

LINEAR GENERALISED COMPLEX STRUCTURES

M. HEUER AND M. JOTZ LEAN

ABSTRACT. This paper studies linear generalised complex structures over vector bundles, as a generalised geometry version of holomorphic vector bundles. In an adapted linear splitting, a linear generalised complex structure on a vector bundle $E \rightarrow M$ is equivalent to a \mathbb{C} -multiplication j in the fibers of $TM \oplus E^*$ and \mathbb{C} -Lie algebroid structure on $TM \oplus E^*$.

Generalised complex Lie algebroids (or Glanon algebroids) are then studied in this context, and expressed as a pair of complex conjugated Lie bialgebroids.

CONTENTS

1. Introduction	2
2. Background	7
2.1. Courant algebroids and generalised complex structures	7
2.2. The generalised tangent bundle of a vector bundle and Dorfman connections	8
2.3. VB-Courant algebroids	11
2.4. The generalised tangent bundle of a Lie algebroid	11
3. Holomorphic vector bundles and holomorphic Lie algebroids	12
3.1. Linear almost complex structures via connections	12
3.2. Holomorphic Lie algebroids and infinitesimal ideal systems	17
4. Linear generalised complex structures and Dorfman connections	18
4.1. Adapted Dorfman connections	20
4.2. Integrability	21
4.3. Generalised Kähler structures on vector bundles	23
4.4. Complex VB-Dirac structures	24
5. Generalised complex structures on Lie algebroids	27
5.1. The degenerate generalised complex structure on $A \oplus T^*M$	29
5.2. The complex A -Manin pair	30
5.3. The Lie bialgebroid (U_{\pm}, K_{\mp}) ; proof of Theorem 1.5	31
6. Generalised complex structures in VB-Courant algebroids	32
Appendix A. Relation with the adapted generalised connections in [4]	35
Appendix B. Proof of Lemma 3.1	37

1. INTRODUCTION

This paper studies linear generalised complex structures on vector bundles, and on Lie algebroids. Generalised complex geometry was introduced by Hitchin in [14] as a unification of symplectic and complex geometry. It was further developed by Gualtieri in his thesis [9, 11]. Since they simultaneously unify symplectic and complex structures, generalised complex structures have been studied for their relation to T-duality – a concept arising in string theory – by Cavalcanti and Gualtieri in [3]. Gualtieri also defined generalised Kähler structures in [9, 12]. These have been studied for example in [15] and in [2].

The relation between generalised complex geometry and Lie algebroids and Lie groupoids was first studied by Crainic in [5], and generalised complex structures on Lie groupoids and Lie algebroids were studied in [21]. In particular, [21] proves that multiplicative generalised complex structures on source simply connected Lie groupoids are equivalent to generalised complex integrable Lie algebroids. This paper studies in more detail the obtained generalised complex Lie algebroids (or *Glanon algebroids*), and in particular the underlying special case of generalised complex vector bundles.

The notion of generalised holomorphic bundles has been introduced by Gualtieri [11, 10] as a complex vector bundle E over a generalised complex manifold M , equipped with a flat L -connection $\bar{\partial}: \Gamma(E) \rightarrow \Gamma(L^* \otimes E)$, where L is the $+i$ -eigenbundle of the generalised complex structure on M . This is in analogy to how a holomorphic vector bundle structure on E over a complex manifold M is equivalent to a flat Dolbeault operator $\bar{\partial}_E$. However, while a holomorphic vector bundle $E \rightarrow M$ amounts to a linear complex structure on the smooth manifold E , in Gualtieri’s definition of a generalised holomorphic vector bundle the manifold E itself does not carry a generalised complex structure.

This paper instead adopts the point of view that a “generalised holomorphic vector bundle” should be a smooth vector bundle equipped with a *linear* generalised complex structure, and explores the property of such an object – here, it is now the manifold M which does not automatically inherit a generalised complex structure. Note that symplectic vector bundles, i.e. smooth vector bundles equipped with a linear symplectic form, and Poisson holomorphic vector bundles are natural examples of this notion of generalised holomorphic vector bundle.

The following sections describe holomorphic vector bundles as linear complex structures on vector bundles, and explain how the obtained infinitesimal structures in this setting can be recovered in the more general context of linear generalised complex structures.

Holomorphic vector bundles and linear complex structures. If $q: E \rightarrow M$ is a holomorphic vector bundle over a complex manifold M , then $TE \rightarrow E$ and $TM \rightarrow M$ are also holomorphic vector bundles. The multiplication by i in the fibers of TM , when seen as a real vector bundle, is the complex structure $J_M: TM \rightarrow TM$ – i.e. a vector bundle morphism over the identity that squares to $-\text{id}_{TM}$. In the same manner $J_E: TE \rightarrow TE$ is the complex structure of TE , when seen as a real vector bundle over E . It is easy to see (see Section 3.1)

that $J_E: TE \rightarrow TE$ is a morphism of double vector bundles

$$(1) \quad \begin{array}{ccccc} TE & \xrightarrow{J_E} & TE & & \\ \downarrow & \searrow & \downarrow & \searrow & \\ & E & \xrightarrow{\text{id}_E} & E & \\ \downarrow & \downarrow & \downarrow & \downarrow & \\ TM & \xrightarrow{J_M} & TM & & \\ \downarrow & \searrow & \downarrow & \searrow & \\ & M & \xrightarrow{\text{id}_M} & M & \end{array}$$

with side morphisms $J_M: TM \rightarrow TM$ and $\text{id}_E: E \rightarrow E$ and with core morphism $j_E: E \rightarrow E$, the multiplication by i in the fibers of E .

Theorem 1.1. *A holomorphic structure on a smooth vector bundle $E \rightarrow M$ is equivalent to an integrable linear almost complex structure J_E as in (1).*

Although this result seems evident, so far it does not seem to have been worked out in the literature. The integrability of J_E implies immediately the integrability of J_M . Therefore, if J_E is integrable then (M, J_M) is a complex manifold. The integrability of J_E can then be reduced to the existence of a \mathbb{C} -linear TM -connection ∇ on E , the complexification $D: \Gamma(TM_{\mathbb{C}}) \times \Gamma(E) \rightarrow \Gamma(E)$ of which decomposes as

$$D = D^{0,1} + D^{0,1},$$

with $D^{0,1}$ a flat $T^{0,1}M$ -connection on E . Following classical results, see e.g. [22], $D^{0,1}$ is then the $\bar{\partial}$ -operator of a holomorphic structure on E , such that J_E becomes the multiplication by i in the fibers of TE . For the convenience of the reader, and since this approach to holomorphic vector bundles is on the one hand new, and motivates on the other hand this paper's study of generalised complex vector bundles, the proof of Theorem 1.1 is carried out in detail in Section 3.

Linear generalised complex structures. This paper studies the generalisation of this description of holomorphic vector bundles, to vector bundles endowed with a *linear generalised complex structure*. Since the terminology of *generalised holomorphic vector bundles* is already used in the literature for a different generalisation of holomorphic vector bundles, here vector bundles endowed with a linear generalised complex structure are simply called *generalised complex vector bundles*.

Let $E \rightarrow M$ be a smooth vector bundle. The generalised tangent bundle $\mathbb{T}E = TE \oplus T^*E$ is then a double vector bundle

$$\begin{array}{ccc} TE \oplus T^*E & \longrightarrow & E \\ \downarrow & & \downarrow \\ TM \oplus E^* & \longrightarrow & M \end{array}$$

with core $E \oplus T^*M$. The vector bundle $TE \oplus T^*E \rightarrow E$ is naturally equipped with the standard Courant algebroid structure over the manifold E .

Definition 1.2. *A generalised complex structure \mathcal{J} on a vector bundle $E \rightarrow M$ is called **linear** if $\mathcal{J}: TE \oplus T^*E \rightarrow TE \oplus T^*E$ is a morphism of double vector bundles over a side*

morphism $j: TM \oplus E^* \rightarrow TM \oplus E^*$ and with a core morphism $j_C: E \oplus T^*M \rightarrow E \oplus T^*M$.

$$(2) \quad \begin{array}{ccccc} TE \oplus T^*E & \xrightarrow{\mathcal{J}} & TE \oplus T^*E & & \\ \downarrow & \searrow & \downarrow & \searrow & \\ & E & \xrightarrow{\text{id}_E} & E & \\ \downarrow & \downarrow & \downarrow & \downarrow & \\ TM \oplus E^* & \xrightarrow{j} & TM \oplus E^* & \xrightarrow{\text{id}_M} & M \end{array} .$$

Consider first a linear generalised complex structure on a vector space V (i.e. on a vector bundle over a point). In this case the tangent and cotangent bundle are canonically split, $TV \simeq V \times V$ and $T^*V \simeq V \times V^*$. The linearity condition on the generalised complex structure \mathcal{J} is equivalent to \mathcal{J} being determined by the maps in the fibres, that is by the side morphism $j_{V^*}: V^* \rightarrow V^*$ and the core morphism $j_C: V \rightarrow V$. These have to be in negative duality to each other, that is $j = -j_C^t$, since the generalised complex structure is orthogonal with respect to the canonical pairing. Therefore, a linear generalised complex structure on a vector space in this sense is equivalent to the choice of an ordinary complex structure in the vector space.

Back to the general case, this paper shows that after the choice of an adequate linear splitting of $TE \oplus T^*E$, the generalised complex structure is equivalent to a special complex Lie algebroid structure on $TM \oplus E^*$, as in the following definition – the bundle $TM \oplus E^*$ is seen as a complex vector bundle with $j: TM \oplus E^* \rightarrow TM \oplus E^*$ the multiplication by i .

Definition 1.3. Let $Q \rightarrow M$ be a vector bundle with complex fibers, hence with a vector bundle morphism $j: Q \rightarrow Q$ such that $j^2 = -\text{id}_Q$. A **complex Lie algebroid structure** on (Q, j) is a \mathbb{C} -bilinear Lie algebra bracket $[\cdot, \cdot]$ on sections of Q and a morphism $\lambda: Q \rightarrow T_{\mathbb{C}}M$ of complex vector bundles that anchors the bracket: $[q_1, f q_2] = \lambda(q_1)(f)q_2 + f[q_1, q_2]$ for all $f \in C^\infty(M, \mathbb{C})$ and $q_1, q_2 \in \Gamma(Q)$.

A complex Lie algebroid $(Q \rightarrow M, j, \lambda, [\cdot, \cdot])$ is **quasi-real** if there exists

(1) a real vector bundle morphism $\rho: Q \rightarrow TM$ such that $\lambda = \rho_j: Q \rightarrow T_{\mathbb{C}}M$ defined by

$$(3) \quad \rho_j(q) := \frac{1}{2}(\rho(q) \otimes 1 - \rho(jq) \otimes i)$$

for all $q \in \Gamma(Q)$, and

(2) a dull bracket $[[\cdot, \cdot]]$ on (Q, ρ) such that the complexification $(Q_{\mathbb{C}} \rightarrow M, \rho_{\mathbb{C}}, [[\cdot, \cdot]]_{\mathbb{C}})$ restricts to $(Q^{1,0} \rightarrow M, \rho_{1,0}, [[\cdot, \cdot]]_{1,0})$ on the i -eigenspace of $j_{\mathbb{C}}$, which coincides with the complex Lie algebroid $(Q \rightarrow M, \rho, j, [\cdot, \cdot])$ via the canonical isomorphism

$$Q \rightarrow Q^{1,0}, \quad q \mapsto \frac{1}{2}(q \otimes 1 - j(q) \otimes i).$$

The definition above of a complex algebroid follows the one in [34] but assumes that the complex-linear anchor is induced by a real anchor as in (3). The following theorem is the main result of this paper.

Theorem 1.4. Let $E \rightarrow M$ be a smooth vector bundle. A linear generalised complex structure on E with side $j: TM \oplus E^* \rightarrow TM \oplus E^*$ is equivalent to a quasi-real complex Lie algebroid structure on $(TM \oplus E^*, j)$ with anchor $\text{pr}_{TM, j}: TM \oplus E^* \rightarrow T_{\mathbb{C}}M$, $\nu \mapsto \frac{1}{2}(\text{pr}_{TM} \nu \otimes 1 - \text{pr}_{TM}(j\nu) \otimes i)$.

In other words, a linear generalised complex structure with side j on a vector bundle E is equivalent to a special complex Lie algebroid structure on the complex vector bundle $(TM \oplus E^* \rightarrow M, j)$. The equivalence in Theorem 1.4 is of course the most important part of the statement. It is explained along the introduction of the necessary tools: via this equivalence, a quasi-real Lie algebroid structure on $(TM \oplus E^*, j)$ is sent to a generalised complex structure $\mathcal{J}: TE \oplus T^*E \rightarrow TE \oplus T^*E$ such that any dull bracket on $TM \oplus E^*$ realising the complex Lie bracket on $(TM \oplus E^*, j)$ is *adapted to \mathcal{J}* .

In the case of a holomorphic vector bundle, the complex Lie algebroid found in Theorem 1.4 is simply $T^{1,0}M \oplus (E^{0,1})^* \rightarrow M$, with the bracket defined by the complex Lie algebroid bracket on $T^{1,0}M$ (since M is a complex manifold), and the flat $T^{1,0}M$ -connection on $E^{0,1}$ that is complex conjugated to the $\bar{\partial}$ -operator $\bar{\partial}: \Gamma(T^{0,1}M) \times \Gamma(E^{1,0}) \rightarrow \Gamma(E^{1,0})$. See Example 4.16.

Section 6 extends the results of Section 4 to the more general case of linear generalised complex structures in VB-Courant algebroids. Making use of the correspondence between VB-Courant algebroids and Lie 2-algebroids [24, 18], this leads after the choice of a linear splitting to a definition of generalised complex structures in split Lie 2-algebroids.

Generalised complex Lie algebroids. In Section 5 the vector bundle is equipped with the additional structure of a Lie algebroid, and the linear generalised complex structure is required to be compatible with the Lie algebroid structure. The obtained *generalised complex Lie algebroids* were already studied in [21], where they are called ‘‘Glanon algebroids’’. The paper [21] gives a correspondence between *multiplicative* generalised complex structures on Lie groupoids and compatible generalised complex structures on Lie algebroids. Hence in order to better understand generalised complex Lie groupoids it is useful to study generalised complex Lie algebroids in this sense. The goal of this section is a deeper study of the properties of generalised complex Lie algebroids, in the spirit of the study of holomorphic Lie algebroids done in [23]: that paper studies holomorphic Lie algebroids in detail and shows an equivalence between holomorphic Lie algebroid structures on $A \rightarrow M$ and linear holomorphic Poisson structures on the complex dual $\text{Hom}_{\mathbb{C}}(A, \mathbb{C})$. Moreover, it shows that a holomorphic Lie algebroid structure on A is equivalent to a matched pair [28, 27] of the complex Lie algebroids $T^{0,1}M$ and $A^{1,0}$. Additionally, for a complex manifold M , the Lie algebroids $T^{1,0}M$ and $T^{0,1}M$ form a matched pair of complex Lie algebroids with matched sum $T_{\mathbb{C}}M$ and more generally, for a holomorphic Lie algebroid A the Lie algebroids $A^{1,0}$ and $A^{0,1}$ form a matched pair with matched sum $A_{\mathbb{C}}$.

[21] proves that a Poisson holomorphic Lie algebroid is equivalent to a holomorphic Lie bialgebroid. More generally, Section 5 proves the following theorem:

Theorem 1.5. *Let $A \rightarrow M$ be a Lie algebroid. Let $\mathcal{J}: TA \oplus T^*A \rightarrow TA \oplus T^*A$ be a linear generalised complex structure on A with side $j: TM \oplus A^* \rightarrow TM \oplus A^*$. Then (A, \mathcal{J}) is a Glanon algebroid if and only if the quasi-real complex Lie algebroid structure on $(TM \oplus A^*, j)$ found in Theorem 1.4 fits in a complex Lie bialgebroid $(TM \oplus A^*, K_-)$ over M .*

Here, the complex Lie algebroid $TM \oplus A^*$ is identified with U_+ , the i -eigenspace of $j_{\mathbb{C}}$ in $(TM \oplus A^*)_{\mathbb{C}}$ equipped with the complexification of a dull bracket realising the one on $(TM \oplus A^*, j)$, as in (2) of Definition 1.3. The space K_- is then the i -eigenspace of $(-j^t)_{\mathbb{C}}$, hence $K_- \simeq (A \oplus T^*M)_{\mathbb{C}}/K_+ \simeq U_+^*$, since K_+ is the annihilator of U_+ . The Lie algebroid structure on A induces a *degenerate* Courant algebroid structure on $A \oplus T^*M$ (see Section 5.1), the complexification of which has K_- and K_+ as Dirac structures.

The Drinfeld double Courant algebroid of the obtained complex Lie bialgebroid is isomorphic to a Courant algebroid C_{\pm} , which is obtained via a construction with complex A -Manin pairs from A and U_{\pm} (as in [17]). In the initially studied special case of holomorphic Lie algebroids, where the generalised complex structure is induced by a complex structure, these Courant algebroids C_{\pm} decompose as a direct sum of Courant algebroids, $C_T^{1,0} \oplus C_A^{0,1}$, or $C_T^{0,1} \oplus C_A^{1,0}$, respectively. The Courant algebroid structure in C_{\pm} is then the same as the matched pair Courant algebroid structure on these bundles already given in [8]. This matched pair of Courant algebroids arises from the aforementioned matched pair of Lie algebroids $(T^{0,1}M, A^{1,0})$ of [23]. The Courant algebroids C_{\pm} therefore are the generalised version of this matched sum Courant algebroid in the special case of a holomorphic Lie algebroid.

Methodology. The key to the results in this paper is the equivalence of linear splittings of $TE \oplus T^*E$ with $TM \oplus E^*$ -Dorfman connections on $E \oplus T^*M$, in a similar way as linear splittings of TE are equivalent to linear TM -connections on E ; see [16]. Section 2.2 recalls this correspondence. Proposition 4.7 establishes the existence of an adapted Dorfman connection to any linear generalised almost complex structure \mathcal{J} on E . This is a Dorfman connection the horizontal sections of which are preserved by the generalised complex structure \mathcal{J} .

This allows then a description of the properties of the generalised complex structures \mathcal{J} in terms of this adapted Dorfman connection and the side morphism $j: TM \oplus E^* \rightarrow TM \oplus E^*$, see Theorem 4.10. Theorem 1.4 follows immediately from the study of the integrability of the linear generalised complex structure in terms of its side morphism and an adapted Dorfman connection.

Outline of the paper. Section 2 recalls some necessary background on Courant algebroids and generalised complex structures, on linear splittings of VB-Courant algebroids and Dorfman connections and on morphisms of 2-representations of Lie algebroids. Section 3 studies holomorphic vector bundles and holomorphic Lie algebroids, describing holomorphic vector bundles via linear complex structures on a real vector bundle.

Section 4 then studies linear generalised complex structures on vector bundles in detail. For the convenience of the reader the basic definitions and properties of generalised complex structures are given as well in Section 2.1. This section proves the existence of an adapted Dorfman connection and uses it to construct the complex Lie algebroid in Theorem 1.4.

In Section 5, the vector bundle is endowed with the additional structure of a Lie algebroid and the linear generalised complex structure \mathcal{J} is assumed to be compatible with the Lie algebroid structure. This is equivalent to \mathcal{J} defining a morphism of 2-representations. Some results of [23] are expanded in this more general framework.

Finally, Section 6 studies the more general case of a linear generalised complex structure in a VB-Courant algebroid. Appendix A compares for completeness this paper's adapted Dorfman connections with the adapted generalised connections in [4].

Prerequisites and notation. All manifolds and vector bundles in this paper are smooth and real. The reader is referred to Section 2.3 of [16] for the definition of a double vector bundle, their morphisms and induced core morphisms and for the necessary background on linear and core sections, and on their linear splittings and dualisations. Section 2.3 of [16] recalls the definition of a VB-algebroid, and also the equivalence of 2-term representations up to homotopy (called here *2-representations* for short) with linear decompositions of VB-algebroids [7]. The notation used here is the same as in [16]. In particular, a linear splitting of a double

vector bundle (D, A, B, M) is written $\Sigma: A \times_M B \rightarrow D$, and the corresponding horizontal lifts are then $\sigma := \sigma_A: \Gamma(A) \rightarrow \Gamma_B^l(D)$ and $\sigma := \sigma_B: \Gamma(B) \rightarrow \Gamma_A^l(D)$. The reader is invited to consult also [29, 26, 7] for more details on double vector bundles.

Vector bundle projections are written $q_E: E \rightarrow M$, and $p_M: TM \rightarrow M$ for tangent bundles. Given a section ε of E^* , the map $\ell_\varepsilon: E \rightarrow \mathbb{R}$ is the linear function associated to it, i.e. the function defined by $e_m \mapsto \langle \varepsilon(m), e_m \rangle$ for all $e_m \in E$. The set of global sections of a vector bundle $E \rightarrow M$ is denoted by $\Gamma(E)$. Elements $e \otimes (a + ib)$ of the complexification $E_{\mathbb{C}}$ may be written as $ae + ibe$ if there is no confusion with a complex structure of the vector bundle E itself. The dual of a vector bundle morphism φ is written as φ^t , to avoid confusion with pullbacks.

Acknowledgements. The authors thank Ping Xu and Chenchang Zhu for helpful comments and useful questions on an early version of this paper. They are also grateful to Vicente Cortés for pointing them to the reference [1].

2. BACKGROUND

This section recalls basic notions and results, in particular on linear splittings of the generalised tangent bundle of a vector bundle [16].

2.1. Courant algebroids and generalised complex structures. Let $(\mathbb{E} \rightarrow M, \rho, \langle \cdot, \cdot \rangle, \llbracket \cdot, \cdot \rrbracket)$ be a Courant algebroid. That is [25, 30], $\rho: \mathbb{E} \rightarrow TM$ is a vector bundle morphism over the identity on M , $\langle \cdot, \cdot \rangle: \mathbb{E} \times_M \mathbb{E} \rightarrow \mathbb{R}$ is a non-degenerate bilinear pairing and $\llbracket \cdot, \cdot \rrbracket$ is an \mathbb{R} -bilinear bracket on $\Gamma(\mathbb{E})$ such that

- (1) $\llbracket e_1, \llbracket e_2, e_3 \rrbracket \rrbracket = \llbracket \llbracket e_1, e_2 \rrbracket, e_3 \rrbracket + \llbracket e_2, \llbracket e_1, e_3 \rrbracket \rrbracket$,
- (2) $\rho(e_1)\langle e_2, e_3 \rangle = \langle \llbracket e_1, e_2 \rrbracket, e_3 \rangle + \langle e_2, \llbracket e_1, e_3 \rrbracket \rangle$ and
- (3) $\llbracket e_1, e_2 \rrbracket + \llbracket e_2, e_1 \rrbracket = \rho^t \mathbf{d}\langle e_1, e_2 \rangle$

for all $e_1, e_2, e_3 \in \Gamma(\mathbb{E})$ and $f \in C^\infty(M)$. On the right-hand side of the third equation, \mathbb{E} is identified with \mathbb{E}^* via the pairing. The identity $\rho\llbracket e_1, e_2 \rrbracket = [\rho(e_1), \rho(e_2)]$ follows from the equations above for all $e_1, e_2 \in \Gamma(\mathbb{E})$ [33].

The vector bundle $T\mathbb{M} := TM \oplus T^*M$ over a smooth manifold M together with the natural anchor $\rho := \text{pr}_{TM}$, the symmetric pairing $\langle (v_p, \theta_p), (w_p, \omega_p) \rangle = \theta_p(w_p) + \omega_p(v_p)$ and the **Courant-Dorfman bracket** $\llbracket (X, \theta), (Y, \omega) \rrbracket = ([X, Y], \mathcal{L}_X \omega - \iota_Y \mathbf{d}\theta)$ is the prototype of a Courant algebroid. It is called here the **standard Courant algebroid over M** .

Definition 2.1. A *generalised almost complex structure* in \mathbb{E} is a vector bundle morphism $\mathcal{J}: \mathbb{E} \rightarrow \mathbb{E}$ over id_M such that $\mathcal{J}^2 = -1$ and \mathcal{J} is orthogonal with respect to the pairing, i.e. $\langle \mathcal{J}(e_1), \mathcal{J}(e_2) \rangle = \langle e_1, e_2 \rangle$, for all sections $e_1, e_2 \in \Gamma(\mathbb{E})$.

Definition 2.2. A *generalised almost complex structure* $\mathcal{J}: \mathbb{E} \rightarrow \mathbb{E}$ is called a **generalised complex structure** in \mathbb{E} if the Nijenhuis tensor of \mathcal{J} vanishes:

$$0 = N_{\mathcal{J}}(e_1, e_2) := \llbracket e_1, e_2 \rrbracket - \llbracket \mathcal{J}(e_1), \mathcal{J}(e_2) \rrbracket + \mathcal{J}(\llbracket \mathcal{J}(e_1), e_2 \rrbracket + \llbracket e_1, \mathcal{J}(e_2) \rrbracket),$$

for all sections $e_1, e_2 \in \Gamma(\mathbb{E})$.

A generalised complex structure in the standard Courant algebroid $TM \oplus T^*M$ is simply called a **generalised complex structure on M** .

Example 2.3. Given an almost complex structure $J: TM \rightarrow TM$ the map $\mathcal{J}: TM \rightarrow TM$

$$\mathcal{J}_J = \begin{pmatrix} J & 0 \\ 0 & -J^t \end{pmatrix}$$

is a generalised almost complex structure. It is a generalised complex structure if and only if J is a complex structure on M .

Equivalently, generalised complex structures \mathcal{J} in \mathbb{E} can be described by pairs of transversal, complex conjugated Dirac structures in $\mathbb{E}_{\mathbb{C}}$, given by the $\pm i$ -eigenbundles of \mathcal{J} . In fact, this was the original definition in [14], see also [9, 11].

2.2. The generalised tangent bundle of a vector bundle and Dorfman connections.

Let $q_E: E \rightarrow M$ be a vector bundle. Then the tangent bundle TE has two vector bundle structures; one as the tangent bundle of the manifold E , and the second as a vector bundle over TM . The structure maps of $TE \rightarrow TM$ are the derivatives of the structure maps of $E \rightarrow M$. The space TE is a double vector bundle with core bundle $E \rightarrow M$. Linear splittings of TE are equivalent to linear horizontal subspaces $H \subseteq TE$, which in turn are equivalent to linear TM -connections ∇ in E . For details on these double vector bundles, their core and linear sections, on linear splittings and on connections, consult [26, 16].

Dualising TE as a vector bundle over E gives the cotangent bundle T^*E , which is itself a double vector bundle with sides E and E^* and core T^*M , see [26].

$$\begin{array}{ccc} TE & \xrightarrow{p_E} & E \\ Tq_E \downarrow & & \downarrow q_E \\ TM & \xrightarrow{p_M} & M \end{array} \quad \begin{array}{ccc} T^*E & \xrightarrow{c_E} & E \\ r_E \downarrow & & \downarrow q_E \\ E^* & \xrightarrow{q_{E^*}} & M \end{array}$$

Consider the direct sum over E of these two double vector bundles,

$$\begin{array}{ccc} TE \oplus T^*E & \xrightarrow{\pi_E} & E \\ \Phi_E \downarrow & & \downarrow q_E \\ TM \oplus E^* & \xrightarrow{q_{TM \oplus E^*}} & M \end{array}$$

with $\Phi_E = Tq_E \oplus r_E$. A subbundle $L \subseteq TE \oplus T^*E$ that is closed under the addition of $TE \oplus T^*E \rightarrow TM \oplus E^*$, and complementary to $T^qE \oplus (T^qE)^\circ$, is called a **linear horizontal subspace in $TE \oplus T^*E$** .

In the following, for any section (e, θ) of $E \oplus T^*M$, the vertical section $(e, \theta)^\dagger \in \Gamma_E(T^{q_E}E \oplus (T^{q_E}E)^\circ)$ is the pair defined by

$$(e, \theta)^\dagger(e'_m) = \left(\frac{d}{dt} \Big|_{t=0} e'_m + te(m), (T_{e'_m} q_E)^t \theta(m) \right)$$

for all $e'_m \in E$. By construction this is a core section of $TE \oplus T^*E \rightarrow E$. For any section $\tilde{\phi}$ of $\text{Hom}(E, E \oplus T^*M)$, the core-linear section $\tilde{\phi} \in \Gamma_E^l(T^{q_E}E \oplus (T^{q_E}E)^\circ)$ is defined by $\tilde{\phi}(e(m)) = (\phi(e))^\dagger(e(m))$ for all $e \in \Gamma(E)$. The double vector bundle $TE \oplus T^*E$ is described in more detail in [16], where also an equivalence of linear splittings of $TE \oplus T^*E$ with Dorfman connections is established.

A **Dorfman $TM \oplus E^*$ -connection on $E \oplus T^*M$** is an \mathbb{R} -bilinear map

$$\Delta: \Gamma(TM \oplus E^*) \times \Gamma(E \oplus T^*M) \rightarrow \Gamma(E \oplus T^*M)$$

satisfying [16]

- (1) $\Delta_\nu(f \cdot \tau) = f \cdot \Delta_\nu \tau + \mathcal{L}_{\text{pr}_{TM}(\nu)}(f) \cdot \tau$,
- (2) $\Delta_{f \cdot \nu} \tau = f \cdot \Delta_\nu \tau + \langle \nu, \tau \rangle \cdot (0, \mathbf{d}f)$, and
- (3) $\Delta_\nu(0, \mathbf{d}f) = (0, \mathbf{d}(\mathcal{L}_{\text{pr}_{TM} \nu} f))$

for all $\nu \in \Gamma(TM \oplus E^*)$, $\tau \in \Gamma(E \oplus T^*M)$ and $f \in C^\infty(M)$. By the first axiom, Δ defines a map $\Delta: \nu \mapsto \Delta_\nu \in \text{Der}(E \oplus T^*M)$. The dual of this map in the sense of derivations defines a **dull bracket on sections of $TM \oplus E^*$** , i.e. an \mathbb{R} -bilinear map

$$[\cdot, \cdot]_\Delta: \Gamma(TM \oplus E^*) \times \Gamma(TM \oplus E^*) \rightarrow \Gamma(TM \oplus E^*)$$

satisfying

- (1) $\text{pr}_{TM}[\nu_1, \nu_2]_\Delta = [\text{pr}_{TM} \nu_1, \text{pr}_{TM} \nu_2]$,
- (2) $[[f_1 \nu_1, f_2 \nu_2]_\Delta] = f_1 f_2 [\nu_1, \nu_2]_\Delta + f_1 \mathcal{L}_{\text{pr}_{TM} \nu_1}(f_2) \nu_2 - f_2 \mathcal{L}_{\text{pr}_{TM} \nu_2}(f_1) \nu_1$

for all $\nu_1, \nu_2 \in \Gamma(TM \oplus E^*)$ and $f_1, f_2 \in C^\infty(M)$.

Since the vector bundle $TM \oplus E^*$ is anchored by the morphism $\text{pr}_{TM}: TM \oplus E^* \rightarrow TM$, the TM -part of $[\nu_1, \nu_2]_\Delta + [\nu_2, \nu_1]_\Delta$ is trivial and this sum can be seen as an element of $\Gamma(E^*)$. Let $\mathbf{Jac}_\Delta \in \Omega^3(TM \oplus E^*, TM \oplus E^*)$ be the Jacobiator of the dull bracket $[\cdot, \cdot]_\Delta$. Then

$$\mathbf{Jac}_\Delta(\nu_1, \nu_2, \nu_3) := [[\nu_1, \nu_2]_\Delta, \nu_3]_\Delta + \text{cyclic permutations} = R_\Delta(\nu_1, \nu_2)^\dagger(\nu_3),$$

with $R_\Delta \in \Omega^2(TM \oplus E^*, \text{Hom}(E \oplus T^*M, E))$ the curvature of the Dorfman connection. Hence a skew-symmetric dull bracket is a Lie algebroid bracket if and only if the corresponding Dorfman connection is flat.

Linear splittings of $TE \oplus T^*E$ are in bijection with dull brackets on sections of $TM \oplus E^*$, or equivalently with Dorfman connections $\Delta: \Gamma(TM \oplus E^*) \times \Gamma(E \oplus T^*M) \rightarrow \Gamma(E \oplus T^*M)$, see [16]. Choose such a Dorfman connection. The horizontal lift $\sigma := \sigma_{TM \oplus E^*}^\Delta: \Gamma(TM \oplus E^*) \rightarrow \Gamma_E^l(TE \oplus T^*E)$ is given by

$$(4) \quad \sigma(X, \epsilon)(e(m)) = (T_m eX(m), \mathbf{d}\ell_\epsilon(e(m))) - \Delta_{(X, \epsilon)}(e, 0)^\dagger(e(m))$$

for $e \in \Gamma(E)$ and any pair $(X, \epsilon) \in \Gamma(TM \oplus E^*)$. The natural pairing on fibres of $TE \oplus T^*E \rightarrow E$ is then given by [16] $\langle \sigma(\nu_1), \sigma(\nu_2) \rangle = \ell_{[\nu_1, \nu_2]_\Delta + [\nu_2, \nu_1]_\Delta}$, $\langle \sigma(\nu), \tau^\dagger \rangle = q_E^* \langle \nu, \tau \rangle$, and $\langle \tau_1^\dagger, \tau_2^\dagger \rangle = 0$ for $\nu, \nu_1, \nu_2 \in \Gamma(TM \oplus E^*)$ and $\tau, \tau_1, \tau_2 \in \Gamma(E \oplus T^*M)$. The following equations follow for $\varphi, \psi \in \Gamma(\text{Hom}(E, E \oplus T^*M))$, $\nu \in \Gamma(TM \oplus E^*)$ and $\tau \in \Gamma(E \oplus T^*M)$

$$(5) \quad \langle \widetilde{\varphi}, \sigma^\Delta(\nu) \rangle = \ell_{\varphi^*(\nu)}, \quad \langle \widetilde{\varphi}, \tau^\dagger \rangle = 0, \quad \langle \widetilde{\varphi}, \widetilde{\psi} \rangle = 0.$$

The Courant-Dorfman bracket on sections of $TE \oplus T^*E \rightarrow E$ is given by [16]

- (1) $[[\sigma(\nu_1), \sigma(\nu_2)]] = \sigma[[\nu_1, \nu_2]_\Delta] - R_\Delta(\widetilde{\nu_1, \nu_2}) \circ \iota_E$,
- (2) $[[\sigma(\nu), \tau^\dagger]] = (\Delta_\nu \tau)^\dagger$, and
- (3) $[[\tau_1^\dagger, \tau_2^\dagger]] = 0$

for $\nu, \nu_1, \nu_2 \in \Gamma(TM \oplus E^*)$ and $\tau, \tau_1, \tau_2 \in \Gamma(E \oplus T^*M)$. Here, $\iota_E: E \rightarrow E \oplus T^*M$ is the canonical inclusion.

The anchor $\Theta = \text{pr}_{TE}: TE \oplus T^*E \rightarrow TE$ restricts to the map $\partial_E = \text{pr}_E: E \oplus T^*M \rightarrow E$ on the cores, and defines an anchor $\rho_{TM \oplus E^*} = \text{pr}_{TM}: TM \oplus E^* \rightarrow TM$ on the side. More precisely, the anchor of $(e, \theta)^\dagger$ is $e^\dagger \in \mathfrak{X}^c(E)$ and $\Theta(\sigma(\nu)) = \widehat{\nabla}_\nu^* \in \mathfrak{X}(E)$, where the linear connection $\nabla: \Gamma(TM \oplus E^*) \times \Gamma(E) \rightarrow \Gamma(E)$ is defined by $\nabla_\nu = \text{pr}_E \circ \Delta_\nu \circ \iota_E$ for all $\nu \in \Gamma(TM \oplus E^*)$.

Example 2.4. *Let $q: E \rightarrow M$ be a smooth vector bundle. Since a linear connection $\nabla: \mathfrak{X}(M) \times \Gamma(E) \rightarrow \Gamma(E)$ is equivalent to a linear horizontal space $H_\nabla \subseteq TE$, it also defines a linear horizontal space $H_\nabla \oplus H_\nabla^\circ \subseteq TE \oplus T^*E$. The corresponding Dorfman connection*

$$\Delta: \Gamma(TM \oplus E^*) \times \Gamma(E \oplus T^*M) \rightarrow \Gamma(E \oplus T^*M)$$

is given by

$$\Delta_{(X, \epsilon)}(e, \theta) = (\nabla_X e, \mathcal{L}_X \theta + \langle \nabla^* \epsilon, e \rangle)$$

for $X \in \mathfrak{X}(M)$, $\theta \in \Omega^1(M)$, $e \in \Gamma(E)$ and $\epsilon \in \Gamma(E^*)$. This is the **standard Dorfman connection** defined by ∇ . The corresponding dull bracket is

$$(6) \quad \llbracket (X, \epsilon), (Y, \eta) \rrbracket_\Delta = ([X, Y], \nabla_X^* \eta - \nabla_Y^* \epsilon)$$

for $X, Y \in \mathfrak{X}(M)$ and $\epsilon, \eta \in \Gamma(E^*)$.

The remainder of this section discusses changes of linear splittings of $TE \oplus T^*E$.

Definition 2.5. *Given two $(TM \oplus E^*)$ -Dorfman connections Δ^1 and Δ^2 on $E \oplus T^*M$ with corresponding lifts σ_1 and σ_2 , the **change of splitting** from Δ^1 to Δ^2 is $\Phi_{12} \in \Gamma((TM \oplus E^*)^* \otimes \text{Hom}(E, E \oplus T^*M))$ defined by the equation*

$$\overline{\Phi_{12}(\nu)} := \sigma_2(\nu) - \sigma_1(\nu),$$

for any $\nu \in \Gamma(TM \oplus E^*)$. The change of splitting is called **skew-symmetric** if $\Psi_{12}(\nu_1, \nu_2) := \Phi_{12}(\nu_1)^\dagger(\nu_2)$ is skew-symmetric, that is $\Psi_{12} \in \Omega^2(TM \oplus E^*, E^*)$. The form Ψ_{12} is also called **change of splittings**.

Lemma 2.6. *Given two Dorfman connections Δ^1, Δ^2 as above, their corresponding dull brackets are related by*

$$(7) \quad \llbracket \nu_1, \nu_2 \rrbracket_{\Delta^2} = \llbracket \nu_1, \nu_2 \rrbracket_{\Delta^1} + (0, \Psi_{12}(\nu_1, \nu_2)).$$

Proof. The definition of the change of splittings together with the correspondence between lifts and Dorfman connections as in (4) immediately gives that for any $\nu \in \Gamma(TM \oplus E^*)$ and $\tau \in \Gamma(E \oplus T^*M)$

$$\Delta_\nu^2 \tau = \Delta_\nu^1 \tau - \Phi_{12}(\nu)(\text{pr}_E \tau).$$

Again dualising this equation gives the desired formula (7) for the change of splittings for the corresponding dull brackets. \square

An immediate consequence is the following corollary.

Corollary 2.7. *If the Dorfman connection Δ_1 is skew-symmetric, then Δ_2 is skew-symmetric if and only if the change of splitting is skew-symmetric.*

2.3. VB-Courant algebroids. The linear Courant algebroid on $TE \oplus T^*E$ is a prototype of a VB-Courant algebroid. This section gives the general definition of a VB-Courant algebroid.

A **metric double vector bundle** [16] is a double vector bundle $(D; A, B; M)$ equipped with a symmetric, non-degenerate fibrewise pairing $D \times_B D \rightarrow \mathbb{R}$, such that the induced map $D \rightarrow D_B^*$ is an isomorphism of double vector bundles. In particular the core must be isomorphic to A^* . A **VB-Courant algebroid** $(\mathbb{E}; Q, B; M)$ is a metric double vector bundle

$$\begin{array}{ccc} \mathbb{E} & \longrightarrow & B \\ \downarrow & & \downarrow \\ Q & \longrightarrow & M \end{array},$$

such that $\mathbb{E} \rightarrow B$ is a Courant algebroid, the anchor $\rho_{\mathbb{E}}: \mathbb{E} \rightarrow TB$ is linear, i.e. a morphism of double vector bundles over some morphism $\rho_Q: Q \rightarrow TM$ and the Courant bracket is linear, that is

$$[[\Gamma_B^\ell(\mathbb{E}), \Gamma_B^\ell(\mathbb{E})] \subseteq \Gamma_B^\ell(\mathbb{E}), \quad [[\Gamma_B^\ell(\mathbb{E}), \Gamma_B^c(\mathbb{E})] \subseteq \Gamma_B^c(\mathbb{E}), \quad \text{and} \quad [[\Gamma_B^c(\mathbb{E}), \Gamma_B^c(\mathbb{E})] = 0.$$

Given a VB-Courant algebroid $(\mathbb{E}; Q, B; M)$, a **VB-Dirac structure** in \mathbb{E} is a sub-double vector bundle $(D; U, B; M)$ with $U \subseteq Q$ such that $D \rightarrow B$ is a Dirac structure in $\mathbb{E} \rightarrow B$.

Example 2.8. *The standard Courant algebroid $\mathbb{E} = TE \oplus T^*E$ over a vector bundle E is a VB-Courant algebroid with $Q = TM \oplus E^*$ and $B = E$. The subspaces TE and T^*E are VB-Dirac structures in \mathbb{E} .*

Example 2.9. *The tangent double of a Courant algebroid $E \rightarrow M$ is a VB-Courant algebroid, where $\mathbb{E} = TE$, $Q = E$ and $B = TM$. The anchor of TE is given by $I \circ T\rho_E: TE \rightarrow T(TM)$, where $I: TTM \rightarrow TTM$ is the canonical involution [32, 26], exchanging the two vector bundle structures Tp_M and p_{TM} of $TTM \rightarrow TM$.*

2.4. The generalised tangent bundle of a Lie algebroid. Let here $A \rightarrow M$ be a Lie algebroid. Then for $a \in \Gamma(A)$, the derivations \mathcal{L}_a of $\Gamma(TM \oplus A^*)$ and of $\Gamma(A \oplus T^*M)$ over $\rho(a)$ are defined by

$$\mathcal{L}_a(X, \alpha) := ([\rho(a), X], \mathcal{L}_a \alpha), \quad \mathcal{L}_a(a', \theta) := ([a, a'], \mathcal{L}_{\rho(a)} \theta)$$

for $(X, \alpha) \in \Gamma(TM \oplus A^*)$ and $(a', \theta) \in \Gamma(A \oplus T^*M)$. Fix a skew-symmetric Dorfman connection $\Delta: \Gamma(TM \oplus A^*) \times \Gamma(A \oplus T^*M) \rightarrow \Gamma(A \oplus T^*M)$ and set [16]

$$\Omega: \Gamma(TM \oplus A^*) \times \Gamma(A) \rightarrow \Gamma(A \oplus T^*M), \quad \Omega_{(X, \alpha)} a := \Delta_{(X, \alpha)}(a, 0) - (0, \mathbf{d}\langle \alpha, a \rangle).$$

Define then the **basic connections associated to Δ** by

$$\begin{aligned} \nabla_a^{\text{bas}}(X, \alpha) &:= (\rho, \rho^t)(\Omega_{(X, \alpha)} a) + \mathcal{L}_a(X, \alpha), \\ \nabla_a^{\text{bas}}(a', \theta) &:= \Omega_{(\rho, \rho^t)(a', \theta)} a + \mathcal{L}_a(a', \theta), \end{aligned}$$

for $a \in \Gamma(A)$, $(X, \alpha) \in \Gamma(TM \oplus A^*)$ and $(a', \theta) \in \Gamma(A \oplus T^*M)$. These are ordinary linear A -connections on $TM \oplus A^*$ and $A \oplus T^*M$, respectively, which are dual to each other [16].

The **basic curvature** $R_\Delta^{\text{bas}} \in \Omega^2(A, \text{Hom}(TM \oplus A^*, A \oplus T^*M))$ associated to Δ is defined by

$$(8) \quad R_\Delta^{\text{bas}}(a_1, a_2)\nu := -\Omega_\nu[a_1, a_2] + \mathcal{L}_{a_1}(\Omega_\nu a_2) - \mathcal{L}_{a_2}(\Omega_\nu a_1) + \Omega_{\nabla_{a_2}^{\text{bas}} \nu} a_1 - \Omega_{\nabla_{a_1}^{\text{bas}} \nu} a_2$$

for $a_1, a_2 \in \Gamma(A)$ and $\nu \in \Gamma(TM \oplus A^*)$. [16] shows that $R_{\nabla^{\text{bas}}} = R_\Delta^{\text{bas}} \circ (\rho, \rho^t)$ and $R_{\nabla^{\text{bas}}} = (\rho, \rho^t) \circ R_\Delta^{\text{bas}}$.

Theorem 2.10. [16] *Let $A \rightarrow M$ be a Lie algebroid with anchor ρ and let Δ a skew-symmetric Dorfman $TM \oplus A^*$ -connection on $A \oplus T^*M$. Write Θ for the anchor of the Lie algebroid $\mathbb{T}A$. Then*

- (1) $[\sigma_A^\Delta(a_1), \sigma_A^\Delta(a_2)] = \sigma_A^\Delta([a_1, a_2]) - \widehat{R_\Delta^{\text{bas}}(a_1, a_2)},$
- (2) $[\sigma_A^\Delta(a), \tau^\dagger] = (\nabla_a^{\text{bas}} \tau)^\dagger,$
- (3) $[\tau_1^\dagger, \tau_2^\dagger] = 0,$
- (4) $\Theta(\sigma_A^\Delta(a)) = \widehat{\nabla_a^{\text{bas}}} \in \mathfrak{X}^\ell(TM \oplus A^*),$
- (5) $\Theta(\tau^\dagger) = ((\rho, \rho^\dagger)\tau)^\dagger \in \mathfrak{X}^c(TM \oplus A^*)$

for $a, a_1, a_2 \in \Gamma(A)$ and $\tau, \tau_1, \tau_2 \in \Gamma(A \oplus T^*M)$.

That is [16], the complex $(\rho, \rho^\dagger): (A \oplus T^*M)[0] \rightarrow (TM \oplus A^*)[1]$, the basic connections ∇^{bas} and the basic curvature R_Δ^{bas} define the 2-representation corresponding to the VB-algebroid $(\mathbb{T}A \oplus_A T^*A \rightarrow TM \oplus A^*, A \rightarrow M)$ [16] in the decomposition corresponding to Δ , see [7].

3. HOLOMORPHIC VECTOR BUNDLES AND HOLOMORPHIC LIE ALGEBROIDS

This section proves that a holomorphic structure on a vector bundle is equivalent to a linear complex structure on it. In particular, ‘‘adapted’’ connections are described in the language of linear complex structures. Finally, a holomorphic Lie algebroid A with a choice of adapted connection gives rise to infinitesimal ideal systems in $A_{\mathbb{C}}$.

3.1. Linear almost complex structures via connections. Let E be a vector bundle over a manifold M . A complex structure in the fibres of E is equivalent to a vector bundle morphism $j_E: E \rightarrow E$ over id_M , such that $j_E^2 = -\text{id}_E$. Given such j_E , it defines a complex scalar multiplication by $(\alpha + i\beta) \cdot e = \alpha e + \beta j_E(e)$, and vice versa, if E is a \mathbb{C} -vector bundle, then multiplication by i defines such a morphism j_E .

Consider now such a vector bundle E with fibrewise complex structure j_E . Take any connection $\nabla: \mathfrak{X}(M) \times \Gamma(E) \rightarrow \Gamma(E)$. Then the connection

$$\tilde{\nabla}: \mathfrak{X}(M) \times \Gamma(E) \rightarrow \Gamma(E), \quad \tilde{\nabla}_X e = \frac{1}{2}(\nabla_X e - j_E(\nabla_X(j_E(e))))$$

satisfies $\tilde{\nabla} j_E = 0$. Such a connection is called *complex-linear* connection on E , since it is \mathbb{C} -linear in its second argument.

Consider a holomorphic vector bundle $E \rightarrow M$. Since E is locally generated as a \mathbb{C} -vector bundle by holomorphic sections e_1, \dots, e_k of $q_E: E \rightarrow M$, the real vector bundle E is locally generated by the holomorphic sections $e_1, \dots, e_k, f_1 := i \cdot e_1, \dots, f_k := i \cdot e_k$. Then since these sections are holomorphic, they satisfy

$$(9) \quad J_E \circ T e_l = T e_l \circ J_M \quad \text{and} \quad J_E \circ T f_l = T f_l \circ J_M$$

for $l = 1, \dots, k$. The following lemma is proved in Appendix B.

Lemma 3.1. *Let $q_E: E \rightarrow M$ be a holomorphic vector bundle. If $e \in \mathcal{E}(U)$ is a holomorphic section, then the vector field $e^\dagger \in \mathfrak{X}(E)$ is holomorphic as well, and for any $e \in \Gamma(E)$, the complex structure J_E sends e^\dagger to $(j_E e)^\dagger$.*

That is, the complex structure $J_E: TE \rightarrow TE$ is a double vector bundle morphism over the identity on E , the complex structure $J_M: TM \rightarrow TM$ of the base M , and with core $j_E: E \rightarrow E$ – as in (1).

The goal of this section is the proof of the converse: if a linear almost complex structure as in (1) is integrable, then the induced complex structures on E and on M make $E \rightarrow M$ a holomorphic vector bundle. Consider therefore for the remainder of this section generally a smooth vector bundle $E \rightarrow M$ with an almost complex structure $TE \rightarrow TE$ that is linear as in (1) over an almost complex structure $J_M: TM \rightarrow TM$, with core morphism $j_E: E \rightarrow E$.

Definition 3.2. A linear connection $\nabla: \mathfrak{X}(M) \times \Gamma(E) \rightarrow \Gamma(E)$ is **adapted to J_E** if the corresponding linear splitting $\Sigma: TM \times_M E \rightarrow TE$ of TE satisfies $J_E \Sigma(v, e) = \Sigma(J_M v, e)$ for all $(v, e) \in TM \times_M E$. Equivalently, the corresponding horizontal lift $\sigma^\nabla: \mathfrak{X}(M) \rightarrow \mathfrak{X}^\ell(E)$ lifts the almost complex structure on M to the one on E , that is $\sigma^\nabla(J_M X) = J_E \sigma^\nabla(X)$ for all $X \in \mathfrak{X}(M)$.

Consider a linear splitting of the double vector bundle TE

$$\Sigma: E \times_M TM \rightarrow TE,$$

corresponding to a linear connection $\nabla: \mathfrak{X}(M) \times \Gamma(E) \rightarrow \Gamma(E)$. Then for all $X \in \mathfrak{X}(M)$, the vector field $J_E(\widehat{\nabla}_X)$ is q -related with $J_M(X) \in \mathfrak{X}(M)$. Hence

$$(10) \quad J_E(\widehat{\nabla}_X) = \widehat{\nabla_{J_M X}} + \widetilde{\psi(X)}$$

for a section $\psi(X)$ of $\text{End}(E)$. It is easy to check that $\psi(f \cdot X) = f \cdot \psi(X)$ for $X \in \mathfrak{X}(M)$ and $f \in C^\infty(M)$. Hence (10) defines a form $\psi \in \Omega^1(M, \text{End}(E))$ with

$$-\widehat{\nabla}_X = J_E^2(\widehat{\nabla}_X) = J_E \left(\widehat{\nabla_{J_M X}} + \widetilde{\psi(X)} \right) = \widehat{\nabla_{J_M^2 X}} + \psi(\widehat{J_M X}) + j_E \widetilde{\psi(X)}$$

for all $X \in \mathfrak{X}(M)$, that is, with

$$(11) \quad \psi(J_M X) = -j_E \circ \psi(X)$$

for all $X \in \mathfrak{X}(M)$. Consider $\Sigma': E \times_M TM \rightarrow TE$,

$$\Sigma'(e_m, v_m) = \Sigma(e_m, v_m) +_{TM} (T_m 0^E v_m +_E \frac{1}{2} \overline{j_E \psi(v_m)(e_m)}).$$

Then a simple computation shows that Σ' satisfies the condition of Definition 3.2, hence showing the following proposition.

Proposition 3.3. Let $E \rightarrow M$ be a smooth vector bundle endowed with a linear almost complex structure $J_E: TE \rightarrow TE$ over $J_M: TM \rightarrow TM$, with core morphism $j_E: E \rightarrow E$. Then there exists a linear TM -connection on E that is adapted to J_E .

By definition, a linear connection $\nabla: \mathfrak{X}(M) \times \Gamma(E) \rightarrow \Gamma(E)$ is adapted if and only if the following identity holds for all $e \in \Gamma(E)$, $X \in \mathfrak{X}(M)$:

$$\begin{aligned} J_E(T_m e X_m) - \frac{d}{dt} \Big|_{t=0} e_m + t j_E(\nabla_X e)(m) &= J_E(\widehat{\nabla}_X(e(m))) = \widehat{\nabla_{J_M X}}(e(m)) \\ &= T_m e J_M X_m - \frac{d}{dt} \Big|_{t=0} e_m + t(\nabla_{J_M X} e)(m). \end{aligned}$$

In particular, if $E \rightarrow M$ is a holomorphic vector bundle, and e a holomorphic section, then $J_E \circ T e = T e \circ J_M$. Hence, ∇ is adapted to J_E if and only if

$$(12) \quad j_E \nabla_X e = \nabla_{J_M X} e$$

for all $X \in \mathfrak{X}(M)$ and all *holomorphic sections* $e \in \Gamma(E)$. This shows also that for any linear TM -connection ∇ on E , the connection $\nabla': \mathfrak{X}(M) \times \Gamma(E) \rightarrow \Gamma(E)$ defined by

$$(13) \quad \nabla'_X e = \frac{1}{2} (\nabla_X e - j_E \nabla_{J_M X} e)$$

on $X \in \mathfrak{X}(M)$ and *holomorphic sections* $e \in \Gamma(E)$, is adapted to J_E . Assume additionally that $\nabla j_E = 0$, which due to the arguments above is always possible. Then

$$\nabla'_X (j_E(e)) = \frac{1}{2} (\nabla_X (j_E(e)) - j_E \nabla_{J_M X} (j_E(e))) = \frac{1}{2} (j_E \nabla_X e - j_E^2 \nabla_{J_M X} e) = j_E \nabla'_X e$$

for holomorphic sections of $e \in \Gamma(E)$ and all $X \in \mathfrak{X}(M)$. Since E has local (real) frames of holomorphic sections, this shows that $\nabla' j_E = 0$ as well.

In general the following theorem shows that the existence of such a \mathbb{C} -linear adapted connection follows easily from the integrability of the linear almost complex structure J_E .

Theorem 3.4. *Let $E \rightarrow M$ be a smooth vector bundle endowed with a linear almost complex structure $J_E: TE \rightarrow TE$ over $J_M: TM \rightarrow TM$, with core morphism $j_E: E \rightarrow E$. Then J_E is integrable if and only if*

- (1) J_M is integrable and
- (2) there exists a \mathbb{C} -linear connection $\nabla: \mathfrak{X}(M) \times \Gamma(E) \rightarrow \Gamma(E)$ that is adapted to J_E ,
- (3) the form $N_{R_\nabla, j_E} \in \Omega^2(M, E)$ defined by

$$N_{R_\nabla, j_E}(X, Y) := R_\nabla(X, Y) - R_\nabla(J_M X, J_M Y) + j_E R_\nabla(J_M X, Y) + j_E R_\nabla(X, J_M Y)$$

vanishes.

Remark 3.5. (1) *In the setting above, a linear connection $\nabla: \mathfrak{X}(M) \times \Gamma(E) \rightarrow \Gamma(E)$ is adapted to J_E if and only if*

$$J_E(H_\nabla) = H_\nabla.$$

It satisfies $\nabla j_E = 0$ if and only if

$$H_\nabla = T j_E(H_\nabla).$$

- (2) *Let $E \rightarrow M$ be a holomorphic vector bundle and $J_E: TE \rightarrow TE$ the linear complex structure as in Lemma 3.1. Then by (12) and the \mathbb{C} -linearity of ∇ ,*

$$N_{R_\nabla, j_E}(X, Y)e = R_\nabla(X, Y)e - j_E^2 R_\nabla(X, Y)e + j_E^2 R_\nabla(X, Y)e + j_E^2 R_\nabla(X, Y)e = 0$$

for $X, Y \in \mathfrak{X}_U(M)$ and a holomorphic section e of E . Since N_{R_∇, j_E} is a tensor, it vanishes then identically.

Proof of Theorem 3.4. Choose a linear connection $\nabla: \mathfrak{X}(M) \times \Gamma(E) \rightarrow \Gamma(E)$ that is adapted to J_E as in Definition 3.2 – the existence of such a connection is given by Proposition 3.3. That is, the linear vector fields $\widehat{\nabla}_X$, for $X \in \mathfrak{X}(M)$, all satisfy

$$J_E \circ \widehat{\nabla}_X = \widehat{\nabla}_{J_M X}.$$

Then

$$(14) \quad \begin{aligned} N_{J_E}(\widehat{\nabla}_X, \widehat{\nabla}_Y) &= [\widehat{\nabla}_X, \widehat{\nabla}_Y] - [J_E \widehat{\nabla}_X, J_E \widehat{\nabla}_Y] + J_E [J_E \widehat{\nabla}_X, \widehat{\nabla}_Y] + J_E [\widehat{\nabla}_X, J_E \widehat{\nabla}_Y] \\ &= \nabla_{N_{J_M}(X, Y)} - N_{R_\nabla, j_E}(X, Y), \\ N_{J_E}(\widehat{\nabla}_X, e^\uparrow) &= [\widehat{\nabla}_X, e^\uparrow] - [\widehat{\nabla}_{J_M X}, (j_E e)^\uparrow] + (j_E \nabla_{J_M X} e)^\uparrow + (j_E \nabla_X j_E e)^\uparrow \\ &= (\nabla_X e - \nabla_{J_M X} j_E e + j_E \nabla_{J_M X} e + j_E \nabla_X j_E e)^\uparrow. \end{aligned}$$

and

$$N_{J_E} \left(e_1^\uparrow, e_2^\uparrow \right) = 0$$

for all $X, Y \in \mathfrak{X}(M)$ and $e, e_1, e_2 \in \Gamma(E)$. Since $\mathfrak{X}(E)$ is spanned as a $C^\infty(E)$ -module by these linear and core sections, N_{J_E} vanishes if and only if $N_{J_M}(X, Y) = 0$ (by projecting (14) to M), $N_{R_{\nabla}, J_E}(X, Y) = 0$ and

$$(15) \quad \nabla_X e - \nabla_{J_M X} j_E e + j_E \nabla_{J_M X} e + j_E \nabla_X j_E e = 0$$

for all $X, Y \in \mathfrak{X}(M)$ and $e \in \Gamma(E)$. Therefore, if $N_{J_M} = 0$, $N_{R_{\nabla}, j} = 0$ and $\nabla j_E = 0$, the almost complex structure J_E is integrable.

Assume conversely that $N_{J_E} = 0$. Then $N_{J_M} = 0$ and J_M is integrable. Define now $\nabla': \mathfrak{X}(M) \times \Gamma(E) \rightarrow \Gamma(E)$ by

$$\nabla'_X e = \frac{1}{2} \nabla_X e - \frac{1}{2} j_E \nabla_X j_E e.$$

Clearly $\nabla' j_E = 0$, so ∇' is \mathbb{C} -linear. Additionally, the difference between ∇ and ∇' is the form $\omega \in \Omega^1(M, \text{End}(E))$, given for $X \in \mathfrak{X}(M)$ and $e \in \Gamma(E)$ by

$$\omega(X)(e) = \frac{1}{2} (\nabla_X e + j_E \nabla_X j_E e).$$

By (15) $j_E \circ (\omega(X)) = \omega(J_M X)$ and hence

$$J_E \left(\widehat{\nabla'_X} \right) = J_E \left(\widehat{\nabla_X} - \widehat{\omega(X)} \right) = \widehat{\nabla_{J_M X}} - \widehat{j_E \omega(X)} = \widehat{\nabla_{J_M X}} - \widehat{\omega(J_M X)} = \widehat{\nabla'_{J_M X}},$$

so ∇' is still adapted to J_E . Therefore, by (14), $N_{R_{\nabla'}, j_E} = 0$. \square

In the situation of the previous theorem, consider a complex-linear connection $\nabla: \mathfrak{X}(M) \times \Gamma(E) \rightarrow \Gamma(E)$, that is adapted to J_E . Consider the complexification $D: \Gamma(T_{\mathbb{C}}M) \times \Gamma(E) \rightarrow \Gamma(E)$ of ∇ in the first argument. Then D is still \mathbb{C} -linear and its curvature $R_D \in \Omega^2(T_{\mathbb{C}}M, \text{End}_{\mathbb{C}}(E))$ is the complexification of R_{∇} in the first two arguments.

Theorem 3.6. *Let $E \rightarrow M$ be a smooth vector bundle endowed with a linear almost complex structure $J_E: TE \rightarrow TE$ over $J_M: TM \rightarrow TM$, with core morphism $j_E: E \rightarrow E$. Assume that J_M is integrable. Then a complex linear connection $\nabla: \mathfrak{X}(M) \times \Gamma(E) \rightarrow \Gamma(E)$ adapted to J_E satisfies $N_{R_{\nabla}, j_E} = 0$ if and only if the complexification $D: \Gamma(T_{\mathbb{C}}M) \times \Gamma(E) \rightarrow \Gamma(E)$ equals*

$$(16) \quad D = D^{1,0} + D^{0,1}$$

with $D^{1,0}: \Gamma(T^{1,0}M) \times \Gamma(E) \rightarrow \Gamma(E)$ a linear connection and $D^{0,1}: \Gamma(T^{0,1}M) \times \Gamma(E) \rightarrow \Gamma(E)$ a flat connection.

Remark 3.7. *If $E \rightarrow M$ is holomorphic, this statement is again immediate since for all $X \in \Gamma(T^{0,1}U)$ and holomorphic sections e of E , (12) implies*

$$i \nabla_X e = j_E \nabla_X e \stackrel{(12)}{=} \nabla_{J_M X} e = \nabla_{-iX} e = -i \nabla_X e,$$

which shows that $\nabla_X e = 0 = \bar{\partial}_X e$. The flat connection $D^{0,1}$ is hence simply the $\bar{\partial}$ -operator.

The complexified connection D can then be in addition compatible with a Hermitian metric on E . In that case, it is the Chern connection associated to the Hermitian holomorphic bundle. In general, a connection as in Theorem 3.6 could be called a Chern connection of the holomorphic vector bundle. In [1] such a connection is simply called adapted to the holomorphic structure.

Proof of Theorem 3.6. Assume that ∇ is \mathbb{C} -linear and adapted to J_E . Then for all $X, Y \in \Gamma(T^{0,1}M)$:

$$\begin{aligned} N_{R_{\nabla, j_E}}^{\mathbb{C}}(X, Y) &= N_{R_D, j_E}(X, Y) \\ &= R_D(X, Y) - R_D(-iX, -iY) + iR_D(-iX, Y) + iR_D(X, -iY) = 4R_D(X, Y). \end{aligned}$$

Hence $R_{D^{0,1}} = 0$ if and only if $N_{R_{\nabla, j_E}}^{\mathbb{C}}(X, Y) = 0$ for all $X, Y \in \Gamma(T^{0,1}M)$. If $N_{R_{\nabla, j_E}} = 0$, then obviously $N_{R_{\nabla, j_E}}^{\mathbb{C}} = 0$ and so $R_{D^{0,1}} = 0$. Conversely, assume that $R_{D^{0,1}} = 0$ and consider arbitrary vector fields $X, Y \in \mathfrak{X}(M)$. Then $X + iJ_M(X), Y + iJ_M(Y) \in \Gamma(T^{0,1}M)$ and a simple computation shows

$$0 = N_{R_{\nabla, j_E}}^{\mathbb{C}}(X + iJ_M(X), Y + iJ_M(Y)) = 4N_{R_{\nabla, j_E}}(X, Y).$$

That is, if $D^{0,1}$ is flat then $N_{R_{\nabla, j_E}} = 0$. \square

Remark 3.8. Let $j_{\mathbb{C}}: E_{\mathbb{C}} \rightarrow E_{\mathbb{C}}$ be the complexification of $j_E: E \rightarrow E$, and let $E^{1,0}$ and $E^{0,1}$ be as usual the eigenspaces of $j_{\mathbb{C}}$ to the eigenvalues i and $-i$, respectively. The complexification of D in the second argument, or of ∇ in both arguments, written $\nabla^{\mathbb{C}}: \Gamma(T_{\mathbb{C}}M) \times \Gamma(E_{\mathbb{C}}) \rightarrow \Gamma(E_{\mathbb{C}})$ clearly preserves $j_{\mathbb{C}}$. As a consequence, $\nabla^{\mathbb{C}}$ preserves $E^{1,0}$ and $E^{0,1} \subseteq E_{\mathbb{C}}$. Denote by $\theta: E \rightarrow E^{1,0}$ the canonical isomorphism of \mathbb{C} -vector bundles, given by $\theta(e) = \frac{1}{2}(e - ij_E(e))$. Then the equality

$$\nabla_X^{\mathbb{C}}\theta(e) = \theta(D_X e)$$

is immediate for all $e \in \Gamma(E)$ and $X \in \Gamma(T_{\mathbb{C}}M)$. That is, modulo the isomorphism θ , the restriction of $\nabla^{\mathbb{C}}$ to $E^{1,0}$ coincides with D .

Theorems 3.4 and 3.6 and the following result yield together Theorem 1.1.

Theorem 3.9 ((3.5) and (3.7) in [22]). *Let $E \rightarrow M$ be a complex vector bundle over a complex manifold. Then $E \rightarrow M$ is a holomorphic vector bundle if and only if there exists a connection $D = D^{1,0} + D^{0,1}: \Gamma(T_{\mathbb{C}}M) \times \Gamma(E) \rightarrow \Gamma(E)$ such that $D^{0,1}: \Gamma(T^{0,1}M) \times \Gamma(E) \rightarrow \Gamma(E)$ is flat.*

In that case, $D^{0,1}$ is the $\bar{\partial}$ -operator of the holomorphic vector bundle:

$$D_X^{0,1}e = \bar{\partial}_X e = 0$$

for all $X \in \Gamma(T^{0,1}M)$ if and only if e is holomorphic.

Proof of Theorem 1.1. Assume first that E is holomorphic. Then by Lemma 3.1, J_E is a double vector bundle morphism with sides $J_M: TM \rightarrow TM$ and $\text{id}_E: E \rightarrow E$, and with core morphism the multiplication $j_E: E \rightarrow E$ by the complex number i in the fibers of E . Since E is a complex manifold, J_E is integrable.

Conversely, consider a linear complex structure $J_E: TE \rightarrow TE$ with side morphisms $J_M: TM \rightarrow TM$ and $\text{id}_E: E \rightarrow E$ and with core morphism $j_E: E \rightarrow E$. Since $j^2 = -\text{id}_E$, it defines a complex structure in the fibers of E . By Theorem 3.4 and Theorem 3.6, M is a complex manifold with complex structure J_M , and there exists a \mathbb{C} -linear connection $\nabla: \mathfrak{X}(M) \times \Gamma(E) \rightarrow \Gamma(E)$, the complexification $D: \Gamma(T_{\mathbb{C}}M) \times \Gamma(E) \rightarrow \Gamma(E)$ of which equals

$$D = D^{0,1} + D^{1,0}$$

with $D^{0,1}$ a flat $T^{0,1}M$ -connection on E . By Theorem 3.9, the vector bundle $E \rightarrow M$ carries a holomorphic structure over the complex manifold (M, J_M) . It remains to show that $J_E: TE \rightarrow TE$ is the complex structure of the complex manifold E . In order to do that, it is sufficient to prove that if $e \in \Gamma(E)$ is $D^{0,1}$ -flat, i.e. if e is a holomorphic section on E , then

$$[\xi, e^{\uparrow}]_{\mathbb{C}} = 0$$

for all $\xi \in \Gamma_E(T^{0,1}E)$. Here, $e^\dagger \in \mathfrak{X}(E)$ is identified with $e^\dagger \in \Gamma(T^{1,0}E)$ via the canonical \mathbb{C} -linear isomorphism $TE \simeq T^{1,0}E$ as in Remark 3.8.

Consider therefore the complexification $(TE)_{\mathbb{C}}$ of TE (as a vector bundle over E). It is a double vector bundle with sides E and $T_{\mathbb{C}}M$, and with core $E_{\mathbb{C}}$. For $X \in \mathfrak{X}(M)$, $e \in \Gamma(E)$ and $z \in \mathbb{C}$ the equality

$$(e \otimes z)^\dagger = e^\dagger \otimes z$$

is immediate and it is easy to check that

$$\widehat{\nabla_{X \otimes z}^{\mathbb{C}}} = \widehat{\nabla_X} \otimes z.$$

Since ∇ is adapted to J_E , the respective complexifications satisfy

$$J_E^{\mathbb{C}} \circ \widehat{\nabla_X^{\mathbb{C}}} = \widehat{\nabla_{J_M^{\mathbb{C}} X}}$$

for all $X \in \Gamma(T_{\mathbb{C}}M)$, and in particular $\widehat{\nabla_X^{\mathbb{C}}} \in \Gamma_E(T^{0,1}E)$ for $X \in \Gamma(T^{0,1}M)$. Since J_E is linear over J_M , $T^{0,1}E$ is a linear subbundle of $TE_{\mathbb{C}}$ over $T^{0,1}M$ and E , and with core $E^{0,1}$. It is spanned as a vector bundle over E by the sections $\widehat{\nabla_X^{\mathbb{C}}}$ and e^\dagger for $X \in \Gamma(T^{0,1}M)$ and $e \in \Gamma(E^{0,1})$.

Since the Lie bracket of two core vector fields always vanishes, the only equality to check is

$$\left[\widehat{\nabla_X^{\mathbb{C}}}, \theta(e)^\dagger \right]_{\mathbb{C}} = 0$$

for all $X \in \Gamma(T^{0,1}M)$ and $e \in \Gamma(E)$ a $D^{0,1}$ -flat section. This bracket is easily seen to be

$$(17) \quad \left[\widehat{\nabla_X^{\mathbb{C}}}, \theta(e)^\dagger \right]_{\mathbb{C}} = (\nabla_X^{\mathbb{C}} \theta(e))^\dagger = \theta(D_X e)^\dagger = \theta(D_X^{0,1} e)^\dagger = 0.$$

using Remark 3.8. □

3.2. Holomorphic Lie algebroids and infinitesimal ideal systems. Given a complex manifold M , let Θ_M denote the sheaf of holomorphic vector fields on M .

Let $A \rightarrow M$ be a holomorphic vector bundle and let $\rho: A \rightarrow TM$ be a holomorphic vector bundle map, called the anchor. Assume that the sheaf \mathcal{A} of holomorphic sections of $A \rightarrow M$ is a sheaf of complex Lie algebras, the anchor map ρ induces a homomorphism of sheaves of complex Lie algebras from \mathcal{A} to Θ_M , and the Leibniz identity

$$[X, fY] = \mathcal{L}_{\rho(X)} f \cdot Y + f[X, Y]$$

holds for all $X, Y \in \mathcal{A}(U)$, $f \in \mathcal{O}_M(U)$, and all open subsets U of M . Then A is a *holomorphic Lie algebroid*, see e.g. [23] and references therein.

Since the sheaf \mathcal{A} locally generates the $C^\infty(M)$ -module of all smooth sections of A , each holomorphic Lie algebroid structure on a holomorphic vector bundle $A \rightarrow M$ determines a unique smooth real Lie algebroid structure on A . Since $\rho: A \rightarrow TM$ is holomorphic, it satisfies

$$J_{TM} \circ T\rho = T\rho \circ J_A.$$

Restricting this to the cores gives $J_M \circ \rho = \rho \circ j_A$, which implies $\rho_{\mathbb{C}}(A^{0,1}) \subseteq T^{0,1}M$.

Choose a TM -connection on A as in Theorem 3.4 and its complexification as in Remark 3.8. Denote by $\nabla^{0,1}$ the restriction

$$\nabla^{0,1}: \Gamma(T^{0,1}M) \times \Gamma(A^{1,0}) \rightarrow \Gamma(A^{1,0}).$$

Recall that by Remark 3.8 it is just $D^{0,1}$ via the canonical \mathbb{C} -isomorphism $A \simeq A^{1,0}$. That is, the $\nabla^{0,1}$ -flat sections $a \in \Gamma(A^{1,0})$ are exactly the holomorphic sections of $A \simeq A^{1,0}$, hence the elements of \mathcal{A} via this isomorphism. If $a, b \in \Gamma_U(A^{1,0})$ are $\nabla^{0,1}$ -flat (on $U \subseteq M$ open), then

- (i) $[a, b]_{\mathbb{C}}$ is again holomorphic so $\nabla^{0,1}$ -flat,
- (ii) $\rho(a) \in \Theta_M(U)$ is a holomorphic vector field, so $\bar{\partial}$ -flat, where

$$\bar{\partial}: \Gamma(T^{0,1}M) \times \Gamma(T^{1,0}M) \rightarrow \Gamma(T^{1,0}M), \quad \bar{\partial}_X Y = \text{pr}_{T^{1,0}M}[X, Y]_{\mathbb{C}}$$

is just the $\bar{\partial}$ -operator of the holomorphic vector bundle $TM \rightarrow M$.

- (iii) $[a, \nu]_{\mathbb{C}} \in \Gamma(A^{0,1})$ for all $\nu \in \Gamma(A^{0,1})$.

The first two assertions are immediate. For the third one, consider $a \in \mathcal{A}_U$. Then a defines $a^{1,0} = \frac{1}{2}(a - ij(a)) \in \Gamma_U(A^{1,0})$ and $a^{0,1} = \frac{1}{2}(a + ij(a)) \in \Gamma_U(A^{0,1})$. Further, $A_U^{0,1}$ is spanned as a $C^\infty(U)$ -module by sections $b^{0,1}$ defined in this manner. Since for $a, b \in \mathcal{A}(U)$

$$[a^{1,0}, b^{0,1}]_{\mathbb{C}} = \frac{1}{4}[a - j(a) \otimes i, b + j(b) \otimes i]_{\mathbb{C}} = \frac{1}{4}([a, b] + j[a, b] \otimes i - j[a, b] \otimes i - [a, b]) = 0,$$

where $[a, j(b)] = [a, ib] = i[a, b] = j[a, b]$ follows from the fact that \mathcal{A} is a sheaf of complex Lie algebras. Since $A^{1,0} = A_{\mathbb{C}}/A^{0,1}$, this shows that $(T^{0,1}M, A^{0,1}, \nabla^{0,1})$ is a complex infinitesimal ideal system [20] in the complex Lie algebroid $A_{\mathbb{C}}$.

Theorem 3.10. *Let $A \rightarrow M$ be a holomorphic vector bundle. If $A \rightarrow M$ is a holomorphic Lie algebroid then the triple $(T^{0,1}M, A^{0,1}, \nabla^{0,1})$ defined as above is an infinitesimal ideal system in the complex Lie algebroid $A_{\mathbb{C}}$.*

4. LINEAR GENERALISED COMPLEX STRUCTURES AND DORFMAN CONNECTIONS

Let E be a vector bundle over a smooth manifold M . Fix a skew-symmetric Dorfman connection

$$(18) \quad \Delta: \Gamma(TM \oplus E^*) \times \Gamma(E \oplus T^*M) \rightarrow \Gamma(E \oplus T^*M),$$

and therefore a horizontal lift $\sigma^\Delta: \Gamma(TM \oplus E^*) \rightarrow \Gamma_E^\ell(TE \oplus T^*E)$. Consider a double vector bundle morphism $\mathcal{J}: TE \oplus T^*E \rightarrow TE \oplus T^*E$ as in (2). This section gives conditions on j and $j_{\mathbb{C}}$ for the morphism \mathcal{J} to be a generalised complex structure.

Lemma 4.1. *Given a double vector bundle morphism \mathcal{J} over j as in (2) and a skew-symmetric Dorfman connection Δ as in (18), there is a section $\Phi \in \Gamma((TM \oplus E^*)^* \otimes E^* \otimes (E \oplus T^*M))$ such that for any $\nu \in \Gamma(TM \oplus E^*)$*

$$\mathcal{J}(\sigma^\Delta(\nu)) = \sigma^\Delta(j(\nu)) + \widetilde{\Phi(\nu)}.$$

Here, $(TM \oplus E^*)^* \otimes E^* \otimes (E \oplus T^*M)$ is identified with $\text{Hom}(TM \oplus E^*, \text{Hom}(E, E \oplus T^*M))$.

Proof. Since \mathcal{J} is a vector bundle morphism over j , $\mathcal{J}(\sigma^\Delta(\nu))$ is a linear section over $j(\nu)$. Thus $\mathcal{J}(\sigma^\Delta(\nu)) - \sigma^\Delta(j(\nu))$ is a core-linear section and this gives for every ν a section $\Phi(\nu) \in \Gamma(\text{Hom}(E, E \oplus T^*M))$, such that

$$\mathcal{J}(\sigma^\Delta(\nu)) = \sigma^\Delta(j(\nu)) + \widetilde{\Phi(\nu)}.$$

Since \mathcal{J} and j are vector bundle morphisms and σ^Δ is a morphism of $C^\infty(M)$ -modules,

$$\begin{aligned} \widetilde{\Phi(f\nu)} &= \mathcal{J}(\sigma^\Delta(f\nu)) - \sigma^\Delta(j(f\nu)) = \mathcal{J}(q_E^* f \sigma^\Delta(\nu)) - \sigma^\Delta(j(f\nu)) \\ &= q_E^* f \mathcal{J}(\sigma^\Delta(\nu)) - q_E^* f \sigma^\Delta(j(\nu)) = q_E^* f \widetilde{\Phi(\nu)} = \widetilde{f\Phi(\nu)} \end{aligned}$$

for $f \in C^\infty(M)$ and $\nu \in \Gamma(TM \oplus E^*)$. That is, $\Phi(f\nu) = f\Phi(\nu)$ and $\Phi \in \Gamma((TM \oplus E^*)^* \otimes E^* \otimes (E \oplus T^*M))$. \square

Remark 4.2. *In the situation of Lemma 4.1, the following equations hold for $\tau \in \Gamma(E \oplus T^*M)$, $\nu \in \Gamma(TM \oplus E^*)$ and $\varphi \in \Gamma(\text{Hom}(E, E \oplus T^*M))$.*

- (1) $\mathcal{J}(\tau^\uparrow) = \overline{j_C(\tau)^\uparrow}$. This follows directly from the definition of the core morphism j_C .
- (2) $\mathcal{J}(\widetilde{\varphi}) = \overline{j_C \circ \varphi}$. This is an easy calculation using the first equation: For $\epsilon \in \Gamma(E^*)$ and $\tau \in \Gamma(E \oplus T^*M)$, $\mathcal{J}(\widetilde{\epsilon \otimes \tau}) = \mathcal{J}(\ell_\epsilon \cdot \tau^\uparrow) = \ell_\epsilon \cdot \mathcal{J}(\tau^\uparrow) = \ell_\epsilon \cdot \overline{j_C(\tau)^\uparrow} = \overline{j_C \circ (\epsilon \otimes \tau)}$.

The map \mathcal{J} is a generalised complex structure if and only if

- (1) $\mathcal{J}^2 = -\text{id}_{TE \oplus T^*E}$,
- (2) \mathcal{J} is orthogonal, and
- (3) the Nijenhuis tensor of \mathcal{J} vanishes.

The following two lemmas give conditions on j , j_C and Φ for the map $\mathcal{J}: TE \oplus T^*E \rightarrow TE \oplus T^*E$ to satisfy the first two properties. The third property is studied in the next section.

Lemma 4.3. *In the situation of Lemma 4.1, $\mathcal{J}^2 = -\text{id}_{TE \oplus T^*E}$ if and only if*

- (1) $j^2 = -\text{id}_{TM \oplus E^*}$,
- (2) $j_C^2 = -\text{id}_{E \oplus T^*M}$,
- (3) $\Phi(j(\nu)) = -j_C \circ (\Phi(\nu))$ for all $\nu \in \Gamma(TM \oplus E^*)$.

Proof. It is sufficient to check that $\mathcal{J}^2 = -\text{id}_{TE \oplus T^*E}$ on lifts $\sigma^\Delta(\nu)$ for any $\nu \in \Gamma(TM \oplus E^*)$ and on core sections τ^\uparrow for any $\tau \in \Gamma(E \oplus T^*M)$, as those sections span $\Gamma_{TM \oplus E^*}(TE \oplus T^*E)$ as a $C^\infty(TM \oplus E^*)$ -module. (2) is immediate by evaluating \mathcal{J} on core sections, using Remark 4.2.

For linear sections, the definition of Φ and Remark 4.2 yield

$$(19) \quad \begin{aligned} \mathcal{J}^2(\sigma^\Delta(\nu)) &= \mathcal{J}(\sigma^\Delta(j(\nu)) + \widetilde{\Phi(\nu)}) \\ &= \sigma^\Delta(j^2(\nu)) + \widetilde{\Phi(j(\nu))} + \overline{j_C \circ (\Phi(\nu))}. \end{aligned}$$

If $\mathcal{J}^2 = -\text{id}_{TE \oplus T^*E}$, then the side morphism j has to satisfy $j^2 = -\text{id}_{TM \oplus E^*}$. Now (19) implies that

$$(20) \quad \Phi(j(\nu)) = -j_C \circ (\Phi(\nu)).$$

Conversely, if $j^2 = -\text{id}_{TM \oplus E^*}$ and $\Phi(j(\nu)) = -j_C \circ (\Phi(\nu))$, then (19) yields immediately $\mathcal{J}^2(\sigma^\Delta(\nu)) = -\sigma^\Delta(\nu)$ for all $\nu \in \Gamma(TM \oplus E^*)$. \square

Lemma 4.4. *A double vector bundle morphism \mathcal{J} as in (2) is orthogonal if and only if for any skew-symmetric Dorfman connection Δ as in (18):*

- (1) $(j_C)^t = j^{-1}$,
- (2) $\Phi(\nu_2)^t(j\nu_1) = -\Phi(\nu_1)^t(j\nu_2)$ for all $\nu_1, \nu_2 \in \Gamma(TM \oplus E^*)$.

Proof. Again it is sufficient to check the orthogonality of \mathcal{J} on core sections and on horizontal lifts. For core sections this is immediate, since the pairing of two core sections always vanishes. For the pairing of a lift with a core section, use (5) to obtain that

$$\langle \mathcal{J}\sigma^\Delta(\nu), \mathcal{J}(\tau^\uparrow) \rangle = \langle \sigma^\Delta(\nu), \tau^\uparrow \rangle,$$

if and only if $\langle j(\nu), j_C(\tau) \rangle = \langle \nu, \tau \rangle$. Since ν and τ are arbitrary, this is equivalent to $(j_C)^t = j^{-1}$. For the pairing of two lifts compute

$$\langle \mathcal{J}(\sigma^\Delta(\nu_1)), \mathcal{J}(\sigma^\Delta(\nu_2)) \rangle = \ell_{\Phi(\nu_2)^t(j(\nu_1))} + \ell_{\Phi(\nu_1)^t(j(\nu_2))},$$

and on the other hand $\langle \sigma^\Delta(\nu_1), \sigma^\Delta(\nu_2) \rangle = 0$. Hence \mathcal{J} is orthogonal if and only if additionally

$$\Phi(\nu_2)^t(j(\nu_1)) = -\Phi(\nu_1)^t(j(\nu_2)). \quad \square$$

Define $\Psi, \Psi^j, (j^*\Psi) \in \Gamma((TM \oplus E^*)^* \otimes (TM \oplus E^*)^* \otimes E^*)$ by

$$\Psi(\nu_1, \nu_2) := \Phi(\nu_1)^t(\nu_2), \quad \Psi^j(\nu_1, \nu_2) := \Phi(\nu_1)^t(j\nu_2), \quad (j^*\Psi)(\nu_1, \nu_2) := \Psi(j\nu_1, j\nu_2).$$

Then Lemma 4.3 and Lemma 4.4 can be combined as follows

Proposition 4.5. *A morphism \mathcal{J} as in (2) is a generalised almost complex structure on E , if and only if for every skew-symmetric Dorfman connection Δ as in (18):*

- (1) $j^2 = -1$,
- (2) $j = -(j_C)^t$,
- (3) Ψ is skew-symmetric, i.e. $\Psi \in \Omega^2(TM \oplus E^*, E^*)$,
- (4) $\Psi(\nu_1, \nu_2) = -j^*\Psi(\nu_1, \nu_2)$.

Proof. Under assumption of the properties $j^2 = -1$ and $j_C^t = j^{-1}$ the condition on Φ given in Lemma 4.3 can be reformulated as $\Psi(j\nu_1, \nu_2) = \Psi(\nu_1, j\nu_2)$, whereas the condition on Φ given in Lemma 4.4 is then equivalent to $\Psi(\nu_2, j\nu_1) = -\Psi(\nu_1, j\nu_2)$, in both cases for all sections ν_1, ν_2 of $TM \oplus E^*$. Applying the former equation on the latter and then replacing ν_2 by $j\nu_3$ shows skew-symmetry of Ψ , again using $j^2 = -1$. The former equation is shown to be equivalent to $\Psi = -j^*\Psi$, again by replacing ν_2 with $j\nu_3$. \square

4.1. Adapted Dorfman connections. This section shows that given a linear generalised almost complex structure \mathcal{J} on $E \rightarrow M$, there is a Dorfman connection Δ which is **adapted to \mathcal{J}** , i.e. such that σ^Δ satisfies $\mathcal{J}(\sigma^\Delta(\nu)) = \sigma^\Delta(j\nu)$ for all $\nu \in \Gamma(TM \oplus E^*)$. Equivalently, the corresponding tensor Φ defined by Lemma 4.1 vanishes. The choice of such an adapted Dorfman connection vastly simplifies all the following computations in this paper.

Recall from Section 2 that a change of skew-symmetric Dorfman connection (from Δ_1 to Δ_2) is equivalent to a 2-form $\Psi_{12} \in \Omega^2(TM \oplus E^*, E^*)$.

Lemma 4.6. *Let \mathcal{J} be a linear generalised almost complex structure over E and choose two skew-symmetric Dorfman connections Δ_1 and Δ_2 as above with change of splitting Ψ_{12} . Then*

$$(21) \quad \Psi_2(\nu_1, \nu_2) = \Psi_1(\nu_1, \nu_2) - \Psi_{12}(\nu_1, j\nu_2) - \Psi_{12}(j\nu_1, \nu_2),$$

where Ψ_1, Ψ_2 are the 2-forms defined as in Proposition 4.5 by \mathcal{J} and Δ_1 and Δ_2 , respectively.

Proof. A computation yields

$$\widehat{\Phi_2(\nu)} = \mathcal{J}(\sigma_2(\nu)) - \sigma_2(j\nu) = \widehat{\Phi_1(\nu)} + \widehat{j_C \circ \Phi_{12}(\nu)} - \widehat{\Phi_{12}(j\nu)}.$$

Dualising this equality leads to

$$\Psi_2(\nu_1, \nu_2) = \Psi_1(\nu_1, \nu_2) - \Psi_{12}(\nu_1, j\nu_2) - \Psi_{12}(j\nu_1, \nu_2),$$

using $j_C^t = -j$, see Proposition 4.5. \square

Now (21) is used to find the existence of a Dorfman connection adapted to \mathcal{J} .

Proposition 4.7. *For every linear generalised complex structure \mathcal{J} on E there is a skew-symmetric $(TM \oplus E^*)$ -Dorfman connection Δ on $E^* \oplus TM$ such that $\mathcal{J}(\sigma^\Delta(\nu)) = \sigma^\Delta(j\nu)$ for all $\nu \in \Gamma(TM \oplus E^*)$.*

Proof. Fix any skew-symmetric Dorfman connection $TM \oplus E^*$ -Dorfman connection Δ_1 on $E^* \oplus TM$ and denote the corresponding lift by σ_1 . Proposition 4.5 defines a two-form $\Psi_1 \in \Omega^2(TM \oplus E^*, E^*)$ such that $\mathcal{J}(\sigma_1(\nu)) = \sigma_1(j\nu) + \overline{\Psi_1(\nu, \cdot)}$. Let now $\Psi_{12}(\nu_1, \nu_2) := -\frac{1}{2}\Psi_1(\nu_1, j\nu_2)$. By Proposition 4.5 this form is skew-symmetric and therefore the dull bracket defined by (7) is skew-symmetric again. Now according to (21) and using the properties from Proposition 4.5 the corresponding 2-form Ψ_2 vanishes. Hence the Dorfman connection $\Delta := \Delta_2$ satisfies $\Psi_2 = 0$. By definition of Ψ_2 , this is equivalent to $\mathcal{J} \circ \sigma^\Delta = \sigma^\Delta \circ j$. \square

The remainder of this section characterises the set of Dorfman connections adapted to \mathcal{J} .

Definition 4.8. *Let $E \rightarrow M$ be a smooth vector bundle and let $j: TM \oplus E^* \rightarrow TM \oplus E^*$ be an arbitrary vector bundle morphism. Two skew-symmetric $(TM \oplus E^*)$ -Dorfman connection Δ_1 and Δ_2 on $E^* \oplus TM$ are **j -equivalent**, if their change of splittings Ψ_{12} defined by (7) satisfies*

$$\Psi_{12}(\nu_1, \nu_2) = \Psi_{12}(j\nu_1, j\nu_2)$$

for all $\nu_1, \nu_2 \in \Gamma(TM \oplus E^*)$.

The following lemma shows that if a Dorfman connection Δ_1 is adapted to \mathcal{J} , then a second Dorfman connection Δ_2 is adapted to \mathcal{J} if and only if they are j -equivalent.

Lemma 4.9. *Let \mathcal{J} be a linear generalised almost complex structure \mathcal{J} on a vector bundle $E \rightarrow M$ and Δ_1 and Δ_2 be two skew-symmetric $(TM \oplus E^*)$ -Dorfman connections on $E \oplus T^*M$. Denote the two-forms given by Proposition 4.5 corresponding to Δ_1 and Δ_2 by Ψ_1 and Ψ_2 , respectively. Then Δ_1 and Δ_2 are j -equivalent if and only if $\Psi_1 = \Psi_2$.*

Proof. Denote the change of splittings again by Ψ_{12} . Δ_1 is j -equivalent to Δ_2 if and only if for all $\nu_1, \nu_2 \in \Gamma(TM \oplus E^*)$

$$(22) \quad \Psi_{12}(\nu_1, j\nu_2) = -\Psi_{12}(j(j\nu_1), j\nu_2) = -\Psi_{12}(j\nu_1, \nu_2).$$

By (21), Ψ_1 and Ψ_2 are related by

$$\Psi_2(\nu_1, \nu_2) = \Psi_1(\nu_1, \nu_2) - \Psi_{12}(\nu_1, j\nu_2) - \Psi_{12}(j\nu_1, \nu_2).$$

So $\Psi_1 = \Psi_2$ if and only if (22) holds, that is, if and only if Δ_1 and Δ_2 are j -equivalent. \square

4.2. Integrability. Consider a linear generalised almost complex structure \mathcal{J} on E as in (2) and fix a skew-symmetric Dorfman connection Δ as in (18), which is adapted to \mathcal{J} . In particular j, j_C satisfy the conditions of Proposition 4.5, and $\mathcal{J}(\sigma^\Delta(\nu)) = \sigma^\Delta(j\nu)$ for all $\nu \in \Gamma(TM \oplus E^*)$. Evaluated at two core sections the Nijenhuis tensor of \mathcal{J} vanishes trivially, since the Courant-Dorfman bracket of two core sections vanishes and the double vector bundle morphism \mathcal{J} sends core sections to core sections.

For the Nijenhuis tensor of \mathcal{J} evaluated at a horizontal lift $\sigma^\Delta(\nu)$ for $\nu \in \Gamma(TM \oplus E^*)$ and a core section τ^\uparrow for $\tau \in \Gamma(E \oplus T^*M)$, compute

$$N_{\mathcal{J}}(\sigma^\Delta(\nu), \tau^\uparrow) = (\Delta_\nu \tau)^\uparrow - (\Delta_{j(\nu)} j_C(\tau))^\uparrow + (j_C(\Delta_{j(\nu)} \tau))^\uparrow + (j_C(\Delta_\nu j_C(\tau)))^\uparrow.$$

Thus the Nijenhuis tensor of \mathcal{J} vanishes for any such pair of a lift $\sigma^\Delta(\nu)$ and a core section τ^\uparrow if and only if for all $\nu \in \Gamma(TM \oplus E^*)$ and $\tau \in \Gamma(E \oplus T^*M)$

$$(23) \quad \Delta_\nu \tau - \Delta_{j(\nu)} j_C(\tau) + j_C(\Delta_{j(\nu)} \tau) + j_C(\Delta_\nu j_C(\tau)) = 0.$$

As the pairing is non-degenerate, this condition can be dualised by pairing it with a second section $\nu_2 \in \Gamma(TM \oplus E^*)$. Recall that Δ is dual to a dull bracket $[\![\cdot, \cdot]\!]_\Delta$. Then the properties of j and j_C obtained in Proposition 4.5 lead to

$$\left\langle \Delta_{\nu_1} \tau - \Delta_{j(\nu_1)} j_C(\tau) + j_C(\Delta_{j(\nu_1)} \tau) + j_C(\Delta_{\nu_1} j_C(\tau)), \nu_2 \right\rangle = \left\langle \tau, -N_{j, [\![\cdot, \cdot]\!]_\Delta}(\nu_1, \nu_2) \right\rangle.$$

Thus the Nijenhuis tensor of a generalised almost complex structure \mathcal{J} vanishes when evaluated at a pair of any lift $\sigma^\Delta(\nu)$ and any core section τ^\uparrow if and only if the Nijenhuis tensor of j with respect to the dull bracket $[\![\cdot, \cdot]\!]_\Delta$ vanishes.

Finally, compute for $\nu_1, \nu_2 \in \Gamma(TM \oplus E^*)$:

$$\begin{aligned} N_{\mathcal{J}}(\sigma^\Delta(\nu_1), \sigma^\Delta(\nu_2)) &= \sigma^\Delta(N_{j, [\![\cdot, \cdot]\!]_\Delta}(\nu_1, \nu_2)) + \overline{R_\Delta(j(\nu_1), j(\nu_2))(\cdot, 0)} - \overline{R_\Delta(\nu_1, \nu_2)(\cdot, 0)} \\ &\quad - \overline{j_C \circ R_\Delta(j(\nu_1), \nu_2)(\cdot, 0)} - \overline{j_C \circ R_\Delta(\nu_1, j(\nu_2))(\cdot, 0)}. \end{aligned}$$

Recall that since the dull bracket on $TM \oplus E^*$ is anchored by pr_{TM} and $\Delta_{(X, \varepsilon)}(0, \theta) = (0, \mathcal{L}_X \theta)$ the curvature $R_\Delta(\nu_1, \nu_2)(0, \theta)$ for $\theta \in \Gamma(T^*M)$ always vanishes. Therefore the terms with R_Δ above evaluated at $(e, 0)$ vanish if and only if the corresponding terms vanish evaluated at (e, θ) for any $\theta \in \Gamma(T^*M)$. Thus the Nijenhuis tensor of \mathcal{J} vanishes for all sections if and only if the Nijenhuis tensor of j with respect to $[\![\cdot, \cdot]\!]_\Delta$ vanishes and additionally the curvature of the adapted Δ satisfies

$$0 = R_\Delta(j(\nu_1), j(\nu_2))(\tau) - R_\Delta(\nu_1, \nu_2)(\tau) - j_C\left(R_\Delta(j(\nu_1), \nu_2)(\tau)\right) - j_C\left(R_\Delta(\nu_1, j(\nu_2))(\tau)\right),$$

for all $\nu_1, \nu_2 \in \Gamma(TM \oplus E^*)$ and $\tau \in \Gamma(E \oplus T^*M)$. By pairing the right hand side of this equation with a third section ν_3 of $TM \oplus E$, obtain equivalently

$$\begin{aligned} (24) \quad 0 &= \mathbf{Jac}_\Delta(j\nu_1, j\nu_2, \nu_3) + \mathbf{Jac}_\Delta(j\nu_1, \nu_2, j\nu_3) + \mathbf{Jac}_\Delta(\nu_1, j\nu_2, j\nu_3) - \mathbf{Jac}_\Delta(\nu_1, \nu_2, \nu_3) \\ &= [\![\nu_1, [\![\nu_2, \nu_3]\!]_\Delta]\!]_\Delta - [\![j\nu_1, [\![j\nu_2, \nu_3]\!]_\Delta]\!]_\Delta - [\![j\nu_1, [\![\nu_2, j\nu_3]\!]_\Delta]\!]_\Delta - [\![\nu_1, [\![j\nu_2, j\nu_3]\!]_\Delta]\!]_\Delta \\ &\quad + \text{cyclic permutations in } 1, 2, 3 \\ &= [\![\nu_1, [\![\nu_2, \nu_3]\!]_\Delta] - [\![j\nu_2, j\nu_3]\!]_\Delta]_\Delta + [\![j\nu_1, -[\![\nu_2, j\nu_3]\!]_\Delta - [\![j\nu_2, \nu_3]\!]_\Delta]\!]_\Delta \\ &\quad + \text{cyclic permutations in } 1, 2, 3. \end{aligned}$$

Define a bracket \mathbb{A} on $\Gamma(TM \oplus E^*)$ by

$$(25) \quad \mathbb{A}(\nu_1, \nu_2) := \frac{1}{2} \left([\![\nu_1, \nu_2]\!]_\Delta - [\![j\nu_1, j\nu_2]\!]_\Delta \right).$$

Then $N_{j, [\![\cdot, \cdot]\!]_\Delta}$ vanishes if and only if \mathbb{A} satisfies

$$(26) \quad \mathbb{A}(\nu_1, j\nu_2) = j\mathbb{A}(\nu_1, \nu_2).$$

Furthermore, (24) is equivalent to the Jacobi identity of this bracket \mathbb{A} .

Note that the bracket \mathbb{A} does not admit a TM -valued anchor on $TM \oplus E^*$, and is thus not a (real) Lie algebroid bracket on $TM \oplus E^*$, since

$$\mathbb{A}(\nu_1, f\nu_2) = f\mathbb{A}(\nu_1, \nu_2) - \frac{1}{2} \text{pr}_{TM}(j\nu_1)(f)j\nu_2 + \frac{1}{2} \text{pr}_{TM}(\nu_1)(f)\nu_2$$

for $f \in C^\infty(M)$ and $\nu_1, \nu_2 \in \Gamma(TM \oplus E^*)$. However, the map j is a fibre-wise complex structure in $TM \oplus E^*$, with respect to which \mathbb{A} is \mathbb{C} -bilinear by (26) and skew-symmetric. And the equation above shows that with the complex anchor $\rho: TM \oplus E^* \rightarrow T_{\mathbb{C}}M$ defined by

$$(27) \quad \rho(\nu) := \text{pr}_{T_{\mathbb{C}}M} \left(\frac{1}{2}(\nu \otimes 1 - (j\nu) \otimes i) \right),$$

$(TM \oplus E^*, \rho, \mathbb{A})$ is a complex Lie algebroid as defined in [34]. Furthermore, it is easy to check that \mathbb{A} is independent of the choice of adapted Dorfman connection Δ .

Theorem 4.10. *Let $E \rightarrow M$ be a smooth vector bundle. A linear generalised complex structure on E is equivalent to*

- (1) *A vector bundle morphism $j: TM \oplus E^* \rightarrow TM \oplus E^*$ such that $j^2 = -\text{id}_{TM \oplus E^*}$, i.e. a complex structure in the fibers of $TM \oplus E^*$, and*
- (2) *a choice of j -equivalence class of Dorfman $TM \oplus E^*$ -connections on $E \oplus T^*M$ such that $TM \oplus E^* \rightarrow M$, equipped with the bracket \mathbb{A} defined in (25) by the corresponding dull brackets and the anchor ρ in (27) becomes a complex Lie algebroid.*

Proof. Given a linear generalised almost complex structure \mathcal{J} on E as in (2), Proposition 4.5 and Proposition 4.7 define the vector bundle morphism j and an adapted Dorfman connection Δ , in turn uniquely defining the bracket \mathbb{A} on $\Gamma(TM \oplus E^*)$ by (25). By the arguments above, integrability of \mathcal{J} implies that \mathbb{A} is a complex Lie algebroid bracket with anchor given by (27).

Conversely, given j and a complex Lie algebroid structure where the bracket \mathbb{A} comes from a Dorfman connection Δ as above, define a double vector bundle morphism $\mathcal{J}: TE \oplus T^*E \rightarrow TE \oplus T^*E$ by

$$\mathcal{J}(\tau^\uparrow) := (-j^t(\tau))^\uparrow, \quad \mathcal{J}(\sigma^\Delta(\nu)) := \sigma^\Delta(j(\nu))$$

for $\tau \in \Gamma(E \oplus T^*M)$ and $\nu \in \Gamma(TM \oplus E^*)$. Again by Proposition 4.5 and the computations above of the Nijenhuis tensor of \mathcal{J} on linear and core sections, this defines a linear generalised complex structure on E .

These two constructions are inverse to each other since neither \mathcal{J} nor \mathbb{A} depend on the choice of the Dorfman connection in the j -equivalence class of Δ . \square

4.3. Generalised Kähler structures on vector bundles. Generalised Kähler structures were introduced in [9, 12]. Any automorphism G of $TM \oplus T^*M$ which is symmetric $G^t = G$, and squares to id defines a symmetric metric on $TM \oplus T^*M$. Such an automorphism is therefore called a *metric*.

Definition 4.11. *A generalised Kähler structure on a manifold is a pair of commuting generalised complex structures \mathcal{J}_1 and \mathcal{J}_2 such that the symmetric non-degenerate metric $G := -\mathcal{J}_1 \circ \mathcal{J}_2$ is positive definite.*

This section shows that in the case of a generalised Kähler structure on a vector bundle E there exists a Dorfman connection which is adapted in the sense of Proposition 4.7 to both generalised complex structures simultaneously. Take a vector bundle $E \rightarrow M$ equipped with a *linear* generalised Kähler structure, i.e. \mathcal{J}_1 and \mathcal{J}_2 are both linear. Denote the side morphisms on $TM \oplus E^*$ by j_1 and j_2 , respectively. Take now any skew-symmetric $TM \oplus E^*$ -Dorfman connection Δ on $E \oplus T^*M$. This gives rise to the corresponding 2-forms Ψ_1 and Ψ_2 as in Proposition 4.5.

Lemma 4.12. *In the setting above, for all $\nu_1, \nu_2 \in TM \oplus E^*$*

$$\Psi_2(j_1\nu_1, \nu_2) + \Psi_2(\nu_1, j_1\nu_2) = \Psi_1(j_2\nu_1, \nu_2) + \Psi_1(\nu_1, j_2\nu_2).$$

Proof. Since \mathcal{J}_1 and \mathcal{J}_2 commute, so do their side morphisms j_1 and j_2 . Thus $\sigma^\Delta(j_1j_2\nu) = \sigma^\Delta(j_2j_1\nu)$ and with the definition of Φ_1 and Φ_2 in Lemma 4.1 this leads by straightforward computation to the equality $j_1^t \circ \Phi_2(\nu) - \Phi_1(j_2\nu) = j_2^t \circ \Phi_1(\nu) - \Phi_2(j_1\nu)$. Pairing this with a second arbitrary section $\nu_2 \in \Gamma(TM \oplus E^*)$ and dualising then gives the desired equality. \square

This lemma easily yields the existence of a Dorfman connection adapted to both generalised complex structures.

Proposition 4.13. *Given two commuting linear generalised complex structures \mathcal{J}_1 and \mathcal{J}_2 on a vector bundle $E \rightarrow M$, there is a $TM \oplus E^*$ -Dorfman connection Δ on $E \oplus T^*M$ which is adapted to both \mathcal{J}_1 and \mathcal{J}_2 .*

Proof. Take a skew-symmetric Dorfman connection Δ_1 , which is adapted to \mathcal{J}_1 as constructed in Proposition 4.7. Thus the 2-form Ψ_1 vanishes. The previous Lemma 4.12 then shows that $\Psi_2(\nu_1, \nu_2) = \Psi_2(j_1\nu_1, j_1\nu_2)$ for all $\nu_1, \nu_2 \in \Gamma(TM \oplus E^*)$. In order to obtain a Dorfman connection Δ_2 adapted to \mathcal{J}_2 , use as change of splitting the form $\Psi_{12} := -\frac{1}{2}\Psi_2(\cdot, j_2\cdot)$ as shown in the proof of Proposition 4.7. But since j_1 and j_2 commute, also this change of splittings satisfies $\Psi_{12}(j_1\nu_1, j_1\nu_2) = \Psi_{12}(\nu_1, \nu_2)$ for all $\nu_1, \nu_2 \in TM \oplus E^*$. Thus Δ_2 is j_1 -equivalent to Δ_1 as in Definition 4.8. Now Lemma 4.9 shows that Δ_2 is still adapted to \mathcal{J}_1 and therefore to both generalised complex structures simultaneously. \square

4.4. Complex VB-Dirac structures. This section shows that a linear generalised complex structure on a vector bundle E is equivalent to a pair of complex conjugated VB-Dirac structures. Consider the complexification of $\mathbb{T}E$ as a vector bundle over E . This is again a double vector bundle $\mathbb{T}_{\mathbb{C}}E \cong T_{\mathbb{C}}E \oplus T_{\mathbb{C}}^*E$ with complexified core and side bundle.

$$\begin{array}{ccc} T_{\mathbb{C}}E \oplus T_{\mathbb{C}}^*E & \longrightarrow & E \\ \downarrow (\pi_{TM \oplus E^*})_{\mathbb{C}} & & \downarrow \\ T_{\mathbb{C}}M \oplus E_{\mathbb{C}}^* & \longrightarrow & M \end{array} \quad \begin{array}{c} \\ \\ \\ \searrow \\ \\ \end{array} \begin{array}{c} \\ \\ \\ E_{\mathbb{C}} \oplus T_{\mathbb{C}}^*M \\ \\ \end{array} \begin{array}{c} \\ \\ \\ \\ \\ \end{array} .$$

It is straightforward to extend the results of [16] complex linearly to characterize linear splittings of the double vector bundle $\mathbb{T}_{\mathbb{C}}E$ which are additionally \mathbb{C} -linear over E , giving a correspondence between such splittings and complex $(T_{\mathbb{C}}M \oplus E_{\mathbb{C}}^*)$ -Dorfman connections on $E_{\mathbb{C}} \oplus T_{\mathbb{C}}^*M$.

Furthermore, the splitting corresponding to the complexification $\Delta^{\mathbb{C}}$ of a real Dorfman connection Δ is the complexification of the splitting Σ^{Δ} , i.e. the complex linear extension in $TM \oplus E^*$.

Let $\mathcal{J}: \mathbb{T}E \rightarrow \mathbb{T}E$ be a linear generalised complex structure on E over $j: TM \oplus E^* \rightarrow TM \oplus E^*$ and with core morphism $j_{\mathbb{C}}$ as in Definition 1.2. The $\pm i$ -eigenbundles of the complexification $\mathcal{J}_{\mathbb{C}}: \mathbb{T}_{\mathbb{C}}E \rightarrow \mathbb{T}_{\mathbb{C}}E$ of \mathcal{J} build a pair of complex conjugated complex Dirac structures $D_{\pm} \subset \mathbb{T}_{\mathbb{C}}E$:

$$D_+ = \{\xi - i\mathcal{J}(\xi) \mid \xi \in \mathbb{T}E\} \quad D_- = \{\xi + i\mathcal{J}(\xi) \mid \xi \in \mathbb{T}E\}.$$

It is easy to see that these are two sub-double vector bundles of $\mathbb{T}_{\mathbb{C}}E$

$$\begin{array}{ccc} D_{\pm} & \longrightarrow & E \\ (\pi_{TM \oplus E^*})_{\mathbb{C}}|_{D_{\pm}} \downarrow & & \downarrow q_E \\ U_{\pm} & \longrightarrow & M \end{array},$$

where U_{\pm} is the $\pm i$ -eigenbundle of the complexification $j_{\mathbb{C}}: (TM \oplus E^*)_{\mathbb{C}} \rightarrow (TM \oplus E^*)_{\mathbb{C}}$ of j . The core of this double vector bundle is $K_{\pm} \subseteq (E \oplus T^*M)_{\mathbb{C}}$, the $\pm i$ -eigenbundle of the complexification $j_{\mathbb{C}, \mathbb{C}}$ of $j_{\mathbb{C}}$.

Lemma 4.14. *In the situation above,*

$$U_{\pm}^{\circ} = K_{\pm},$$

where U_{\pm}° denotes the annihilator of U_{\pm} in $(TM \oplus E^*)^* \cong E \oplus T^*M$.

Proof. This follows directly from the property $j^t = -j_{\mathbb{C}}$ of Proposition 4.5. \square

Now consider the complex linear extension $\mathbb{A}_{\mathbb{C}}$ of \mathbb{A} defined in (25).

Proposition 4.15. *The restriction to U_{\pm} of the complexified bracket $\mathbb{A}_{\mathbb{C}}$ coincides with the restriction of the dull bracket $[[\cdot, \cdot]]_{\Delta^{\mathbb{C}}}$ corresponding to the complexification of an adapted Dorfman connection Δ and defines a \mathbb{C} -Lie algebroid structure on U_{\pm} with anchor $\text{pr}_{T_{\mathbb{C}}M}|_{U_{\pm}}$.*

It is isomorphic as complex Lie algebroid to $(TM \oplus E^, \rho, \mathbb{A})$ of Theorem 4.10 via the canonical isomorphism $\nu \mapsto \frac{\nu \mp ij\nu}{2}$.*

Proof. By (26) the complexified bracket $\mathbb{A}_{\mathbb{C}}$ restricts to the two eigenbundles U_{\pm} of $j_{\mathbb{C}}$.

It follows directly from the definition of \mathbb{A} in (25) that the restriction of $\mathbb{A}_{\mathbb{C}}$ to U_{\pm} coincides with the complexification of $[[\cdot, \cdot]]_{\Delta}$ for any adapted Dorfman connection Δ . Therefore it is anchored by the restriction of $\text{pr}_{T_{\mathbb{C}}M}$ and satisfies the Leibniz identity. Skew-symmetry and Jacobi identity follow from the corresponding properties for \mathbb{A} .

It is easy to check that $TM \oplus E^* \rightarrow U_{\pm}$, $\nu \mapsto \nu \mp ij\nu$ is indeed an isomorphism of complex Lie algebroids, where the fibre-wise complex structures are j on one side and induced from the complexification on the other. \square

Proposition 4.15 and Theorem 4.10 now imply Theorem 1.4.

Proof of Theorem 1.4. The complex Lie algebroid is induced as in Theorem 4.10 by a linear generalised complex structure is quasi-real by Proposition 4.15.

Conversely, let $E \rightarrow M$ be a smooth vector bundle with a vector bundle morphism $j: TM \oplus E^* \rightarrow TM \oplus E^*$ such that $j^2 = -\text{id}_{TM \oplus E^*}$. Assume that $(TM \oplus E^*, \text{pr}_{TM}, j, [\cdot, \cdot])$ is a quasi-real complex Lie algebroid as in Definition 1.3. Then there is a dull bracket $[[\cdot, \cdot]]$ on sections of $TM \oplus E^*$, that is anchored by pr_{TM} and such that the canonical isomorphism $TM \oplus E^* \rightarrow (TM \oplus E^*)^{1,0}$ is an isomorphism of the complex Lie algebroids $((TM \oplus E^*)^{1,0}, \text{pr}_{TM}^{1,0}, j_{\mathbb{C}}, [[\cdot, \cdot]]_{1,0})$ and $(TM \oplus E^*, \text{pr}_{TM}, j, [\cdot, \cdot])$. The compatibility of the anchors is immediate by definition of $\text{pr}_{TM, j}$.

Then for all $\nu_1, \nu_2 \in \Gamma(TM \oplus E^*)$,

$$(28) \quad \frac{1}{2}([\nu_1, \nu_2] \otimes 1 - j[\nu_1, \nu_2] \otimes i) = \frac{1}{4}[[\nu_1 \otimes 1 - j(\nu_1) \otimes i, \nu_2 \otimes 1 - j(\nu_2) \otimes i]].$$

The right-hand side of this equation is

$$\frac{1}{2} \left(\frac{[[\nu_1, \nu_2]] - [[j\nu_1, j\nu_2]]}{2} \otimes 1 - \frac{[[j\nu_1, \nu_2]] + [[\nu_1, j\nu_2]]}{2} \otimes i \right),$$

which, compared with its left-hand side, gives

$$\begin{cases} [\nu_1, \nu_2] &= \frac{[[\nu_1, \nu_2]] - [[j\nu_1, j\nu_2]]}{2} \\ j[\nu_1, \nu_2] &= \frac{[[j\nu_1, \nu_2]] + [[\nu_1, j\nu_2]]}{2} \end{cases}.$$

Since $\nu_1, \nu_2 \in \Gamma(TM \oplus E^*)$ were arbitrary, these equations yield that $[[\cdot, \cdot]]$ has vanishing Nijenhuis tensor with respect to j : $N_{j, [[\cdot, \cdot]]} = 0$ and $[\cdot, \cdot]$ is defined by $[[\cdot, \cdot]]$ and j as in (25).

Finally, assume that $[[\cdot, \cdot]]$ and $[[\cdot, \cdot]]'$ both satisfy (28) and consider the form $\Psi \in \Omega^2(TM \oplus E^*, E^*)$ defined by

$$[[\nu_1, \nu_2]]' = [[\nu_1, \nu_2]] + (0, \Psi(\nu_1, \nu_2))$$

for all $\nu_1, \nu_2 \in \Gamma(TM \oplus E^*)$, see (7). Then

$$\frac{[[\nu_1, \nu_2]]' - [[j\nu_1, j\nu_2]]'}{2} = [\nu_1, \nu_2] = \frac{[[\nu_1, \nu_2]] - [[j\nu_1, j\nu_2]]}{2}$$

and so

$$\begin{aligned} [[\nu_1, \nu_2]] - [[j\nu_1, j\nu_2]] &= [[\nu_1, \nu_2]]' - [[j\nu_1, j\nu_2]]' \\ &= [[\nu_1, \nu_2]] + (0, \Psi(\nu_1, \nu_2)) - [[j\nu_1, j\nu_2]] - (0, \Psi(j\nu_1, j\nu_2)), \end{aligned}$$

which implies $\Psi(\nu_1, \nu_2) = \Psi(j\nu_1, j\nu_2)$ for all $\nu_1, \nu_2 \in \Gamma(TM \oplus E^*)$. Hence, the Dorfman connections defined by the two dull brackets are j -equivalent as in Definition 4.8.

By Theorem 4.10, E is thus equipped with a linear generalised complex structure \mathcal{J} with core j and such that $[[\cdot, \cdot]]$ is adapted to \mathcal{J} . \square

The following example discusses how Theorem 1.4 specialises in the case of holomorphic vector bundles.

Example 4.16. *Let $E \rightarrow M$ be a holomorphic vector bundle. As shown in Section 3 this corresponds to a linear complex structure J_E on E over $J_M: TM \rightarrow TM$ with core morphism $j_E: E \rightarrow E$. The corresponding generalised complex structure \mathcal{J} , its side morphism j and core morphism j_C are*

$$(29) \quad \mathcal{J} = \begin{pmatrix} J_E & 0 \\ 0 & -J_E^t \end{pmatrix}, \quad j = \begin{pmatrix} J_M & 0 \\ 0 & -j_E^t \end{pmatrix}, \quad j_C = \begin{pmatrix} j_E & 0 \\ 0 & -j_M^t \end{pmatrix}.$$

The equality $j = -j_C^t$ is immediate.

The eigenbundles of the complexified morphisms are given by

$$(30) \quad \begin{aligned} U_+ &= T^{1,0}M \oplus (E^{0,1})^*, & K_+ &= E^{1,0} \oplus (T^{0,1}M)^*, \\ U_- &= T^{0,1}M \oplus (E^{1,0})^*, & K_- &= E^{0,1} \oplus (T^{1,0}M)^*, \end{aligned}$$

where $T^{1,0}M$, $T^{0,1}M$, $E^{1,0}$ and $E^{0,1}$ are the $\pm i$ -eigenbundles of J_M and j_E , respectively.

A linear generalised complex structure on E is simply a holomorphic structure on E if and only if the generalised complex structure is of the form above, see Section 3. In that case the complex Lie algebroid structure on $TM \oplus E^*$ with fibrewise complex structure $j = (J_M, -j_E^t)$ is given by the anchor

$$\rho: TM \oplus E^* \rightarrow T^{1,0}M \subseteq T_{\mathbb{C}}M, \quad \rho(X, \varepsilon) = \frac{1}{2}(X \otimes 1 - (J_M X) \otimes i),$$

and, by (6), the bracket

$$\mathbb{A}((X_1, \varepsilon_1), (X_2, \varepsilon_2)) = \frac{1}{2} \left([X_1, X_2] - [J_M X_1, J_M X_2], \nabla_{X_1}^* \varepsilon_2 - \nabla_{X_2}^* \varepsilon_1 + \nabla_{J_M X_1}^* j_E^* \varepsilon_2 - \nabla_{J_M X_2}^* j_E^* \varepsilon_1 \right)$$

where ∇^* is a flat TM -connection on E^* , dual to a TM -connection ∇ on E which is adapted to the linear complex structure J_E , see Section 3.

Now TM is \mathbb{C} -isomorphic to $T^{1,0}M$ and E^* is \mathbb{C} -isomorphic to $(E^{0,1})^*$, since the complex structure on E^* is taken to be $-j_E^t$. After these identifications, ∇^* is dual to the $T^{1,0}M$ -connection ∇ on $E^{0,1}$, defined as complex conjugate to the flat $T^{0,1}M$ -connection $\bar{\partial}$ on $E^{1,0} \cong E$ corresponding to the holomorphic structure. That is for $X \in \Gamma(T^{1,0}M)$ and $e \in \Gamma(E^{0,1})$,

$$\nabla_X e := \overline{\partial_{\bar{X}} \bar{e}}.$$

Hence $TM \oplus E^*$ is isomorphic as a complex Lie algebroid to the Lie algebroid $T^{1,0}M \oplus (E^{0,1})^*$ which is induced by this connection, with bracket $([X_1, X_2], \nabla_{X_1}^* \varepsilon_2 - \nabla_{X_2}^* \varepsilon_1)$ for $X_1, X_2 \in \Gamma(T^{1,0})$ and $\varepsilon_1, \varepsilon_2 \in \Gamma((E^{0,1})^*)$. Complexification of \mathbb{A} and restriction to these eigenbundles gives precisely this bracket.

Example 4.17. In the case where the generalised complex structure is induced by a linear symplectic structure $\omega^b: TE \rightarrow T^*E$, the side morphism is coming from an isomorphism $\tau: TM \rightarrow E^*$. The fibrewise complex structure j on $TM \oplus E^*$ is after identification of E^* with TM via τ simply the complex structure of $T_{\mathbb{C}}M$, the anchor is then given by $\rho = \frac{1}{2} \text{id}_{T_{\mathbb{C}}M}$ and the bracket \mathbb{A} is given by $\frac{1}{2}[\cdot, \cdot]_{\mathbb{C}}$. This follows from the fact that $TM \oplus E^*$ is isomorphic as complex Lie algebroid to $U_+ = \text{graph}(-\tau_{\mathbb{C}})$ according to Proposition 4.15, and the bracket of $\text{graph}(-\tau)$ is shown in [16] to be the Lie bracket of vector fields. The factor of $\frac{1}{2}$ comes here from the isomorphism $TM \oplus E^* \rightarrow U_+$.

The description of VB-Dirac structures in $\mathbb{T}E$ via adapted Dorfman connections of [16] can directly be extended to complex VB-Dirac structures and adapted complex Dorfman connections, by simply demanding complex linearity where appropriate. This leads to the following adaptation of a theorem in [16].

Theorem 4.18. Let D be a sub-double vector bundle of $\mathbb{T}_{\mathbb{C}}E$ over E and $U \subseteq T_{\mathbb{C}}M \oplus E_{\mathbb{C}}^*$, with core $K \subseteq E_{\mathbb{C}} \oplus T_{\mathbb{C}}^*M$ such that D is a complex subbundle of $\mathbb{T}_{\mathbb{C}}E \rightarrow E$. Let Δ be a complex $(T_{\mathbb{C}}M \oplus E_{\mathbb{C}}^*)$ -Dorfman connection on $E_{\mathbb{C}} \oplus T_{\mathbb{C}}^*M$ which is adapted to D . Then D is a complex VB-Dirac structure if and only if $U = K^\circ$ and $(U, \text{pr}_{T_{\mathbb{C}}M}|_U, [\cdot, \cdot]_{\Delta}|_U)$ is a complex Lie algebroid.

This description of complex VB-Dirac structures together with Theorem 4.10 then leads to the following description of linear generalised complex structures.

Corollary 4.19. A linear generalised complex structure \mathcal{J} on a vector bundle E is equivalent to a pair of transverse, complex conjugated complex VB-Dirac structures D_{\pm} in $\mathbb{T}_{\mathbb{C}}E$.

5. GENERALISED COMPLEX STRUCTURES ON LIE ALGEBROIDS

Let $A \rightarrow M$ be a Lie algebroid with anchor $\rho: A \rightarrow TM$. In this case the generalised tangent bundle $\mathbb{T}A$ is itself a Lie algebroid over the side $TM \oplus A^*$, as described for example in [24] and [16]. This section considers a linear generalised complex structure on A , that is also compatible with the Lie algebroid structure on $\mathbb{T}A$.

Definition 5.1. [21] *A generalised complex Lie algebroid is a Lie algebroid $A \rightarrow M$ equipped with a linear generalised complex structure $\mathcal{J}: \mathbb{T}A \rightarrow \mathbb{T}A$ which is also a Lie algebroid morphism over the side morphism $j: TM \oplus A^* \rightarrow TM \oplus A^*$.*

In the situation of the last definition, choose a Dorfman connection $\Delta: \Gamma(TM \oplus A^*) \times \Gamma(A \oplus T^*M) \rightarrow \Gamma(A \oplus T^*M)$ that is adapted to \mathcal{J} . It follows directly from the symmetry of the linear splitting that also the lift $\sigma^\Delta: \Gamma(A) \rightarrow \Gamma_{TM \oplus A^*}^\ell(\mathbb{T}A)$ is compatible with \mathcal{J} , that is

$$\mathcal{J}(\sigma^\Delta(a)) = \sigma^\Delta(a) \circ j$$

for all $a \in \Gamma(A)$.

Hence, by the results in [6], \mathcal{J} is a Lie algebroid morphism if and only if $(j, -j^t, 0)$ is an automorphism of the 2-representation $((\rho, \rho^t), \nabla_a^{\text{bas}}, \nabla_a^{\text{bas}}, R_\Delta^{\text{bas}})$ corresponding to the VB-algebroid structure on $TA \oplus T^*A$ in the linear splitting defined by the adapted skew-symmetric Dorfman connection Δ , see Theorem 2.10. That is,

- (1) $j \circ (\rho, \rho^t) = -(\rho, \rho^t) \circ j^t$,
- (2) $\nabla_a^{\text{bas}, \text{Hom}}(j, -j^t) = 0$ for all $a \in \Gamma(A)$, and
- (3) $-j^t \circ R_\Delta^{\text{bas}} = R_\Delta^{\text{bas}} \circ j$.

Since the basic connections defined by a skew-symmetric Dorfman connection are dual to each other, the second equality reduces to $\nabla_a^{\text{bas}} \circ j = j \circ \nabla_a^{\text{bas}}$ for all $a \in \Gamma(A)$. As observed before, j and $j_C = -j^t$ are fibrewise complex structures on $TM \oplus A^*$ and $A \oplus T^*M$, respectively. The properties above simply state that (ρ, ρ^t) , ∇_a^{bas} and R_Δ^{bas} are all complex linear. Together with Theorem 4.10 this immediately give the following characterisation of a generalised complex Lie algebroid.

Theorem 5.2. *Let A be a Lie algebroid over M with anchor ρ . A linear generalised complex structure $(j, [\Delta])$ on A (see Thm 4.10) is compatible with the Lie algebroid structure in the sense of Definition 5.1 if and only if the basic connections and basic curvature induced by any representative Δ in $[\Delta]$, as well as the map (ρ, ρ^t) are complex linear with respect to the complex structures j on $TM \oplus A^*$ and $-j^t$ on $A \oplus T^*M$.*

A straightforward complex extension of the corresponding result in [17] yields the following description of complex LA-Dirac structures in terms of an adapted splitting.

Corollary 5.3. *A complex VB-Dirac structure $D \subseteq \mathbb{T}_\mathbb{C}A$ with side $U \subseteq T_\mathbb{C}M \oplus A_\mathbb{C}^*$ and core $K \subseteq A_\mathbb{C} \oplus T_\mathbb{C}^*M$ is additionally a Lie subalgebroid of $\mathbb{T}_\mathbb{C}A \rightarrow T_\mathbb{C}M \oplus A_\mathbb{C}^*$ if and only for an adapted complex $(T_\mathbb{C}M \oplus A_\mathbb{C}^*)$ -Dorfman connection Δ on $A_\mathbb{C} \oplus T_\mathbb{C}^*M$ the following conditions are satisfied for all $a, b \in \Gamma(A)$, $u \in \Gamma(U)$:*

- (1) $(\rho, \rho^t)_\mathbb{C}(K) \subseteq U$,
- (2) $\nabla_a^{\text{bas}} u \in \Gamma(U)$,
- (3) $R_\Delta^{\text{bas}}(a, b)u \in \Gamma(K)$.

Note that here ∇_a^{bas} and R_Δ^{bas} denote the complex basic connection and complex basic curvature defined by complex linear extension of the formulas in the real case.

The following description of generalised complex Lie algebroids in terms of LA-Dirac structures is an immediate consequence of this description of complex LA-Dirac structures, together with Theorem 5.3 and Corollary 4.19.

Corollary 5.4. *A generalised complex structure on a Lie algebroid is equivalent to a pair of transverse, complex LA-Dirac structures in $\mathbb{T}_\mathbb{C}A$.*

Proof. According to Corollary 4.19 a linear generalised complex structure \mathcal{J} on A is equivalent to a pair of transversal, complex VB-Dirac structures D_{\pm} . Choose a Dorfman connection adapted to \mathcal{J} as in Proposition 4.7. Then the complexification of Δ is adapted to D_+ and to D_- simultaneously. According to the considerations above, \mathcal{J} is a Lie algebroid morphism if and only if

- (1) $(\rho, \rho^t) \circ j_C = j \circ (\rho, \rho^t)$,
- (2) $\nabla_a^{\text{bas}} \circ j = j \circ \nabla_a^{\text{bas}}$,
- (3) $j_C \circ R_{\Delta}^{\text{bas}}(a, b) = R_{\Delta}^{\text{bas}}(a, b) \circ j$.

The first condition is equivalent to $(\rho, \rho^t)(K_{\pm}) \subseteq U_{\pm}$, the second condition is equivalent to $\nabla_a^{\text{bas}} u_{\pm} \in \Gamma(U_{\pm})$ for any $a \in \Gamma(A)$ and $u_{\pm} \in \Gamma(U_{\pm})$ and the third condition is equivalent to $R_{\Delta}^{\text{bas}}(a, b)(u_{\pm}) \in \Gamma(K_{\pm})$ for all $a, b \in \Gamma(A)$ and $u_{\pm} \in \Gamma(U_{\pm})$. According to Theorem 4.18 these conditions are equivalent to D_{\pm} being complex LA-Dirac structures. \square

5.1. The degenerate generalised complex structure on $A \oplus T^*M$. Recall that the vector bundle $A \oplus T^*M$ can be equipped with the structure of a degenerate Courant algebroid as described in [17]. The anchor is given by $\rho \circ \text{pr}_A: A \oplus T^*M \rightarrow TM$, the (possibly degenerate) pairing and the bracket are given by

$$(31) \quad \begin{aligned} \langle (a, \theta), (b, \eta) \rangle_d &:= \langle \rho(a), \eta \rangle + \langle \rho(b), \theta \rangle, \\ \llbracket (a, \theta), (b, \eta) \rrbracket_d &:= ([a, b], \mathcal{L}_{\rho(a)}\eta - i_{\rho(b)}\mathbf{d}\theta). \end{aligned}$$

where $a, b \in \Gamma(A)$ and $\theta, \eta \in \Gamma(T^*M)$.

This anchor, bracket and pairing satisfy all properties of a Courant algebroid except for the non-degeneracy of the pairing. As shown in [17], the bracket can equivalently be described in terms of the Dorfman connection Δ :

$$(32) \quad \llbracket \tau_1, \tau_2 \rrbracket_d = \Delta_{(\rho, \rho^t)\tau_1} \tau_2 - \nabla_{\text{pr}_A}^{\text{bas}} \tau_2 \tau_1$$

for $\tau_1, \tau_2 \in \Gamma(A \oplus T^*M)$.

Proposition 5.5. *The core morphism $j_C: A \oplus T^*M \rightarrow A \oplus T^*M$ of \mathcal{J} satisfies $j_C^2 = -\text{id}$, is orthogonal with respect to $\langle \cdot, \cdot \rangle_d$ and the Nijenhuis torsion of j_C with respect to $\llbracket \cdot, \cdot \rrbracket_d$ vanishes. Hence, j_C is a **degenerate generalised complex structure in the degenerate Courant algebroid $A \oplus T^*M$.***

Proof. Recall from Proposition 4.5 that $j_C^2 = -\text{id}$ and $j = -j_C^t$. Together with the property $j \circ (\rho, \rho^t) = -(\rho, \rho^t) \circ j^t = (\rho, \rho^t) \circ j_C$, see Theorem 5.2, it is easy to check that j_C is orthogonal with respect to the degenerate pairing.

Theorem 5.2 gives as well the equality $\nabla_a^{\text{bas}} \circ j_C = j_C \circ \nabla_a^{\text{bas}}$. Using the formula (32) it is then easy to compute the Nijenhuis torsion of j_C :

$$N_{j_C, \llbracket \cdot, \cdot \rrbracket_d}(\tau_1, \tau_2) = \Delta_{(\rho, \rho^t)\tau_1} \tau_2 - \Delta_{(\rho, \rho^t)j_C\tau_1} j_C\tau_2 + j_C\Delta_{(\rho, \rho^t)j_C\tau_1} \tau_2 + j_C\Delta_{(\rho, \rho^t)\tau_1} j_C\tau_2,$$

which vanishes for any linear generalised complex structure according to (23) with $\nu = (\rho, \rho^t)\tau_1$ and $\tau = \tau_2$. \square

Proposition 5.6. *Let $A \rightarrow M$ be a Lie algebroid. The restriction of the degenerate Courant algebroid structure on $A_{\mathbb{C}} \oplus T_{\mathbb{C}}^*M$ induces a complex Lie algebroid structure on K_{\pm} .*

Proof. According to Proposition 5.5 the morphism j_C is a generalised complex structure in $A \oplus T^*M$. The vanishing of the Nijenhuis tensor and \mathbb{C} -linearity of the complexified bracket imply that the bracket restricts to the $\pm i$ -eigenbundle K_{\pm} of $j_{C, \mathbb{C}}$. Since $(\rho, \rho^t)_{\mathbb{C}}$ sends

K_{\pm} to $U_{\pm} = K_{\pm}^{\circ}$, the pairing restricted to K_{\pm} vanishes and thus the restricted bracket is skew-symmetric and defines a Lie algebroid structure on K_{\pm} . \square

5.2. The complex A -Manin pair. [17] defines A -Manin pairs for a given Lie algebroid A over M and constructs an equivalence between A -Manin pairs and Dirac bialgebroids over A . Again, the complex linear extension of these results is straightforward.

Definition 5.7. *Let $A \rightarrow M$ be a Lie algebroid. A **complex A -Manin pair** consists of a complex Courant algebroid C over M , a complex Dirac structure $U \rightarrow M$ in C , with a morphism $\iota: U \hookrightarrow T_{\mathbb{C}}M \oplus A_{\mathbb{C}}^*$ such that $\rho_U = \text{pr}_{T_{\mathbb{C}}M} \circ \iota$ and a morphism of (degenerate) complex Courant algebroids $\Phi: A_{\mathbb{C}} \oplus T_{\mathbb{C}}^*M \rightarrow C$ such that*

$$\Phi(A_{\mathbb{C}} \oplus T_{\mathbb{C}}^*M) + U = C$$

and $\langle u, \Phi(\tau) \rangle_C = \langle \iota(u), \tau \rangle$ for all $(u, \tau) \in U \times_M (A_{\mathbb{C}} \oplus T_{\mathbb{C}}^*M)$.

[17] shows that the Courant algebroid structure on C and the morphism Φ can be recovered from the Lie algebroid structures on A , U and ι . All the arguments can be extended complex linearly to obtain the following straightforward consequence.

Proposition 5.8. *Let U_{\pm} be the $\pm i$ -eigenbundles of the side morphism j of a generalised complex structure \mathcal{J} on a Lie algebroid $A \rightarrow M$, and let $K_{\pm} = U_{\mp}^{\circ}$. Define*

$$C_{\pm} := \frac{U_{\pm} \oplus (A_{\mathbb{C}} \oplus T_{\mathbb{C}}^*M)}{\text{graph}(-(\rho, \rho^t)_{\mathbb{C}}|_{K_{\pm}})},$$

and define an anchor map, a \mathbb{C} -bilinear pairing and a bracket as follows. For $u, u_1, u_2 \in \Gamma(U_{\pm})$, $\tau, \tau_1, \tau_2 \in \Gamma(A_{\mathbb{C}} \oplus T_{\mathbb{C}}^*M)$ define the anchor by

$$c_{\pm}(u \oplus \tau) := \rho_{U_{\pm}}(u) + (\rho_A)_{\mathbb{C}} \circ \text{pr}_{A_{\mathbb{C}}} \tau,$$

the pairing by

$$(33) \quad \langle u_1 \oplus \tau_1, u_2 \oplus \tau_2 \rangle_{C_{\pm}} := \langle u_1, \tau_2 \rangle + \langle u_2, \tau_1 \rangle + \langle \tau_1, (\rho, \rho^t)_{\mathbb{C}}(\tau_2) \rangle,$$

and the bracket by

$$(34) \quad \begin{aligned} \llbracket u_1 \oplus \tau_1, u_2 \oplus \tau_2 \rrbracket_{C_{\pm}} := & \left([u_1, u_2]_{U_{\pm}} + \nabla_{\text{pr}_{A_{\mathbb{C}}} \tau_1}^{\text{bas}, \mathbb{C}} u_2 - \nabla_{\text{pr}_{A_{\mathbb{C}}} \tau_2}^{\text{bas}, \mathbb{C}} u_1 \right) \\ & \oplus \left(\llbracket \tau_1, \tau_2 \rrbracket_{d, \mathbb{C}} + \Delta_{u_1}^{\mathbb{C}} \tau_2 - \Delta_{u_2}^{\mathbb{C}} \tau_1 + (0, \mathbf{d}_{\mathbb{C}} \langle \tau_1, u_2 \rangle) \right). \end{aligned}$$

Then C_{\pm} are both complex Courant algebroids and (C_{\pm}, U_{\pm}) together with $\iota: U_{\pm} \hookrightarrow T_{\mathbb{C}}M \oplus A_{\mathbb{C}}^*$ and $\Phi: A_{\mathbb{C}} \oplus T_{\mathbb{C}}^*M \rightarrow C$ the canonical inclusions are complex A -Manin pairs.

Recall that U_+ with its complex Lie algebroid structure is isomorphic to the complex Lie algebroid $(TM \oplus A^*, j, \rho_j, \mathbb{A})$ (see Proposition 4.15). Hence the result above realises the latter Courant algebroid as a Dirac structure in the complex Courant algebroid C_+ .

Next, the generalised complex structure on A induces generalised complex structures J_{\pm} in the Courant algebroids C_{\pm} defined by Proposition 5.8.

Proposition 5.9. *Let $u \oplus \tau \in \Gamma(C_{\pm})$. Then*

$$J_{\pm}(u \oplus \tau) := j_{\mathbb{C}}u \oplus j_{C, \mathbb{C}}\tau$$

is well-defined and a generalised complex structure in C_{\pm} .

Proof. Take any element $(-\rho, \rho^t)_{\mathbb{C}}k \oplus k$ of $\text{graph}(-(\rho, \rho^t)_{\mathbb{C}}|_{K_{\pm}})$. Then

$$J_{\pm}((-\rho, \rho^t)_{\mathbb{C}}k \oplus k) = \pm i((-\rho, \rho^t)_{\mathbb{C}}k \oplus k),$$

which is again an element of $\text{graph}(-(\rho, \rho^t)_{\mathbb{C}}|_{K_{\pm}})$. Thus the map J_{\pm} is well-defined on C_{\pm} . It is clear that $J_{\pm}^2 = -1$. Orthogonality follows from an easy computation using $j^2 = -1$, $j_{\mathbb{C}}^t = -j_{\mathbb{C}, \mathbb{C}}$ and $(\rho, \rho^t)_{\mathbb{C}} \circ j_{\mathbb{C}, \mathbb{C}} = j_{\mathbb{C}} \circ (\rho, \rho^t)_{\mathbb{C}}$.

The last remaining condition is the vanishing of the Nijenhuis torsion of J_{\pm} with respect to the bracket on C_{\pm} . Lengthy, but straightforward computations making use of the previously proven facts that $\nabla^{\text{bas}, \mathbb{C}}$ preserves U_{\pm} , $\Delta_u^{\mathbb{C}}$ preserves K_{\pm} and that the Nijenhuis torsion of $j_{\mathbb{C}}$ with respect to $\llbracket \cdot, \cdot \rrbracket_d$ vanishes (Proposition 5.5), establish this condition. Hence J_{\pm} defines a generalised complex structure in C_{\pm} . \square

5.3. The Lie bialgebroid (U_{\pm}, K_{\mp}) ; proof of Theorem 1.5. This section shows that the pair (U_{\pm}, K_{\mp}) forms a Lie bialgebroid with Drinfeld double Courant algebroid isomorphic to C_{\pm} . First, observe that the identity $U_{\pm} = K_{\pm}$ induces isomorphisms $U_{\pm}^* \cong K_{\mp}$ and $K_{\pm}^* \cong U_{\mp}$. The following theorem establishes then Theorem 1.5, since U_{\pm} with its complex Lie algebroid structure is isomorphic to the complex Lie algebroid $(TM \oplus A^*, j, \rho_j, \mathbb{A})$ (see Proposition 4.15).

Theorem 5.10. *Let $(A \rightarrow M, \mathcal{J})$ be a generalised complex Lie algebroid. There is an isomorphism of vector bundles*

$$(35) \quad F: U_{\pm} \oplus K_{\mp} \rightarrow C_{\pm}, \quad (u, k) \mapsto u \oplus k.$$

This equips $U_{\pm} \oplus K_{\mp}$ with the structure of a Courant algebroid, in which the complex Lie algebroids U_{\pm} and K_{\mp} are transversal Dirac structures. Thus the pair (U_{\pm}, K_{\mp}) is a complex Lie bialgebroid. F is an isomorphism of Courant algebroids where $U_{\pm} \oplus K_{\mp}$ is the Drinfeld double Courant algebroid of this Lie bialgebroid.

Proof. It is easy to verify that

$$u \oplus \tau \mapsto \left(u + \frac{1}{2}(\rho, \rho^t)_{\mathbb{C}}(\tau \mp ij_{\mathbb{C}}\tau), \frac{1}{2}(\tau \pm ij_{\mathbb{C}}\tau) \right).$$

is well-defined and defines an inverse to F .

The Courant algebroid structure of C_{\pm} induces via this isomorphism a Courant algebroid structure on the bundle $U_{\pm} \oplus K_{\mp}$. The following shows that the Lie algebroids U_{\pm} and K_{\mp} are Dirac structures in C_{\pm} . Liu, Weinstein and Xu showed in [25] that two transversal Dirac structures in a Courant algebroid are equivalent to a Lie bialgebroid. Thus (U_{\pm}, K_{\mp}) is a Lie bialgebroid, which induces the Drinfeld double Courant algebroid on $U_{\pm} \oplus K_{\mp}$. It remains then to show that the pairing and bracket of C_{\pm} are equal to the pairing and bracket of this Drinfeld double and that they are thus isomorphic as Courant algebroids with the isomorphism given by the map F defined in (35).

With the definition of the bracket in C_{\pm} in (34), it follows for $k_1, k_2 \in \Gamma(K_{\mp})$ directly that

$$\llbracket (0 \oplus k_1), (0 \oplus k_2) \rrbracket_{C_{\pm}} = 0 \oplus \llbracket k_1, k_2 \rrbracket_{d, \mathbb{C}}.$$

Similarly, for two sections $u_1, u_2 \in \Gamma(U_{\pm})$, the bracket is $\llbracket (u_1 \oplus 0), (u_2 \oplus 0) \rrbracket_{C_{\pm}} = [u_1, u_2]_{U_{\pm}} \oplus 0$.

From the definition of the pairing in C_{\pm} in (33) it is easy to see that both $U_{\pm} \oplus 0$ and $0 \oplus K_{\mp}$ are maximally isotropic with respect to this pairing and thus Dirac structures in C_{\pm} . Thus by the argument in [25] (U_{\pm}, K_{\mp}) form complex Lie bialgebroids.

The anchor and pairing are easily seen to be equal to the anchor and the pairing in the Drinfeld double Courant algebroid. The bracket in the Drinfeld double is defined for $u_1, u_2 \in \Gamma(U_\pm)$ and $k_1, k_2 \in \Gamma(K_\mp)$ by

$$\llbracket (u_1, k_1), (u_2, k_2) \rrbracket = \left([u_1, u_2] + \mathcal{L}_{k_1}^K u_2 - \iota_{k_2} \mathbf{d}_K u_1, [k_1, k_2] + \mathcal{L}_{u_1}^U k_2 - \iota_{u_2} \mathbf{d}_U k_1 \right).$$

It only remains to be shown that the brackets of elements of the form $(u, 0)$ with $(0, k)$ coincide, the rest follows by bilinearity, since the brackets on U_\pm and K_\mp were already shown to be inherited from the bracket in C_\pm . A straightforward computation using (32) shows

$$\llbracket u \oplus 0, 0 \oplus k \rrbracket_{C_\pm} = -\nabla_{\text{pr}_{A_C}}^{\text{bas}, C} u \oplus \Delta_u^C k = -\iota_k \mathbf{d}_K u \oplus \mathcal{L}_u^U k,$$

see also [13] for details. Hence F defines indeed an isomorphism of Courant algebroids from C_\pm to the Drinfeld double $U_\pm \oplus K_\mp$. \square

Example 5.11. *In the situation of Example 4.16, if $A \rightarrow M$ is equipped with a Lie algebroid structure and a compatible linear complex structure, then the eigenbundles are Lie algebroids and thus also define Drinfeld double Courant algebroids*

$$\begin{aligned} C_T^{1,0} &= T^{1,0}M \oplus (T^{1,0}M)^*, & C_A^{1,0} &= A^{1,0} \oplus (A^{1,0})^*, \\ C_T^{0,1} &= T^{0,1}M \oplus (T^{0,1}M)^*, & C_A^{0,1} &= A^{0,1} \oplus (A^{0,1})^*, \end{aligned}$$

induced by the Lie bialgebroid structure where T_C^*M and A_C^* are endowed with trivial Lie algebroid structures. That is, the brackets on $C_T^{1,0}$ and $C_T^{0,1}$ are given by

$$\llbracket (X, \theta), (Y, \eta) \rrbracket = ([X, Y], \mathcal{L}_X \eta - \iota_Y \mathbf{d}\theta),$$

and analogously on $C_A^{1,0}$ and $C_A^{0,1}$. (30) shows that as vector bundles $C_+ = C_T^{1,0} \oplus C_A^{0,1}$ and $C_- = C_T^{0,1} \oplus C_A^{1,0}$. The computations in [13] show that these are orthogonal decompositions with respect to the pairings in C_\pm and that the brackets in $C_T^{1,0}$, $C_T^{0,1}$, $C_A^{1,0}$ and $C_A^{0,1}$ coincide with the respective restrictions of the brackets in C_\pm . In other words, they form matched pairs of Courant algebroids, a notion introduced in [8].

6. GENERALISED COMPLEX STRUCTURES IN VB-COURANT ALGEBROIDS

In this section the results of Section 4 are extended to general VB-Courant algebroids $(\mathbb{E}; Q, B; M)$. This leads to a definition of generalised complex structures in split Lie 2-algebroids.

A linear splitting Σ of the double vector bundle \mathbb{E} is called **Lagrangian** if the image of Σ is isotropic in \mathbb{E} . The paper [18] shows that a change of Lagrangian splittings corresponds to a skew-symmetric element $\Phi_{12} \in \Gamma(Q^* \otimes B^* \otimes Q^*)$.

Only the description of linear splittings with Dorfman connections relies on the special case of $TE \oplus T^*E$. The other results of Section 4 only use the abstract structure of a metric double vector bundle and Lagrangian lifts. They therefore generalise to VB-Courant algebroids in the following way.

Fix a Lagrangian splitting Σ of \mathbb{E} and denote the corresponding lift by $\sigma: \Gamma(Q) \rightarrow \Gamma_B^\ell(\mathbb{E})$. Consider a double vector bundle morphism $\mathcal{J}: \mathbb{E} \rightarrow \mathbb{E}$ over id_B and $j: Q \rightarrow Q$ with core morphism $j_C: Q^* \rightarrow Q^*$. As in Lemma 4.1, the following definition of Φ depends on the choice of the splitting.

Lemma 6.1. *Given a double vector bundle morphism $\mathcal{J}: \mathbb{E} \rightarrow \mathbb{E}$ over j and id_B there is $\Phi \in \Gamma(Q^* \otimes B^* \otimes Q^*)$ defined by setting for any $q \in \Gamma(Q)$*

$$\mathcal{J}(\sigma(q)) = \sigma(jq) + \widetilde{\Phi(q)}.$$

Furthermore, the following lemmas generalise the description of generalised almost complex structures on a vector bundle in Section 4.

Lemma 6.2. *A double vector bundle morphism $\mathcal{J}: \mathbb{E} \rightarrow \mathbb{E}$ satisfies $\mathcal{J}^2 = -\text{id}_{TE \oplus T^*E}$ if and only if for any Lagrangian splitting and corresponding Φ , and for any $q \in \Gamma(Q)$:*

- (1) $j^2 = -\text{id}_Q$,
- (2) $j_C^2 = -\text{id}_{Q^*}$,
- (3) $\Phi(j(q)) = -j_C \circ (\Phi(q))$.

Lemma 6.3. *A double vector bundle morphism $\mathcal{J}: \mathbb{E} \rightarrow \mathbb{E}$ such that additionally $\mathcal{J}^2 = -1$, is orthogonal if and only if for any Lagrangian splitting*

- (1) $j = -(j_C)^t$,
- (2) $\langle j(q_1), \Phi(q_2)(b) \rangle = -\langle j(q_2), \Phi(q_1)(b) \rangle$

for all $b \in \Gamma(B)$ and $q_1, q_2 \in \Gamma(Q)$.

Now define a 2-form $\Psi \in \Omega^2(Q, B^*)$ by setting $\Psi(q_1, q_2) := \Phi(q_1)^t(q_2)$. This definition yields the following, as in Proposition 4.5.

Proposition 6.4. *A morphism $\mathcal{J}: \mathbb{E} \rightarrow \mathbb{E}$ is a generalised almost complex structure in \mathbb{E} , if and only if for any Lagrangian splitting*

- (1) $j^2 = -1$,
- (2) $j = -(j_C)^t$,
- (3) Ψ is skew-symmetric, that is $\Psi \in \Omega^2(Q, B^*)$,
- (4) $\Psi(q_1, q_2) = -j^* \Psi(q_1, q_2)$ for $q_1, q_2 \in \Gamma(Q)$.

Also in this case, a Lagrangian splitting can be adapted to the generalised almost complex structure. As mentioned before it was shown in [18] that such a change of splittings corresponds to a skew-symmetric element $\Phi_{12} \in \Gamma(Q^* \otimes B^* \otimes Q^*)$.

Proposition 6.5. *Given a generalised almost complex structure \mathcal{J} in a VB-Courant algebroid $(\mathbb{E}; Q, B; M)$ with side morphism $j: Q \rightarrow Q$, there is a Lagrangian lift $\sigma: \Gamma(Q) \rightarrow \Gamma_B^\ell(\mathbb{E})$, such that for any $q \in \Gamma(Q)$*

$$\mathcal{J}(\sigma(q)) = \sigma(jq).$$

Proof. Fix any Lagrangian lift σ_1 of \mathbb{E} . This defines by Lemma 6.1 a tensor $\Phi_1 \in \Gamma(Q^* \otimes B^* \otimes Q^*)$. Define another tensor $\Phi_{12} \in \Gamma(Q^* \otimes B^* \otimes Q^*)$ by setting for any $q \in \Gamma(Q)$ and $b \in \Gamma(B)$

$$\Phi_{12}(q)(b) := \frac{1}{2} j_C(\Phi_1(q)(b)).$$

By Lemma 6.2 and Lemma 6.3, Φ_{12} is skew-symmetric. Define a new Lagrangian lift by $\sigma_2(q) := \sigma_1(q) - \widetilde{\Phi_{12}(q)}$. This lift satisfies the desired property. \square

Using this existence of an adapted Lagrangian splitting, use the correspondence of VB-Courant algebroid structures to split Lie 2-algebroids proved in [18]. Fix such an adapted Lagrangian splitting as in Proposition 6.5. Then the VB-Courant algebroid structure is

equivalent to a split Lie 2-algebroid structure $(\rho_Q, \partial_B^t, \llbracket \cdot, \cdot \rrbracket_\Delta, \nabla, \omega)$ on $Q \oplus B^*$, where the bracket in \mathbb{E} is described by the dull bracket on Q and the dual Dorfman connection as follows.

$$\begin{aligned} \llbracket \sigma(q_1), \sigma(q_2) \rrbracket &= \sigma(\llbracket q_1, q_2 \rrbracket_\Delta) - \overline{R_\omega(q_1, q_2)} \\ \llbracket \sigma(q), \tau^\dagger \rrbracket &= (\Delta_q \tau)^\dagger \quad \text{and} \quad \llbracket \tau_1^\dagger, \tau_2^\dagger \rrbracket = 0. \end{aligned}$$

Here $R_\omega(q_1, q_2) := \omega(q_1, q_2, \cdot)^t \in \Gamma(\text{Hom}(B, Q^*))$

This description of the Courant algebroid bracket yields similar computations and results for the Nijenhuis tensor of core sections and lifts for a linear generalised almost complex structure in the VB-Courant algebroid \mathbb{E} as in Section 4.2 in the special case of $TE \oplus T^*E$.

First, analogously to the computations in Section 4.2, the section $N_{\mathcal{J}}(\sigma(q), \tau^\dagger)$ vanishes for any $q \in \Gamma(Q)$ and $\tau \in \Gamma(Q^*)$ if and only if $N_{j, \llbracket \cdot, \cdot \rrbracket_\Delta}$ vanishes. Second, the analogous computation for the Nijenhuis tensor of two lifts gives

$$\begin{aligned} N_{\mathcal{J}}(\sigma(q_1), \sigma(q_2)) &= \sigma^\Delta(N_{j, \llbracket \cdot, \cdot \rrbracket_Q}(q_1, q_2)) + \overline{R_\omega(j(q_1), j(q_2))} - \overline{R_\omega(q_1, q_2)} \\ &\quad - \overline{j_C \circ R_\omega(j(q_1), q_2)} - \overline{j_C \circ R_\omega(q_1, j(q_2))} \end{aligned}$$

Dualising the property

$$R_\omega(j(q_1), j(q_2)) - R_\omega(q_1, q_2) - j_C \circ R_\omega(j(q_1), q_2) - j_C \circ R_\omega(q_1, j(q_2)) = 0,$$

by evaluating at any $b \in \Gamma(B)$ and then pairing with q_3 gives as an equivalent condition on $\omega \in \Omega^3(Q, B^*)$ the following:

$$\omega(q_1, q_2, q_3) - \omega(jq_1, jq_2, q_3) - \omega(jq_1, q_2, jq_3) - \omega(q_1, jq_2, jq_3) = 0.$$

This yields the following proposition.

Proposition 6.6. *A linear generalised almost complex structure \mathcal{J} in \mathbb{E} over $j: Q \rightarrow Q$ is integrable if and only if for any adapted Lagrangian splitting of the corresponding split Lie 2-algebroid,*

- (1) $N_{j, \llbracket \cdot, \cdot \rrbracket_\Delta}(q_1, q_2) = 0,$
- (2) $\omega(q_1, q_2, q_3) - \omega(jq_1, jq_2, q_3) - \omega(jq_1, q_2, jq_3) - \omega(q_1, jq_2, jq_3) = 0$

for any $q_1, q_2, q_3 \in \Gamma(Q)$.

As before the vector bundle morphism $j: Q \rightarrow Q$ defines an equivalence relation on the Lagrangian splittings.

Definition 6.7. *Given a VB-Courant algebroid $(\mathbb{E}; Q, B; M)$ and a vector bundle morphism $j: Q \rightarrow Q$, two Lagrangian splittings Σ_1 and Σ_2 are j -**equivalent** if the corresponding change of splittings $\Psi \in \Omega^2(Q, B^*)$ satisfies $\Psi(q_1, q_2) = \Psi(jq_1, jq_2)$ for any $q_1, q_2 \in \Gamma(Q)$.*

Analogously to Lemma 4.9, given a splitting Σ_1 which is adapted to a linear generalised almost complex structure (\mathcal{J}, j) , then a second splitting Σ_2 is also adapted to (\mathcal{J}, j) if and only if Σ_1 and Σ_2 are j -equivalent. This allows a formulation of the analogue of Theorem 4.10 in the general case.

Theorem 6.8. *A linear generalised complex structure \mathcal{J} in a VB-Courant algebroid \mathbb{E} is equivalent to a vector bundle morphism $j: Q \rightarrow Q$ and a j -equivalence class of linear splittings such that in the corresponding split Lie 2-algebroid $(\rho_Q, \partial_B^t, \llbracket \cdot, \cdot \rrbracket_\Delta, \nabla, \omega)$ over $Q \oplus B^*$*

- (1) $j^2 = -\text{id}_Q,$
- (2) $N_{j, \llbracket \cdot, \cdot \rrbracket_\Delta} = 0,$

$$(3) \quad \omega(q_1, q_2, q_3) - \omega(jq_1, jq_2, q_3) - \omega(jq_1, q_2, jq_3) - \omega(q_1, jq_2, jq_3) = 0$$

for any $q_1, q_2, q_3 \in \Gamma(Q)$.

Analogously to the case of $TE \oplus T^*E$, a bracket \mathbb{A} on $\Gamma(Q)$ can be defined by $\mathbb{A}(q_1, q_2) = \frac{1}{2}(\llbracket q_1, q_2 \rrbracket_\Delta - \llbracket jq_1, jq_2 \rrbracket_\Delta)$. The vanishing of $N_{j, \llbracket \cdot, \cdot \rrbracket_\Delta}$ is equivalent to complex bilinearity of \mathbb{A} and the condition on ω in Theorem 6.8 implies the Jacobi identity for \mathbb{A} . This defines a complex Lie algebroid $(Q, \rho, \mathbb{A}$ with the complex anchor $\rho: Q \rightarrow T_{\mathbb{C}}M$ given by

$$\rho(q) = \frac{1}{2}(\rho_Q(q) - ij\rho_Q(q)).$$

But if the core-anchor ∂_B is not surjective, then the condition on ω in Theorem 6.8 is stronger than the Jacobi identity of this bracket, since $\mathbf{Jac}_{\llbracket \cdot, \cdot \rrbracket_\Delta} = \partial_B^t \circ \omega$. Therefore – unlike in the special case of $TE \oplus T^*E$ – here the complex Lie algebroid structure is not sufficient to describe the conditions on the linear generalised complex structure.

Theorem 6.8 suggests that a generalised complex structure in a split Lie 2-algebroid should be defined as a tuple of maps $(\rho_Q, \partial_B^t, \llbracket \cdot, \cdot \rrbracket, \nabla, \omega)$ over $Q \oplus B^*$ is a vector bundle morphism $j: Q \rightarrow Q$, such that for any $q_1, q_2, q_3 \in \Gamma(Q)$

- (1) $j^2 = -\text{id}_Q$,
- (2) $N_{j, \llbracket \cdot, \cdot \rrbracket} = 0$,
- (3) $\omega(q_1, q_2, q_3) - \omega(jq_1, jq_2, q_3) - \omega(jq_1, q_2, jq_3) - \omega(q_1, jq_2, jq_3) = 0$.

APPENDIX A. RELATION WITH THE ADAPTED GENERALISED CONNECTIONS IN [4]

The equality $\mathcal{J}(\sigma^\Delta(\nu)) = \sigma^\Delta(j\nu)$ in Proposition 4.7 for $\nu \in \Gamma(TM \oplus E^*)$ is equivalent to

$$(36) \quad \mathcal{J}(L_\Delta) = L_\Delta$$

for the horizontal space $L_\Delta \subseteq TE \oplus T^*E$ corresponding to Δ .

Let $\mathbb{E} \rightarrow M$ be an arbitrary Courant algebroid. A *generalised connection* on \mathbb{E} is a linear connection $\nabla: \Gamma(\mathbb{E}) \times \Gamma(\mathbb{E}) \rightarrow \Gamma(\mathbb{E})$, which is compatible with the pairing in \mathbb{E} ([10]). For instance, if $\nabla: \mathfrak{X}(M) \times \Gamma(\mathbb{E}) \rightarrow \Gamma(\mathbb{E})$ is an ordinary metric linear connection, then $\nabla^\rho: \Gamma(\mathbb{E}) \times \Gamma(\mathbb{E}) \rightarrow \Gamma(\mathbb{E})$ defined by $\nabla_e^\rho e' = \nabla_{\rho(e)} e'$ for $e, e' \in \Gamma(\mathbb{E})$, is a generalised connection.

Let $\mathcal{J}: \mathbb{E} \rightarrow \mathbb{E}$ be a generalised almost complex structure. The paper [4] shows that there exists a metric linear connection $\nabla: \mathfrak{X}(M) \times \Gamma(\mathbb{E}) \rightarrow \Gamma(\mathbb{E})$ that is *adapted* to \mathcal{J} :

$$(37) \quad \nabla \cdot \mathcal{J} = 0.$$

The pullback ∇^ρ is then a generalised connection *adapted* to \mathcal{J} and its intrinsic torsion relative to the connection is studied in [4] in relation with the integrability of \mathcal{J} – generalising the fact that an almost complex structure $J: TM \rightarrow TM$ on a smooth manifold M is integrable if and only if there exists a complex-linear torsion-free connection $\nabla: \mathfrak{X}(M) \times \mathfrak{X}(M) \rightarrow \mathfrak{X}(M)$. The condition (37) is equivalent to the generalised complex structure $T\mathcal{J}: T\mathbb{E} \rightarrow T\mathbb{E}$ over TM preserving the horizontal space $H_\nabla \subseteq T\mathbb{E}$ defined by ∇ :

$$(38) \quad T\mathcal{J}(H_\nabla) = H_\nabla.$$

The notion of adapted generalised connection in [4] seems in general different from the notion of adapted Dorfman connection in Proposition 4.7. However, as the similarity of (36) with (38) suggests, they are equivalent at least in a special situation, which is explained in the remainder of this section.

Let $\mathbb{E} \rightarrow M$ be a Courant algebroid and denote the co-anchor of \mathbb{E} by ρ^* , which is defined by composing ρ^t with the isomorphism between \mathbb{E} and \mathbb{E}^* induced by the pairing in \mathbb{E} . Consider a generalised connection $\nabla: \Gamma(\mathbb{E}) \times \Gamma(\mathbb{E}) \rightarrow \Gamma(\mathbb{E})$ such that $\nabla_{\rho^*}\theta = 0$ for all $\theta \in \Omega^1(M)$. It is easy to see that

$$(39) \quad \Delta_{e_1}e_2 = \llbracket e_1, e_2 \rrbracket + \nabla_{e_2}e_1$$

for all $e_1, e_2 \in \Gamma(\mathbb{E})$, defines a Dorfman connection $\Delta: \Gamma(\mathbb{E}) \times \Gamma(\mathbb{E}) \rightarrow \Gamma(\mathbb{E})$, see also [19]. Conversely a Dorfman \mathbb{E} -connection on \mathbb{E} defines a generalised connection by (39), that must satisfy

$$(40) \quad \nabla_{\rho^*}\theta e = -\llbracket e, \rho^*\theta \rrbracket + \Delta_e\rho^*\theta = \rho^*(-\mathcal{L}_{\rho(e)}\theta + \mathcal{L}_{\rho(e)}\theta) = 0$$

for all $e \in \Gamma(\mathbb{E})$ and all $\theta \in \Omega^1(M)$. Hence Dorfman \mathbb{E} -connections on \mathbb{E} are equivalent with linear \mathbb{E} -connections on \mathbb{E} satisfying (40). In particular, since $\rho \circ \rho^* = 0$ (see [31]), the pullbacks of TM -connections are equivalent to a class of Dorfman \mathbb{E} -connections on \mathbb{E} . A metric TM -connection on \mathbb{E} is equivalent to a Lagrangian splitting of the tangent prolongation of \mathbb{E} , which is a VB-Courant algebroid. The induced dull bracket on $\Gamma(\mathbb{E})$ is the degree 1 part of the splitting of the corresponding Lie 2-algebroid, see [18].

Equations (36) and (38) can in fact be related in the case of the standard Courant algebroid $TM \oplus T^*M$ over a smooth manifold M . A computation shows that the canonical isomorphism

$$(41) \quad \begin{array}{ccccc} T(TM \oplus T^*M) & \xrightarrow{\mathcal{I}} & T(TM) \oplus T^*(TM) & & \\ \downarrow & \searrow & \downarrow & \searrow & \\ TM & \xrightarrow{\text{id}} & TM & \xrightarrow{\text{id}} & TM \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ TM \oplus T^*M & \xrightarrow{\text{id}} & TM \oplus T^*M & \xrightarrow{\text{id}} & M \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ M & \xrightarrow{\text{id}} & M & \xrightarrow{\text{id}} & M \end{array}$$

arising from the canonical involution $I: TTM \rightarrow TTM$ (see e.g. [21] and references therein) sends $H_{\nabla} \subseteq T(TM \oplus T^*M)$ to $L_{\Delta} \subseteq T(TM) \oplus T^*(TM)$, if and only if Δ and ∇^{ρ} are related by (39).

Consider an almost complex structure $J: TM \rightarrow TM$, as well as a torsion-free linear connection $\nabla: \mathfrak{X}(M) \times \mathfrak{X}(M) \rightarrow \mathfrak{X}(M)$. Consider the TM -connection $\tilde{\nabla}: \mathfrak{X}(M) \times \Gamma(TM \oplus T^*M) \rightarrow \Gamma(TM \oplus T^*M)$, $\tilde{\nabla}_X(Y, \theta) = (\nabla_X Y, \nabla_X^*\theta)$. Let \mathcal{J}_J be the generalised almost complex structure defined as in Example 2.3. The generalised connection $\tilde{\nabla}^{\rho}$ satisfies

$$\tilde{\nabla}^{\rho}\mathcal{J}_J = 0, \text{ or in other words } T\mathcal{J}_J(H_{\tilde{\nabla}}) = H_{\tilde{\nabla}},$$

if and only if $\nabla \cdot J = 0$, i.e. if and only if $TJ(H_{\nabla}) = H_{\nabla}$.

Here, an easy computation shows that Δ and $\tilde{\nabla}^{\rho}$ are related by (39) if and only if Δ is the standard Dorfman connection defined by ∇ as in Example 2.4. As a consequence

$$\mathcal{I}(H_{\tilde{\nabla}}) = L_{\Delta}.$$

The canonical isomorphism \mathcal{I} also transforms $T\mathcal{J}_J$ into the linear generalised almost complex structure \mathcal{J}_{J_T} , where J_T is the almost complex structure $I \circ TJ \circ I$ on the vector bundle TM seen as a manifold. Then $\tilde{\nabla} \cdot \mathcal{J}_J = 0$ if and only if

$$\mathcal{J}_{J_T}(L_{\Delta}) = L_{\Delta},$$

where $\Delta: \Gamma(TM \oplus T^*M) \times \Gamma(TM \oplus T^*M) \rightarrow \Gamma(TM \oplus T^*M)$ is the standard Dorfman connection defined by ∇ as in Example 2.4. That is,

$$T\mathcal{J}_J(H_{\tilde{\nabla}}) = H_{\tilde{\nabla}} \quad \text{if and only if} \quad \mathcal{J}_{J_T}(L_\Delta) = L_\Delta.$$

In other words, $\tilde{\nabla}$ is adapted to \mathcal{J}_J in the sense of [4] if and only if Δ is adapted to $\mathcal{J}_{I \circ T J \circ I}$ in the sense of Proposition 4.7.

More generally, let $\mathcal{J}: TM \oplus T^*M \rightarrow TM \oplus T^*M$ be a generalised almost complex structure and let $\nabla: \Gamma(TM) \times \Gamma(TM \oplus T^*M) \rightarrow \Gamma(TM \oplus T^*M)$ be a linear connection adapted to \mathcal{J} . As before $T\mathcal{J}: T(TM \oplus T^*M) \rightarrow T(TM \oplus T^*M)$ is a linear generalised almost complex structure over the identity on the base TM , and $\mathcal{J}: TM \oplus T^*M \rightarrow TM \oplus T^*M$ on the side. The isomorphism \mathcal{I} in (41) transforms $T\mathcal{J}$ into a linear generalised almost complex structure $\mathcal{J}_{TM}: T(TM) \oplus T^*(TM) \rightarrow T(TM) \oplus T^*(TM)$ in the standard VB-Courant algebroid over TM . Then since ∇ is adapted to \mathcal{J} :

$$T\mathcal{J}(H_\nabla) = H_\nabla,$$

which is again equivalent to

$$\mathcal{J}_{TM}(L_\Delta) = L_\Delta,$$

where ∇^ρ and Δ are equivalent via (39).

APPENDIX B. PROOF OF LEMMA 3.1

Since $E \rightarrow M$ is a holomorphic vector bundle, it has local holomorphic frames. Let $p \in M$. Then there exists $U \subseteq M$ open with $p \in U$ and holomorphic sections e_1, \dots, e_k of $E|_U \rightarrow U$ such that $e_1(q), \dots, e_k(q)$ span $E(q)$ as a \mathbb{C} -vector space for each $q \in U$. Then, if $\varphi: U \rightarrow \tilde{U} \subseteq \mathbb{C}^m$ is a holomorphic chart for M , the smooth map

$$\tilde{\varphi}: E|_U \rightarrow \tilde{U} \times \mathbb{C}^k \subseteq \mathbb{C}^{m+k}, \quad e_q = \sum_{j=1}^k \alpha_j e_j(q) \mapsto (\varphi(q), \alpha_1, \dots, \alpha_k)$$

is a linear holomorphic chart for E . A section $e \in \Gamma_U(E)$ is holomorphic if and only if $e = \sum_{j=1}^k f_j e_j$ with f_j holomorphic functions on U . In the coordinates $E|_U \simeq \tilde{U} \times \mathbb{C}^k$ and $T(E|_U) \simeq \tilde{U} \times \mathbb{C}^m \times \mathbb{C}^k \times \mathbb{C}^k$, the vector field $e^\dagger \in \mathfrak{X}(E|_U)$ is defined by $\tilde{U} \times \mathbb{C}^k \rightarrow \tilde{U} \times \mathbb{C}^m \times \mathbb{C}^k \times \mathbb{C}^k$,

$$(q, z_1, \dots, z_k) \mapsto (q, 0, z_1, \dots, z_k, (f_1 \circ \varphi^{-1})(q), \dots, (f_k \circ \varphi^{-1})(q)).$$

This map is clearly holomorphic. The map $J_E: TE \rightarrow TE$ is, modulo the isomorphism $\tilde{\varphi}$, just the multiplication by i in the fibers of $T(\tilde{U} \times \mathbb{C}^k) \rightarrow \tilde{U} \times \mathbb{C}^k$:

$$(q, w_1, \dots, w_m, z_1, \dots, z_k, \zeta_1, \dots, \zeta_k) \mapsto (q, iw_1, \dots, iw_m, z_1, \dots, z_k, i\zeta_1, \dots, i\zeta_k)$$

Let $e = \sum_{j=1}^k f_j e_j$ with $f_j \in C^\infty(U, \mathbb{C})$ – hence e is a section of $E|_U \rightarrow U$, not necessarily holomorphic. Then, again in the chart,

$$J_E(e^\dagger(q, z)) = (q, i \cdot 0, z_1, \dots, z_k, i(f_1 \circ \varphi^{-1})(q), \dots, i(f_k \circ \varphi^{-1})(q)) = (i \cdot e)^\dagger(q, z) = (je)^\dagger(q, z)$$

for any $(q, z) \in \tilde{U} \times \mathbb{C}^k$.

REFERENCES

- [1] N. Abe and S. Kurosu. A decomposition of a holomorphic vector bundle with connection and its applications to complex affine immersions. *Results Math.*, 44(1-2):3–24, 2003.
- [2] H. Bursztyn, G. R. Cavalcanti, and M. Gualtieri. Generalized Kähler and hyper-Kähler quotients. In *Poisson geometry in mathematics and physics*, volume 450 of *Contemp. Math.*, pages 61–77. Amer. Math. Soc., Providence, RI, 2008.
- [3] G. R. Cavalcanti and M. Gualtieri. Generalized complex geometry and T -duality. In *A celebration of the mathematical legacy of Raoul Bott*, volume 50 of *CRM Proc. Lecture Notes*, pages 341–365. Amer. Math. Soc., Providence, RI, 2010.
- [4] V. Cortés and L. David. Generalized connections, spinors, and integrability of generalized structures on courant algebroids. *arXiv e-prints*, page arXiv:1905.01977, May 2019.
- [5] M. Crainic. Generalized complex structures and Lie brackets. *Bull. Braz. Math. Soc. (N.S.)*, 42(4):559–578, 2011.
- [6] T. Drummond, M. Jotz Lean, and C. Ortiz. VB-algebroid morphisms and representations up to homotopy. *Differential Geom. Appl.*, 40:332–357, 2015.
- [7] A. Gracia-Saz and R. A. Mehta. Lie algebroid structures on double vector bundles and representation theory of Lie algebroids. *Adv. Math.*, 223(4):1236–1275, 2010.
- [8] M. Grützmann and M. Stiénon. Matched pairs of Courant algebroids. *Indag. Math. (N.S.)*, 25(5):977–991, 2014.
- [9] M. Gualtieri. *Generalized complex geometry*. PhD thesis, University of Oxford, Jan 2004. Available at <https://arxiv.org/abs/math/0401221v1>.
- [10] M. Gualtieri. Branes on Poisson varieties. In *The many facets of geometry*, pages 368–394. Oxford Univ. Press, Oxford, 2010.
- [11] M. Gualtieri. Generalized complex geometry. *Ann. of Math. (2)*, 174(1):75–123, 2011.
- [12] M. Gualtieri. Generalized Kähler geometry. *Comm. Math. Phys.*, 331(1):297–331, 2014.
- [13] M. Heuer. *Multiple vector bundles and linear generalised complex structures*. PhD thesis, University of Sheffield, Jun 2019. Available at <http://etheses.whiterose.ac.uk/24878>.
- [14] N. Hitchin. Generalized Calabi-Yau manifolds. *Q. J. Math.*, 54(3):281–308, 2003.
- [15] N. Hitchin. Instantons, Poisson structures and generalized Kähler geometry. *Comm. Math. Phys.*, 265(1):131–164, 2006.
- [16] M. Jotz Lean. Dorfman connections and Courant algebroids. *J. Math. Pures Appl. (9)*, 116:1–39, 2018.
- [17] M. Jotz Lean. Dirac groupoids and Dirac bialgebroids. *J. Symplectic Geom.*, 17(1):179–238, 2019.
- [18] M. Jotz Lean. Lie 2-algebroids and matched pairs of 2-representations: a geometric approach. *Pacific J. Math.*, 301(1):143–188, 2019.
- [19] M. Jotz Lean. On LA-Courant Algebroids and Poisson Lie 2-Algebroids. *Math. Phys. Anal. Geom.*, 23(3):31, 2020.
- [20] M. Jotz Lean and C. Ortiz. Foliated groupoids and infinitesimal ideal systems. *Indag. Math. (N.S.)*, 25(5):1019–1053, 2014.
- [21] M. Jotz Lean, M. Stiénon, and P. Xu. Glanon groupoids. *Math. Ann.*, 364(1-2):485–518, 2016.
- [22] S. Kobayashi. *Differential geometry of complex vector bundles*. Princeton Legacy Library. Princeton University Press, Princeton, NJ, [2014]. Reprint of the 1987 edition [MR0909698].
- [23] C. Laurent-Gengoux, M. Stiénon, and P. Xu. Holomorphic Poisson manifolds and holomorphic Lie algebroids. *Int. Math. Res. Not. IMRN*, pages Art. ID rnn 088, 46, 2008.

- [24] D. S. Li-Bland. *LA-Courant algebroids and their applications*. ProQuest LLC, Ann Arbor, MI, 2012. Thesis (Ph.D.)—University of Toronto (Canada).
- [25] Z.-J. Liu, A. Weinstein, and P. Xu. Manin triples for Lie bialgebroids. *J. Differential Geom.*, 45(3):547–574, 1997.
- [26] K. C. H. Mackenzie. *General theory of Lie groupoids and Lie algebroids*, volume 213 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2005.
- [27] K. C. H. Mackenzie. Ehresmann doubles and Drinfel’d doubles for Lie algebroids and Lie bialgebroids. *J. Reine Angew. Math.*, 658:193–245, 2011.
- [28] T. Mokri. Matched pairs of Lie algebroids. *Glasgow Math. J.*, 39(2):167–181, 1997.
- [29] J. Pradines. *Fibres vectoriels doubles et calcul des jets non holonomes*, volume 29 of *Esquisses Mathématiques [Mathematical Sketches]*. Université d’Amiens, U.E.R. de Mathématiques, Amiens, 1977.
- [30] D. Roytenberg. *Courant algebroids, derived brackets and even symplectic supermanifolds*. ProQuest LLC, Ann Arbor, MI, 1999. Thesis (Ph.D.)—University of California, Berkeley.
- [31] D. Roytenberg. On the structure of graded symplectic supermanifolds and Courant algebroids. In *Quantization, Poisson brackets and beyond (Manchester, 2001)*, volume 315 of *Contemp. Math.*, pages 169–185. Amer. Math. Soc., Providence, RI, 2002.
- [32] W. M. Tulczyjew. *Geometric formulations of physical theories*, volume 11 of *Monographs and Textbooks in Physical Science. Lecture Notes*. Bibliopolis, Naples, 1989. Statics and dynamics of mechanical systems.
- [33] K. Uchino. Remarks on the definition of a courant algebroid. *Lett. Math. Phys.*, 60(2):171–175, 2002.
- [34] A. Weinstein. The integration problem for complex Lie algebroids. In *From geometry to quantum mechanics*, volume 252 of *Progr. Math.*, pages 93–109. Birkhäuser Boston, Boston, MA, 2007.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF HAMBURG, GERMANY

E-mail address: malte.heuer@uni-hamburg.de

MATHEMATISCHES INSTITUT, GEORG-AUGUST UNIVERSITÄT GÖTTINGEN, GERMANY

E-mail address: madeleine.jotz-lean@mathematik.uni-goettingen.de