu(V_1) + \cdots + \mu(V_N) \leq \sum_{n=1}^{\infty} \mu(E_n) + \varepsilon$$

Since the above inueqality holds for every funtion $f \subseteq \chi_U$, and $E \subseteq U$, we see that

$$\mu(E) \leq \sum_{n=1}^{\infty} \mu(E_n) + \varepsilon$$

But this inequality holds whenever $\varepsilon > 0$, so the result follows. □

Proposition 9.13 *Let $K \subseteq X$ be compact. Then $\mu(K) \leq \Lambda f$ whenever $f \geq \chi_K$, and $K \in \Omega_F$.*

Proof: Let $0 < a < 1$, and choose $f \in C_c(X)$ such that $f \geq \chi_K$. Write

$$V_a = \{x \in X \mid f(x) > a\}$$

Then $K \subseteq V_a$, and $ag \leq f$ whenever $f \leq \chi_{V_a}$. Therefore

$$\mu(K) \leq \mu(V_a) = \sup\{\Lambda g \mid g \leq \chi_{V_a}\} \leq a^{-1}\Lambda f$$

Since this inequaltity holds whenever $0 < a < 1$, it follows that $\mu(K) \leq \Lambda f$. It follows that $\mu(K) < \infty$, and so $K \in \Omega_F$. □

Lemma 9.14 *Let $K \subseteq X$ be compact. Then*

$$\mu(K) = \inf\{\Lambda f \mid \chi_K \leq f\}$$

Proof: Let $\varepsilon > 0$. Then there is an open set $U \supseteq K$ such that $\mu(U) < \mu(K)+\varepsilon$. By Urysohn's lemma there is a continuous function $f\colon [0,1] \to X$ such that $\chi_K \leq f \leq \chi_U$. It follows that

$$\Lambda f \leq \mu(U) < \mu(K) + \varepsilon$$

The result follows from the above inequality combined with the previous proposition. □

Proposition 9.15 *Let $K_1, \dots K_N$ be disjoint compact sets. Then*

$$\mu(K_1 \cup \cdots \cup K_N) \leq \mu(K_1) + \cdots + \mu(K_N)$$

Proof: Let $N = 2$. We can find an open set U such that $U \supseteq K_1$ and $U \cap K_2 = \emptyset$. It follows by Urysohn's lemma that we can find a compactly supported function $u\colon X \to [0,1]$ such that $u(x) = 1$ whenever $x \in K_1$, and $u(x) = 0$ whenever $x \in K_2$.

Let $\varepsilon > 0$. By lemma 9.14 there is a function $g \in C_c(X)$ such that

$$\chi_{K_1 \cup K_2} \leq g \qquad \Lambda g \leq \mu(K_1 + K_2) + \varepsilon$$

Observe that

$$\chi_{K_1} \leq fg \qquad \chi_{K_2} \leq (1-f)g$$

Hence

$$\mu(K_1) + \mu(K_2) \leq \Lambda(fg) + \Lambda(g - fg) \leq \mu(K_1 \cup K_2) + \varepsilon$$

Since the above inequality holds whenever $\varepsilon > 0$, the desired result follows when $N = 2$. The general result follows by induction. □

Lemma 9.16 *Let $E_1, E_2, E_3, \dots$ be pairwise disjoint members of the collection Ω_F. Write*

$$E = \bigcup_{n=1}^{\infty} E_n$$

Then

$$\mu(E) = \sum_{n=1}^{\infty} \mu(E_n)$$

Further, if $\mu(E) < \infty$, then $E \in \Omega_F$.

Proof: Observe that the result follows from lemma 9.12 when $\mu(E) = \infty$. Let us therefore assume that $\mu(E) < \infty$. Choose $\varepsilon > 0$. Since $E_n \in \Omega_F$, we can find a compact set $K_n \subseteq E_n$ such that

$$\mu(K_n) > \mu(E_n) - 2^{-n}\varepsilon$$

for each n. Let $H_N = K_1 \cup \cdots \cup K_N$. Then by the above propositiion:

$$\mu E) \geq \mu(H_N) = \sum_{n=1}^{N} \mu(K_n) > \sum_{n=1}^{N} \mu(E_n) - \varepsilon$$

Since the above inequality holds whenever $\varepsilon > 0$, combining it with the inequality in lemma 9.12, we see that

$$\mu(E) = \sum_{n=1}^{\infty} \mu(E_n)$$

Now, if $\mu(E) < \infty$, and $\varepsilon > 0$, then we can find N such that

$$\mu(E) \leq \sum_{n=1}^{N} \mu(E_n) + \varepsilon$$

It follows that $\mu(E) \leq \mu(H_N) + 2\varepsilon$, and so $E \in \Omega_F$. □

Proposition 9.17 *Let $E \subseteq \Omega_F$, and let $\varepsilon > 0$. Then there is a compact set K and an open set V such that $K \subseteq E \subseteq V$, and $\mu(V \backslash K) < \varepsilon$.*

Proof: by definition of the collection Ω_F, we can find a compact set $K \subseteq E$ and an open set $U \supseteq E$ such that

$$\mu(V) - \frac{\varepsilon}{2} < \mu(E) < \mu(K) + \frac{\varepsilon}{2}$$

By lemma 9.10, we see that $V \backslash K \in \Omega_F$. By lemma 9.16, we see

$$\mu(K) + \mu(U \backslash K) = \mu(U) < \mu(K) + \varepsilon$$

and we are done. □

Proposition 9.18 *Let $A, B \in \Omega_F$. Then the sets $A \backslash B$, $A \cup B$, and $A \cap B$ belong to the collection Ω_F.*

Proof: By the above proposition, there are compact sets K and K', and open sets U and U' such that

$$K \subseteq A \subseteq U \qquad K' \subseteq B \subseteq U'$$

and

$$\mu(U \backslash K) < \varepsilon \qquad \mu(U' \backslash K') < \varepsilon$$

Observe

$$A \backslash B \subseteq U \backslash K' \subseteq U \backslash K \cup K \backslash U' \cup U' \backslash K'$$

Hence by lemme 9.12:

$$\mu(A \backslash B) \subseteq \mu(K \backslash V') + 2\varepsilon$$

Further, the set $K\backslash V'$ is compact, so the above inequality tells us that $A\backslash B \in \Omega_F$.

But $A \cup B = (A\backslash B) \cup B$, so $A \cup B \in \Omega_F$ by lemma 9.16. Finally, $A \cap B = A\backslash(A\backslash B)$, so $A \cap B \in \Omega_F$ by the above calculation. □

We are now nearly done, and can prove a slightly less technical result.

Theorem 9.19 *The set Ω is a σ-algebra containing all Borel sets.*

Proof: Let $K \subseteq X$ be compact. If $A \in \Omega$, then $X\backslash A \cap K = K\backslash(A \cap K)$, so $X\backslash A \cap K \in \Omega_F$ by the above proposition, and $X\backslash A \in \Omega$.

Suppose that

$$A = \bigcup_{n=1}^{\infty} A_n \qquad A_n \in \Omega$$

Let $B_1 = A_1 \cap K$, and

$$B_n = (A_n \cap K)\backslash(B_1 \cup \cdots \cup B_n) \qquad n \geq 2$$

Then the collection $\{B_n \mid n = 1, 2, \ldots\}$ is a pairwise disjoint, and $B_n \in \Omega_F$ for all n by the above lemma. But $A \cap K = \bigcup_{n=1}^{\infty} B_n$, so $A \cap K \in \Omega_F$ by lemma 9.16. It follows that $A \in \Omega$.

We have proved that the collection Ω is a σ-algebra. If $C \subseteq X$ is a closed subset, then the intersection $K \cap C$ is compact. Thus $C \cap K \in \Omega_F$, and so $C \in \Omega$. Thus every closed set belongs to the collection Ω. It follows that the σ-algebra Ω contains all Borel sets. □

Lemma 9.20

$$\Omega_F = \{E \in \Omega \mid \mu(E) < \infty\}$$

Proof: Let $E \in \Omega_F$. Then by lemmas 9.14 and 9.16, we see $E \cap K \in \Omega_F$ whenever $K \subseteq X$ is compact. Then $E \in \Omega$. By definition of the set Ω_F, $\mu(E) < \infty$.

Conversely, suppose that $E \in \Omega$ and $\mu(E) < \infty$. Let $\varepsilon > 0$. We can certainly find an open set $U \supseteq E$ such that $\mu(E) < \infty$. By propositions 9.10 and 9.17, there is a compact set $K \subseteq U$ such that $\mu(U\backslash K) < \varepsilon$.

We know that $E \cap K \in \Omega_F$. There is therefore a compact set $H \subseteq E \cap K$ such that

$$\mu(E \cap K) < \mu(H) + \varepsilon$$

But $E \subseteq (E \cap K) \cup (U\backslash K)$. Therefore

$$\mu(E) \subseteq \mu(E \cap K) + \mu(V\backslash K) < \mu(H) + \varepsilon$$

and we see that $E \in \Omega_F$. □

We can now prove our main result.

Theorem 9.21 *The function μ is a measure on the σ-algebra Ω. It is the unique measure with the property*

$$\Lambda f = \int_X f(x)\ d\mu(x)$$

for all $f \in C_c(X)$*.*

Proof: It follows immediately that μ is a measure from lemmas 9.16 and 9.20. Our next step is to prove the inequality

$$\Lambda f \leq \int_X f(x)\ d\mu(x)$$

for every real-valued compactly supported function f. To do this, let $K = Supp(f)$, and choose $a, b \in \mathbb{R}$ such that $f[K] \subseteq [a, b]$. Let $\varepsilon > 0$, and choose $y_0, \ldots, y_N$ such that

$$a = y_0 < \cdots < y_N \qquad y_n - y_{n-1} < \varepsilon \text{ for all } n$$

We can form Borel sets

$$E_n := \{x \in X \mid y_{n-1} < f(x) \leq y_n\}$$

The sets E_n are pairwise disjoint with union K. We can find open sets $U_n \supseteq E_n$ such that

$$\mu(U_k) < \mu(E_k) + \frac{\varepsilon}{n} \qquad f(x) < y_n + \varepsilon$$

whenever $x \in U_n$.

By theorem 9.4, we can choose a partition of unity $\{u_1, \ldots, u_N\}$ subordinate to the open cover $\{U_1, \ldots, U_N\}$. It follows that

$$f = \sum_{n=1}^{N} u_n f$$

and by lemma 9.14

$$\mu(K) \leq \Lambda\left(\sum_{n=1}^{N} u_n\right) = \sum_{n=1}^{N} \Lambda(u_n)$$

But by construction $u_n f \leq (y + n + \varepsilon)u_n$, and $y_n - \varepsilon < f(x)$ for all $x \in E_n$, so

$$\Lambda f \leq \sum_{n=1}^{N} (y_k + \varepsilon)\Lambda(u_n) = \sum_{n=1}^{N} (|a| + y_k + \varepsilon)\Lambda(u_n) - |a| \sum_{n=1}^{N} \Lambda(u_n)$$

and

$$\Lambda f \leq \sum_{n=1}^{N} (|a| + y_k + \varepsilon)(\mu(E_n) + \varepsilon/n) - |a|\mu(K)$$

Multiplying out, we see that

$$\Lambda f \leq \sum_{n=1}^{N} (y_n - \varepsilon)\mu(E_n) + 2\varepsilon\mu(K)\frac{\varepsilon}{n} \sum_{n=1}^{N} (|a| + y_n + \varepsilon)$$

so by construction of the integral

$$\Lambda f \leq \int_X f\ d\mu + \varepsilon(2\mu(K) + |a| + b + \varepsilon)$$

Since the above inequality must hold for every choice of $\varepsilon > 0$, we see that

$$\Lambda f \leq \int_X f(x)\ d\mu(x)$$

as required.

Now, if we replace the function f by the function $-f$, we see that

$$-\Lambda f \leq -\int_X f(x)\ d\mu(x)$$

Combining the above two inequalities, we have the equation

$$\Lambda f = \int_X f(x)\ d\mu(x)$$

for every real-valued compactly supported function f. The proof of the above equation for complex-valued functions follows by splitting such a function into real and imaginary parts, and using linearity.

All that remains is to show uniqueness. Let μ' be a measure such that the eqation

$$\Lambda f = \int_X f(x)\ d\mu'(x)$$

holds for every compactly supported function f. Let K be a compact set. By theorem 9.4, given an open set $U \supseteq K$, there is a compactly supported function g such that $\chi_K \leq g \leq \chi_U$. Hence

$$\mu'(K) \leq \int_X f d\mu' \leq \mu'(U)$$

and

$$\mu'(U) = \sup\{\Lambda f \mid f \leq \chi_U\} = \mu(U)$$

It follows that $\mu(B) = \mu'(B)$ whenever B is a Borel set, and we are done. □

10 Integration of Continuous Functions

We would like to use the Riesz representation theorem to define a measure on the real line $\mathbb{R}$ that gives the usual integral expected from elementary calculus. To apply the Reisz representation theorem, we need a sensible definition of the integral of a continuous compactly supported function.

Let us consider a continuous function $f\colon [a, b] \to \mathbb{R}$. Let n be a positive integer. Then the interval $[a, b]$ can be divided into S^n equal-sized pieces:

$$a < a + 2^{-n}(b-a) < a + 2(2^{-n})(b-a) < \cdots < a + (2^n - 1)(2^{-n})(b-a) < b$$

Let us define

$$\mu_{n,r} = \inf\{f(x) \mid a + r2^{-n}(b-a) \leq f(x) < a + (r+1)2^{-n}(b-a)$$

and

$$I_n(f) = \sum_{r=0}^{2^n-1} 2^{-n}(b-a)\mu_{n,r}$$

The following observations are clear.

- The sequence $(I_n(f))$ is monotonically increasing
- Since the interval $[a, b]$ is compact, and the function f is continuous, there is a constant C such that $f(x) \leq C$ for all $x \in [a, b]$. Hence $I_n(f) \leq C(b - a)$ for all n.

It follows that we have a well-defined limit

$$\Lambda(f) := \lim_{n \to \infty} I_n(f)$$

We would like to extend the definition of the function Λ. There are two stages to this extension.

- Let $f\colon [a, b] \to \mathbb{C}$ be a continuous function. Write $f(x) = u(x) + iv(x)$, where $u, v\colon [a, b] \to \mathbb{R}$, and define

$$\Lambda(f) = \Lambda(u) + i\Lambda(v)$$

- Let $f \in C_c(\mathbb{R})$. Let $[a, b] \supseteq Supp(f)$. Then we define

$$\Lambda(f) = \Lambda(f|_{[a,b]})$$

The following result is straightforward to check.

Proposition 10.1 *The map Λ is a positive linear functional on the space $C_c(\mathbb{R})$.* □

Definition 10.2 Let $f \in C_c(\mathbb{R})$. Then the number $\Lambda(f)$ is called the *Riemann integral* of f.

11 The Lebesgue Measure on $\mathbb{R}$

Definition 11.1 Let $\Lambda\colon C_c(\mathbb{R}) \to \mathbb{C}$ be the Riemann integral. Then the *Lebesgue measure* on $\mathbb{R}$ is the unique measure such that

$$\int_{\mathbb{R}} f \; d\mu = \Lambda(f)$$

whenever $f \in C_c(\mathbb{R})$.

By the Riesz representation, the Lebesgue measure exists and is unique on the collection of all Borel sets. The integral of a Borel measurable function with respect to the Lebesgue measure is termed the *Lebesgue integral*. We will normally write

$$\int_a^b f \; d\mu := \int_{\mathbb{R}} f\chi_{(a,b)} \; d\mu$$

Proposition 11.2 *Let $a < b$ be real numbers. Then $\mu(a, b) = b - a$.*

Proof: Let $[c,d] \subseteq (a,b)$ be a compact interval. By Urysohn's lemma, there is a function $f \in C_c(\mathbb{R})$ such that $\chi_{[c,d]} \leq f \leq \chi_{(a,b)}$.

By definition of the Riemann integral:

$$d - c \leq \int_{\mathbb{R}} f \leq b - a$$

Let $c \to a$ and $d \to b$. Then $f \to \chi_{(a,b)}$ and by the dominated convergence theroem,

$$\int_{\mathbb{R}} f \to \mu(a,b)$$

It follows that $\mu(a,b) = b - a$, and we are done. □

A similar computation tells us that

$$\mu[a,b] = \mu[a,b) = \mu(a,b] = b - a$$

whenever $a < b$.

The next fundamental property of the Lebesgue measure follows from a topological property of the real line, which we will state without proof.

Proposition 11.3 *Every open subset of the real line $\mathbb{R}$ is a countable disjoint union of open intervals.* □

Corollary 11.4 *Let $E \subseteq \mathbb{R}$ be a Borel set. Then $\mu(X + E) = \mu(E)$ whenever $x \in \mathbb{R}$.* □

We conclude with a general characterisation of sets of measure zero, or *null sets.*

Theorem 11.5 *Let $E \subseteq \mathbb{R}$ be a set such that every subset of A is measurable. Then $\mu(A) = 0$.*

Proof: The set $\mathbb{R}$ is an Abelian group under the operation of addition, and the set $\mathbb{Q}$ is a subgroup. Let E be a set of real numbers containing precisely one element of each coset $x + \mathbb{Q} \in \mathbb{R}/\mathbb{Q}$.

We claim:

- $(r + E) \cap (s + E) = \emptyset$ whenever $r, s \in \mathbb{Q}$, $r \neq s$.
- Let $x \in \mathbb{R}$. Then we can find an element $r \in \mathbb{Q}$ such that $x \in r + E$.

To see the first claim, suppose that $x \in (r + E) \cap (s + E)$, where $r, s \in \mathbb{Q}$. Then there are elements $y, z \in E$ such that $r + y = s + z$, and so $y - z \in \mathbb{Q}$. But the definition of the set E means that $r = s$.

As for the second claim, let $x \in \mathbb{R}$. Construction of the set E means that we can find a point $y \in E$ such that $x - y \in \mathbb{Q}$. But $x = y + (x - y)$ so the claim is established.

We now use the above to claims to prove the theorem. Let $t \in \mathbb{Q}$, and define $A_t := A \cap (t + E)$. The set A_t is measurable since it is a subset of the set A. Consider a compact subset $K \subseteq A_t$, and let

$$H = \bigcup_{r \in \mathbb{Q} \cap [0,1]} (r + K)$$

Then the set H is bounded and measurable, so $\mu(H) < \infty$. The first of the above claims tells us that the sets $r + K$ are pair-wise disjoint, so

$$\mu(H) = \sum_{r \in \mathbb{Q} \cap [0,1]} \mu(r + K) = \sum_{r \in \mathbb{Q} \cap [0,1]} \mu(K)$$

by corollary 11.4. It follows that $\mu(K) = 0$ whenever $K \subseteq A_t$ is compact.

So $\mu(A_t) = 0$. But

$$A = \bigcup_{t \in \mathbb{Q}} A_t$$

and it follows that $\mu(A) = 0$. □

Corollary 11.6 *Any countable subset of the space* $\mathbb{R}$ *has measure zero.* □

Corollary 11.7 *There are non-measurable subsets of the space* $\mathbb{R}$. □

12 The Fundamental Theorem of Calculus

By convention, when $a < b$ are real numbers, and μ is the Lebesgue measure on the space $\mathbb{R}$, we simplfy our notation slightly and write just

$$\int_a^b f(x)\ dx := \int_a^b f\ d\mu$$

If $b < a$, we write

$$\int_a^b f(x)\ dx := -\int_b^a f(x)\ dx$$

Linearity of the integral gives us the equation

$$\int_a^b f(x)\ dx = \int_a^c f(x)\ dx + \int_c^b f(x)\ dx$$

whenever $a, b, c \in \mathbb{R}$.

This new notation is convenient when integating a concrete function given by some definite formula.

In this section we will focus on one major result, which is of absolutely vital importance when trying to calculate integrals. This result is termed the em fundamental theorem of calculus.

Theorem 12.1 *Let* $f\colon [a,b] \to \mathbb{C}$ *be a continuous function. Define a function* $F\colon [a,b] \to \mathbb{C}$ *by the formula*

$$F(x) = \int_a^x f(y)\ dy$$

Then the function F *is differentiable on the open interval* (a,b)*, and has a one-sided derivative at the end-points* a *and* b*. In all cases, the derivative is given by the formula*

$$F'(x) = f(x)$$

Proof: Let $\varepsilon > 0$, and let $x \in [a, b]$. Since the function f is continuous, we can choose $\delta > 0$ such that $|f(x+h) - f(x)| < \varepsilon$ whenever $|h| < \delta$ and $x + h \in [a, b]$.

Let $x \in [a, b]$, and $x + h \in [a, b]$. Observe:

$$F(x + h) - F(x) = \int_x^{h+h} f(y)\ dy$$

and

$$hf(x) = f(x)\mu(x, x + h) = \int_x^{x+h} f(x)\ dy$$

Suppose that $|h| < \delta$. Then $|f(y) - f(x)| < \varepsilon$ whenever $y \in [x, x + h]$, and so:

$$\left| \int_x^{x+h} f(y) - f(x)\ dy \right| \leq \int_x^{x+h} |f(y) - f(x)|\ dy \leq \varepsilon |h|$$

Thus:

$$|F(x + h) - F(x) - hf(x)| \leq \varepsilon |h|$$

whenver $|h| < \delta$. It follows that the function F is differentiable, and $F'(x) = f(x)$ as claimed. □

In actual fact, the more useful form of the fundmantal theorem of calculus is a variation of the above formula.

Corollary 12.2 *Let $F\colon [a, b] \to \mathbb{C}$ be a function with a continuous derivative f. Then*

$$\int_a^b f(x)\ dx = F(b) - F(a)$$

Proof: Define

$$F_0(x) = \int_a^x f(y)\ dy$$

Then by the above version of the fundamental theorem of calculus, $F_0'(x) = f(x)$ whenever $x \in [a, b]$. Hence $F_0'(x) = F'(x)$ whenever $x \in [a, b]$, so there is a constant C such that $F_0(x) = F(x) + C$ for all $x \in [a, b]$.

We know that $F_0(a) = 0$. Therefore $C = -F(a)$. We see that

$$int_a^b f(x)\ dx = F_0(b) = F(b) - F(a)$$

as claimed. □

The various integration formulae, such as integration by parts and the change of variable formula, come from the fundamental theorem of calculus along with the corresponding formulae for differentives, such as the derivative of a product and the derivative of a composition.

13 Product Measures

Let Ω_1 and Ω_2 be measure spaces, with measures μ_1 and μ_2 on σ-algebras $\mathcal{M}_1$ and $\mathcal{M}_2$ respectively.

Definition 13.1 We call a subset of the form $A \times B \subseteq X \times Y$, where $A \in \mathcal{M}_1$ and $B \in \mathcal{M}_2$ a *measurable rectangle*. A finite union of measurable rectangles is called an *elementary set*.

We write $\mathcal{M}_{12}$ to denote the smallest σ-algebra in the set $\Omega_1 \times \Omega_2$ that contains every measurable rectangle.

We want to define a measure on the σ-algebra $\mathcal{M}_{12}$. Before we can do this, we need some technical constructions.

Definition 13.2 Let $\mathcal{C}$ be a collection of subsets of some set. Suppose that the following two conditions hold:

- Let (A_n) be a sequence of sets in the collection $\mathcal{C}$ such that $A_n \subseteq A_{n+1}$ for all n. Then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{C}$.
- Let (B_n) be a sequence of sets in the collection $\mathcal{C}$ such that $B_n \supseteq B_{n+1}$ for all n. Then $\bigcup_{n=1}^{\infty} B_n \in \mathcal{C}$.

Then we call the collection $\mathcal{C}$ a *monotone class*.

The proof of the following lemma is elementary, but rather abstract. We omit it.

Lemma 13.3 *The σ-algebra $\mathcal{M}_{12}$ is the smallest monotone class in the product $\Omega_1 \times \Omega_2$ which contains all elementary sets.* □

Given a subset $E \subseteq \Omega_1 \times \Omega_2$, and points $x \in Omega_1$ and $y \in \Omega_2$, let us write

$$E_x = \{y \in \Omega_2 \mid (x, y) \in E\} \qquad E^y = \{x \in \Omega_1 \mid (x, y) \in E\}$$

Proposition 13.4 *Let $E \in \mathcal{M}_{12}$. Then $E_x\ in \mathcal{M}_1$ and $E^y \in \mathcal{M}_2$ whenever $x \in \Omega_1$ and $y \in \Omega_2$.*

Proof: Let $x \in \Omega_1$. Let $\mathcal{M}$ be the collection of all elements $E \in \Omega_1 \times \Omega_2$ such that $E_x \in \Omega_2$. It is straightforward to check that $\mathcal{M}$ is a σ-algebra that contains every measurable rectangle. Therefore $\mathcal{M}_{12} \subseteq \mathcal{M}$, and we see that $E_x \in \mathcal{M}_2$ for every measurable set $E \subseteq \Omega_1 \times \Omega_2$ and point $x \in \Omega_2$.

The corresponding statement concerning sets of the form E^y is proved in the same way. □

Corollary 13.5 *Let X be a topological space, and let $f\colon \Omega_1 \times \Omega_2 \to X$ be a measurable function. Choose points $x \in \Omega_1$ and $y \in \Omega_2$. Then the functions*

$$f(x, -)\colon \Omega_2 \to X \qquad f(-, y)\colon \Omega_1 \to X$$

are measurable. □

Definition 13.6 A measure space Ω is called *σ-finite* if it is a countable union of spaces of finite measure.

Example 13.7 The space $\mathbb{R}$, equipped with the standard Lebesgue measure, is σ-finite.

The following result lets us define measures on products of σ-finite measure spaces.

Theorem 13.8 *Let Ω_1 and Ω_2 be σ-finite measure spaces. Let $E \subseteq \Omega_1 \times \Omega_2$ be a measurable subset. Then we can define measurable functions $f: \Omega_1 \to [0, \infty]$ and $g: \Omega_1 \to [0, \infty]$ by the formulae*

$$f_E(x) = \mu_2(E_x) \qquad g_E(y) = \mu_1(E^y)$$

respectively. Further,

$$\int_{\Omega_1} f_E = \int_{\Omega_2} g_E$$

Proof: Measurability of the functions f_E and g_E associated as above to a measurable set $E \subseteq X \times Y$ follows from the above proposition and corollary; all that remains it to prove the main equation.

Let $\mathcal{M}$ be the set of all measurable subsets $E \subseteq \Omega_1 \times \Omega_2$ such that the equation

$$\int_{\Omega_1} f_E = \int_{\Omega_2} g_E$$

holds.

Let $E = A \times B$ be a measurable rectangle. Then $f_E = \mu_2(B)\chi_A$ and $g_E = \mu_1(A)\chi_B$. It follows that

$$\int_{\Omega_1} f_E = \int_A \mu_2(B) = \mu_1(A)\mu_2(B) \qquad \int_{\Omega_1} g_E = \int_B \mu_1(A) = \mu_1(A)\mu_2(B)$$

so $E \in \mathcal{M}$.

Let (E_n) be a sequence of sets in the collection $\mathcal{M}$ such that $E_n \subseteq E_{n+1}$ for all n. Write

$$E = \bigcup_{n=1}^{\infty} E_n$$

Then the sequences of functions (f_{E_n}) and (g_{E_n}) are monotonic increasing, with limits f_E and g_E respectively. We know that $E_n \in \mathcal{M}$ for all n, so that the equation

$$\int_{\Omega_1} f_{E_n} = \int_{\Omega_2} g_{E_n}$$

holds for all n. The monotone convergence theorem gives us the equation

$$\int_{\Omega_1} f_E = \int_{\Omega_2} g_E$$

and so tells us that $E \in \mathcal{M}$.

As a consequence of the above calculation, we can easily show that the union of a discrete sequence of measurable sets in the set $\mathcal{M}$ also belongs to the set $\mathcal{M}$. Let (E_n) be a sequence of sets in the collection $\mathcal{M}$ such that $E_1 \subseteq A \times B$, where $\mu_1(A) < \infty$, $\mu_2(B) < \infty$, and $E_n \supseteq E_{n+1}$ for all n. Write

$$E = \bigcup_{n=1}^{\infty} E_n$$

Then an argument similar to the above one, only using the dominated convergence theorem rather than the monotone convergence theorem, tells us that the set E belongs to the collection $\mathcal{M}$.

Now, let $\Omega_1 = \cup_{n=1}^{\infty}\Omega_1^{(n)}$ and $\Omega_2 = \cup_{n=1}^{\infty}\Omega_2^{(n)}$, where $\mu_1(\Omega_1^{(m)}) < \infty$ and $\mu_2(\Omega_2^{(n)}) < \infty$ for all $m, n \in \mathbb{N}$. Given a set $E \subseteq \Omega_1 \times \Omega_2$, let us write

$$E_{mn} = E \cap (\Omega_1^{(m)} \times \Omega_2^{(n)}$$

Let $\mathcal{C}$ be the collection of all measurable sets $E \subseteq \Omega_1 \times \Omega_2$ such that $E_{mn} \in \mathcal{M}$ for all natural numbers m and n. Then the above calculations tell us that the collection $\mathcal{C}$ is a monotone class that contains every elementary rectangle. It follows from lemma 13.3 $\mathcal{M}_{12} \subseteq \mathcal{C}$, and we are done. $\square$

To paraphrase the above theorem, the equation

$$\int_{\Omega_1}\left(\int_{\Omega_2}\chi_E(x,y)\ d\mu_2(y)\right)\ d\mu_1(x) = \int_{\Omega_2}\left(\int_{\Omega_1}\chi_E(x,y)\ d\mu_1(x)\right)\ d\mu_2(y)$$

holds for every measurable set $E \subseteq \Omega_1 \times \Omega_2$.

Definition 13.9 Let Ω_1 and Ω_2 be σ-finite measure sets. Then we define a measure μ on the product $\Omega_1 \times \Omega_2$ by writing

$$\mu(E) := \int_{\Omega_1}\left(\int_{\Omega_2}\chi_E(x,y)\ d\mu_2(y)\right)\ d\mu_1(x) = \int_{\Omega_2}\left(\int_{\Omega_1}\chi_E(x,y)\ d\mu_1(x)\right)\ d\mu_2(y)$$

whenever the set $E \subseteq \Omega_1 \times \Omega_2$ is measurable.

It is easy to check that the above definition satisfies the axioms required of a measure. As a special case of the above definition, we can now define a Lebesgue measure on the space $\mathbb{R}^n$ by viewing it as a product of copies of the space $\mathbb{R}$. This measure is defined on every Borel set, and the measure of the n-dimensional cuboid

$$[a_1, b_1] \times \cdots \times [a_n, b_n]$$

is the product

$$(b_1 - a_1)(b_2 - a_2)\cdots(b_n - a_n)$$

14 Fubini's Theorem

In the previous section, we saw how to define measures on products of σ-finite measure spaces. We can therefore integrate on such spaces. The purpose of this section is two state two results on the integrability of such functions, and how they are integrated. These results are usually put together, and referred to in one piece as *Fubini's theorem*.

Theorem 14.1 *Let Ω_1 and Ω_2 be σ-finite measure spaces, and let $f\colon \Omega_1 \times \Omega_2 \to \mathbb{C}$ be an integrable function. Then the functions $f(x,-)$ and $f(-,y)$ are integrable almost everywhere, and the functions*

$$x \mapsto \int_{\Omega_2} f(x,y)d\mu_2(y) \qquad y \mapsto \int_{\Omega_2} f(x,y)d\mu_2(y)$$

are integrable. Moreover,

$$\int_{\Omega_1 \times \Omega_2} f(x,y) d\mu(x,y) = \int_{\Omega_1} \left(\int_{\Omega_2} f(x,y)\ d\mu_2(y) \right)\ d\mu_1(x) = \int_{\Omega_2} \left(\int_{\Omega_1} f(x,y)\ d\mu_1(x) \right)\ d\mu_2(y)$$

Proof: Let $s\colon \Omega_1 \times \Omega_2 \to \mathbb{C}$ be a simple function. Then the functions $s(x,-)$ and $s(-,y)$ are integrable almost everywhere, the functions

$$x \mapsto \int_{\Omega_2} s(x,y) d\mu_2(y) \qquad y \mapsto \int_{\Omega_2} s(x,y) d\mu_2(y)$$

are integrable, and the equation

$$\int_{\Omega_1 \times \Omega_2} s(x,y) d\mu(x,y) = \int_{\Omega_1} \left(\int_{\Omega_2} s(x,y)\ d\mu_2(y) \right)\ d\mu_1(x) = \int_{\Omega_2} \left(\int_{\Omega_1} s(x,y)\ d\mu_1(x) \right)\ d\mu_2(y)$$

holds by theorem 13.8 and the definition of the product measure.

Now, suppose that $f(x,y) \geq 0$ for all points $(x,y) \in \Omega_1 \times \Omega_2$. Since the function f is measurable, by proposition 5.2 there is a monotonically increasing sequence, (s_n), of simple functions, with point-wise limit f. The result therefore follows in this case by the monotone convergence theorem.

By splitting a real-valued function into positive and negative parts, we see that the result holds for all real-valued functions. We can deduce the result for complex-valued functions by splitting such a function into real and imaginary parts. □

For the above theorem to be useful, we would like a criterion for a function $f\colon \Omega_1 \times \Omega_2 \to \mathbb{C}$ to be integrable. Fortunately, such a condition forms the second half of Fubini's theorem, which is also sometimes referred to as Tonelli's theorem.

Theorem 14.2 *Let Ω_1 and Ω_2 be σ-finite measure spaces, and let $f\colon \Omega_1 \times \Omega_2 \to \mathbb{C}$ be an integrable function. Suppose that*

$$\int_{\Omega_1} \left(\int_{\Omega_2} |f(x,y)|\ d\mu_2(y) \right)\ d\mu_1(x) < \infty$$

or

$$\int_{\Omega_2} \left(\int_{\Omega_1} |f(x,y)|\ d\mu_1(x) \right)\ d\mu_2(y) < \infty$$

Then the function $f\colon \Omega_1 \times \Omega_2 \to \mathbb{C}$ is integrable.

Proof: The result is obvious if the function f is simple. A similar argument to the proof of Fubini's theorem gives us the result in general. □

Combining the two theorems in this section (ie: the two halves of Fubini's theorem), we have the following handy result on swapping the order of integration.

Corollary 14.3 *Let Ω_1 and Ω_2 be σ-finite measure spaces, and let $f\colon \Omega_1 \times \Omega_2 \to \mathbb{C}$ be an integrable function. Suppose that*

$$\int_{\Omega_1} \left(\int_{\Omega_2} |f(x,y)|\ d\mu_2(y) \right)\ d\mu_1(x) < \infty$$

Then

$$\int_{\Omega_2} \left(\int_{\Omega_1} f(x,y)\ d\mu_1(x) \right)\ d\mu_2(y) < \infty = \int_{\Omega_1} \left(\int_{\Omega_2} f(x,y)\ d\mu_2(y) \right)\ d\mu_1(x) < \infty$$

□