N-MANIFOLDS OF DEGREE 2 AND METRIC DOUBLE VECTOR BUNDLES.

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Abstract. This paper shows the equivalence of the categories of N-manifolds of degree 2 with the category of double vector bundles endowed with a linear metric.

Split Poisson N-manifolds of degree 2 are shown to be equivalent to *self-dual representations up to homotopy.* As a consequence, the equivalence above induces an equivalence between so called *metric VB-algebroids* and Poisson N-manifolds of degree 2.

Then a new, simple description of split Lie 2-algebroids is given, as well as their "duals", the *Dorfman 2-representations*. We show that Dorfman 2-representations are equivalent in a simple manner to *Lagrangian splittings* of VB-Courant algebroids. This yields the equivalence of the categories of Lie 2-algebroids and of VB-Courant algebroids. We give several natural classes, some of them new, of examples of split Lie 2-algebroids and of the corresponding VB-Courant algebroids.

We then show that a split Poisson Lie 2-algebroid is equivalent to the "matched pair" of a Dorfman 2-representation with a self-dual representation up to homotopy. We deduce a new proof of the equivalence of categories of LA-Courant algebroids and Poisson Lie 2-algebroids. We show that the core of an LA-Courant algebroid inherits naturally the structure of a degenerate Courant algebroid, as a kind of bicrossproduct of the Poisson bracket and the Dorfman connection defined by a Lagrangian splitting. This yields a new formula to retrieve in a direct manner the Courant algebroid found by Roytenberg and Severa to correspond to a symplectic Lie 2-algebroid.

Finally we study VB- and LA-Dirac structures in VB- and LA-Courant algebroids. As an application, we extend Li-Bland's results on pseudo-Dirac structures and we construct a Manin pair associated to an LA-Dirac structure.

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Contents

1. Introduction

Lie and Courant algebroids are differential geometric objects that are useful in encoding infinitesimal symmetry. Lie algebroids generalise tangent bundles and Lie algebras. They were originally defined by Jean Pradines [31] as the infinitesimal counterpart of Lie groupoids, just as Lie algebras describe Lie groups infinitesimally. The integration of Lie algebroids to Lie groupoids has only been fully understood around ten years ago, in seminal papers of Crainic and Fernandes [8] and Cattaneo and Felder [5].

The standard Courant algebroid over a manifold was discovered in the late eighties by (Ted) Courant in his study of Dirac structures [6]. He had earlier discovered Dirac structures with his adviser Weinstein [7]; Dirac structures arose from Dirac's theory of constraints and encompass Poisson structures, closed two-forms, and regular foliations [6]. A few years later, Liu, Weinstein and Xu defined general Courant algebroids and showed that the "bicrossproducts" of Lie bialgebroids are examples of Courant algebroids [22]. The standard Courant algebroids over smooth manifolds turned out to be crucial in Hitchin's definition of generalised complex structures [15], which were then further developed by his students Cavalcanti and Gualtieri [14]. Severa described in [36] how the standard Courant algebroid describes symmetries of variational problems.

Despite a great and growing interest in Courant algebroids, still little is known about them. The two main goals in their study are now their integration, and understanding their representation theory. The main tool for investigations in that direction is the well-known, but rather complicated, equivalence between Courant algebroids and symplectic Lie 2-algebroids. One of the features of this paper is a thorough explanation of this equivalence, in a very simple and innovative manner.

Lie 1*- and Lie* 2*-algebroids.* For clarity let us recall a few definitions. An **$\mathbb{N}$-graded manifold** $\mathcal{M}$ of degree n (an N-manifold of degree n for short) and dimension $(p; r_1, \ldots, r_n)$ is a smooth p-dimensional manifold M endowed with a sheaf $C^\infty(\mathcal{M})$ of $\mathbb{N}$-graded commutative associative unital $\mathbb{R}$-algebras, which can locally be written as

$$C^\infty_U(M)[\xi_1^1, \ldots, \xi_1^{r_1}, \xi_2^1, \ldots, \xi_2^{r_2}, \ldots, \xi_n^1, \ldots, \xi_n^{r_n}]$$

with ξ_i^j of degree i. For instance, an $\mathbb{N}$-graded manifold of degree 1 is just a locally free and finitely generated sheaf of $C^\infty(M)$-modules, hence canonicaly isomorphic to the set of sections of a vector bundle: $C^\infty(\mathcal{M}) = \Gamma(\wedge^\bullet E^*)$ for a vector bundle $E \to M$. On a local chart U of M trivialising E, we have coordinates $x_1, \ldots, x_n$, i.e. functions of degree 0 on $\mathcal{M}$. We also have local basis sections $e_1, \ldots, e_r$ of E and the dual local basis sections $\xi_1, \ldots, \xi_r$ of E^*; which are seen as functions of degree 1 on $\mathcal{M}$.

A vector field of degree j on an $\mathbb{N}$-graded manifold is a graded derivation that increases the degree by j. An NQ-manifold of degree 1 is an $\mathbb{N}$-graded manifold $\mathcal{M}$ of degree 1, with a vector field $\mathcal{Q}$ of degree 1 that commutes with itself; $[\mathcal{Q}, \mathcal{Q}] = 0$. The vector field $\mathcal{Q}$ is then called a **homological vector field**. Take again an N-manifold of degree 1, i.e. $C^\infty(\mathcal{M}) = \Gamma(\wedge^\bullet E^*)$ for a vector bundle E over M and take a trivialising chart $U \subseteq M$ for E. The vector fields ∂_{x_i} have degree 0 and the vector fields ∂_{ξ_i} have degree -1. Assume that E has a Lie algebroid structure with anchor $\rho\colon E \to TM$ and bracket $[\cdot\,,\cdot]$, and define $\mathcal{Q}$ on $\mathcal{M}$ by

$$\mathcal{Q}|_U = \sum_{i=1}^{n}\sum_{j=1}^{r} \rho(e_j)(x_i)\xi_j\partial_{x_i} - \sum_{i<j}^{r}\sum_{k=1}^{r}\langle [e_i, e_j], \xi_k\rangle \xi_i\xi_j\partial_{\xi_k}. \tag{1}$$

A quick degree count shows that $\mathcal{Q}$ has degree 1. Let us check that $\mathcal{Q}\circ\mathcal{Q} = 0$, i.e. that $\mathcal{Q}$ is a homological vector field. First note that $f \in C^\infty(U)$ is sent by $\mathcal{Q}$ to $\mathcal{Q}(f) = \sum_{j=1}^{r}\rho(e_j)(f)\xi_j = \rho^*\mathbf{d}f$. Then

$$\begin{aligned}\mathcal{Q}^2(f) &= \mathcal{Q}\left(\sum_{j=1}^{r}\rho(e_j)(f)\xi_j\right)\\ &= \sum_{j=1}^{r}\rho^*\mathbf{d}(\rho(e_j)(f))\xi_j - \sum_{j=1}^{r}\rho(e_j)(f)\sum_{s<t}\langle [e_s, e_t], \xi_j\rangle\xi_s\xi_t\\ &= \sum_{t=1}^{r}\sum_{s=1}^{r}\rho(e_s)\rho(e_t)(f)\xi_s\xi_t - \sum_{s<t}\rho[e_s, e_t](f)\xi_s\xi_t\end{aligned}$$

which vanishes since $\rho[e_s, e_t] = [\rho(e_s), \rho(e_t)]$. Then we find that $\mathcal{Q}(\xi_k)$ equals $-\sum_{i<j}^{r}\langle [e_i, e_j], \xi_k\rangle\xi_i\xi_j$. As an element of $\Gamma(\wedge^2 E^*)$, this is $\mathbf{d}_E\xi_k$, where $\mathbf{d}_E$ is the operator defined on $\Gamma(\wedge^\bullet E^*)$ by the Lie algebroid structure on E. A similar computation as the one above shows that the Jacobi identity implies $\mathcal{Q}^2(\xi_k) = 0$ for all k. Thus we have found that a Lie algebroid structure on E defines a homological vector field $\mathcal{Q} = \mathbf{d}_E$ on the corresponding N-manifold of degree 1.

Conversely, any homological vector field $\mathcal{Q}$ on a degree 1 graded manifold can be written as $\sum_{ij} f^{ij}\xi_i\partial_{x_j} + \sum_{ijk} g^{ijk}\xi_i\xi_j\partial_{\xi_k}$ with smooth functions f^{ij}, $g^{ijk} \in C^\infty(U)$. Setting $f^{ij} = \rho(e_j)(x_i)$ and $g^{ijk} = -\langle [e_i, e_j], \xi_k\rangle$ and extending using the Leibniz identities *defines* then locally a Lie algebroid structure on $E|_U$, which can further be checked to be global since $\mathcal{Q}$ does not depend on the local coordinates. Hence, NQ-manifolds of degree 1 are equivalent to Lie algebroids. This result, due to Arkady Vaintrob [39], is the reason why NQ-manifolds of degree 1 are called Lie 1-algebroids and NQ-manifolds of degree $n \geq 1$ are called Lie n-algebroids. Let us describe yet another way to get the Lie algebroid structure from the homological vector field $\mathcal{Q}$. A study of Equation (1) shows that $[\mathcal{Q}, e](f) = \rho(e)(f)$ and $[[\mathcal{Q}, e], e'] = [e, e']$, if $e \in \Gamma(E)$ is identified with the graded vector field e of degree -1 that sends $\xi \in \Gamma(E^*)$ to $\langle e, \xi\rangle$ and $f \in C^\infty(M)$ to 0. The Lie algebroid structure is hence *derived* from the homological vector field.

We now turn to the case of degree 2 N-manifolds, and in particular the one of Lie 2-algebroids. The equivalence between *symplectic* degree 2 N-manifolds and vector bundles with a metric, and the one between *symplectic* Lie 2-algebroids and Courant algebroids are due to Dmitry Roytenberg [35] and Pavol Severa [36]. Li-Bland established in his thesis correspondences between Lie 2-algebroids and VB-Courant algebroids and between *Poisson* Lie 2-algebroids and LA-Courant algebroids [19].

These correspondence are full of insight, but are not explained in the literature as concretely as the one of NQ-manifolds of degree 1 with Lie algebroids. This paper geometrises N-manifolds of degree 2: We find an equivalence between the category of N-manifolds of degree 2 and double vector bundles endowed with a linear metric. Then we deduce correspondences between geometric structures on both sides.

Metric double vector bundles. A double vector bundle is a commutative square

$$\begin{tikzcd} D \arrow[r, "\pi_B"] \arrow[d, "\pi_A"'] & B \arrow[d, "q_B"] \\ A \arrow[r, "q_A"'] & M \end{tikzcd}$$

of vector bundles such that the structure maps of the vertical bundles define morphisms of the horizontal bundles. Each of the vector bundles $D \to A$ and $D \to B$ has very useful sections to work with; the *linear* and the *core sections*. Take a triple A, B, C of vector bundles over a smooth manifold M. Then the fibre-product $A \times_M B \times_M C$ has a vector bundle structure over A given by $(a,b,c) +_A (a,b',c') = (a, b+b', c+c')$, and similarly a vector bundle structure over B. This defines a commutative square as above and thus a double vector bundle structure on $A \times_M B \times_M C \to B$. This type of double vector bundle is called *decomposed*. Any double vector bundle is non-canonically isomorphic to a decomposed one. A **VB-Courant algebroid** is a double vector bundle (D, B, A, M) with a Courant algebroid structure on $D \to B$ that is linear, i.e. compatible with the other vector bundle structure on D. An **LA-Courant algebroid** is a VB-Courant algebroid with an additional Lie algebroid structure on $D \to A$ that is linear, i.e. compatible with the vector bundle structure on $D \to B$, and that is also compatible with the Courant algebroid structure in a sense that is explained in detail in this paper.

Up to now, the homological vector fields and the symplectic and Poisson structures corresponding to Courant algebroids and VB-Courant algebroids have never been written in coordinates as we do it in (1) for Lie 1-algebroids [1], and even the construction of these structures as derived structures from the homological vector field was not straightforward in practice. We remedy to this by giving a simple description of the VB-Courant algebroid that is defined by a Lie 2-algebroid, and vice-versa. More precisely, this paper explains in detail how to construct a *decomposed* VB-Courant algebroid from a *split* Lie 2-algebroid. In order to do this, we give a simplified definition of split Lie 2-algebroids.

Our main result, advertised in the title of this paper, is an explicit equivalence between the category of degree 2 N-manifolds with a category of double vector bundles with a linear non-degenerate pairing over one of their sides. We call the latter **metric double vector bundles**. From this theorem follow many enlightening results on geometric structures on degree 2 N-manifolds and on their counterparts on metric double vector bundles, reflecting equivalences between the category of Poisson degree 2 N-manifolds and the category of metric VB-algebroids, between the category of Lie 2-algebroids and the category of VB-Courant algebroids, and between the category of Poisson Lie 2-algebroids and the category of LA-Courant algebroids.

Self-dual representations up to homotopy, Dorfman 2-representations, etc. We prove along the way that split Poisson N-manifolds of degree 2 are equivalent to *self-dual 2-term representations up to homotopy*, that split Lie 2-algebroids are dual to *Dorfman 2-representations*, which resemble very much 2-term representations up to homotopy and demonstrate how Lie 2-algebroids can really be understood as

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[1] Note that in [35], Roytenberg wrote in coordinates the symplectic form and the Hamiltonian function corresponding to the homological vector field defining a symplectic Lie 2-algebroid.

Lie algebroids up to homotopy. We then find a new way of checking if a Poisson bracket of degree -2 on an N-manifold of degree 2 is invariant under a homological vector field, making the N-manifold into a Poisson Lie 2-algebroid. In any splitting of the underlying N-manifold, the self-dual representation up to homotopy and the Dorfman 2-representation defined by the Poisson structure and the homological vector field, respectively, have to form a matched pair, the definition of which resembles the one of matched pairs of 2-term representations up to homotopy [12]. This provides on the classical side a new and much simpler way of defining LA-Courant algebroids, and using the result in [12], shows immediately that LA-Dirac structures in LA-Courant algebroids are double Lie algebroids.

We exhibit several new examples of (Poisson) Lie 2-algebroids, and show in particular that the bicrossproduct of a matched pair of 2-term representations up to homotopy is a split Lie 2-algebroid (see Theorem 55), just as the bicrossproduct of a matched pair of Lie algebroid representations is a Lie algebroid. This result is independently interesting, because it finally unifies in a natural framework the two notions of double of a matched pair of representations. A matched pair of representations of two Lie algebroids A and B over M defines a Lie algebroid structure on $A \oplus B$ [30], sometimes called the double of the matched pair, but called here the bicrossproduct of the matched pair. The matched pair defines also a double Lie algebroid $A \times B$ with sides A and B and trivial core. This double Lie algebroid is called the double of the matched pair. Similarly, we know now that a matched pair of 2-term representations up to homotopy has a split Lie 2-algebroid as bicrossproduct, and a decomposed double Lie algebroid as double. The split Lie 2-algebroid is exactly the N-geometric counterpart of the VB-Courant algebroid that is equivalent to the double Lie algebroid. The case of matched pair of representations and their bicrossproducts and doubles are in fact a special (degenerate) case of this more general equivalence; namely the one of a linear Courant algebroid over a trivial base.

We find as well that a matched pair of 2-term representations up to homotopy defines in a less conventional, but still very natural manner two Poisson Lie 2-algebroids.

In Figure 1 is a table of all the classical differential geometric objects that we encounter in this paper. The ordinary arrows are the obvious forgetful functors. The snake arrows correspond to constructions in §6.4.4 (and §2.4.2) and §7.3.3. We do not discuss these arrows as functors, but it would be a good exercise to do so. Note that the triangle on the bottom left does not commute; we exhibit two different constructions of a VB-Courant algebroid and of an LA-Courant algebroid from a double Lie algebroid. The hooked arrows are embeddings. In Figure 2 is a very similar table, with the N-geometric counterparts of the objects in Figure 1.

The range of applications of our results culminates in Theorem 7.9 with a new method exhibiting explicitly the Courant algebroid structure arising from a symplectic Lie 2-algebroid, and thus contradicting the common wisdom that the Courant bracket can only be obtained as a derived bracket from a symplectic Lie 2-algebroid. The core of the LA-Courant algebroid corresponding to a Poisson Lie 2-algebroid inherits the structure of a degenerate Courant algebroid, which looks like the correction of a Dorfman connection (one of the ingredients of the Dorfman 2-representation) by the Poisson bracket. In the symplectic case, the Courant algebroid is non-degenerate, hence a Courant algebroid in the ordinary sense, and

FIGURE 1. Diagrammatic table of the (classical, double) geometric objects in this paper.

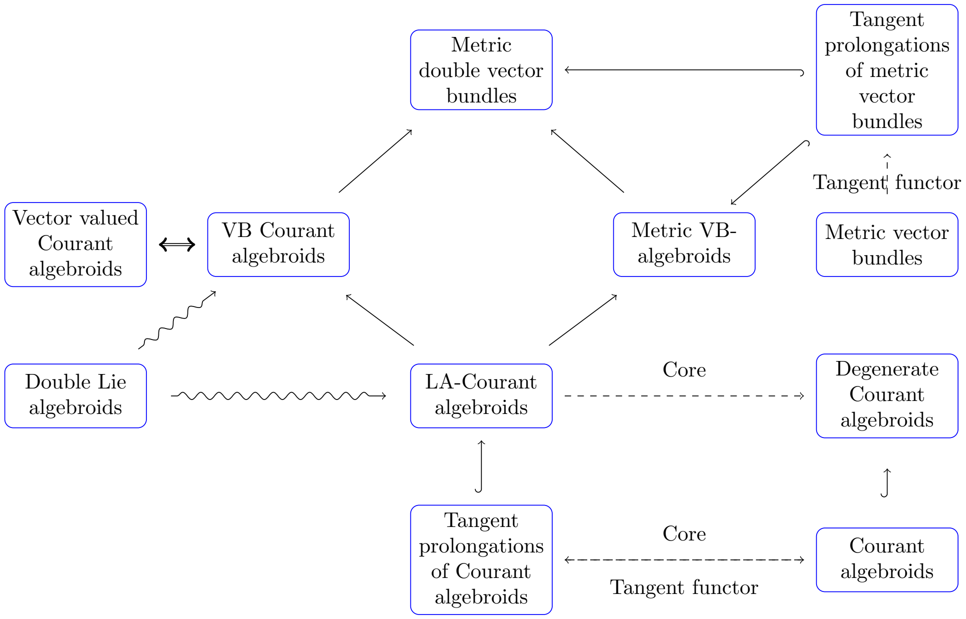
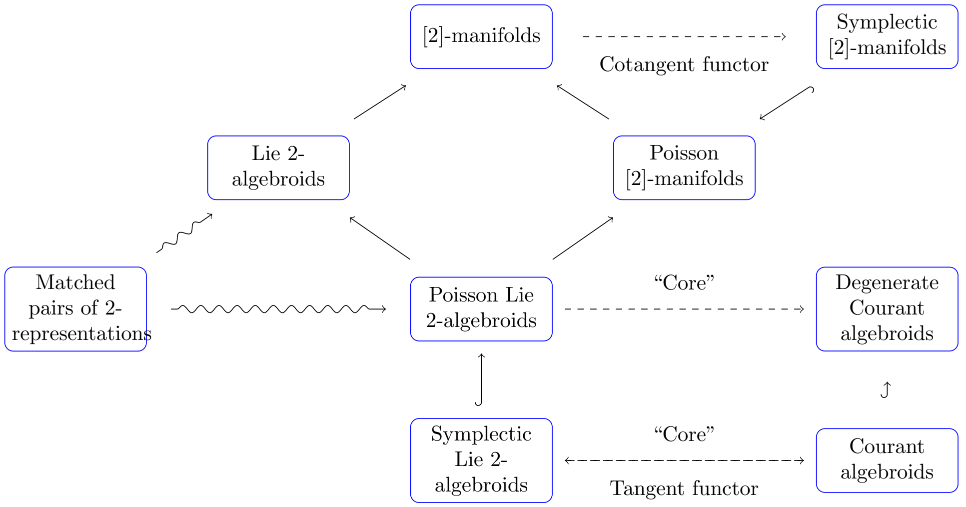


FIGURE 2. Diagrammatic table of the supergeometric objects in this paper.



exactly the one that is found following Roytenberg's approach to be equivalent to the symplectic Lie 2-algebroid. Unsurprisingly, the ambient LA-Courant algebroid is nothing else than the tangent double of the Courant algebroid.

This explains what might seem at first sight confusing in comparing Li-Bland's work with Roytenberg's results. As symplectic manifolds are special (non-degenerate) Poisson manifolds, symplectic Lie 2-algebroids form a special class of Poisson Lie

2-algebroids. Hence, the “classical” counterparts of symplectic Lie 2-algebroids should form a subcategory of the classical counterparts of Poisson Lie 2-algebroids. We adopt the view (and prove) that symplectic Lie 2-algebroids are in fact equivalent to tangent doubles of Courant algebroids, and that the Courant algebroids are in fact the core structures emanating from those.

Original motivation. Let us explain in more detail our methodology and our original motivation. A VB-Lie algebroid is a double vector bundle with one side $D \to B$ endowed with a Lie algebroid bracket and an anchor that are *linear*. The side $A \to M$ then inherits a natural Lie algebroid structure. Take for simplicity a decomposed VB-algebroid $(A \times_M B \times_M C \to B, A \to M)$. Here, linearity of the bracket means that the bracket of two core sections[2] is zero, the bracket of a core section with a linear section is a core section, and the bracket of two linear sections is again linear. In formulae

$$[c_1^\dagger, c_2^\dagger] = 0, \quad [a^l, c^\dagger] = \nabla_a c^\dagger, \quad [a_1^l, a_2^l] = [a_1, a_2]^l - \widetilde{R(a_1, a_2)},$$

defining so a linear connection $\nabla\colon \Gamma(A) \times \Gamma(C) \to \Gamma(C)$ and a vector valued two-form $R \in \Omega^2(A, \operatorname{Hom}(B, C))$. Linearity of the anchor means that the anchor of a linear section is a linear vector field on the base B, and that the anchor of a core section is a vertical vector field on B. In formulae again

$$\rho_D(c^\dagger) = (\partial c)^\uparrow, \qquad \rho_D(a^l) = \widehat{\nabla_a},$$

defining so a linear connection $\nabla\colon \Gamma(A) \times \Gamma(B) \to \Gamma(B)$ and a vector bundle morphism $\partial\colon C \to B$. Gracia-Saz and Mehta prove in [13] that the two connections, the bundle map and the vector-valued 2-form are the ingredients of a super-representation, also known as 2-term representation up to homotopy.

The definition of a VB-Courant algebroid is very similar to the one of VB-algebroids. The Courant bracket, the anchor and the non-degenerate pairing all have to be linear. In 2012 we proved that the standard Courant algebroid over a vector bundle can be decomposed into a connection, a Dorfman connection, a curvature term and a vector bundle map, in a manner that resembles very much the one in [13]. There our original motivation was to prove that, as linear splittings of the tangent space TE of a vector bundle E are equivalent to linear connections on the vector bundle, linear splittings of the Pontryagin bundle $TE \oplus T^*E$ over E are equivalent to a certain class of Dorfman connections [16].

It was then very natural to show that this was in fact a very special case of a general decomposition of VB-Courant algebroids, in the spirit of Gracia-Saz and Mehta's result. For very long we worked with general splittings of VB-Courant algebroids, exhibiting very promising formulae and objects looking like the “Courant counterpart” to 2-term representations up to homotopy. Nonetheless these objects were hard to grasp conceptually and to relate to Lie 2-algebroids, which were already known to correspond to VB-Courant algebroids. It was only when we implemented the idea of working only with maximally isotropic horizontal spaces in the metric double vector bundles underlying VB-Courant algebroids, that the

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[2] The sections of A, B and $C \to M$ define very useful sections of the double vector bundle $A \times_M B \times_M C$: a section $a \in \Gamma(A)$ defines a linear section a^l of $A \times_M B \times_M C \to B$: $a^l(b_m) = (a(m), b_m, 0^C_m)$. The map $\tilde{a}\colon B \to A \times_M B \times_M C$ is then a vector bundle morphism over $a\colon M \to A$. A section $c \in \Gamma(C)$ defines a core section $c^\dagger$ of $A \times_M B \times_M C \to B$: $c^\dagger(b_m) = (0^A_m, b_m, c(m))$. For $\phi \in \Gamma(\operatorname{Hom}(B, C))$, the linear section $\widetilde{\phi}$ sends b_m to $(0^A_m, b_m, \phi(b_m))$.

geometric objects that we had found simplified to what could be called Lie derivatives up to homotopy, or Lie 2-derivatives, in duality to Lie 2-algebroids. For simplicity we called them Dorfman 2-representations, because Dorfman representations (skew-symmetric and flat Dorfman connections) are exactly the Lie derivatives in duality with Lie algebroids [16]. The next main consequence of this new method was our discovery of the equivalence of metric double vector bundles with N-manifolds of degree 2, which is now the core of this paper.

Outline, main results and applications. This paper is organised as follows.
Section 2. We start by quickly recalling how vector bundle morphisms are equivalent to morphisms of the sheaves of sections of the dual bundles. We then discuss in detail the necessary background on double vector bundles and their dualisation and splittings. We recall how linear splittings of VB-algebroids induce 2-term representations up to homotopy [13] and how matched pairs of 2-term representations up to homotopy correspond in this manner to linear splittings of double Lie algebroids [12].
Section 3. In this section we recall the definition of $\mathbb{N}$-graded manifolds (N-manifolds in short) and we recall the equivalence of N-manifolds of degree 1 with vector bundles. Then we define metric double vector bundles and their Lagrangian splittings, the existence of which we prove. We describe the morphisms in the category of metric double vector bundles, and we construct an equivalence of this category with the category of N-manifolds of degree 2.
Section 4. We study Poisson structures of degree -2 on N-manifolds of degree 2. We show how a Poisson structure of degree -2 on a split N-manifold of degree 2 is equivalent to a 2-term representation up to homotopy that is dual to itself. Then we give the geometrisation of Poisson N-manifolds of degree 2; namely linear Lie algebroids structures on metric double vector bundles, that are compatible with the metric. We prove that the equivalence of categories established in the previous section induces an equivalence of the category of Poisson N-manifolds of degree 2 with the category of metric VB-algebroids. Finally, we discuss some examples of Poisson N-manifolds of degree 2, and the corresponding metric VB-algebroids. We discuss in detail symplectic N-manifolds of degree 2, and how they correspond to tangent doubles of Euclidean vector bundles.
Section 5. In this section we start by recalling necessary background on Courant algebroids, Dirac structures and Dorfman connections. Then we formulate in our own manner Sheng and Zhu's definition of split Lie 2-algebroids [37], before dualising it and obtaining the notion of Dorfman 2-representation. Then we write in coordinates the homological vector field corresponding to a split Lie 2-algebroid, showing where the components of the Dorfman 2-representation (or equivalently of the split Lie 2-algebroid) appear. In Section 5.4, we give several classes of examples of split Lie 2-algebroids, introducing in particular the standard split Lie 2-algebroids defined by a vector bundle, and the bicrossproduct Lie 2-algebroid of a matched pair of 2-term representations up to homotopy. Finally we describe morphisms of split Lie 2-algebroids.
Section 6. In this section we give the definition of VB-Courant algebroids [19] and we relate Dorfman 2-representations with Lagrangian splittings of VB-Courant algebroids, in the spirit of Gracia-Saz and Mehta's description of split VB-algebroids via 2-term representations up to homotopy [13]. Then we describe the VB-Courant algebroids corresponding to the examples of split Lie 2-algebroids found in the

previous section, and we prove that the equivalence of categories established in Section 3 induces an equivalence of the category of VB-Courant algebroids with the category of Lie 2-algebroids.

Section 7. We define the matching of self-adjoint 2-representations with Dorfman 2-representations. Then we show how split Poisson Lie 2-algebroids define such matched pairs, and how decomposed LA-Courant algebroids define such matched pairs. We use this to deduce an equivalence of the categories of Poisson Lie 2-algebroids and LA-Courant algebroids. Then we describe examples of LA-Courant algebroids and Poisson Lie 2-algebroids.

Next we focus on the core of an LA-Courant algebroid; we prove that it inherits a natural structure of degenerate Courant algebroid ("natural" in the sense that this structure does not depend on any Lagrangian splitting). Symplectic Lie 2-algebroids correspond via the equivalence above to the tangent doubles of ordinary Courant algebroids. The core structure in this class of LA-Courant algebroids is just the underlying ordinary Courant algebroid. Hence, the Courant bracket defined by a symplectic Lie 2-algebroid can be recovered, after any choice of splitting, as a kind of correction of a Dorfman connection by the Poisson bracket.

Section 8. In the last section we discuss VB-Dirac structures (with support) in VB-Courant algebroids, maximally isotropic subalgebroids in metric VB-algebroids, and LA-Dirac structures (with support) in LA-Courant algebroids. We find in each case, after the choice of an adequate splitting, conditions on the sides and core of the double subbundles with the (matching) Dorfman 2-representation and self-adjoint 2-representation for the double subbundle to be a VB-Dirac structure, maximally isotropic subalgebroid or LA-Dirac structure. We find that the integrability of a wide VB-Dirac structure is completely encoded in the restriction to its side of the dull bracket defined by any splitting adapted to the Dirac structure. We prove that LA-Dirac structures are automatically double Lie algebroids.

Next we explain in the language developed in this paper the notion of pseudo-Dirac connection defined by Li-Bland in [20], and we explain the equivalence that he finds between pseudo-Dirac connections and VB-Dirac structures in tangent doubles of Courant algebroids. We extend his result to an equivalence of "quadratic" pseudo-Dirac connections with LA-Dirac structures in the tangent double of a Courant algebroid.

Finally we prove that an LA-Dirac structure defines a Manin pair over its double base. The construction of the Courant algebroid resembles a semi-direct product of a Lie algebroid with the degenerate Courant algebroid on the core of the ambient LA-Courant algebroid. These Manin pairs will be the subject of future studies of the author. In particular, we will relate them to the infinitesimal description of Dirac groupoids in [21].

Appendices. To increase the readability of this paper, we give in the appendix the proofs of its three most technical theorems. In Section A we prove the theorem describing decomposed VB-Courant algebroids via Dorfman 2-representations. In Section B we prove that the Dorfman 2-representations and the self-adjoint 2-representations encoding the two sides of a split LA-Courant algebroid form a matched pair, and vice-versa. This proof is very long and very technical, but contains some interesting constructions. In Section C we prove the theorem on the Manin pair associated to an LA-Dirac structure.

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After this work was completed, the author learned that Fernando del Carpio-Marek has independently found, mostly through different methods, results similar to some of hers in his PhD thesis in preparation [9].

Notation and conventions. We write $p_M\colon TM \to M$, $q_E\colon E \to M$ for vector bundle mapsand $\pi_A\colon D \to A$ and $\pi_B\colon D \to B$ for the two vector bundle projections of a double vector bundle. For a vector bundle $Q \to M$ we often identify without further mentioning the vector bundle $(Q^*)^*$ with Q via the canonical isomorphism. We write $\langle\cdot\,,\cdot\rangle$ or $\langle\cdot\,,\cdot\rangle_M$ the canonical pairing of a vector bundle with its dual; i.e. $\langle a_m, \alpha_m\rangle = \alpha_m(a_m)$ for $a_m \in A$ and $\alpha_m \in A^*$. We will use many different pairings; in general, which pairing will be used will be clear from its arguments. Given a section ξ of E^*, we will always write $\ell_\xi\colon E \to \mathbb{R}$ for the linear function associated to it, i.e. the function defined by $e_m \mapsto \langle \xi(m), e_m\rangle$ for all $e_m \in E$.

Let M be a smooth manifold. We will denote by $\mathfrak{X}(M)$ and $\Omega^1(M)$ the spaces of (local) smooth sections of the tangent and the cotangent bundle, respectively. For an arbitrary vector bundle $E \to M$, the space of (local) sections of E will be written $\Gamma(E)$. Let $f\colon M \to N$ be a smooth map between two smooth manifolds M and N. Then two vector fields $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$ are said to be **f-related** if $Tf \circ X = Y \circ f$ on $\mathrm{Dom}(X) \cap f^{-1}(\mathrm{Dom}(Y))$. We write then $X \sim_f Y$.

We write "$[n]$-manifold" for "N-manifolds of degree n". This notation is to avoid confusions with n-manifolds, which are usually understood as smooth manifolds of *dimension* n. We will say 2-representations for 2-term representations up to homotopy.

2. Background and definitions on double vector bundles and VB-algebroids

We collect in this section background notions on ordinary vector bundles and their morphisms, on double vector bundles and their linear splittings and dualisations, on VB-algebroids and double Lie algebroids, and on 2-term representations up to homotopy and how they encode split VB-algebroids. Further references will be given throughout the text.

2.1. **Vector bundles and morphisms.**

Let $A \to M$ and $B \to N$ be vector bundles and $\omega\colon A \to B$ a morphism of vector bundles over a smooth map $\omega_0\colon M \to N$. First we set a few notations. We will say that $a \in \Gamma_M(A)$ is ω-related to $b \in \Gamma_N(B)$ if

$$\omega(a(m)) = b(\omega_0(m))$$

for all $m \in M$. We will then write $a \sim_\omega b$.

We will write $\omega_0^*B \to M$ for the pullback of B under ω_0; for $m \in M$, elements of $(\omega_0^*B)(m)$ are pairs $(m, b_{\omega_0(m)})$ with $b_{\omega_0(m)} \in B(\omega_0(m))$. The morphism ω induces then a morphism $\omega^!\colon A \to \omega_0^*B$, $\omega^!(a_m) = (m, \omega(a_m))$ over the identity on M. For a section $b \in \Gamma_N(B)$, we get in a similar manner a section $\omega_0^! b \in \Gamma_M(\omega_0^*B)$; defined by $(\omega_0^!b)(m) = (m, b(\omega_0(m)))$ for all $m \in M$. We have then $\omega^\star(b) = (\omega^!)^*(\omega_0^! b)$ for all $b \in \Gamma_N(B)$.

The dual of a morphism $\omega\colon A \to B$ over $\omega_0\colon M \to N$ is in general not a morphism of vector bundles, but a relation $R_{\omega^*} \subseteq A^* \times B^*$ defined as follows:

$$R_{\omega^*} = \{(\omega_m^*\beta_{\omega_0(m)}, \beta_{\omega_0(m)}) \mid m \in M, \beta_{\omega_0(m)} \in B^*_{\omega_0(m)}\},$$

where $\omega_m\colon A_m \to B_{\omega_0(m)}$ is the morphism of vector spaces. This relation induces a morphism $\omega^\star$ of modules over $\omega_0^*\colon C^\infty(N) \to C^\infty(M)$:

$$\omega^\star\colon \Gamma_N(B^*) \to \Gamma_M(A^*), \qquad \omega^\star(\beta)(m) = \omega_m^*\beta_{\omega_0(m)} \tag{2}$$

for all $\beta \in \Gamma_N(B^*)$ and $m \in M$. That is, $(\omega^\star(\beta)(m), \beta(\omega_0(m))) \in R_{\omega^*}$ for all $\beta \in \Gamma_N(B^*)$ and $m \in M$. We prove the following lemma.

Lemma 2.1. *The map $\cdot^\star$, that sends a morphism of vector bundles $\omega\colon A \to B$ over $\omega_0\colon M \to N$ to the morphism $\omega^\star\colon \Gamma_N(B^*) \to \Gamma_M(A^*)$ of modules over $\omega_0^*\colon C^\infty(N) \to C^\infty(M)$, is a bijection.*

Proof. First we have to check that $\omega^\star$ is well-defined, that is, that the image under $\omega^\star$ of a smooth section of B^* is again smooth. Consider the pullback of B under ω_0, i.e. the vector bundle $\omega_0^*B \to M$. The morphism ω of vector bundles induces a morphism $\omega^!\colon A \to \omega_0^*B$ of smooth vector bundles over the identity on M: $\omega^!$ is defined by $\omega^!(a_m) = (m, \omega(a_m))$ for all a_m in the fiber of A over $m \in M$. Now we have $(\omega_0^*B)^* = \omega_0^*B^*$ and for each section $\beta \in \Gamma_N(B^*)$, we define $\beta^! \in \Gamma_M(\omega_0^*B^*)$ by $\beta^!(m) = (m, \beta(\omega_0(m)))$. The smoothness of $\omega^\star(\beta)$ follows from the equality $\omega^\star(\beta) = (\omega^!)^*\beta^!$: for each $m \in M$, and each $a_m \in A_m$, we have

$$\begin{aligned}\langle((\omega^!)^*\beta^!)(m), a_m\rangle &= \langle \beta^!(m), \omega^!(a_m)\rangle = \langle (m, \beta(\omega_0(m))), (m, \omega(a_m))\rangle \\ &= \langle \beta(\omega_0(m)), \omega(a_m)\rangle = \langle \omega^\star(\beta)(m), a_m\rangle.\end{aligned}$$

The map $\omega^\star$ is obviously a morphism of modules over ω_0^*: for $\beta \in \Gamma(B^*)$ and $f \in C^\infty(N)$, we find

$$\omega^\star(f\beta) = \omega_0^* f\, \omega^\star(\beta).$$

Next we need to show that a morphism $\mu^\star\colon \Gamma_N(B^*) \to \Gamma_M(A^*)$ of modules over $\mu_0^*\colon C^\infty(N) \to C^\infty(M)$, for $\mu_0\colon M \to N$ smooth, induces a morphism $A \to B$ of vector bundles over $\mu_0\colon M \to N$. Choose a_m in the fiber of A over m and define $\mu(a_m) \in B_{\mu_0(m)}$ by

$$\langle \beta(\mu_0(m)), \mu(a_m)\rangle = \langle \mu^\star(\beta)(m), a_m\rangle$$

for all $\beta \in \Gamma(B^*)$. To prove that μ is a vector bundle morphism[3], we need to check that $\langle \beta(\mu_0(m)), \mu(a_m)\rangle$ only depends on the value of β at $\mu_0(m)$, or in other words, that if β vanishes at $\mu_0(m)$, then $\langle \beta(\mu_0(m)), \mu(a_m)\rangle = 0$. If β vanishes at $\mu_0(m)$, then β can be written as $f \cdot \beta'$ with $\beta' \in \Gamma(B^*)$ and $f \in C^\infty(N)$ with $f(\mu_0(m)) = 0$. But then

$$\langle \beta(\mu_0(m)), \mu(a_m)\rangle = \langle f(\mu_0(m))\mu^\star(\beta')(m), a_m\rangle = 0.$$

The morphism μ of vector bundles clearly induces $\mu^\star$ on the sets of sections of the duals, and vice-versa. ☐

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[3] The smoothness of μ is seen as follows: let $b_1, \ldots, b_n$ be local basis fields for B and let $\beta_1, \ldots, \beta_n$ be the dual basis fields. Then for eah $a_m \in A$, $\mu(a_m)$ can be written $\sum_{i=1}^n \langle \mu(a_m), \beta_i(\mu_0(m))\rangle b_i(\mu_0(m))$. Since each $\langle \mu(a_m), \beta_i(\mu_0(m))\rangle$ equals $\ell_{\mu^\star(\beta_i)}(a_m)$, we find that locally, μ can be written $\mu = \sum_{i=1}^n \ell_{\mu^\star(\beta_i)} \cdot (b_i \circ \mu_0 \circ q_A)$.

2.2. **Double vector bundles.** We briefly recall the definitions of double vector bundles, of their *linear* and *core* sections, and of their *linear splittings* and *lifts*. We refer to [33, 27, 13] for more details.

A **double vector bundle** $(D; A, B; M)$ is a commutative square

$$\begin{tikzcd} D \arrow[r, "\pi_B"] \arrow[d, "\pi_A"'] & B \arrow[d, "q_B"] \\ A \arrow[r, "q_A"'] & M \end{tikzcd}$$

satisfying the following four conditions:

1. all four sides are vector bundles;
2. π_B is a vector bundle morphism over q_A;
3. $+_B : D \times_B D \to D$ is a vector bundle morphism over $+ : A \times_M A \to A$, where $+_B$ is the addition map for the vector bundle $D \to B$, and
4. the scalar multiplication $\mathbb{R} \times D \to D$ in the bundle $D \to B$ is a vector bundle morphism over the scalar multiplication $\mathbb{R} \times A \to A$.

The corresponding statements for the operations in the bundle $D \to A$ follow. Note that the condition that each addition in D is a morphism with respect to the other is exactly

$$(d_1 +_A d_2) +_B (d_3 +_A d_4) = (d_1 +_B d_3) +_A (d_2 +_B d_4) \tag{3}$$

for $d_1, d_2, d_3, d_4 \in D$ with $\pi_A(d_1) = \pi_A(d_2)$, $\pi_A(d_3) = \pi_A(d_4)$ and $\pi_B(d_1) = \pi_B(d_3)$, $\pi_B(d_2) = \pi_B(d_4)$.

Given a double vector bundle $(D; A, B; M)$, the vector bundles A and B are called the **side bundles**. The **core** C of a double vector bundle is the intersection of the kernels of π_A and of π_B. It has a natural vector bundle structure over M, the projection of which we call $q_C : C \to M$. The inclusion $C \hookrightarrow D$ is usually denoted by $C_m \ni c \longmapsto \overline{c} \in \pi_A^{-1}(0_m^A) \cap \pi_B^{-1}(0_m^B)$.

Given a double vector bundle $(D; A, B; M)$, the space of sections $\Gamma_B(D)$ is generated as a $C^\infty(B)$-module by two distinguished classes of sections (see [28]), the *linear* and the *core sections* which we now describe. For a smooth section $c : M \to C$, the corresponding **core section** $c^\dagger : B \to D$ is defined as

$$c^\dagger(b_m) = \tilde{0}_{b_m} +_A \overline{c(m)}, \; m \in M, \, b_m \in B_m. \tag{4}$$

We denote the corresponding core section $A \to D$ by $c^\dagger$ also, relying on the argument to distinguish between them. The space of core sections of D over B will be written $\Gamma_B^c(D)$.

A section $\xi \in \Gamma_B(D)$ is called **linear** if $\xi : B \to D$ is a bundle morphism from $B \to M$ to $D \to A$ over a section $a \in \Gamma(A)$. The space of linear sections of D over B is denoted by $\Gamma_B^\ell(D)$. A section $\psi \in \Gamma(B^* \otimes C)$ defines a linear section $\widetilde{\psi} : B \to D$ over the zero section $0^A : M \to A$ by

$$\widetilde{\psi}(b_m) = \tilde{0}_{b_m} +_A \overline{\psi(b_m)} \tag{5}$$

for all $b_m \in B$. We call $\widetilde{\psi}$ a **core-linear section**.

Example 2.2. Let A, B, C be vector bundles over M and consider $D = A \times_M B \times_M C$. With the vector bundle structures $D = q_A^!(B \oplus C) \to A$ and $D = q_B^!(A \oplus C) \to B$, one finds that $(D; A, B; M)$ is a double vector bundle called the *decomposed double vector bundle with sides A and B and core C*. The core sections are given by

$c^\dagger\colon b_m \mapsto (0^A_m, b_m, c(m))$, where $m \in M$, $b_m \in B_m$, $c \in \Gamma(C)$, and similarly for $c^\dagger\colon A \to D$. The space of linear sections $\Gamma^\ell_B(D)$ is naturally identified with $\Gamma(A) \oplus \Gamma(B^* \otimes C)$ via

$$(a, \psi) : b_m \mapsto (a(m), b_m, \psi(b_m)), \text{ where } \psi \in \Gamma(B^* \otimes C),\ a \in \Gamma(A).$$

In particular, the fibered product $A \times_M B$ is a double vector bundle over the sides A and B and its core is the trivial bundle over M.

2.2.1. *Linear splittings and lifts.* A **linear splitting** of $(D; A, B; M)$ is an injective morphism of double vector bundles $\Sigma\colon A \times_M B \hookrightarrow D$ over the identity on the sides A and B. That every double vector bundle admits local linear splittings was proved by [11]. Local linear splittings are equivalent to double vector bundle charts. Pradines originally defined double vector bundles as topological spaces with an atlas of double vector bundle charts [32] (see Definition 3.18). Using a partition of unity, he proved that (provided the double base is a smooth manifold) this implies the existence of a global double splitting [33]. Hence, any double vector bundle in the sense of our definition admits a (global) linear splitting.

Note that a linear splitting of D is equivalent to a **decomposition** of D, i.e. an isomorphism $\mathbb{I}\colon A \times_M B \times_M C \to D$ of double vector bundles over the identities on the sides and inducing the identity on the core. Given a linear splitting Σ, the corresponding decomposition $\mathbb{I}$ is given by $\mathbb{I}(a_m, b_m, c_m) = \Sigma(a_m, b_m) +_B (\tilde{0}_{b_m} +_A \overline{c_m})$. Given a decomposition $\mathbb{I}$, the corresponding linear splitting Σ is given by $\Sigma(a_m, b_m) = \mathbb{I}(a_m, b_m, 0^C_m)$.

A linear splitting Σ of D is also equivalent to a splitting σ_A of the short exact sequence of $C^\infty(M)$-modules

$$0 \longrightarrow \Gamma(B^* \otimes C) \hookrightarrow \Gamma^\ell_B(D) \longrightarrow \Gamma(A) \longrightarrow 0, \tag{6}$$

where the third map is the map that sends a linear section (ξ, a) to its base section $a \in \Gamma(A)$. The splitting σ_A will be called a **horizontal lift** or simply a **lift**. Given Σ, the horizontal lift $\sigma_A\colon \Gamma(A) \to \Gamma^\ell_B(D)$ is given by $\sigma_A(a)(b_m) = \Sigma(a(m), b_m)$ for all $a \in \Gamma(A)$ and $b_m \in B$. By the symmetry of a linear splitting, we find that a lift $\sigma_A\colon \Gamma(A) \to \Gamma^\ell_B(D)$ is equivalent to a lift $\sigma_B\colon \Gamma(B) \to \Gamma^\ell_A(D)$. Given a lift $\sigma_A\colon \Gamma(A) \to \Gamma^\ell_B(D)$, the corresponding lift $\sigma_B\colon \Gamma(B) \to \Gamma^\ell_A(D)$ is given by $\sigma_B(b)(a(m)) = \sigma_A(a)(b(m))$ for all $a \in \Gamma(A)$, $b \in \Gamma(B)$.

Note finally that two linear splittings $\Sigma^1, \Sigma^2\colon A \times_M B \to D$ differ by a section ϕ_{12} of $A^* \otimes B^* \otimes C \simeq \operatorname{Hom}(A, B^* \otimes C) \simeq \operatorname{Hom}(B, A^* \otimes C)$ in the following sense. For each $a \in \Gamma(A)$ the difference $\sigma^1_A(a) -_B \sigma^2_A(a)$ of horizontal lifts is the core-linear section defined by $\phi_{12}(a) \in \Gamma(B^* \otimes C)$. By symmetry, $\sigma^1_B(b) -_A \sigma^2_B(b) = \widetilde{\phi_{12}(b)}$ for each $b \in \Gamma(B)$.

The space of linear sections is a locally free and finitely generated $C^\infty(M)$-module (this follows from the existence of local splittings). Hence, there is a vector bundle $\widehat{A}$ over M such that $\Gamma^l_B(D)$ is isomorphic to $\Gamma(\widehat{A})$ as $C^\infty(M)$-modules. The vector bundle $\widehat{A}$ is sometimes called the **fat vector bundle** defined by $\Gamma^l_B(D)$. The short exact sequence (6) induces a short exact sequence of vector bundles

$$0 \longrightarrow B^* \otimes C \hookrightarrow \widehat{A} \longrightarrow A \longrightarrow 0. \tag{7}$$

2.2.2. *The tangent double of a vector bundle.* Let $q_E\colon E\to M$ be a vector bundle. Then the tangent bundle TE has two vector bundle structures; one as the tangent bundle of the manifold E, and the second as a vector bundle over TM. The structure maps of $TE\to TM$ are the derivatives of the structure maps of $E\to M$. The space TE is a double vector bundle with core bundle $E\to M$. The map $\bar{\ }\colon E\to p_E^{-1}(0^E)\cap(Tq_E)^{-1}(0^{TM})$ sends $e_m\in E_m$ to $\bar{e}_m=\frac{d}{dt}\big\rvert_{t=0}te_m\in T_{0^E_m}E$.

$$\begin{tikzcd}
TE \arrow[r, "p_E"] \arrow[d, "Tq_E"'] & E \arrow[d, "q_E"] \\
TM \arrow[r, "p_M"'] & M
\end{tikzcd}$$

The core vector field corresponding to $e\in\Gamma(E)$ is the vertical lift $e^\uparrow\colon E\to TE$, i.e. the vector field with flow $\phi\colon E\times\mathbb{R}\to E$, $\phi_t(e'_m)=e'_m+te(m)$. An element of $\Gamma^\ell_E(TE)=\mathfrak{X}^\ell(E)$ is called a **linear vector field**. It is well-known (see e.g. [27]) that a linear vector field $\xi\in\mathfrak{X}^l(E)$ covering $X\in\mathfrak{X}(M)$ is equivalent to a derivation $D_\xi^*\colon\Gamma(E^*)\to\Gamma(E^*)$ over $X\in\mathfrak{X}(M)$, and hence to the dual derivation $D_\xi\colon\Gamma(E)\to\Gamma(E)$ over $X\in\mathfrak{X}(M)$. The precise correspondence is given by[4]

$$\xi(\ell_\varepsilon)=\ell_{D^*_\xi(\varepsilon)}\quad\text{and}\quad\xi(q_E^*f)=q_E^*(X(f))\tag{8}$$

for all $\varepsilon\in\Gamma(E^*)$ and $f\in C^\infty(M)$. Here ℓ_ε is the linear function $E\to\mathbb{R}$ corresponding to ε. We will write $\widehat{D}$ for the linear vector field in $\mathfrak{X}^l(E)$ corresponding in this manner to a derivation D of $\Gamma(E)$. The choice of a linear splitting Σ for $(TE;TM,E;M)$ is equivalent to the choice of a connection on E: Since a linear splitting gives us a linear vector field $\sigma_{TM}(X)\in\mathfrak{X}^l(E)$ for each $X\in\mathfrak{X}(M)$, we can define $\nabla\colon\mathfrak{X}(M)\times\Gamma(E)\to\Gamma(E)$ by $\sigma_{TM}(X)=\widehat{\nabla_X}$ for all $X\in\mathfrak{X}(M)$. Conversely, a connection $\nabla\colon\mathfrak{X}(M)\times\Gamma(E)\to\Gamma(E)$ defines a lift σ^∇_{TM} for $(TE;TM,E;M)$ and a linear splitting $\Sigma^\nabla\colon TM\times_M E\to TE$.

We recall as well the relation between the connection and the Lie bracket of vector fields on E. Given ∇, it is easy to see using the equalities in (8) that, writing σ for σ^∇_{TM}:

(9)
$$[\sigma(X),\sigma(Y)]=\sigma[X,Y]-\widetilde{R_\nabla(X,Y)},\qquad\left[\sigma(X),e^\uparrow\right]=(\nabla_Xe)^\uparrow,\qquad\left[e^\uparrow,e'^\uparrow\right]=0,$$

for all $X,Y\in\mathfrak{X}(M)$ and $e,e'\in\Gamma(E)$. That is, the Lie bracket of vector fields on M and the connection encode completely the Lie bracket of vector fields on E.

Now let us have a quick look at the other structure on the double vector bundle TE. The lift $\sigma^\nabla_E\colon\Gamma(E)\to\Gamma^\ell_{TM}(TE)$ is given by

$$\sigma^\nabla_E(e)(v_m)=T_me(v_m)+_{TM}\left(T_m0^E(v_m)-_E\overline{\nabla_{v_m}e}\right),\ v_m\in TM,\ e\in\Gamma(E).$$

Further, for $e\in\Gamma(E)$, the core section $e^\times\in\Gamma_{TM}(TE)$ is given by

$$e^\times(v_m)=T_m0^E(v_m)+_E\left.\frac{d}{dt}\right|_{t=0}te(m).\tag{10}$$

---

[4] Since its flow is a flow of vector bundle morphisms, a linear vector field sends linear functions to linear functions and pullbacks to pullbacks.

2.2.3. *Dualisation and lifts.* Double vector bundles can be dualised in two distinct ways. We denote by $D \ast A$ the dual of D as a vector bundle over A and likewise for $D \ast B$. The dual $D \ast A$ is again a double vector bundle[5], with side bundles A and C^* and core B^* [26, 28].

$$\begin{tikzcd} D \arrow[r, "\pi_B"] \arrow[d, "\pi_A"'] & B \arrow[d, "q_B"] \\ A \arrow[r, "q_A"'] & M \end{tikzcd} \qquad \begin{tikzcd} D \ast A \arrow[r, "\pi_{C^*}"] \arrow[d, "\pi_A"'] & C^* \arrow[d, "q_{C^*}"] \\ A \arrow[r, "q_A"'] & M \end{tikzcd} \qquad \begin{tikzcd} D \ast B \arrow[r, "\pi_B"] \arrow[d, "\pi_{C^*}"'] & B \arrow[d, "q_B"] \\ C^* \arrow[r, "q_{C^*}"'] & M \end{tikzcd}$$

By dualising again $D \ast A$ over C^*, we get

$$\begin{tikzcd} D \ast A \ast C^* \arrow[r, "\pi_{C^*}"] \arrow[d, "\pi_B"'] & C^* \arrow[d, "q_{C^*}"] \\ B \arrow[r, "q_B"'] & M, \end{tikzcd}$$

with core A^*. In the same manner, we get a double vector bundle $D \ast B \ast C^*$ with sides A and C^* and core B^*.

The vector bundles $D \ast B \to C^*$ and $D \ast A \to C^*$ are, up to a sign, naturally in duality to each other [27]. The pairing

$$\boldsymbol{\langle} \cdot\,, \cdot \boldsymbol{\rangle} \colon (D \ast A) \times_{C^*} (D \ast B) \to \mathbb{R}$$

is defined as follows: for $\Phi \in D \ast A$ and $\Psi \in D \ast B$ projecting to the same element γ_m in C^*, choose $d \in D$ with $\pi_A(d) = \pi_A(\Phi)$ and $\pi_B(d) = \pi_B(\Psi)$. Then $\langle \Phi, d \rangle_A - \langle \Psi, d \rangle_B$ does not depend on the choice of d and we set $\boldsymbol{\langle} \Phi, \Psi \boldsymbol{\rangle} = \langle \Phi, d \rangle_A - \langle \Psi, d \rangle_B$.

This implies in particular that $D \ast A$ is canonically (up to a sign) isomorphic to $D \ast B \ast C^*$ and $D \ast B$ is isomorphic to $D \ast A \ast C^*$.

Each linear section $\xi \in \Gamma_B(D)$ over $a \in \Gamma(A)$ induces a linear section $\xi^{\sqcap} \in \Gamma^{\ell}_{C^*}(D \ast B \ast C^*)$ over a. Namely ξ induces a function $\ell_\xi \colon D \ast B \to \mathbb{R}$ which is fibrewise linear over B and, using the definition of the addition in $D \ast B \to C^*$, it follows that ℓ_ξ is also linear over C^*. The corresponding linear section of $D \ast B \ast C^* \to C^*$ is denoted $\xi^{\sqcap}$ [28]. Thus

$$\langle \xi^{\sqcap}(\gamma), \Phi \rangle_{C^*} = \ell_\xi(\Phi) = \langle \Phi, \, \xi(b) \rangle_B \tag{11}$$

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[5] The projection $\pi_{C^*} : D \ast A \to C^*$ is defined as follows: if $\Phi \in D \ast A$ projects to $\pi_A(\Phi) = a_m$, then $\pi_{C^*}(\Phi) \in C^*_m$ is defined by

$$\pi_{C^*}(\Phi)(c_m) = \Phi(0^D_{a_m} +_B \overline{c_m})$$

for all $c_m \in C_m$. The addition in the fibers of the vector bundle $D \ast A \to C^*$ is defined as follows: if Φ_1 and $\Phi_2 \in D \ast A$ satisfy

$$\pi_{C^*}(\Phi_1) = \pi_{C^*}(\Phi_2) \qquad \pi_A(\Phi_1) = a^1_m \qquad \pi_A(\Phi_2) = a^2_m,$$

then $\Phi_1 +_{C^*} \Phi_2$ is defined by

$$(\Phi_1 +_{C^*} \Phi_2)(d_1 +_B d_2) = \Phi_1(d_1) + \Phi_2(d_2)$$

for all $d_1, d_2 \in D$ with $\pi_B(d_1) = \pi_B(d_2)$ and $\pi_A(d_1) = a^1_m$, $\pi_A(d_2) = a^2_m$. The core element $\overline{\beta_m} \in D \ast A$ defined by $\beta_m \in B^*$ is defined by $\overline{\beta_m}(d) = \beta_m(\pi_B(d))$ for all $d \in D$ with $\pi_A(d) = 0^A_m$. By playing with the vector bundle structures on $D \ast A$ and (3), one can show that each core element of $D \ast A$ is of this form. We encourage the reader who is not familiar with the dualisations of double vector bundles to check this, and also to find out where the projection to C^* is relevant in the definition of the addition over C^*. See [28].

for $\Phi \in D \mathbin{\ast} B$ such that $\pi_B(\Phi) = b$ and $\pi_{C^*}(\Phi) = \gamma$.

Given a linear splitting $\Sigma\colon A \times_M B \to D$ of D, we get hence a linear splitting $\Sigma^{\star,B}\colon C^* \times_M A \to D \mathbin{\ast} B \mathbin{\ast} C^*$, defined by the horizontal lift $\sigma_A^{\star,B}\colon \Gamma(A) \to \Gamma_{C^*}^{\ell}(D \mathbin{\ast} B \mathbin{\ast} C^*)$:

$$\sigma_A^{\star,B}(a) = (\sigma_A(a))^{\sqcap} \tag{12}$$

for all $a \in \Gamma(A)$.

Now we use the (canonical up to a sign) isomorphism of $D \mathbin{\ast} A$ with $D \mathbin{\ast} B \mathbin{\ast} C^*$ to construct a canonical linear splitting of $D \mathbin{\ast} A$ given a linear splitting of D. We identify $D \mathbin{\ast} A$ with $D \mathbin{\ast} B \mathbin{\ast} C^*$ using $-\boldsymbol{\langle}\cdot\,,\cdot\boldsymbol{\rangle}$. Thus we define the horizontal lift $\sigma_A^{\star}\colon \Gamma(A) \to \Gamma_{C^*}^{\ell}(D \mathbin{\ast} A)$ by

$$\boldsymbol{\langle}\sigma_A^{\star}(a), \cdot\boldsymbol{\rangle} = -\sigma_A^{\star,B}(a) \tag{13}$$

for all $a \in \Gamma(A)$. The choice of sign in (13) is necessary for $\sigma_A^{\star}(a)$ to be a linear section of $D \mathbin{\ast} A$ over a (and not over $-a$).

By (skew-)symmetry, given the lift $\sigma_B\colon \Gamma(B) \to \Gamma_A^{\ell}(D)$, we identify $D \mathbin{\ast} B$ with $D \mathbin{\ast} A \mathbin{\ast} C^*$ using $\boldsymbol{\langle}\cdot,\cdot\boldsymbol{\rangle}$ and define the lift $\sigma_B^{\star}\colon \Gamma(B) \to \Gamma_{C^*}^{\ell}(D \mathbin{\ast} B)$ by $\boldsymbol{\langle}\sigma_B^{\star}(b), \cdot\boldsymbol{\rangle} = \sigma_B^{\star,A}(b)$ for all $b \in \Gamma(B)$. (This time, we do not need the minus sign.) As a summary, we have the equations:

$$\boldsymbol{\langle}\sigma_A^{\star}(a), \sigma_B^{\star}(b)\boldsymbol{\rangle} = 0, \qquad \boldsymbol{\langle}\sigma_A^{\star}(a), \alpha^{\dagger}\boldsymbol{\rangle} = -q_{C^*}^{*}\langle\alpha, a\rangle, \qquad \boldsymbol{\langle}\beta^{\dagger}, \sigma_B^{\star}(b)\boldsymbol{\rangle} = q_{C^*}^{*}\langle\beta, b\rangle, \tag{14}$$

for all $a \in \Gamma(A)$, $b \in \Gamma(B)$, $\alpha \in \Gamma(A^*)$ and $\beta \in \Gamma(B^*)$. Furthermore, the following Lemma shows that the horizontal lift $\sigma_A^{\star}\colon \Gamma(A) \to \Gamma_{C^*}^{l}(D \mathbin{\ast} A)$ is very natural.

Lemma 2.3. *Given a horizontal lift $\sigma_A\colon \Gamma(A) \to \Gamma_B^{l}(D)$, the "dual" horizontal lift $\sigma_A^{\star}\colon \Gamma(A) \to \Gamma_{C^*}^{l}(D \mathbin{\ast} A)$ can alternatively be defined by*

$$\langle\sigma_A^{\star}(a)(\gamma_m), \sigma_A(a)(b_m)\rangle_A = 0, \qquad \langle\sigma_A^{\star}(a)(\gamma_m), c^{\dagger}(a(m))\rangle_A = \langle\gamma_m, c(m)\rangle$$

for all $a \in \Gamma(A)$, $c \in \Gamma(C)$, $b_m \in B$ and $\gamma_m \in C^*$.

Proof. By (12) and (11), we have

$$\boldsymbol{\langle}\sigma_A^{\star}(a)(\gamma_m), \Phi\boldsymbol{\rangle} = -\langle(\sigma_A(a))^{\sqcap}(\gamma_m), \Phi\rangle = -\langle\Phi, \sigma_A(a)(b_m)\rangle \tag{15}$$

for $\Phi \in D \mathbin{\ast} B$ with $\pi_B(\Phi) = b_m$ and $\pi_{C^*}(\Phi) = \gamma_m$. Since for $b \in \Gamma(B)$ with $b(m) = b_m$, $d = \sigma_A(a)(b_m) = \sigma_B(b)(a(m)) \in D$ is an element with $\pi_A(d) = a(m)$ and $\pi_B(d) = b(m) = b_m$, the pairing on the left can also be written

$$\boldsymbol{\langle}\sigma_A^{\star}(a)(\gamma_m), \Phi\boldsymbol{\rangle} = \langle\sigma_A^{\star}(a)(\gamma_m), \sigma_A(a)(b(m))\rangle_B - \langle\Phi, \sigma_A(a)(b(m))\rangle. \tag{16}$$

(15) and (16) together show that $\langle\sigma_A^{\star}(a)(\gamma_m), \sigma_A(a)(b(m))\rangle = 0$. Further, we find for $\alpha \in \Gamma(B^*)$ and any $c \in \Gamma(C)$:

$$-\langle\alpha, a\rangle(m) = \boldsymbol{\langle}\sigma_A^{\star}(a)(\gamma_m), \alpha^{\dagger}(\gamma_m)\boldsymbol{\rangle} = \langle\sigma_A^{\star}(a)(\gamma_m), c^{\dagger}(a(m))\rangle - \langle\alpha^{\dagger}(\gamma_m), c^{\dagger}(a(m))\rangle.$$

In order to pair $\alpha^{\dagger}(\gamma_m)$ with $c^{\dagger}(a(m))$ over $0_m^B \in B$, we write them as

$$\alpha^{\dagger}(\gamma_m) = 0_{\gamma_m}^{D \mathbin{\ast} B} +_B \overline{\alpha(m)}, \qquad c^{\dagger}(a(m)) = 0_{a(m)}^{D} +_B \overline{c(m)}.$$

We get

$$-\langle\alpha, a\rangle(m) = \langle\sigma_A^{\star}(a)(\gamma_m), c^{\dagger}(a(m))\rangle - \langle\gamma_m, c(m)\rangle - \langle a, \alpha\rangle(m),$$

and so $\langle\sigma_A^{\star}(a)(\gamma_m), c^{\dagger}(a(m))\rangle = \langle\gamma_m, c(m)\rangle$ for all $a \in \Gamma(A)$, $c \in \Gamma(C)$ and $\gamma_m \in C^*$. □

2.3. **VB-algebroids and double Lie algebroids.** Let $(D;A,B;M)$ be a double vector bundle

$$\begin{tikzcd} D \arrow[r, "\pi_B"] \arrow[d, "\pi_A"'] & B \arrow[d, "q_B"] \\ A \arrow[r, "q_A"'] & M \end{tikzcd}$$

with core C. Then $(D\to B;A\to M)$ is a **VB-algebroid** ([25]; see also [13]) if $D\to B$ has a Lie algebroid structure the anchor of which is a bundle morphism $\Theta_B\colon D\to TB$ over $\rho_A\colon A\to TM$ and such that the Lie bracket is linear:

$$[\Gamma^\ell_B(D),\Gamma^\ell_B(D)]\subset\Gamma^\ell_B(D),\qquad [\Gamma^\ell_B(D),\Gamma^c_B(D)]\subset\Gamma^c_B(D),\qquad [\Gamma^c_B(D),\Gamma^c_B(D)]=0.$$

The vector bundle $A\to M$ is then also a Lie algebroid, with anchor ρ_A and bracket defined as follows: if $\xi_1,\xi_2\in\Gamma^\ell_B(D)$ are linear over $a_1,a_2\in\Gamma(A)$, then the bracket $[\xi_1,\xi_2]$ is linear over $[a_1,a_2]$. We also say that the Lie algebroid structure on $D\to B$ is linear over the Lie algebroid $A\to M$.

Since the anchor Θ_B is linear, it sends a core section $c^\dagger$, $c\in\Gamma(C)$ to a vertical vector field on B. This defines the **core-anchor** $\partial_B\colon C\to B$; for $c\in\Gamma(C)$ we have $\Theta_B(c^\dagger)=(\partial_B c)^\uparrow$ (see [24]).

Example 2.4. It is easy to see from the considerations in §2.2.2 that the tangent double $(TE;E,TM;M)$ of a vector bundle $E\to M$ has a VB-algebroid structure $(TE\to E,TM\to M)$. (The Lie algebroid structures are the tangent Lie algebroid structures.)

If D is a VB-algebroid with Lie algebroid structures on $D\to B$ and $A\to M$ the dual vector bundle $D\mathbin{|\!\!\!\ast}B\to B$ has a *Lie-Poisson structure* (a linear Poisson structure), and the structure on $D\mathbin{|\!\!\!\ast}B$ is also Lie-Poisson with respect to $D\mathbin{|\!\!\!\ast}B\to C^*$ [28, 3.4]. Dualising this bundle gives a Lie algebroid structure on $D\mathbin{|\!\!\!\ast}B\mathbin{|\!\!\!\ast}C^*\to C^*$. This equips the double vector bundle $(D\mathbin{|\!\!\!\ast}B\mathbin{|\!\!\!\ast}C^*;C^*,A;M)$ with a VB-algebroid structure. Using the isomorphism defined by $-\langle\cdot,\cdot\rangle$, the double vector bundle $(D\mathbin{|\!\!\!\ast}A\to C^*;A\to M)$ is also a VB-algebroid. In the same manner, if $(D\to A,B\to M)$ is a VB-algebroid then $(D\mathbin{|\!\!\!\ast}B\to C^*;B\to M)$ is a VB-algebroid.

A **double Lie algebroid** [28] is a double vector bundle $(D;A,B;M)$ with core denoted C, and with Lie algebroid structures on each of $A\to M$, $B\to M$, $D\to A$ and $D\to B$ such that each pair of parallel Lie algebroids gives D the structure of a VB-algebroid, and such that the pair $(D\mathbin{|\!\!\!\ast}A,D\mathbin{|\!\!\!\ast}B)$ with the induced Lie algebroid structures on base C^* and the pairing $\langle\cdot,\cdot\rangle$, is a Lie bialgebroid.

2.4. **Representations up to homotopy and VB-algebroids.** Let $A\to M$ be a Lie algebroid and consider an A-connection ∇ on a vector bundle $E\to M$. Then the space $\Omega^\bullet(A,E)$ of E-valued Lie algebroid forms has an induced operator $\mathbf{d}_\nabla$ given by:

$$\begin{aligned}\mathbf{d}_\nabla\omega(a_1,\ldots,a_{k+1})=&\sum_{i<j}(-1)^{i+j}\omega([a_i,a_j],a_1,\ldots,\hat{a}_i,\ldots,\hat{a}_j,\ldots,a_{k+1})\\ &+\sum_i(-1)^{i+1}\nabla_{a_i}(\omega(a_1,\ldots,\hat{a}_i,\ldots,a_{k+1}))\end{aligned}$$

for all $\omega\in\Omega^k(A,E)$ and $a_1,\ldots,a_{k+1}\in\Gamma(A)$. The connection is flat if and only if $\mathbf{d}_\nabla=0$.

Let now $\mathcal{E} = \bigoplus_{k\in\mathbb{Z}} E_k[k]$ be a graded vector bundle. Consider the space $\Omega(A, \mathcal{E})$ with grading given by

$$\Omega(A, \mathcal{E})[k] = \bigoplus_{i+j=k} \Omega^i(A, E_j).$$

Definition 2.5. [1] *A representation up to homotopy of A on $\mathcal{E}$ is a map $\mathcal{D}\colon \Omega(A, \mathcal{E}) \to \Omega(A, \mathcal{E})$ with total degree 1 and such that $\mathcal{D}^2 = 0$ and*

$$\mathcal{D}(\alpha \wedge \omega) = \mathbf{d}_A\alpha \wedge \omega + (-1)^{|\alpha|}\alpha \wedge \mathcal{D}(\omega), \text{ for } \alpha \in \Gamma(\wedge A^*),\ \omega \in \Omega(A, \mathcal{E}),$$

where $\mathbf{d}_A\colon \Gamma(\wedge A^*) \to \Gamma(\wedge A^*)$ is the Lie algebroid differential.

Note that Gracia-Saz and Mehta [13] defined this concept independently and called them "superrepresentations".

Let A be a Lie algebroid. The representations up to homotopy which we will consider are always on graded vector bundles $\mathcal{E} = E_0[0] \oplus E_1[1]$ concentrated on degrees 0 and 1, so called 2-*term graded vector bundles.* These representations are equivalent to the following data (see [1, 13]):

1. a map $\partial\colon E_0 \to E_1$,
2. two A-connections, ∇^0 and ∇^1 on E_0 and E_1, respectively, such that $\partial \circ \nabla^0 = \nabla^1 \circ \partial$,
3. an element $R \in \Omega^2(A, \mathrm{Hom}(E_1, E_0))$ such that $R_{\nabla^0} = R \circ \partial$, $R_{\nabla^1} = \partial \circ R$ and $\mathbf{d}_{\nabla^{\mathrm{Hom}}}R = 0$, where ∇^{Hom} is the connection induced on $\mathrm{Hom}(E_1, E_0)$ by ∇^0 and ∇^1.

For brevity we will call such a 2-term representation up to homotopy a **2-representation**.

Let $(D \to B, A \to M)$ be a VB-Lie algebroid and choose a linear splitting $\Sigma\colon A \times_M B \to D$. Since the anchor of a linear section is linear, for each $a \in \Gamma(A)$ the vector field $\Theta_B(\sigma_A(a))$ defines a derivation of $\Gamma(B)$ with symbol $\rho(a)$ (see §2.2.2). This defines a linear connection $\nabla^{AB}\colon \Gamma(A) \times \Gamma(B) \to \Gamma(B)$:

$$\Theta_B(\sigma_A(a)) = \widehat{\nabla_a^{AB}}$$

for all $a \in \Gamma(A)$. Since the bracket of a linear section with a core section is again a core section, we find a linear connection $\nabla^{AC}\colon \Gamma(A) \times \Gamma(C) \to \Gamma(C)$ such that

$$[\sigma_A(a), c^\dagger] = (\nabla_a^{AC}c)^\dagger$$

for all $c \in \Gamma(C)$ and $a \in \Gamma(A)$. The difference $\sigma_A[a_1, a_2] - [\sigma_A(a_1), \sigma_A(a_2)]$ is a core-linear section for all $a_1, a_2 \in \Gamma(A)$. This defines a vector valued Lie algebroid form $R \in \Omega^2(A, \mathrm{Hom}(B, C))$ such that

$$[\sigma_A(a_1), \sigma_A(a_2)] = \sigma_A[a_1, a_2] - \widetilde{R(a_1, a_2)},$$

for all $a_1, a_2 \in \Gamma(A)$. See [13] for more details on these constructions. The following theorem is proved in [13].

Theorem 2.6. *Let $(D \to B; A \to M)$ be a VB-algebroid and choose a linear splitting $\Sigma\colon A \times_M B \to D$. The triple $(\nabla^{AB}, \nabla^{AC}, R)$ defined as above is a 2-representation of A on the complex $\partial_B\colon C \to B$, where ∂_B is the core-anchor.*

Conversely, let $(D; A, B; M)$ be a double vector bundle such that A has a Lie algebroid structure and choose a linear splitting $\Sigma\colon A \times_M B \to D$. Then if $(\nabla^{AB}, \nabla^{AC}, R)$ is a 2-representation of A on a complex $\partial_B\colon C \to B$, then the three equations above and the core-anchor ∂_B define a VB-algebroid structure on $(D \to B; A \to M)$.

In the situation of the previous theorem, we have

$$\left[\sigma_A(a), \widetilde{\phi}\right] = \widetilde{\nabla^{\mathrm{Hom}}_a \phi} \qquad \text{and} \qquad \left[c^\dagger, \widetilde{\phi}\right] = (\phi(\partial_B c))^\dagger$$

for all $a \in \Gamma(A)$, $\phi \in \Gamma(\mathrm{Hom}(B, C))$ and $c \in \Gamma(C)$, see for instance [12].

Remark 2.7. If $\Sigma_1, \Sigma_2 \colon A \times_M B \to D$ are two linear splittings of a VB-algebroid $(D \to B, A \to M)$ and $\phi_{12} \in \Gamma(A^* \otimes B^* \otimes C)$ is the change of splitting, then the two corresponding 2-representations are related by the following identities [13].

$$\nabla^{B,2}_a = \nabla^{B,1}_a + \partial_B \circ \phi_{12}(a), \quad \nabla^{C,2}_a = \nabla^{C,1}_a + \phi_{12}(a) \circ \partial_B$$

and

$$\begin{aligned} R^2(a_1, a_2) =& R^1(a_1, a_2) + (\mathbf{d}_{\nabla^{\mathrm{Hom}(B,C)}} \phi_{12})(a_1, a_2) \\ &+ \phi_{12}(a_1)\partial_B \phi_{12}(a_2) - \phi_{12}(a_2)\partial_B \phi_{12}(a_1) \end{aligned}$$

for all $a, a_1, a_2 \in \Gamma(A)$.

Example 2.8. Choose a linear connection $\nabla \colon \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$ and consider the corresponding linear splitting Σ^∇ of TE as in Section 2.2.2. The description of the Lie bracket of vector fields in (9) shows that the 2-representation induced by Σ^∇ is the 2-representation of TM on $\mathrm{Id}_E \colon E \to E$ given by $(\nabla, \nabla, R_\nabla)$.

Example 2.9 (The tangent of a Lie algebroid)**.** Let $(A \to M, \rho, [\cdot\,,\cdot])$ be a Lie algebroid. Then the tangent $TA \to TM$ has a Lie algebroid structure with bracket defined by $[Ta_1, Ta_2] = T[a_1, a_2]$, $[Ta_1, a_2^\dagger] = [a_1, a_2]^\dagger$ and $[a_1^\dagger, a_2^\dagger] = 0$ for all $a_1, a_2 \in \Gamma(A)$. The anchor of Ta is $\widehat{[\rho(a), \cdot]} \in \mathfrak{X}(TM)$ and the anchor of $a^\dagger$ is $\rho(a)^\uparrow$ for all $a \in \Gamma(A)$. This defines a VB-algebroid structure $(TA \to TM; A \to M)$ on $(TA; TM, A; M)$.

Given a TM-connection on A, and so a linear splitting Σ^∇ of TA as in Section 2.2.2, the 2-representation of A on $\rho \colon A \to TM$ encoding this VB-algebroid is the **adjoint 2-representation** $(\nabla^{\mathrm{bas}}, \nabla^{\mathrm{bas}}, R^{\mathrm{bas}}_\nabla)$, where the connections are defined by

$$\nabla^{\mathrm{bas}} \colon \Gamma(A) \times \mathfrak{X}(M) \to \mathfrak{X}(M), \qquad \nabla^{\mathrm{bas}}_a X = [\rho(a), X] + \rho(\nabla_X a)$$

and

$$\nabla^{\mathrm{bas}} \colon \Gamma(A) \times \Gamma(A) \to \Gamma(A), \qquad \nabla^{\mathrm{bas}}_{a_1} a_2 = [a_1, a_2] + \nabla_{\rho(a_2)} a_1,$$

and $R^{\mathrm{bas}}_\nabla \in \Omega^2(A, \mathrm{Hom}(TM, A))$ is given by

$$R^{\mathrm{bas}}_\nabla(a_1, a_2)X = -\nabla_X [a_1, a_2] + [\nabla_X a_1, a_2] + [a_1, \nabla_X a_2] + \nabla_{\nabla^{\mathrm{bas}}_{a_2} X} a_1 - \nabla_{\nabla^{\mathrm{bas}}_{a_1} X} a_2$$

for all $X \in \mathfrak{X}(M)$, $a, a_1, a_2 \in \Gamma(A)$.

2.4.1. *Dualisation and 2-representations.* Let $(D \to B, A \to M)$ be a VB-algebroid. Let $\Sigma \colon A \times_M B \to D$ be a linear splitting of D and denote by (∇^B, ∇^C, R) the 2-representation of the Lie algebroid A on $\partial_B \colon C \to B$. We have seen above that $(D \mathbin{\mid\!\!\!\ast} A \to C^*, A \to M)$ has an induced VB-algebroid structure, and we have shown that the linear splitting Σ induces a linear splitting $\Sigma^\star \colon A \times_M C^* \to D \mathbin{\mid\!\!\!\ast} A$ of $D \mathbin{\mid\!\!\!\ast} A$. The 2-representation of A that is associated to this splitting is then $(\nabla^{C^*}, \nabla^{B^*}, -R^*)$ on the complex $\partial_B^* \colon B^* \to C^*$. This is easy to verify, and proved in the appendix[6]

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[6] The construction of the "dual" linear splitting of $D \mathbin{\mid\!\!\!\ast} A$, given a linear splitting of D, is done in [10] by dualising the decomposition and taking its inverse. The resulting linear splitting of $D \mathbin{\mid\!\!\!\ast} A$ is the same.

of [10]. For the proof we only need to recall that, by construction, $\ell_{\sigma^\star_A(a)}$ equals $\ell_{\sigma_A(a)}$ as a function on $D \mathbin{\ast\!\!\!|} B$.

2.4.2. *Double Lie algebroids and matched pairs of* 2*-representations.*

Definition 2.10. [12] *Let* $(A \to M, \rho_A, [\cdot\,,\cdot])$ *and and* $(B \to M, \rho_B, [\cdot\,,\cdot])$ *be two Lie algebroids and assume that* A *acts on* $B \oplus C$ *up to homotopy via* $(\partial_B \colon C \to B, \nabla^{AB}, \nabla^{AC}, R_{AB})$ *and* B *acts on* $A \oplus C$ *up to homotopy via* $(\partial_A \colon C \to A, \nabla^{BA}, \nabla^{BC}, R_{BA})$[7]*. Then we say that the two representations up to homotopy form a matched pair if*

(1) $\nabla_{\partial_A c_1} c_2 - \nabla_{\partial_B c_2} c_1 = -\nabla_{\partial_A c_2} c_1 + \nabla_{\partial_B c_1} c_2$,
(2) $[a, \partial_A c] = \partial_A(\nabla_a c) - \nabla_{\partial_B c} a$,
(3) $[b, \partial_B c] = \partial_B(\nabla_b c) - \nabla_{\partial_A c} b$,
(4) $\nabla_b \nabla_a c - \nabla_a \nabla_b c - \nabla_{\nabla_b a} c + \nabla_{\nabla_a b} c = R_{BA}(b, \partial_B c)a - R_{AB}(a, \partial_A c)b$,
(5) $\partial_A(R_{AB}(a_1, a_2)b) = -\nabla_b[a_1, a_2] + [\nabla_b a_1, a_2] + [a_1, \nabla_b a_2] + \nabla_{\nabla_{a_2} b} a_1 - \nabla_{\nabla_{a_1} b} a_2$,
(6) $\partial_B(R_{BA}(b_1, b_2)a) = -\nabla_a[b_1, b_2] + [\nabla_a b_1, b_2] + [b_1, \nabla_a b_2] + \nabla_{\nabla_{b_2} a} b_1 - \nabla_{\nabla_{b_1} a} b_2$,

for all $a, a_1, a_2 \in \Gamma(A)$*,* $b, b_1, b_2 \in \Gamma(B)$ *and* $c, c_1, c_2 \in \Gamma(C)$*, and*

(7) $\mathbf{d}_{\nabla^A} R_{BA} = \mathbf{d}_{\nabla^B} R_{AB} \in \Omega^2(A, \wedge^2 B^* \otimes C) = \Omega^2(B, \wedge^2 A^* \otimes C)$*, where* R_{AB} *is seen as an element of* $\Omega^1(A, \wedge^2 B^* \otimes C)$ *and* R_{AB} *as an element of* $\Omega^1(B, \wedge^2 A^* \otimes C)$*.*

Remark 2.11. From these equations follow $\rho_A \circ \partial_A = \rho_B \circ \partial_B$ and $[\rho_A(a), \rho_B(b)] = \rho_B(\nabla_a b) - \rho_A(\nabla_b a)$ for all $a \in \Gamma(A)$ and $b \in \Gamma(B)$.

The vector bundle C inherits a Lie algebroid structure with anchor $\rho_A \circ \partial_A = \rho_B \circ \partial_B$ and with bracket given by $[c_1, c_2] = \nabla_{\partial_A c_1} c_2 - \nabla_{\partial_B c_2} c_1$ for all $c_1, c_2 \in \Gamma(C)$. The proof of the Jacobi identity is a not completely straightforward computation; it follows from (2), (3) and (4) and can be done just as the proof of Theorem 7.7.

Consider a double vector bundle $(D; A, B; M)$ with core C and a VB-Lie algebroid structure on each of its sides. After the choice of a splitting $\Sigma \colon A \times_M B \to D$, the Lie algebroid structures on the two sides of D are described as above by two 2-representations; the Lie algebroid $D \to B$ is described by $(\partial_B, \nabla^{AB}, \nabla^{AC}, R_{AB} \in \Omega^2(A, \operatorname{Hom}(B, C)))$. The Lie algebroid structure of $D \to A$ is described by $(\partial_A, \nabla^{BA}, \nabla^{BC}, R_{BA} \in \Omega^2(B, \operatorname{Hom}(A, C)))$, where $\partial_A \colon C \to A$, $\nabla^{BA} \colon \Gamma(B) \times \Gamma(A) \to \Gamma(A)$, $\nabla^{BC} \colon \Gamma(B) \times \Gamma(C) \to \Gamma(C)$ are connections and R_{BA} is the curvature term.

We prove in [18] that $(D; A, B, M)$ is a double Lie algebroid if and only if, for any decomposition of D, the two induced 2-representations above form a matched pair.

3. [2]-MANIFOLDS AND METRIC DOUBLE VECTOR BUNDLES

In this section we recall the definition of N-manifolds of degree 2. Then we introduce linear metrics on double vector bundles, and we show how the category of N-manifolds of degree 2 is equivalent to the category of metric double vector bundles.

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[7] For the sake of simplicity, we write in this definition ∇ for all the four connections. It will always be clear from the indexes which connection is meant. We write ∇^A for the A-connection induced by ∇^{AB} and ∇^{AC} on $\wedge^2 B^* \otimes C$ and ∇^B for the B-connection induced on $\wedge^2 A^* \otimes C$.

3.1. **N-manifolds.**

Here we give the definitions of N-manifolds. We are particularly interested in N-manifolds of degree 2. We refer to [3] for more details.

Definition 3.1. *An* ***N-manifold*** *or* $\mathbb{N}$***-graded manifold*** $\mathcal{M}$ *of degree* n *and dimension* $(p; r_1, \ldots, r_n)$ *is a smooth* p*-dimensional manifold* M *endowed with a sheaf* $\mathcal{A} = C^\infty(\mathcal{M})$ *of* $\mathbb{N}$*-graded commutative associative unital* $\mathbb{R}$*-algebras, whose degree 0 term is* $\mathcal{A}^0 = C^\infty(M)$ *and which is locally freely generated as a sheaf of* $C^\infty(M)$*-algebras by* $r_1 + \ldots + r_n$ *graded commutative generators* $\xi_1^1, \ldots, \xi_1^{r_1}, \xi_2^1, \ldots, \xi_2^{r_2}, \ldots, \xi_n^1, \ldots, \xi_n^{r_n}$ *with* ξ_i^j *of degree* i *for* $i \in \{1, \ldots, n\}$ *and* $j \in \{1, \ldots, r_i\}$.

A morphism of N-manifolds $\mu : \mathcal{M} \to \mathcal{N}$ *is a smooth map* $\mu_0 \colon M \to N$ *of the underlying smooth manifolds together with a morphism* $\mu^\star \colon C^\infty(\mathcal{N}) \to C^\infty(\mathcal{M})$ *of sheaves of graded algebras such that* $\mu^\star(f) = \mu_0^* f$ *for all* $f \in C^\infty(N)$.

We will call $[n]$**-manifold** an N-manifold of degree $n \in \mathbb{N}$. We will write $\mathcal{A}^i$ for the elements of degree i in $\mathcal{A}$, and we will write $|\xi|$ for the degree of a homogeneous element $\xi \in \mathcal{A}$, i.e. an element which can be written as a sum of functions of the same degree. Note that a [1]**-manifold** over a smooth manifold M is equivalent to a locally finitely generated sheaf of $C^\infty(M)$-modules.

Let us quickly introduce the notion of vector field on an N-manifold. Let $\mathcal{M}$ be an $[n]$-manifold and write as before $\mathcal{A} = C^\infty(\mathcal{M})$. A **vector field** of degree j on $\mathcal{M}$ is a degree j derivation ϕ of $\mathcal{A}$ such that

$$|\phi(\xi)| = j + |\xi|$$

for a homogeneous element $\xi \in \mathcal{A}$. As in [3], we write $\operatorname{Der}^\bullet \mathcal{A}$ for the sheaf of graded derivations of $\mathcal{A}$.

The vector fields on $\mathcal{M}$ and their Lie bracket defined by $[\phi, \psi] = \phi\psi - (-1)^{|\phi||\psi|}\psi\phi$ satisfy the following conditions:

1. $\phi(\xi\eta) = \phi(\xi)\eta + (-1)^{|\phi||\xi|}\xi\phi(\eta)$,
2. $[\phi, \psi] = (-1)^{1+|\phi||\psi|}[\psi, \phi]$,
3. $[\phi, \xi\psi] = \phi(\xi)\psi + (-1)^{|\phi||\xi|}\xi[\phi, \psi]$,
4. $(-1)^{|\phi||\gamma|}[\phi, [\psi, \gamma]] + (-1)^{|\psi||\phi|}[\psi, [\gamma, \phi]] + (-1)^{|\gamma||\psi|}[\gamma, [\phi, \psi]] = 0$

for ϕ, ψ, γ homogeneous elements of $\operatorname{Der}^\bullet \mathcal{A}$ and ξ, η homogeneous elements of $\mathcal{A}$. For instance, the derivation $\partial_{\xi_j^i}$ of $\mathcal{A}(U)$ sends ξ_j^i to 1 and the other local generators to 0. It is hence a derivation of degree $-j$. Locally, $\operatorname{Der}^\bullet \mathcal{A}(U)$ is generated as a $\mathcal{A}(U)$-module by ∂_{x_k}, $k = 1, \ldots, p$ and $\partial_{\xi_j^i}$, $j = 1, \ldots, n$, $i = 1, \ldots, r_j$.

Our goal in this chapter is to prove that [2]-manifolds are equivalent to double vector bundles endowed with a linear metric (Theorem 3.17). We begin with a few observations on the equivalence of locally free and finitely generated sheaves of C^∞-modules with smooth vector bundles. Theorem 3.17 will in a sense generalise this result.

3.1.1. *Vector bundles and* [1]*-manifolds.*

Here we recall the equivalence of categories between degree [1]-manifolds (or locally free and finitely generated sheaves of C^∞-modules) and smooth vector bundles (see for instance [40, Theorem II.1.13]). This section can be seen as introductory to the methods in Section 3.3.

Let VB be the category of smooth vector bundles, with the following morphisms. Let $E \to M$ and $F \to N$ be vector bundles. Then a morphism $\Phi \colon F \dashrightarrow E$ of vector bundles is a vector bundle map $\phi \colon F^* \to E^*$ over $\phi_0 \colon N \to M$. Recall from

Lemma 2.1 that this is equivalent to a map $\phi^\star\colon \Gamma(E) \to \Gamma(F)$ defined as in (2) and satisfying

$$\phi^\star(f \cdot e) = \phi_0^* f \cdot \phi^\star(e)$$

for all $f \in C^\infty(M)$ and $e \in \Gamma(E)$.

Let [1]-Man be the category of [1]-manifolds, or equivalently the category of locally free and finitely generated sheaves of $C^\infty(M)$-modules, for smooth manifolds M. The morphisms in this category are defined as follows. Let $\mathcal{A}_M$ and $\mathcal{A}_N$ be two sheaves of $C^\infty(M)$, respectively $C^\infty(N)$-modules, for two smooth manifolds M and N. A morphism $\mu\colon \mathcal{A}_N \dashrightarrow \mathcal{A}_M$ is a pair of a morphism $\mu^\star\colon \mathcal{A}_M \to \mathcal{A}_N$ of sheaves of modules and a smooth map $\mu_0\colon N \to M$, such that

$$\mu^\star(f \cdot a) = \mu_0^* f \cdot \mu^\star(a)$$

for all $f \in C^\infty(U)$ and $a \in \mathcal{A}_M(U)$, U open in M.

We now establish the equivalence between VB and [1]-Man. The functor $\Gamma(\cdot)\colon$ VB $\to$ [1]-Man sends a vector bundle $E \to M$ to its set of sections $\Gamma(E)$, a locally free and finitely generated sheaf of $C^\infty(M)$-modules. $\Gamma(\cdot)$ sends a morphism $\Phi = (\phi, \phi_0)\colon F \dashrightarrow E$ as above to the morphism $\phi^\star\colon \Gamma(E) \to \Gamma(F)$ over $\phi_0^*\colon C^\infty(M) \to C^\infty(N)$.

Next choose a [1]-manifold $\mathcal{A}$ over a smooth manifold M. There exists a maximal covering $\{U_\alpha\}$ of M such that $\mathcal{A}(U_\alpha)$ is finitely generated by generators $\xi_1^\alpha, \ldots, \xi_m^\alpha$. For two indices α, β such that $U_\alpha \cap U_\beta \neq \emptyset$, we can write each generator in an unique manner as $\xi_j^\beta = \sum_{i=1}^m \psi_{\alpha\beta}^{ij} \xi_i^\alpha$ with smooth functions $\psi_{\alpha\beta}^{ij} \in C^\infty(U_\alpha \cap U_\beta)$. We define $A_{\alpha\beta} \in C^\infty(U_\alpha \cap U_\beta, \mathrm{Gl}(\mathbb{R}^m))$ by $A_{\alpha\beta}(x) = (\psi_{\alpha\beta}^{ij})_{i,j=1,\ldots,m}$. We have then immediately

$$A_{\gamma\alpha} \cdot A_{\alpha\beta} = A_{\gamma\beta}, \tag{17}$$

where $\cdot$ is the pointwise multiplication of matrices. Next we consider the disjoint union $\tilde{E} = \bigsqcup_\alpha U_\alpha \times \mathbb{R}^m$ and identify for $x \in U_\alpha \cap U_\beta \neq \emptyset$

$$(x, v) \in U_\beta \times \mathbb{R}^m \quad \text{with} \quad (x, A_{\alpha\beta}(x)v) \in U_\alpha \times \mathbb{R}^m.$$

By (17), this defines an equivalence relation on $\tilde{E}$ and the quotient $E = E(\mathcal{A}^1)$ has a smooth vector bundle structure with vector bundle charts given by the inclusions $U_\alpha \times \mathbb{R}^m \hookrightarrow E$, and changes of charts the cocycles $A_{\alpha\beta}$. Note that the maps $e_i^\alpha\colon U_\alpha \to U_\alpha \times \mathbb{R}^m$, $x \mapsto (x, e_i)$ define smooth local sections of E and $e_i^\beta = \sum_{j=1}^n \psi_{\alpha\beta}^{ji} e_j^\alpha$ for α, β such that $U_\alpha \cap U_\beta \neq \emptyset$. Hence, we can identify ξ_i^α with the section e_i^α and we see that a morphism $\mu\colon \mathcal{A}_N^1 \dashrightarrow \mathcal{A}_M^1$ over $\mu_0\colon N \to M$ defines a morphism $E(\mu)^\star\colon \Gamma(E(\mathcal{A}_M^1)) \to \Gamma(E(\mathcal{A}_N^1))$ of modules over $\mu_0^*\colon C^\infty(M) \to C^\infty(N)$, and so by Lemma 2.1 a vector bundle morphism $E^*(\mathcal{A}_N^1) \to E^*(\mathcal{A}_M^1)$ over $\mu_0\colon N \to M$. Hence we have constructed a functor $E(\cdot)\colon$ [1]-Man $\to$ VB.

Next we show that the two functors build together an equivalence of categories. The functor $\mathbb{E}(\cdot) \circ \mathcal{A}(\cdot)\colon$ VB $\to$ VB sends a vector bundle to the abstract vector bundle defined by its trivialisations and cocycles. There is an obvious natural isomorphism between this functor and the identity functor VB $\to$ VB.

The functor $\mathcal{A}(\cdot) \circ E(\cdot)\colon$ [1]-Man $\to$ [1]-Man sends a [1]-manifold over M with local generators ξ_i^α and cocycles $A_{\alpha\beta}$ to the sheaf of sections of $E(\mathcal{A})$, with local basis sections e_i^α and cocycles $A_{\alpha\beta}$. There is an obvious natural isomorphism between this functor and the identity functor [1]-Man $\to$ [1]-Man.

Finally note that it would be more natural to define a (contravariant!) functor from the category of [1]-manifolds to the category of vector bundles with the usual notion of vector bundle morphism, by sending a vector bundle $E \to M$ to the sheaf of sections of its dual E^* (i.e. the sheaf over M of linear functions on E), and by sending a [1]-manifold $\mathcal{A}$ to the dual of $E(\mathcal{A})$ constructed above. In the remainder of this section we will extend the less natural equivalence to an equivalence of metric double vector bundles with [2]-manifolds. In that case, the equivalence of categories will be more natural in this manner.

3.1.2. *Split N-manifolds.* Next we quickly discuss split N-manifolds and we recall how each N-manifold is noncanonically isomorphic to a split N-manifold of the same degree and of the same dimension.

Example 3.2. (1) Let E be a smooth vector bundle of rank r over a smooth manifold M of dimension p and assign the degree n to the fiber coordinates of E. This defines $E[-n]$, an $[n]$-manifold of dimension $(p; r_1 = 0, \ldots, r_{n-1} = 0, r_n = r)$ with $\mathcal{A}^n = \Gamma(E^*)$.

(2) Now let $E_{-1}, E_{-2}, \ldots, E_{-n}$ be smooth vector bundles of finite ranks $r_1, \ldots, r_n$ over M and assign the degree i to the fiber coordinates of E_{-i}, for each $i = 1, \ldots, n$. The direct sum $E = E_{-1} \oplus \ldots \oplus E_{-n}$ is a graded vector bundle with grading concentrated in degrees $-1, \ldots, -n$. The $[n]$-manifold $E_{-1}[-1] \oplus \ldots \oplus E_{-n}[-n]$ has local basis sections of ${E_{-i}}^*$ as local generators of degree i, for $i = 1, \ldots, n$, and so dimension $(p; r_1, \ldots, r_n)$. An $[n]$-manifold $\mathcal{M} = E_{-1}[-1] \oplus \ldots \oplus E_{-n}[n]$ defined in this manner by a graded vector bundle is called a **split** $[n]$**-manifold**.

In this paper, we are exclusively interested in the cases $n = 2$ and $n = 1$. Choose two vector bundles E_{-1} and E_{-2} of ranks r_1 and r_2 over a smooth manifold M. Set $E = E_{-1} \oplus E_{-2}$ and consider $\mathcal{M} = E_{-1}[-1] \oplus E_{-2}[-2]$. We find $\mathcal{A}^0 = C^\infty(M)$, $\mathcal{A}^1 = \Gamma({E_{-1}}^*)$ and $\mathcal{A}^2 = \Gamma({E_{-2}}^* \oplus \wedge^2 {E_{-1}}^*)$.

A morphism $\mu\colon E_{-1}[-1] \oplus E_{-2}[-2] \to F_{-1}[-1] \oplus F_{-2}[-2]$ of split [2]-manifolds over the bases M and N, respectively, consists of a smooth map $\mu_0\colon M \to N$, three vector bundle morphisms $\mu_1\colon E_{-1} \to F_{-1}$, $\mu_2\colon E_{-2} \to F_{-2}$ and $\mu_{12}\colon \wedge^2 E_{-1} \to F_{-2}$ over μ_0. The map $\mu^\star$ sends a degree 1 function $\xi \in \Gamma(F_{-1})$ to

$$\mu_1{}^\star \xi \in \Gamma(E_{-1}), \qquad (\mu_1{}^\star \xi)(m) = {\mu_1}^*_m(\xi(\mu_0(m))) \quad \text{for all } m \in M, \tag{18}$$

and a degree 2-function $\xi \in \Gamma(F^*_{-2})$ to

$$\mu_2{}^\star \xi + \mu_{12}^\star \xi \in \Gamma({E_{-2}}^* \oplus \wedge^2 {E_{-1}}^*). \tag{19}$$

Any N-manifold is non-canonically diffeomorphic to a split N-manifold. Further, the categories of split N-manifolds and of N-manifolds are equivalent. This is proved for instance in [3]. The super-version of this theorem is due to Batchelor [2] and is known as Batchelor's theorem.

Theorem 3.3. *Any $[n]$-manifold is non-canonically diffeomorphic to a split $[n]$-manifold.*

We give here the proof by [3] in the case $n = 2$. We are especially interested in the morphism of split [2]-manifolds induced by a change of splitting of a [2]-manifold and will emphasize this in the proof.

Sketch of Proof, [3]. Consider a [2]-manifold $\mathcal{M}$ over a smooth base manifold M. Since $\mathcal{A}^0 = C^\infty(M)$ and $\mathcal{A}^0\mathcal{A}^1 \subset \mathcal{A}^1$, the sheaf $\mathcal{A}^1$ is a locally free and finitely generated sheaf of $C^\infty(M)$-modules and there exists a vector bundle $E \to M$ such that $\mathcal{A}^1 \simeq \Gamma(E)$. Set $E^*_{-1} = E$. Now let $\mathcal{A}_1$ be the subalgebra of $\mathcal{A}$ generated by $\mathcal{A}^0 \oplus \mathcal{A}^1$. We find easily that $\mathcal{A}_1 \simeq \Gamma(\wedge^\bullet E^*_{-1})$ and $\mathcal{A}_1 \cap \mathcal{A}^2 = (\mathcal{A}^1)^2$ is a proper $\mathcal{A}^0$-submodule of $\mathcal{A}^2$. Since the quotient $\mathcal{A}^2/(\mathcal{A}^1)^2$ is a locally free and finitely generated sheaf of $C^\infty(M)$-modules, we have $\mathcal{A}^2/(\mathcal{A}^1)^2 \simeq \Gamma(E^*_{-2})$, where E_{-2} is a vector bundle over M. The short exact sequence

$$0 \to (\mathcal{A}^1)^2 \hookrightarrow \mathcal{A}^2 \to \Gamma(E^*_{-2}) \to 0$$

of $\mathcal{A}^0$-modules is non canonically split. Let us choose a splitting and identify $\Gamma(E^*_{-2})$ with a submodule of $\mathcal{A}^2$:

$$\mathcal{A}^2 \simeq (\mathcal{A}^1)^2 \oplus \Gamma(E^*_{-2}) = \Gamma(\wedge^2 E^*_{-1} \oplus E^*_{-2}) \ .$$

Hence, the considered [2]-manifold is diffeomorphic, modulo the chosen splitting, to the split [2]-manifold $E_{-1}[-1] \oplus E_{-2}[-2]$.

Note finally that a change of splitting is equivalent to a section ϕ of $\operatorname{Hom}(E^*_{-2}, \wedge^2 E^*_{-1})$ and induces an isomorphism of split [2]-manifolds over the identity on M: $\mu^\star(\xi) = \xi + \phi \circ \xi \in \Gamma(E^*_{-2} \oplus \wedge^2 E^*_{-1})$ for all $\xi \in \Gamma(E^*_{-2})$ and $\mu^\star(\xi) = \xi$ for all $\xi \in \Gamma(E^*_{-1})$. □

Note that [1]-manifolds are automatically split. As we have seen in §3.1.1, [1]-manifolds are just vector bundles with a degree shifting in the fibers, i.e. $\mathcal{M} = E[-1]$ for some vector bundle $E \to M$ and $C^\infty(\mathcal{M}) = \Gamma(\bigwedge^\bullet E^*)$, the exterior algebra of E. We finally give the definition of wide [1]-submanifolds of [2]-manifolds.

Definition 3.4. *Let $\mathcal{M}$ be a [2]-manifold of dimension $(p; r_1, r_2)$. A wide [1]-submanifold $\mathcal{N}$ of $\mathcal{M}$ is a [1]-manifold of dimension $(p; r)$, $r \leq r_1$, over the same smooth base M, together with a morphism of N-manifolds $\mu\colon \mathcal{N} \to \mathcal{M}$ over the identity on M and such that locally, $\mu^\star_U\colon C^\infty_U(\mathcal{M}) \to C^\infty_U(\mathcal{N})$, $\mu^\star_U(\xi^j_2) = 0$ for $j = 1, \ldots, r_2$, $\mu^\star_U(\xi^j_1) = 0$ for $j = r+1, \ldots, r_1$ and $\mu^\star_U(\xi^j_1) = \eta^j_1$ for $j = 1, \ldots, r$. Here, ξ^j_1, ξ^k_2 are local generators of $C^\infty(\mathcal{M})$ and η^j_1 are local generators for $C^\infty(\mathcal{N})$.*

More explicitly, if $\mathcal{M}$ splits as $Q[-1] \oplus B^*[-2]$, then $\mathcal{N}$ can be understood as $U[-1]$ for a subbundle $U \subseteq Q$. The map $\mu^\star$ is then locally described as follows. Take a trivialising open set $V \subseteq M$ for both B and Q. Choose local basis fields $u_1, \ldots, u_r$ for U over V, and complete this list to local basis fields $u_1, \ldots, u_r, q_{r+1}, \ldots, q_{r_1}$ for Q. The dual basis fields $\tau_1, \ldots, \tau_{r_1}$ for Q^* satisfy then $\tau_{r+1}, \ldots, \tau_{r_1} \in \Gamma(U^\circ)$. The morphism $\mu^\star_U$ sends τ_j to 0 for $j = r+1, \ldots, r_1$ and to $\bar{\tau}_j = \tau_j + U^\circ \in \Gamma(Q^*/U^\circ) \simeq \Gamma(U^*)$ for $j = 1, \ldots, r$.

Finally note that if an $[n]$-manifold $\mathcal{M}$ splits as $E_1[-1] \oplus E_2[-2] \oplus \ldots \oplus E_n[-n]$, then each section e of E_j defines a derivation $\hat{e}$ of degree $-j$ on $\mathcal{M}$: $\hat{e}(f) = 0$, $\hat{e}(\xi^i_j) = \langle e, \xi^j_j \rangle$, and $\hat{e}(\xi^i_k) = 0$ for $k \neq j$. We find $\widehat{e^i_j} = \partial_{\xi^i_j}$ if $\{e^1_j, \ldots, e^{r_j}_j\}$ is a local basis of E_j and $\{\xi^1_j, \ldots, \xi^{r_j}_j\}$ is the dual basis of E^*_j.

Further, a derivation ϕ of degree 0 on $\mathcal{M}$ can be written as a sum

$$X + D_1 + D_2 + \ldots + D_n,$$

with $X \in \mathfrak{X}(M)$ and each D_i a derivation of E_i^* with symbol $X \in \mathfrak{X}(M)$. The derivation $X + D_1 + \ldots + D_n$ can be written in coordinates as

$$\sum_{i=1}^{p} X(x_i)\partial_{x_i} + \sum_{i,j=1}^{r_1} \langle e_1^i, D_1(\xi_1^j)\rangle \xi_1^i \partial_{\xi_1^j} + \ldots + \sum_{i,j=1}^{r_1} \langle e_n^i, D_n(\xi_n^j)\rangle \xi_n^i \partial_{\xi_n^j}.$$

In particular, if for each j the map $D^j \colon \mathfrak{X}(M) \to \operatorname{Der}(E_j)$ is a morphism of $C^\infty(M)$-modules that sends a vector field X to a derivation $D^j(X)$ over X, then

$$\{X + D^1(X) + \ldots + D^n(X) \mid X \in \mathfrak{X}(M)\} \cup \{\hat{e} \mid e \in \Gamma(E_j) \text{ for some } j\} \tag{20}$$

span $\operatorname{Der}(C^\infty(\mathcal{M}))$ as a $C^\infty(\mathcal{M})$-module.

3.2. **Metric double vector bundles.** Next we introduce linear metrics on double vector bundles.

Definition 3.5. *A* ***metric double vector bundle*** *is a double vector bundle* $(\mathbb{E}, Q; B, M)$ *equipped with a* ***linear symmetric non-degenerate pairing on*** $\mathbb{E} \times_B \mathbb{E} \to \mathbb{R}$, *i.e. such that the map*

$$\begin{tikzcd}
\mathbb{E} \arrow[rr, "\pi_B"] \arrow[dd, "\pi_Q"'] & & B \arrow[dd, "q_B"] & & & \mathbb{E}\overset{*}{|}B \arrow[rr] \arrow[dd] & & B \arrow[dd] \\
& C \arrow[dr] & & {} \arrow[r, "\boldsymbol{\beta}"] & {} & & Q^* \arrow[dr] & \\
Q \arrow[rr, "q_Q"'] & & M & & & C^* \arrow[rr] & & M
\end{tikzcd} \tag{21}$$

defined by the pairing $\langle\cdot,\cdot\rangle$ *is an isomorphism of double vector bundles. In particular, the core* $C \to M$ *of* $\mathbb{E}$ *is canonically isomorphic to* $Q^* \to M$.

Note that equivalently, a linear symmetric non-degenerate pairing $\langle\cdot\,,\cdot\rangle$ on $\mathbb{E} \to B$ is linear if the core of $\mathbb{E}$ is isomorphic to Q^* and, via this isomorphism,

1. $\langle \tau_1^\dagger, \tau_2^\dagger\rangle = 0$ for $\tau_1, \tau_2 \in \Gamma(Q^*)$,
2. $\langle \chi, \tau^\dagger\rangle = q_B^*\langle q, \tau\rangle$ for $\chi \in \Gamma_B^l(\mathbb{E})$ linear over $q \in \Gamma(Q)$ and $\tau \in \Gamma(Q^*)$ and
3. $\langle \chi_1, \chi_2\rangle$ is a linear function on B for $\chi_1, \chi_2 \in \Gamma_B^l(\mathbb{E})$.

In the following, we will always identify with Q^* the core of a metric double vector bundle $(\mathbb{E}, Q; B, M)$.

Note also that the **opposite** $(\overline{\mathbb{E}}; Q; B, M)$ of a metric double vector bundle $(\overline{\mathbb{E}}; B; Q, M)$ is the metric double vector bundle with

$$\langle\cdot\,,\cdot\rangle_{\overline{\mathbb{E}}} = -\langle\cdot\,,\cdot\rangle_{\mathbb{E}}.$$

3.2.1. *Lagrangian decompositions of a metric double vector bundle.*

Definition 3.6. *Let* $(\mathbb{E}, B; Q, M)$ *be a metric double vector bundle. A linear splitting* $\Sigma \colon Q \times_M B \to \mathbb{E}$ *(or equivalently a decomposition of* $\mathbb{E}$*) is said to be* ***Lagrangian*** *if its image is maximal isotropic in* $\mathbb{E} \to B$. *The corresponding horizontal lift* $\sigma_Q \colon \Gamma(Q) \to \Gamma_B^l(\mathbb{E})$ *is then also said to be* ***Lagrangian***.

Note that by definition, a horizontal lift $\sigma_Q \colon \Gamma(Q) \to \Gamma_B^l(\mathbb{E})$ is Lagrangian if and only if $\langle \sigma_Q(q_1), \sigma_Q(q_2)\rangle = 0$ for all $q_1, q_2 \in \Gamma(Q)$.

Recall from Section 2.2.3 that given a linear splitting $\Sigma \colon Q \times_M B \to \mathbb{E}$, one can construct a linear splitting $\Sigma^\star \colon Q^{**} \times_M B \to \mathbb{E}\overset{*}{|}B$.

Lemma 3.7. *Let $(\mathbb{E}; Q, B; M)$ be a metric double vector bundle and choose a linear splitting Σ of $\mathbb{E}$. Then Σ is Lagrangian if and only if the linear map $\boldsymbol{\beta}\colon \mathbb{E} \to \mathbb{E} \ast B$ sends $\sigma_B(b)$ to $\sigma_B^\star(b)$ for all $b \in \Gamma(B)$.*

Proof. Recall that Lemma 2.3 states that given a horizontal lift $\sigma_B\colon \Gamma(B) \to \Gamma^l_Q(\mathbb{E})$, the dual horizontal lift $\sigma_B^\star\colon \Gamma(B) \to \Gamma^l_{Q^{**}}(\mathbb{E} \ast B)$ can be defined by

$$\langle \sigma_B^\star(b)(p_m), \sigma_B(b)(q_m)\rangle_B = 0, \qquad \langle \sigma_B^\star(b)(p_m), \tau^\dagger(b(m))\rangle_B = \langle p_m, \tau(m)\rangle$$

for all $b \in \Gamma(B)$, $\tau \in \Gamma(Q^*)$, $q_m \in Q$ and $p_m \in Q^{**} \simeq Q$.

On the other hand, if $\Sigma\colon B \times_M Q \to \mathbb{E}$ is a Lagrangian splitting, we have

$$\begin{aligned}\langle \boldsymbol{\beta}(\sigma_B(b)(p(m))), \sigma_B(b)(q(m))\rangle_B &= \langle \sigma_B(b)(p(m)), \sigma_B(b)(q(m))\rangle_{\mathbb{E}} \\ &= \langle \sigma_Q(p), \sigma_Q(q)\rangle_{\mathbb{E}}(b(m)) = 0\end{aligned}$$

for all $q, p \in \Gamma(Q)$ and $b \in \Gamma(B)$, and

$$\begin{aligned}\langle \boldsymbol{\beta}(\sigma_B(b)(p(m))), \tau^\dagger(b(m))\rangle_B &= \langle \sigma_B(b)(p(m)), \tau^\dagger(b(m))\rangle_{\mathbb{E}} \\ &= \langle \sigma_Q(p)(b(m)), \tau^\dagger(b(m))\rangle_{\mathbb{E}} = \langle p, \tau\rangle(m)\end{aligned}$$

for all $\tau \in \Gamma(Q^*)$. This proves that $\boldsymbol{\beta}$ sends the linear section $\sigma_B(b) \in \Gamma^l_Q(\mathbb{E})$ to $\sigma_B^\star(b) \in \Gamma^l_{Q^{**}}(\mathbb{E} \ast B)$. It is easy to see from the four equalities above that this condition is necessary for Σ to be Lagrangian. □

Let $\sigma_Q\colon \Gamma(Q) \to \Gamma^l_B(\mathbb{E})$ be an arbitrary horizontal lift. We have seen that by definition of a linear metric on $\mathbb{E} \to B$, the pairing of two linear sections is a linear function on B. This implies with

$\sigma_Q(fq) = q_B^* f \cdot \sigma_Q(q)$ and $\ell_{f\beta} = q_B^* f \cdot \ell_\beta$ for all $f \in C^\infty(M), q \in \Gamma(Q)$ and $\beta \in \Gamma(B^*)$

the existence of a symmetric tensor $\Lambda \in S^2(Q, B^*)$ such that

$$\langle \sigma_Q(q_1), \sigma_Q(q_2)\rangle_{\mathbb{E}} = \ell_{\Lambda(q_1,q_2)}. \tag{22}$$

In particular, $\Lambda(q, \cdot) : Q \to B^*$ is a morphism of vector bundles for each $q \in \Gamma(Q)$. Define a new horizontal lift $\sigma'_Q\colon \Gamma(Q) \to \Gamma^l_B(\mathbb{E})$ by $\sigma'_Q(q) = \sigma_Q(q) - \frac{1}{2}\widetilde{\Lambda(q, \cdot)}^*$ for all $q \in \Gamma(Q)$. Since for $\phi \in \Gamma(\mathrm{Hom}(B, Q^*))$, $\langle \widetilde{\phi}, \chi\rangle = \ell_{\phi^*(q)}$ if $\chi \in \Gamma^l_B(\mathbb{E})$ is linear over $q \in \Gamma(Q)$, we find then

$$\langle \sigma'_Q(q_1), \sigma'_Q(q_2)\rangle_{\mathbb{E}} = \langle \sigma_Q(q_1), \sigma_Q(q_2)\rangle_{\mathbb{E}} - \frac{1}{2}\ell_{\Lambda(q_1,q_2)} - \frac{1}{2}\ell_{\Lambda(q_2,q_1)} = 0$$

for all $q_1, q_2 \in \Gamma(Q)$. This proves the following result.

Theorem 3.8. *Let $(\mathbb{E}, B; Q, M)$ be a metric double vector bundle. Then there exists a Lagrangian splitting of $\mathbb{E}$.*

Next we show that a change of Lagrangian splitting corresponds to a skew-symmetric element of $\Gamma(Q^* \otimes B^* \otimes Q^*)$.

Proposition 3.9. *Let $(\mathbb{E}, B; Q, M)$ be a metric double vector bundle and choose a Lagrangian horizontal lift $\sigma^1_Q\colon \Gamma(Q) \to \Gamma^l_B(\mathbb{E})$. Then a second horizontal lift $\sigma^2_Q\colon \Gamma(Q) \to \Gamma^l_B(\mathbb{E})$ is Lagrangian if and only if the change of lift $\phi_{12} \in \Gamma(Q^* \otimes B^* \otimes Q^*)$ satisfies the following equality:*

$$\langle \phi_{12}(q), q'\rangle = -\langle \phi_{12}(q'), q\rangle \in \Gamma(B^*)$$

for all $q, q' \in \Gamma(Q)$, i.e. if and only if $\phi_{12} \in \Gamma(Q^* \wedge Q^* \otimes B^*)$

Proof. For $q \in \Gamma(Q)$ we have $\langle \widetilde{\phi_{12}(q)}, \chi\rangle = \ell_{\langle \phi_{12}(q), q'\rangle}$ for any linear section $\chi \in \Gamma^l_B(\mathbb{E})$ over $q' \in \Gamma(Q)$. Hence we find

$$\begin{aligned}&\langle \sigma^1_Q(q), \sigma^1_Q(q')\rangle_{\mathbb{E}} - \langle \sigma^2_Q(q), \sigma^2_Q(q')\rangle_{\mathbb{E}}\\ &= \langle \sigma^1_Q(q) - \sigma^2_Q(q), \sigma^1_Q(q')\rangle_{\mathbb{E}} + \langle \sigma^2_Q(q), \sigma^1_Q(q') - \sigma^2_Q(q')\rangle_{\mathbb{E}}\\ &= \ell_{\langle \phi_{12}(q), q'\rangle} + \ell_{\langle q, \phi_{12}(q')\rangle}\end{aligned}$$

and we can conclude. □

Remark 3.10. It is interesting to see that the last proposition implies that not any linear section of $\mathbb{E}$ over B can be obtained as the Lagrangian horizontal lift of a section of Q. This is easy to understand in the following example.

Example 3.11. Let $E \to M$ be a metric vector bundle, i.e. a vector bundle endowed with a symmetric non-degenerate pairing $\langle \cdot\,, \cdot\rangle \colon E \times_M E \to \mathbb{R}$. Then $E \simeq E^*$ and the tangent double is a metric double vector bundle $(TE, E; TM, M)$ with pairing $TE \times_{TM} TE \to \mathbb{R}$ the tangent of the pairing $E \times_M E \to \mathbb{R}$. In particular, we have

$$\langle Te_1, Te_2\rangle_{TE} = \ell_{\mathbf{d}\langle e_1, e_2\rangle}, \quad \langle Te_1, e_2^\dagger\rangle_{TE} = p_M^*\langle e_1, e_2\rangle \quad \text{ and } \langle e_1^\dagger, e_2^\dagger\rangle_{TE} = 0$$

for $e_1, e_2 \in \Gamma(E)$.

Recall from §2.2.2 that linear splittings of TE are equivalent to linear connections $\nabla \colon \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$. We have then for all $e_1, e_2 \in \Gamma(\mathsf{E})$:

$$\left\langle \sigma^\nabla_{\mathsf{E}}(e_1), e_2^\dagger \right\rangle = \left\langle Te_1 - \widetilde{\nabla_\cdot e_1}, e_2^\dagger\right\rangle = p_M^*\langle e_1, e_2\rangle$$

and

$$\left\langle \sigma^\nabla_{\mathsf{E}}(e_1), \sigma^\nabla_{\mathsf{E}}(e_2) \right\rangle = \left\langle Te_1 - \widetilde{\nabla_\cdot e_1}, Te_2 - \widetilde{\nabla_\cdot e_2}\right\rangle = \ell_{\mathbf{d}\langle e_1, e_2\rangle - \langle e_2, \nabla_\cdot e_1\rangle - \langle e_1, \nabla_\cdot e_2\rangle}.$$

The Lagrangian splittings of TE are hence exactly the linear splittings that correspond to **metric** connections, i.e. linear connections $\nabla \colon \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$ that preserve the metric:

$$\langle \nabla_\cdot e_1, e_2\rangle + \langle e_1, \nabla_\cdot e_2\rangle = \mathbf{d}\langle e_1, e_2\rangle \quad \text{ for } e_1, e_2 \in \Gamma(E).$$

Example 3.12. Let $q_E \colon E \to M$ be a vector bundle and consider the double vector bundle

$$\begin{tikzcd} TE \oplus T^*E \arrow[r, "\Phi_E := (q_{E*}, r_E)"] \arrow[d, "\pi_E"'] & TM \oplus E^* \arrow[d] \\ E \arrow[r, "q_E"'] & M \end{tikzcd}$$

with sides E and $TM \oplus E^* \to M$, and with core $E \oplus T^*M \to M$. The projection $r_E \colon T^*E \to E$ is defined by

$$r_E(\theta_{e_m}) \in E^*_m, \qquad \langle r_E(\theta_{e_m}), e'_m\rangle = \left\langle \theta_{e_m}, \left.\frac{d}{dt}\right|_{t=0} e_m + te'_m \right\rangle,$$

and is a fibration of vector bundles over the projection $q_E \colon E \to M$. The core elements are identified in the following manner with elements of $E \oplus T^*M \to M$. For $m \in M$ and $(e_m, \theta_m) \in E_m \times T^*_m M$, the pair

$$\left(\left.\frac{d}{dt}\right|_{t=0} te_m, (T_{0^E_m} q_E)^* \theta_m \right)$$

projects to $(0^{TM}_m, 0^{E^*}_m)$ under Φ_E and to 0^E_m under π_E. Conversely, any element of $TE \oplus T^*E$ in the double kernel can be written in this manner. Next recall that $TE \oplus_E T^*E \to E$ has a natural pairing given by

$$\langle (v^1_{e_m}, \theta^1_{e_m}), (v^2_{e_m}, \theta^2_{e_m}) \rangle = \theta^1_{e_m}(v^2_{e_m}) + \theta^2_{e_m}(v^1_{e_m}), \tag{23}$$

the natural pairing underlying the standard Courant algebroid structure on $TE \oplus_E T^*E \to E$.

We prove in [16] that linear splittings of $TE \oplus_E T^*E$ are in bijection with dull brackets on sections of $TM \oplus E^*$, and so also with Dorfman connections $\Delta\colon \Gamma(TM \oplus E^*) \times \Gamma(E \oplus T^*M) \to \Gamma(E \oplus T^*M)$. Choose such a Dorfman connection. For any pair $(X, \epsilon) \in \Gamma(TM \oplus E^*)$, the horizontal lift $\sigma := \sigma^{\Delta}_{TM\oplus E^*}\colon \Gamma(TM \oplus E^*) \to \Gamma_E(TE \oplus T^*E) = \mathfrak{X}(E) \times \Omega^1(E)$ is given by

$$\sigma(X,\epsilon)(e_m) = (T_m e X(m), \mathbf{d}\ell_\epsilon(e_m)) - \Delta_{(X,\epsilon)}(e,0)^\dagger(e_m)$$

for all $e_m \in E$.

The vector bundle $TM \oplus E^*$ is anchored by the morphism $\mathrm{pr}_{TM}\colon TM \oplus E^* \to TM$. As aconsequence, the TM-part of $[\![q_1, q_2]\!]_\Delta + [\![q_2, q_1]\!]_\Delta$ is trivial and this sum be seen as an element of $\Gamma(E^*)$. We proved the following result in [16].

Theorem 3.13. *Choose $q, q_1, q_2 \in \Gamma(TM \oplus E^*)$ and $\tau, \tau_1, \tau_2 \in \Gamma(E \oplus T^*M)$. The natural pairing on fibres of $TE \oplus T^*E \to E$ is given by*

1. $\langle \sigma(q_1), \sigma(q_2) \rangle = \ell_{[\![q_1,q_2]\!]_\Delta + [\![q_2,q_1]\!]_\Delta}$,
2. $\left\langle \sigma(q), \tau^\dagger \right\rangle = q_E^* \langle q, \tau \rangle$.

As a consequence, the natural pairing on fibres of $TE \oplus T^*E \to E$ is a linear metric on $(TE \oplus T^*E; TM \oplus E^*, E; M)$ and the Lagrangian splittings are equivalent to skew-symmetric dull brackets on sections of the anchored vector bundle $(TM \oplus E^*, \mathrm{pr}_{TM})$.

3.2.2. *The category of metric double vector bundles.* We define a morphism Ω of metric double vector bundles $(\mathbb{E}; Q, B; M)$ and $(\mathbb{F}; P, A; M)$ as the dual of a genuine morphism $\mathbb{E} \ast Q \to \mathbb{F} \ast P$ of double vector bundles.

Definition 3.14. *A morphism*

$$\Omega\colon \mathbb{F} \dashrightarrow \mathbb{E}$$

of metric double vector bundles is an isotropic relation $\Omega \subseteq \overline{\mathbb{F}} \times \mathbb{E}$ that is the dual of a double vector bundle morphism

$$\begin{tikzcd}
\omega\colon \mathbb{F} \ast P \arrow[r] \arrow[d] & P \arrow[d] \arrow[r, phantom, "\to"] & \mathbb{E} \ast Q \arrow[r] \arrow[d] & Q \arrow[d] \\
P^{**} \arrow[r] & N & Q^{**} \arrow[r] & M
\end{tikzcd}$$

over $\omega_0\colon N \to M$.

We write MDVB *for the obtained category of metric double vector bundles.*

Let $(\mathbb{E}, B; Q, M)$ be a metric double vector bundle. Choose a Lagrangian splitting $\Sigma\colon Q \times_M B \to \mathbb{E}$ and set

$$\mathcal{A}^2(\mathbb{E}) := \sigma_B(\Gamma(B)) + \{\tilde{\omega} \mid \omega \in \Gamma(Q^* \wedge Q^*)\}.$$

In other words, $\mathcal{A}^2(\mathbb{E})$ is the $C^\infty(M)$-module generated by all Lagrangian horizontal lifts of sections of B. Note that by Proposition 3.9, $\mathcal{A}^2(\mathbb{E})$ does not depend on the choice of Lagrangian splitting.

Theorem 3.15. *Let* $(\mathbb{E}; B, Q; M)$ *and* $\mathbb{F}; A, P; N)$ *be two metric double vector bundles. A morphism* $\Omega\colon \mathbb{F} \dashrightarrow \mathbb{E}$ *is equivalent to a triple of maps*

$$\omega^\star\colon \mathcal{A}^2(\mathbb{E}) \to \mathcal{A}^2(\mathbb{F}), \quad \omega_P^\star\colon \Gamma(Q^*) \to \Gamma(P^*), \qquad \textit{and} \qquad \omega_0\colon N \to M$$

such that

1. $\omega^\star\left(\widetilde{\tau_1 \wedge \tau_2}\right) = \widetilde{\omega_P^\star\tau_1 \wedge \omega_P^\star\tau_2}$,
2. $\omega^\star(q_Q^* f \cdot \chi) = q_P^*(\omega_0^* f) \cdot \omega^\star(\chi)$ *and*
3. $\omega_P^\star(f \cdot \tau) = \omega_0^* f \cdot \omega_P^\star \tau$

for all $\tau, \tau_1, \tau_2 \in \Gamma(Q^*)$, $f \in C^\infty(M)$ *and* $\chi \in \mathcal{A}^2(\mathbb{E})$.

Proof. By definition, a morphism $\Omega\colon \mathbb{F} \dashrightarrow \mathbb{E}$ is a morphism

$$\omega\colon \mathbb{F}\mathbin{\ast\!\!\!|}P \to \mathbb{E}\mathbin{\ast\!\!\!|}Q$$

of double vector bundles with some further properties. Let $\omega_P\colon P \to Q$ be the induced vector bundle map on the side P of $\mathbb{F}\mathbin{\ast\!\!\!|}P$, and let $\omega_0\colon N \to M$ be the induced smooth map on the double base. The morphism of double vector bundles induces a morphism $\omega^\star\colon \Gamma_Q(\mathbb{E}) \to \Gamma_P(\mathbb{F})$ of modules over $\omega_P^*\colon C^\infty(Q) \to C^\infty(P)$. Since ω_P is a vector bundle map, the pullback $\omega_P^*\colon C^\infty(Q) \to C^\infty(P)$ is completely determined by its value on linear functions and on pullbacks under q_Q of functions on M;

$$\omega_P^*(q_Q^* f) = q_P^*(\omega_0^* f)$$

for all $f \in C^\infty(M)$ and

$$\omega_P^*(\ell_\tau) = \ell_{\omega_P^\star(\tau)}$$

for all $\tau \in \Gamma(Q^*)$. The map $\omega^\star$ sends linear sections to linear sections and core sections to core sections, and it is completely determined by its images on these two sets of sections. We denote by $\omega^\star\colon \Gamma_Q^l(\mathbb{E}) \to \Gamma_P^l(\mathbb{F})$ the induced map. We need to check that the induced map on core sections is given by $\omega^\star(\tau^\dagger) = (\omega_P^\star(\tau))^\dagger$ for all $\tau \in \Gamma(Q^*)$ and that $\omega^\star$ restricts further to a map

$$\omega^\star\colon \mathcal{A}(\mathbb{E}) \to \mathcal{A}(F).$$

The morphism ω of double vector bundles induces a vector bundle map $\omega_{A^*}\colon A^* \to B^*$ on the cores, and so a map $\omega_{A^*}^\star\colon \Gamma(B) \to \Gamma(A)$ of modules over $\omega_0^*\colon C^\infty(M) \to C^\infty(N)$. If $\chi \in \Gamma_Q^l(\mathbb{E})$ is linear over $b \in \Gamma(B)$, then $\omega^\star(\chi) \in \Gamma_P^l(\mathbb{F})$ is linear over $\omega_{A^*}^\star(b) \in \Gamma(A)$. In particular, choose Lagrangian splittings $\Sigma^e\colon Q \times_M B \to \mathbb{E}$, $\Sigma^f\colon P \times_N A \to \mathbb{F}$. Then the image of $\sigma_B^e(b)$ is $\sigma_A^f(\omega_{A^*}^\star(b)) + \widetilde{\psi}$ for some $\psi \in \Gamma(\operatorname{Hom}(P, P^*))$. We have for all $p^1, p^2 \in \Gamma(P)$, $n \in N$:

$$\begin{aligned}
0 &= \langle \sigma_B^e(b)(\omega_P(p_n^1)), \sigma_B^e(b)(\omega_P(p_n^2))\rangle \\
&= \langle \sigma_A^f(\omega_{A^*}^\star(b))(p_n^1) + \widetilde{\psi}(p_n^1), \sigma_A^f(\omega_{A^*}^\star(b))(p_n^2) + \widetilde{\psi}(p_n^2)\rangle \\
&= \langle \sigma_P^f(p^1)(\omega_{A^*}^\star(b)(n)) + (\psi(p^1))^\dagger(\omega_{A^*}^\star(b)(n)), \sigma_P^f(p^2)(\omega_{A^*}^\star(b)(n)) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + (\psi(p^2))^\dagger(\omega_{A^*}^\star(b)(n))\rangle \\
&= 0 + \langle \psi(p^1(n)), p^2(n)\rangle + \langle \psi(p^2(n)), p^1(n)\rangle + 0.
\end{aligned}$$

This shows that $\psi \in \Gamma(P^* \wedge P^*)$, and so that $\omega^\star(\sigma_B^e(b)) \in \mathcal{A}^2(\mathbb{F})$.

Finally note that for $\nu \in \Gamma(P^*)$ and $p_1, p_2 \in \Gamma(P)$, we have

$$\nu^\dagger(p_1(n)) = \nu^\dagger(0^A(n)) +_A \sigma_P(p_1)(0^A(n))$$

and so

$$\begin{aligned}\langle \nu^\dagger(p_1(n)), \sigma_A(0^A)(p_2(n))\rangle &= \langle \nu^\dagger(0^A(n)) +_A \sigma_P(p_1)(0^A(n)), \sigma_P(p_2)(0^A(n))\rangle \\ &= \langle \nu, p_2\rangle(n).\end{aligned}$$

Hence, since for $\tau \in \Gamma(Q^*)$, $\omega^\star(\tau^\dagger)$ is a core section of $\mathbb{F}$ over P and

$$\begin{aligned}\langle \omega^\star(\tau^\dagger)(p^1(n)), \omega^\star(\sigma_B^e(0^B))(p^2(n))\rangle &= \langle \tau^\dagger(\omega_P(p^1(n))), \sigma_B^e(0^B)(\omega_P(p^2(n)))\rangle \\ &= \langle \tau(\omega_0(n)), \omega_P(p^2(n))\rangle \\ &= \langle \omega_P^* \tau(\omega_0(n)), p^2(n)\rangle = \langle (\omega_P^\star \tau)(n), p^2(n)\rangle,\end{aligned}$$

we find that $\omega^\star(\tau^\dagger) = (\omega_P^\star \tau)^\dagger$.

Now we check the 3 conditions. (2) and (3) follow immediately from the definition of $\omega_P^\star$ and $\omega^\star$. To see (1), write $\widetilde{\tau_1 \wedge \tau_2}$ as $\ell_{\tau_1}\tau_2^\dagger - \ell_{\tau_2}\tau_1^\dagger$. Then

$$\omega^\star\left(\widetilde{\tau_1 \wedge \tau_2}\right) = \ell_{\omega_P^\star \tau_1}(\omega_P^\star \tau_2)^\dagger - \ell_{\omega_P^\star \tau_2}(\omega_P^\star \tau_1)^\dagger = \omega_P^\star \widetilde{\tau_1 \wedge \omega_P^\star} \tau_2.$$

□

Remark 3.16. (1) The morphism

$$\omega_{A^*}^\star \colon \Gamma(B) \to \Gamma(A)$$

of modules over $\omega_0^* \colon C^\infty(M) \to C^\infty(N)$, i.e. the vector bundle morphism $\omega_{A^*} \colon A^* \to B^*$ is induced as follows by the three maps in the theorem. If $\chi \in \Gamma_Q^l(\mathbb{E})$ is linear over $b \in \Gamma(B)$, then $\omega^\star(\chi)$ is linear over $\omega_{A^*}^\star(b)$. To see that $\omega_{A^*}^\star$ is well-defined (i.e. $\omega_{A^*}^\star(b)$ does not depend on the choice of χ over b), use (1) in the theorem. To see that $\omega_{A^*}^\star(f \cdot b) = \omega_0^* f \cdot \omega_{A^*}^\star(b)$ for $f \in C^\infty(M)$, use that $q_Q^* f \cdot \chi$ is linear over $f \cdot b$ and (2) in the theorem.

(2) A morphism $\Omega \colon B_2 \times_{M_2} Q_2 \times_{M_2} Q_2^* \dashrightarrow B_1 \times_{M_1} Q_1 \times_{M_1} Q_1^*$ of decomposed metric double vector bundles is consequently described by $\omega_Q \colon Q_1 \to Q_2$, $\omega_B \colon B_1^* \to B_2^*$ and $\omega_{12} \colon Q_1 \wedge Q_1 \to B_2^*$, all morphisms of vector bundles over a smooth map $\omega_0 \colon M_1 \to M_2$.

For $b \in \Gamma(B_2)$ the section $b^l \in \Gamma_{Q_2}^l(B_2 \times_{M_2} Q_2 \times_{M_2} Q_2^*)$, $b^l(q_m) = (b(m), q_m, 0_m^{Q^*})$, is sent by $\omega^\star$ to $(\omega_B^\star(b))^l + \widetilde{\omega_{12}^\star(b)} \in \Gamma_{Q_1}^l(B_1 \times_{M_1} Q_1 \times_{M_1} Q_1^*)$, where for $\phi \in \Gamma(\operatorname{Hom}(Q_1, Q_1^*))$, $\widetilde{\phi} \in \Gamma_{Q_1}^l(B_1 \times_{M_1} Q_1 \times_{M_1} Q_1^*)$ is defined by $\widetilde{\phi}(q_m) = (0_m^B, q_m, \phi(q_m))$. For $\tau \in \Gamma_{M_2}(Q_2^*)$, the core section $\tau^\dagger \in \Gamma_{Q_2}^c(B_2 \times_{M_2} Q_2 \times_{M_2} Q_2^*)$, $\tau^\dagger(q_m) = (0_m^B, q_m, \tau(m))$, is sent to $(\omega_Q^\star \tau)^\dagger \in \Gamma_{Q_1}^c(B_1 \times_{M_1} Q_1 \times_{M_1} Q_1^*)$.

3.3. **Equivalence of [2]-manifolds and metric double vector bundles.**

In this section our goal is to prove the following theorem.

Theorem 3.17. *The category of metric double vector bundles and the category of* [2]*-manifolds are equivalent.*

3.3.1. *The functor* $\mathcal{M}(\cdot) \colon \mathrm{MDVB} \to [2]\text{-Man}$. Let $(\mathbb{E}, B; Q, M)$ be a metric double vector bundle. We define a [2]-manifold $\mathcal{M}(\mathbb{E})$ as follows. The sheaf $\mathcal{A}(\mathbb{E}) = C^\infty(\mathcal{M}(\mathbb{E}))$ of $\mathbb{N}$-graded commutative associative unital $\mathbb{R}$-algebras is generated by $\mathcal{A}^0(\mathbb{E}) = C^\infty(M)$, $\mathcal{A}^1(\mathbb{E}) = \Gamma(Q^*)$ and the sheaf of degree 2-functions is $\mathcal{A}^2(\mathbb{E})$. The product of τ_1 with $\tau_2 \in \Gamma(Q^*)$ is $\widetilde{\tau_1 \wedge \tau_2} \in \mathcal{A}^2(\mathbb{E})$. The product of $f \in C^\infty(M)$ with $\xi \in \mathcal{A}^2(\mathbb{E})$ is $q_Q^* f \cdot \xi \in \mathcal{A}^2(\mathbb{E})$ and the product of elements of $\mathcal{A}^0$ with elements of $\mathcal{A}^1$ is obvious since $\Gamma(Q^*)$ is a sheaf of $C^\infty(M)$-modules. This proves that $\mathcal{A}(\mathbb{E})$ is well-defined.

Next we check that $\mathcal{A}(\mathbb{E})$ is locally free and finitely generated over $C^\infty(M)$. Choose $m \in M$ and a coordinate neighborhood $U \ni m$ such that Q and B are trivialized on U by the basis frames $(q_1, \ldots, q_k)$ and $(b_1, \ldots, b_l)$. Let $(\tau_1, \ldots, \tau_k)$ be the dual frame to $(q_1, \ldots, q_k)$, i.e. a trivialization for Q^*. Recall also that after the choice of a Lagrangian splitting $\Sigma\colon Q \times_M B \to \mathbb{E}$, each element $\xi \in \mathcal{A}^2(\mathbb{E})$ over $b \in \Gamma(B)$ can be written $\sigma_B(b) + \widetilde{\phi}$ with $\phi \in \Gamma(Q^* \wedge Q^*)$. Since $\sigma_B\left(\sum_{i=1}^l f_i b_i\right) = \sum_{i=1}^l q_Q^* f_i \cdot \sigma_B(b_i)$ for $f_1, \ldots, f_l \in C^\infty(M)$, we conclude that $\mathcal{A}(\mathbb{E})$ is generated on U by $\{\tau_1, \ldots, \tau_k, \sigma_B(b_1), \ldots, \sigma_B(b_l)\}$ over $C^\infty(M)$.

We have constructed a map $\mathcal{M}(\cdot)$ sending metric double vector bundles to [2]-manifolds. By Theorem 3.15 a morphism $\Omega\colon \mathbb{F} \dashrightarrow \mathbb{E}$ of metric double vector bundles is the same as a triple of maps

$$\omega_0\colon N \to M \quad \Leftrightarrow \quad \omega_0^*\colon C^\infty(M) \to C^\infty(N),$$
$$\omega^\star\colon \mathcal{A}^2(\mathbb{E}) \to \mathcal{A}^2(\mathbb{F}) \quad \text{and} \quad \omega_P^\star\colon \Gamma(Q^*) \to \Gamma(P^*)$$

with

$$\omega^\star\left(\widetilde{\tau_1 \wedge \tau_2}\right) = \widetilde{\omega_P^\star \tau_1 \wedge \omega_P^\star \tau_2}, \qquad q_P^* \omega_0^* f \cdot \omega^\star(\chi) = \omega^\star(q_Q^* f \cdot \chi)$$

and

$$\omega_0^* f \cdot \omega_P^\star(\tau) = \omega^\star(f \cdot \tau)$$

for $f \in C^\infty(M) = \mathcal{A}^0(\mathbb{E})$, $\tau \in \Gamma(Q^*) = \mathcal{A}^1(\mathbb{E})$ and $\chi \in \mathcal{A}^2(\mathbb{E})$. Hence we find that the triple $(\omega^\star, \omega_P^\star, \omega_0^*)$ defines a morphism $\mathcal{M}(\Omega)\colon \mathcal{M}(\mathbb{F}) \to \mathcal{M}(\mathbb{E})$ of [2]-manifolds.

We have so defined a functor $\mathcal{M}(\cdot)\colon$ MDVB $\to$ [2]-Man from the category of metric double vector bundles to the category of [2]-manifolds.

3.3.2. *The functor* $\mathcal{G}\colon$ [2]-Man $\to$ MDVB. Conversely, we construct explicitly a metric double vector bundle associated to a given [2]-manifold $\mathcal{M}$. The idea is to adapt the proof of the equivalence between locally free and finitely generated sheaves of $C^\infty(M)$-modules with vector bundles over M (see §3.1.1).

First we give Pradines' original definition of a double vector bundle [33] (in the smooth and finite-dimensional case).

Definition 3.18. [33, C. §1] *Let M be a smooth manifold and D a set with a map $\Pi\colon D \to M$. A **double vector bundle chart** is a quintuple $c = (U, \Theta, V_1, V_2, V_0)$, where U is an open set in M, V_1, V_2, V_3 are three vector spaces and $\Theta\colon \Pi^{-1}(U) \to U \times V_1 \times V_2 \times V_0$ is a bijection such that $\Pi = \mathrm{pr}_1 \circ \Theta$.*

*Two double vector bundle charts c and c' are **compatible** if the "change of chart" $\Theta' \circ \Theta^{-1}$ over $U \cap U'$ has the following form:*

$$(x, v_1, v_2, v_0) \mapsto (x, A_1(x)v_1, A_2(x)v_2, A_0(x)v_0 + \omega(x)(v_1, v_2))$$

with $x \in U \cap U'$, $v_i \in V_i$, $A_i \in C^\infty(M, \mathrm{Gl}(V_i))$ for $i = 0, 1, 2$ and $\omega \in C^\infty(M, \mathrm{Hom}(V_1 \otimes V_2, V_0))$.

*A **double vector bundle atlas** $\mathfrak{A}$ on D is a set of double vector bundle charts of D that are pairwise compatible and such that the set of underlying open sets in M is a covering of M. As usual, two double vector bundle atlases $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are **equivalent** if their union is an atlas. A double vector bundle structure on D is an equivalence class of double vector bundle atlases on D.*

Given a [2]-manifold $\mathcal{M}$, we interpret its functions as the components of a double vector bundle atlas, and show that the obtained double vector bundle has a natural metric structure.

Let M be the smooth manifold underlying $\mathcal{M}$ and assume that $\mathcal{M}$ has dimension $(l; m, n)$. Choose a maximal open covering $\{U_\alpha\}$ of M such that $\mathcal{A}(U_\alpha)$ is freely generated by $\xi_1^\alpha, \ldots, \xi_m^\alpha$ (in degree 1) and $\eta_1^\alpha, \ldots, \eta_n^\alpha$ (degree 2 generators). Choose now α, β such that $U_\alpha \cap U_\beta \neq \emptyset$. Then each generator ξ_i^β can be written in a unique manner as $\sum_{j=1}^m \omega_{\alpha\beta}^{ji} \xi_j^\alpha$ with $\omega^{ji} \in C^\infty(U_\alpha \cap U_\beta)$. Each generator η_i^β can be written

$$\eta_i^\beta = \sum_{j=1}^n \psi_{\alpha\beta}^{ji} \cdot \left(\eta_j^\alpha + \sum_{1 \leq k < l \leq m} \rho_{\alpha\beta}^{jkl} \cdot \xi_k^\alpha \wedge \xi_l^\alpha \right)$$

with $\psi_{\alpha\beta}^{ij}, \rho_{\alpha\beta}^{ikl} \in C^\infty(U_\alpha \cap U_\beta)$. Set $A_1^{\alpha\beta} = (\omega_{\alpha\beta}^{ij})_{i,j} \in C^\infty(M, \mathrm{Gl}(\mathbb{R}^{m*}))$, $A_2^{\alpha\beta} = (\psi_{\alpha\beta}^{ij})_{i,j} \in C^\infty(M, \mathrm{Gl}(\mathbb{R}^n))$. Define $\nu^{\alpha\beta} \in C^\infty(M, \mathrm{Hom}(\mathbb{R}^m \otimes \mathbb{R}^n, \mathbb{R}^{m*}))$ by $\nu^{\alpha\beta}(e_i, e_j)(e_l) = \rho^{jil}$ for $1 \leq i < l \leq m$ and $j = 1, \ldots, n$. Then by construction

$$A_1^{\gamma\alpha} \cdot A_1^{\alpha\beta} = A_1^{\gamma\beta}, \qquad A_2^{\gamma\alpha} \cdot A_2^{\alpha\beta} = A_2^{\gamma\beta} \quad \text{and}$$

$$\begin{aligned} \nu^{\gamma\beta}({A_1^{\beta\gamma}}^*(e_i), A_2^{\gamma\beta}(e_j))(e_l) = \nu^{\gamma\alpha}&({A_1^{\beta\gamma}}^*(e_i), A_2^{\gamma\beta}(e_j), e_l) \\ &+ \nu^{\alpha\beta}({A_1^{\beta\alpha}}^*(e_i), A_2^{\alpha\beta}(e_j))(A_1^{\gamma\alpha *} e_l). \end{aligned} \tag{24}$$

Set $\tilde{\mathbb{E}} = \bigsqcup_\alpha U_\alpha \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^{m*}$ (the disjoint union) and identify

$$(x, v_1, v_2, l_0) \in U_\beta \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^{m*}$$

with

$$\Big(x, (A_1^{\beta\alpha}(x))^*(v_1), A_2^{\alpha\beta}(x)(v_2), A_1^{\alpha\beta}(x)(l_0) + \nu^{\alpha\beta}(x)((A_1^{\beta\alpha}(x))^*(v_1), A_2^{\alpha\beta}(x)(v_2))\Big)$$

in $U_\alpha \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^{m*}$ for $x \in U_\alpha \cap U_\beta$. The cocycle equations (24) imply that this defines an equivalence relation on $\tilde{\mathbb{E}}$. The quotient space is $\mathbb{E}$, a double vector bundle that does not depend anymore on the choice of charts covering M. The map $\Pi \colon \mathbb{E} \to M$, $(x, v_1, v_2, l_0) \mapsto x$ is well-defined and, by construction, the charts $c = (U_\alpha, \Theta_\alpha = \mathrm{Id}, \mathbb{R}^m, \mathbb{R}^n, \mathbb{R}^{m*})$ define a double vector bundle atlas on $\mathbb{E}$. Since the covering was chosen to be maximal, the obtained double atlas of $\mathbb{E}$ does not depend on any choices.

Recall from the proof of Theorem 3.3 that there are two vector bundles E_{-1} and E_{-2} associated canonically to a [2]-manifold $\mathcal{M}$ (only the inclusion of $\Gamma(E_{-2}^*)$ in $\mathcal{A}^2$ is non-canonical). E_{-1} and E_{-2}^* are the sides of $\mathbb{E}$ and E_{-1}^* is the core of $\mathbb{E}$. The vector bundle E_{-1}^* can be defined by the transition functions $A_1^{\alpha\beta}$ and $\tilde{E}_{-1} = \bigsqcup_\alpha U_\alpha \times \mathbb{R}^n$, $(x, v) \sim (x, A_1^{\alpha\beta}(x)(v))$ for $x \in U_\alpha \cap U_\beta$. The vector bundle E_{-2} can be defined in the same manner using $A_2^{\alpha\beta}$ and the model space $\mathbb{R}^m$.

We use this to define a linear metric on $\mathbb{E}$. Over a chart domain U_α we set

$$\langle (x, v_1, v_2, l_0), (x, v_1', v_2, l_0') \rangle = l_0(v_1') + l_0'(v_1).$$

By construction, this does not depend on the choice of α with $x \in U_\alpha$.

Again by definition of the morphisms in the category of [2]-manifolds and in the category of metric double vector bundles, this defines a functor $\mathcal{G} \colon$ [2]-Man $\to$ MDVB between the two categories.

3.3.3. *Equivalence of categories.* Finally we need to prove that the two obtained functors define an equivalence of categories. The functor $\mathcal{G}\circ\mathcal{M}(\cdot)$ is the functor that sends a metric double vector bundle to its maximal double vector bundle atlas, hence it is naturally isomorphic to the identity functor.

The functor $\mathcal{M}(\cdot)\circ\mathcal{G}\colon$ [2]-Man $\to$ [2]-Man sends a [2]-manifold $\mathcal{M}$ over M with degree 1 local generators ξ^i_α and cocycles $A^1_{\alpha\beta}$ and degree 2 generators η^i_α and cocycles $A^2_{\alpha\beta}$ and $\nu_{\alpha\beta}$ to the sheaf of core and Lagrangian linear sections of $\mathcal{G}(\mathcal{M})$. There is an obvious natural isomorphism between this functor and the identity functor [2]-Man $\to$ [2]-Man.

3.3.4. *Correspondence of splittings.* Decomposed metric double vector bundles $B\times_M Q\times_M Q^*$ are equivalent to split [2]-manifolds.

Choose a metric double vector bundle $(\mathbb{E};Q,B;M)$ and the corresponding [2]-manifold $\mathcal{M}$. Each choice of a Lagrangian decomposition $\mathbb{I}$ of $\mathbb{E}$ is equivalent to a choice of splitting $\mathcal{S}$ of the corresponding [2]-manifold, such that the following diagram commutes

$$\begin{tikzcd}
\mathbb{E} \arrow[r, "\mathbb{I}"] \arrow[d, "\mathcal{M}(\cdot)"'] & Q\times_M B\times_M Q^* \arrow[d, "\mathcal{M}(\cdot)"] \\
\mathcal{M}(\mathbb{E}) \arrow[r, "\mathcal{S}"'] & Q[-1]\oplus B^*[-2].
\end{tikzcd}$$

Note also that the category of split [2]-manifolds is equivalent to the category of [2]-manifolds, and the category of decomposed metric double vector bundles is equivalent to the category of metric double vector bundles. We will often use this in the following sections.

4. Poisson [2]-manifolds and metric VB-algebroids.

In this section we study [2]-manifolds endowed with a Poisson structure of degree -2. We show how split Poisson [2]-manifolds are equivalent to a special family of 2-representations. Then we prove that Poisson [2]-manifolds are equivalent to metric double vector bundles endowed with a linear Lie algebroid structure that is compatible with the metric.

Definition 4.1. *A Poisson* [2]*-manifold is a* [2]*-manifold endowed with a Poisson structure of degree* -2*. A morphism of Poisson* [2]*-manifolds is a morphism of* [2]*-manifolds that preserves the Poisson structure.*

Note that a Poisson bracket of degree -2 on a [2]-manifold $\mathcal{M}$ is an $\mathbb{R}$-bilinear map $\{\cdot\,,\cdot\}\colon C^\infty(\mathcal{M})\times C^\infty(\mathcal{M})\to C^\infty(\mathcal{M})$ of the graded sheaves of functions, such that[8] $\deg\{\xi,\eta\}=\deg\xi+\deg\eta-2$ for homogeneous elements $\xi,\eta\in C^\infty_{\mathcal{M}}(U)$. The bracket is graded skew-symmetric; $\{\xi,\eta\}=-(-1)^{|\xi|\,|\eta|}\{\eta,\xi\}$ and satisfies the graded Leibniz and Jacobi identities

$$\{\xi_1,\xi_2\cdot\xi_3\}=\{\xi_1,\xi_2\}\cdot\xi_3+(-1)^{|\xi_1|\,|\xi_2|}\xi_2\cdot\{\xi_1,\xi_3\} \tag{25}$$

and

$$\{\xi_1,\{\xi_2,\xi_3\}\}=\{\{\xi_1,\xi_2\},\xi_3\}+(-1)^{|\xi_1|\,|\xi_2|}\{\xi_2,\{\xi_1,\xi_3\}\} \tag{26}$$

for homogeneous $\xi_1,\xi_2,\xi_3\in C^\infty_{\mathcal{M}}(U)$.

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[8] We will also write $|\xi|$ for the degree of a homogeneous element $\xi\in C^\infty_{\mathcal{M}}(U)$.

A morphism $\mu\colon \mathcal{M}_1 \to \mathcal{M}_2$ of Poisson [2]-manifolds satisfies

$$\mu^\star\{\xi_1, \xi_2\} = \{\mu^\star\xi_1, \mu^\star\xi_2\}$$

for all $\xi_1, \xi_2 \in C^\infty_{\mathcal{M}_2}(U)$, U open in M_2.

4.1. **Split Poisson [2]-manifolds and self-dual 2-representations.** We begin by defining self-dual 2-representations.

Definition 4.2. *Let $(A, \rho, [\cdot\,,\cdot])$ be a Lie algebroid. A 2-representation $(\nabla^Q, \nabla^{Q^*}, R)$ of A on a complex $\partial_Q\colon Q^* \to Q$ is said to be **self-dual** if it equals its dual, i.e.*

1. $\partial_Q = \partial_Q^*$,
2. ∇^Q *and* ∇^{Q^*} *are dual to each other,*
3. *and* $R^* = -R \in \Omega^2(A, \operatorname{Hom}(Q, Q^*))$.

Then we prove the following result.

Theorem 4.3. *There is a bijection between split Poisson 2-manifolds and self-dual 2-representations.*

Proof. First let us consider a split 2-manifold $\mathcal{M} = Q[-1] \oplus B^*[-2]$. That is, Q and B are vector bundles over M and the functions of degree 0 on $\mathcal{M}$ are the elements of $C^\infty(M)$, the functions of degree 1 are sections of Q^* and the functions of degree 2 are sections of $B \oplus Q^* \wedge Q^*$. Let us now take any Poisson bracket $\{\cdot\,,\cdot\}$ of degree -2 on $C^\infty(\mathcal{M})$. In the following, we consider arbitrary $f, f_1, f_2 \in C^\infty(M)$, $\tau, \tau_1, \tau_2 \in \Gamma(Q^*)$, and $b, b_1, b_2 \in \Gamma(B)$.

The brackets $\{f_1, f_2\}$, $\{f, \tau\}$ have degree -2 and -1, respectively, and must hence vanish. The bracket $\{\tau_1, \tau_2\}$ is a function on M because it has degree 0. Since $\{f, \tau\} = 0$ for all $f \in C^\infty(M)$ and $\tau \in \Gamma(Q^*)$, this defines a vector bundle morphism $\partial_Q\colon Q^* \to Q$ by (25): $\langle \tau_2, \partial_Q(\tau_1)\rangle = \{\tau_1, \tau_2\}$. Since $\{\tau_1, \tau_2\} = -(-1)^{|\tau_2|}\{\tau_2, \tau_1\} = \{\tau_2, \tau_1\}$, we find that $\partial_Q^* = \partial_Q$. The Poisson bracket $\{b, f\}$ has degree 0 and is hence an element of $C^\infty(M)$. Again by (25), this defines a derivation $\{b, \cdot\}|_{C^\infty(M)}$ of $C^\infty(M)$, hence a vector field $\rho_B(b) \in \mathfrak{X}(M)$; $\{b, f\} = \rho_B(b)(f)$. By the Leibniz identity (25) for the Poisson bracket and the equality $\{f_1, f_2\} = 0$ for all $f_1, f_2 \in C^\infty(M)$, we get in this manner a vector bundle morphism (an anchor) $\rho_B\colon B \to TM$. The bracket $\{b, \tau\}$ has degree 1 and is hence a section of Q^*. Since $\{b, f\tau\} = f\{b, \tau\} + \{b, f\}\tau = f\{b, \tau\} + \rho_B(b)(f)\tau$ and $\{fb, \tau\} = f\{b, \tau\} + \{f, \tau\}b = f\{b, \tau\}$, we find a linear B-connection ∇ on Q^* by setting $\nabla_b\tau = \{b, \tau\}$. Let us finally look at the bracket $\{b_1, b_2\}$. This function has degree 2 and is hence the sum of a section of B and a section of $Q^* \wedge Q^*$. We write $\{b_1, b_2\} = [b_1, b_2] - R(b_1, b_2)$ with $[b_1, b_2] \in \Gamma(B)$ and $R(b_1, b_2) \in \Gamma(Q^* \wedge Q^*)$. By a similar reasoning as before, we find that this defines a skew-symmetric bracket $[\cdot\,,\cdot]$ on $\Gamma(B)$ that satisfies a Leibniz equality with respect to ρ_B, and an element $R \in \Omega^2(B, \operatorname{Hom}(Q, Q^*))$ such that $R^* = -R$. Note also here that the bracket $\{b, \phi\}$ for $\phi \in \Gamma(Q^* \wedge Q^*) \subseteq \Gamma(\operatorname{Hom}(Q, Q^*))$ is just $\nabla_b^{\operatorname{Hom}}\phi$, where $\nabla^{\operatorname{Hom}}$ is the B-connection induced on $\operatorname{Hom}(Q, Q^*)$ by ∇ and ∇^*.

Now we will show that the dull algebroid structure on B is in reality a Lie algebroid structure, and that (∇, ∇^*, R) is a self-dual 2-representation of B on $\partial_Q\colon Q \to Q^*$. In order to do this, we only need to recall that the Poisson structure $\{\cdot\,,\cdot\}$ satisfies the Jacobi identity. The Jacobi identity for the three functions b_1, b_2, f yields the compatibility of the anchor on B with the bracket on $\Gamma(B)$. The Jacobi identity for b, τ_1, τ_2 yields $\partial_Q \circ \nabla = \nabla^* \circ \partial_Q$, and the Jacobi identity for

b_1, b_2, τ yields $R_\nabla = R \circ \partial_Q$. The equality $R_{\nabla^*} = \partial_Q \circ R$ follows using $\partial_Q = \partial_Q^*$, $R^* = -R$ and $R_\nabla^* = -R_{\nabla^*}$. The Jacobi identity for $b_1, b_2, b_3 \in \Gamma(B)$ yields in a straightforward manner the Jacobi identity for $[\cdot\,,\cdot]$ on sections of $\Gamma(B)$ and the equation $\mathbf{d}_{\nabla^{\mathrm{Hom}}} R = 0$.

Take conversely a self dual 2-representation of a Lie algebroid B on a 2-term complex $\partial_Q\colon Q^* \to Q$ and consider the [2]-manifold $\mathcal{M} = Q[-1] \oplus B^*[-2]$. Then the self-dual 2-representation defines as described above a Poisson bracket of degree -2 on $C^\infty(\mathcal{M})$. □

Next take two split [2]-manifolds $Q_1 \oplus B_1^*$ and $Q_2 \oplus B_2^*$ over M_1 and M_2, respectively, endowed with two Poisson structures of degree -2. By the preceding theorem, we have hence two self-dual 2-representations: B_1 acts on $\partial_{Q_1}\colon Q_1^* \to Q_1$ via $\nabla^1\colon \Gamma(B_1) \times \Gamma(Q_1) \to \Gamma(Q_1)$ and $R_1 \in \Omega^2(B_1, Q_1^* \wedge Q_1^*)$, and B_2 acts on $\partial_{Q_2}\colon Q_2^* \to Q_2$ via $\nabla^2\colon \Gamma(B_2) \times \Gamma(Q_2) \to \Gamma(Q_2)$ and $R_2 \in \Omega^2(B_2, Q_2^* \wedge Q_2^*)$.

A computation shows that a morphism $\mu\colon Q_1 \oplus B_1^* \to Q_2 \oplus B_2^*$ preserves the Poisson structures if and only if its decomposition $\mu_0\colon M_1 \to M_2$, $\mu_Q\colon Q_1 \to Q_2$, $\mu_B\colon B_1^* \to B_2^*$ and $\mu_{QB} \in \Omega^2(Q_1, \omega_0^* B_2)$ define a *morphism of self-dual 2-representations*. That is:

1. $[\mu_B^\star(b_1), \mu_B^\star(b_2)]_1 = \mu_B^\star[b_1, b_2]_2$ for all $b_1, b_2 \in \Gamma(B_2)$,
2. $\rho_{B_1}(\mu_B^\star(b)) \sim_{\mu_0} \rho_{B_2}(b)$ for all $b \in \Gamma(B)$,
3. $\langle \mu_Q^\star(\tau_1), \partial_{Q_1}\mu_Q^\star(\tau_2)\rangle = \langle \tau_1, \partial_{Q_2}\tau_2\rangle$ for all $\tau_1, \tau_2 \in \Gamma(Q_2^*)$,
4. $\mu_Q^\star(\nabla_b^2 \tau) = \nabla^1_{\mu_B^\star(b)}\mu_Q^\star(\tau) - \mu_{QB}^\star(b) \circ \partial_{Q_1}\mu_Q^\star(\tau)$ for all $b \in \Gamma(B_2)$ and $\tau \in \Gamma(Q_2^*)$, and
5. $R_1(\mu_B^\star(b_1), \mu_B^\star(b_2)) - \mu_Q^\star R_2(b_1, b_2) = -\mu_{QB}^\star[b_1, b_2] + \nabla_{\mu_B^\star(b_1)}\mu_{QB}^\star(b_2) - \nabla_{\mu_B^\star(b_2)}\mu_{QB}^\star(b_1) + \mu_{QB}^\star(b_2) \circ \partial_{Q_1}\mu_{QB}^\star(b_1) - \mu_{QB}^\star(b_1) \circ \partial_{Q_1}\mu_{QB}^\star(b_2)$.

In the fourth and fifth equation, $\mu_{QB}^\star(b)$ is seen as a section of $\mathrm{Hom}(Q, Q^*)$. Note that if $M_1 = M_2 = M$ and $\omega_0 = \mathrm{Id}_M$, then (1)–(3) translate to

1. $\mu_B^*\colon B_2 \to B_1$ is a Lie algebroid morphism,
2. $\mu_Q \circ \partial_{Q_1} \circ \mu_Q^* = \partial_{Q_2}$.

4.1.1. *Symplectic* [2]*-manifolds.* Note that an ordinary Poisson manifold $(M, \{\cdot\,,\cdot\})$ is symplectic if and only if the vector bundle morphism $\sharp\colon T^*M \to TM$ defined by $\mathbf{d}f \mapsto X_f$ is surjective, where $X_f \in \mathfrak{X}(M)$ is the derivation $\{f, \cdot\}$. Alternatively, we can say that the Poisson manifold is symplectic if the image of the map $\sharp\colon C^\infty(M) \to \mathfrak{X}(M)$, $f \mapsto \{f, \cdot\}$ generates $\mathfrak{X}(M)$ as a $C^\infty(M)$-module.

In the same manner, if $(\mathcal{M}, \{\cdot\,,\cdot\})$ is a Poisson $[n]$-manifold, the map $\sharp\colon C^\infty(\mathcal{M}) \to \mathrm{Der}(C^\infty(\mathcal{M}))$ sends ξ to $\{\xi, \cdot\}$. Then $(\mathcal{M}, \{\cdot\,,\cdot\})$ is a **symplectic** $[n]$**-manifold** if the image of this map generates $\mathrm{Der}(C^\infty(\mathcal{M}))$ as a $C^\infty(\mathcal{M})$-module.

Let $(q_E\colon E \to M, \langle\cdot\,,\cdot\rangle)$ be an Euclidean vector bundle, i.e. a vector bundle endowed with a nondegenerate fiberwise pairing $\langle\cdot\,,\cdot\rangle\colon E \times_M E \to \mathbb{R}$. Choose a metric connection $\nabla\colon \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$. Then, identifing E with E^* via the metric, we find that the 2-representation $(\mathrm{Id}_E\colon E \to E, \nabla, \nabla, R_\nabla)$ is self-dual (an easy calculation shows that if ∇ is metric, then $\langle R_\nabla(X_1, X_2)e_1, e_2\rangle = -\langle R_\nabla(X_1, X_2)e_2, e_1\rangle$ for all $e_1, e_2 \in \Gamma(E)$ and $X_1, X_2 \in \mathfrak{X}(M)$). Consider the split Poisson [2]-manifold $E[-1] \oplus T^*M[-2]$, with the Poisson bracket given by the metric connection $\nabla\colon \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$. That is, the Poisson bracket is given by

$$\{f_1, f_2\} = 0, \quad \{f, \boldsymbol{\beta}(e)\} = 0, \quad \{\boldsymbol{\beta}(e_1), \boldsymbol{\beta}(e_2)\} = \boldsymbol{\beta}(e_2)(e_1) = \langle e_1, e_2\rangle,$$

$$\{X, \xi\} = \boldsymbol{\beta}(\nabla_X e), \quad \{X, f\} = X(f)$$

and

$$\{X_1, X_2\} = [X_1, X_2] - R_\nabla(X_1, X_2).$$

Recall from (20) the special derivations that we found on split $[n]$-manifolds. The function $\sharp\colon C^\infty(E[-1] \oplus T^*M[-2]) \to \operatorname{Der}(C^\infty(E[-1] \oplus T^*M[-2]))$ sends a function f of degree 0 to $\widehat{\mathbf{d}f}$, a derivation of degree -2. $\sharp$ sends $\boldsymbol{\beta}(e)$ to $\hat{e} + \boldsymbol{\beta}(\nabla_\cdot e)$, which is a derivation of degree -1. Note that $\boldsymbol{\beta}(\nabla_\cdot e)$ can be written as a sum $\sum_i \boldsymbol{\beta}(e_i)\widehat{\mathbf{d}f_i}$ with some sections $e_i \in \Gamma(E)$ and functions $f_i \in C^\infty(M)$. Finally $\sharp$ sends X to $X + \boldsymbol{\beta} \circ \nabla_X \circ \boldsymbol{\beta}^{-1} + [X, \cdot] - R(X, \cdot)$, which is a derivation of degree 0. Note that $R(X, \cdot)$ can be written as $\sum \boldsymbol{\beta}(e_i)\boldsymbol{\beta}(e_j)\widehat{\mathbf{d}f_{ij}}$ for some sections $e_i, e_j \in \Gamma(E)$ and some functions f_{ij} in $C^\infty(M)$. Hence, since the derivations $\widehat{\mathbf{d}f}$, $\hat{e}$ and $X + \boldsymbol{\beta} \circ \nabla_X \circ \boldsymbol{\beta}^{-1} + [X, \cdot]$ for $f \in C^\infty(M)$, $e \in \Gamma(E)$ and $X \in \mathfrak{X}(M)$, span $\operatorname{Der}(C^\infty(E[-1] \oplus T^*M[-2]))$ as a $C^\infty(E[-1] \oplus T^*M[-2])$-module, we find as a consequence that $E[-1] \oplus T^*M[-2]$ is a symplectic [2]-manifold.

More generally, take a split Poisson [2]-manifold $Q[-1] \oplus B^*[-2]$, hence a self-dual 2-representation $(\partial_Q\colon Q^* \to Q, \nabla, \nabla^*, R)$ of a Lie algebroid B. Then $\sharp f = \rho_B^* \mathbf{d}f$ for all $f \in C^\infty(M)$, $\sharp\tau = \widehat{\partial_Q\tau} - \nabla^*_\cdot\tau$ and $\sharp b = \rho_B(b) + \nabla^*_b + [b, \cdot] - R(b, \cdot)$. A discussion as the one above shows that the Poisson structure is symplectic if and only if $\rho_B\colon B \to TM$ is injective and surjective, hence an isomorphism and $\partial_Q\colon Q^* \to Q$ is surjective, hence an isomorphism. The isomorphism ∂_Q identifies then Q with its dual and Q becomes so an Euclidean bundle with the pairing $\langle q_1, q_2\rangle_Q = \langle \partial_Q^{-1}(q_1), q_2\rangle = \{\partial_Q^{-1}q_1, \partial_Q^{-1}q_2\}$. Via the identification $\partial_Q\colon Q^* \xrightarrow{\sim} Q$, the linear connection ∇ is then automatically a metric connection and the self-dual 2-representation is $(\operatorname{Id}_Q\colon Q \to Q, \nabla, \nabla, R_\nabla)$.

We have hence found that split symplectic [2]-manifolds are equivalent to self-dual 2-representation $(\operatorname{Id}_E\colon E \to E, \nabla, \nabla, R_\nabla)$ defined by an Euclidean vector bundle E together with a metric connection ∇, see also [35].

4.2. **Metric VB-algebroids.**

Next we introduce the notion of metric VB-algebroids.

Definition 4.4. *Let $(D; Q, B; M)$ be a metric double vector bundle (with core Q^*) and assume that $(D \to Q, B \to M)$ is a VB-algebroid. Then $(D \to Q, B \to M)$ is a* ***metric VB-algebroid*** *if the isomorphism $\boldsymbol{\beta}\colon D \to D \mathbin{\ast\!\!\!|} B$ is an isomorphism of VB-algebroids.*

A morphism $\Omega\colon \mathbb{E}_2 \to \mathbb{E}_1$ of metric VB-algebroids is a morphism of the underlying metric double vector bundles, such that $\Omega \subseteq \overline{\mathbb{E}_2} \times \mathbb{E}_1$ is a subalgebroid.

Recall from Theorem 2.6 that linear splittings of VB-algebroids define 2-representations. We will prove that Lagrangian splittings of metric VB-algebroids correspond to self-dual 2-representations.

Proposition 4.5. *Let $(\mathbb{E} \to Q, B \to M)$ be a VB-algebroid with core Q^* and assume that $\mathbb{E}$ is endowed with a linear metric. Choose a Lagrangian decomposition of $\mathbb{E}$ and consider the corresponding 2-representation of B on $\partial_Q\colon Q^* \to Q$. This 2-representation is self-dual if and only if $(\mathbb{E} \to Q, B \to M)$ is a metric VB-algebroid.*

Proof. It is easy to see that $\boldsymbol{\beta}\colon \mathbb{E} \to \mathbb{E} \mathbin{\ast\!\!\!|} B$ sends core sections $\tau^\dagger \in \Gamma_Q^c(\mathbb{E})$ to core sections $\tau^\dagger \in \Gamma_Q^c(\mathbb{E} \mathbin{\ast\!\!\!|} B)$. (As always, we identify Q^{**} with Q via the canonical isomorphism.) Let $\Sigma\colon B \times_M Q \to \mathbb{E}$ be a Lagrangian splitting of $\mathbb{E}$. We have seen in Section 2.2.3 that the map $\sigma_B\colon \Gamma(B) \to \Gamma^l_Q(\mathbb{E})$ induces a horizontal lift $\sigma^\star_B\colon \Gamma(B) \to \Gamma^l_Q(\mathbb{E} \ast B)$. Recall from Lemma 3.7 that $\boldsymbol{\beta}$ sends also the linear sections $\sigma_B(b)$ to $\sigma^\star_B(b)$, for all $b \in \Gamma(B)$.

The double vector bundle $\mathbb{E} \ast B$ has a VB-algebroid structure $(\mathbb{E} \ast B \to Q^{**}, B \to M)$ (see §2.3). Given the splitting $\Sigma^\star\colon B \times_M Q^{**} \to \mathbb{E} \ast B$ defined by a Lagrangian splitting $\Sigma\colon B \times_M Q \to \mathbb{E}$, the VB-algebroid structure is given by the dual of the 2-representation $(\partial_Q\colon Q^* \to Q, \nabla^Q, \nabla^{Q^*}, R \in \Omega^2(B, \operatorname{Hom}(Q, Q^*)))$, i.e.

$$\begin{aligned}
\rho_{\mathbb{E} \ast B}(\tau^\dagger) &= (\partial_Q^*\tau)^\uparrow \in \mathfrak{X}^c(Q^{**}),\\
\rho_{\mathbb{E} \ast B}(\sigma^\star_B(b)) &= \widehat{\nabla^{Q^{**}}} \in \mathfrak{X}^l(Q^{**}),\\
\left[\sigma^\star_B(b), \tau^\dagger\right] &= ({\nabla^{Q^*}}_b\tau)^\dagger, \text{ and}\\
[\sigma^\star_B(b_1), \sigma^\star_B(b_2)] &= \sigma^\star_B[b_1, b_2] + R\widetilde{(b_1, b_2)}^*
\end{aligned}$$

(see §2.4.1). This shows immediately that $\boldsymbol{\beta}$ is an isomorphism of VB-algebroids over the canonical isomorphism $Q \to Q^{**}$ if and only if the 2-representation $(\partial_Q\colon Q^* \to Q, \nabla^Q, \nabla^{Q^*}, R \in \Omega^2(B, \operatorname{Hom}(Q, Q^*)))$ is self-dual. □

Recall that a morphism $\Omega \subseteq \overline{\mathbb{E}_2} \times \mathbb{E}_1$ of metric double vector bundles has four components: a smooth map $\omega_0\colon M_1 \to M_2$ of the double bases, two vector bundle morphisms $\omega_Q\colon Q_1 \to Q_2$ and $\omega_B\colon B_1^* \to B_2^*$ over ω_0 and a vector valued 2-form $\omega \in \Omega^2(Q_1, \omega_0^*B_2)$. Choose two Lagrangian splittings $\Sigma^1\colon Q_1 \times_{M_1} B_1 \to \mathbb{E}_1$ and $\Sigma^2\colon Q_2 \times_{M_2} B_2 \to \mathbb{E}_2$ and the corresponding self-dual 2-representations. Using Section 3.2.2, we note that Ω is spanned over $(\operatorname{graph}(\omega_Q) \subseteq Q_1 \times Q_2)$ by sections $\tau^\dagger\colon \operatorname{graph}(\omega_Q) \to \Omega$, $\tau^\dagger(q_m, \omega_Q(q_m)) = (\omega_Q^\star(\tau)^\dagger(q_m), \tau^\dagger(\omega_Q(q_m)))$ for all $\tau \in \Gamma(Q_2^*)$ and $\sigma(b)\colon \operatorname{graph}(\omega_Q) \to \Omega$, $\sigma(b)(q_m, \omega_Q(q_m)) = (\sigma^1_{B_1}(\omega^\star_B(b))(q_m) + \widetilde{\omega^\star(b)}(q_m), \sigma^2_{B_2}(b)(\omega_Q(q_m)))$ for all $b \in \Gamma(B_2)$.

A straightforward computation shows that Ω is a subalgebroid over $\operatorname{graph}(\omega_Q)$ of $\overline{\mathbb{E}_2} \times \mathbb{E}_1$ if and only if $\omega_0, \omega_Q, \omega_B, \omega$ and the two self-dual 2-representations satisfy Conditions (1) to (5) on Page 36:

1. $[\omega^\star_B(b_1), \omega^\star_B(b_2)]_1 = \omega^\star_B[b_1, b_2]_2$ for all $b_1, b_2 \in \Gamma(B_2)$,
2. $\rho_{B_1}(\omega^\star_B(b)) \sim_{\omega_0} \rho_{B_2}(b)$ for all $b \in \Gamma(B)$,
3. $\langle \omega^\star_Q(\tau_1), \partial_{Q_1}\omega^\star_Q(\tau_2)\rangle = \langle \tau_1, \partial_{Q_2}\tau_2\rangle$ for all $\tau_1, \tau_2 \in \Gamma(Q_2^*)$,
4. $\omega^\star_Q(\nabla^2_b\tau) = \nabla^1_{\omega^\star_B(b)}\omega^\star_Q(\tau) - \omega^\star(b) \circ \partial_{Q_1}\omega^\star_Q(\tau)$ for all $b \in \Gamma(B_2)$ and $\tau \in \Gamma(Q_2^*)$, and
5. $R_1(\omega^\star_B(b_1), \omega^\star_B(b_2)) - \omega^\star_Q R_2(b_1, b_2) = -\omega^\star[b_1, b_2] + \nabla_{\omega^\star_B(b_1)}\omega^\star(b_2) - \nabla_{\omega^\star_B(b_2)}\omega^\star(b_1) + \omega^\star(b_2) \circ \partial_{Q_1}\omega^\star(b_1) - \omega^\star(b_1) \circ \partial_{Q_1}\omega^\star(b_2)$.

4.3. **Equivalence of Poisson [2]-manifolds with metric VB-algebroids.** The functors found in Section 3.3 between the category of metric double vector bundles and the category of [2]-manifolds restrict to functors between the category of metric VB-algebroids and the category of Poisson [2]-manifolds.

Theorem 4.6. *The category of Poisson [2]-manifolds is equivalent to the category of metric VB-algebroids.*

Proof. Let $(\mathcal{M}, \{\cdot\,, \cdot\})$ be a Poisson [2]-manifold and consider the corresponding double vector bundle $\mathbb{E}_{\mathcal{M}}$. Choose a splitting $\mathcal{M} \simeq Q[-1] \oplus B^*[-2]$ of $\mathcal{M}$ and consider the corresponding Lagrangian splitting Σ of $\mathbb{E}_{\mathcal{M}}$.

As we have seen in Theorem 4.3, the split Poisson 2-manifold $Q[-1]\oplus B^*[-2]$ is equivalent to a self-dual 2-representation of a Lie algebroid structure on B on a morphism $\partial_Q\colon Q^*\to Q$. This 2-representation defines a VB-algebroid structure on the decomposition of $\mathbb{E}_{\mathcal{M}}$ and so by isomorphism on $\mathbb{E}_{\mathcal{M}}$. Proposition 4.5 implies then that the Lie algebroid structure is compatible with the metric. The metric Lie algebroid structure on $\mathbb{E}_{\mathcal{M}}$ does not depend on the choice of splitting of $\mathcal{M}$. Hence, the functor $\mathcal{G}$ restricts to a functor $\mathcal{G}_{Poi}$ from the category of Poisson [2]-manifolds to metric VB-algebroids.

The discussions at the end of Sections 4.1 and 4.2 show that morphisms of split Poisson [2]-manifolds are sent by $\mathcal{G}$ to morphisms of decomposed metric VB-algebroids.

The functor $\mathcal{F}$ restricts in a similar manner to a functor $\mathcal{F}_{LA}$ from the category of metric VB-algebroids to the category of Poisson [2]-manifolds. The natural transformations found in the proof of Theorem 3.17 restrict to natural transformations $\mathcal{F}_{LA}\mathcal{G}_{Poi}\simeq \mathrm{Id}$ and $\mathcal{G}_{Poi}\mathcal{F}_{LA}\simeq \mathrm{Id}$. □

4.4. **Examples.** We conclude by discussing three important classes of examples.

4.4.1. *Tangent doubles of Euclidean bundles vs symplectic* [2]*-manifolds.* Consider an Euclidean vector bundle $E\to M$ and a metric connection $\nabla\colon \mathfrak{X}(M)\times\Gamma(E)\to\Gamma(E)$. The double tangent

$$\begin{tikzcd} TE \arrow[r, "p_E"] \arrow[d, "Tq_E"'] & E \arrow[d, "q_E"] \\ TM \arrow[r, "p_M"'] & M \end{tikzcd}$$

has a VB-algebroid structure $(TE\to E; TM\to M)$ and a linear metric $\langle\cdot\,,\cdot\rangle\colon TE\times_{TM}TE\to\mathbb{R}$ defined as in Example 3.11.

Recall that Lagrangian linear splittings of TE are equivalent to metric connections $\nabla\colon\mathfrak{X}(M)\times\Gamma(E)\to\Gamma(E)$, i.e. connections that preserve the pairing: $\mathbf{d}\langle e_1,e_2\rangle = \langle\nabla_\cdot e_1,e_2\rangle+\langle e_1,\nabla_\cdot e_2\rangle$ for all $e_1,e_2\in\Gamma(E)$. In other words, $\nabla=\nabla^*$ when E^* is identified with E via the non-degenerate pairing. The 2-representation $(\mathrm{Id}_E\colon E\to E,\nabla,\nabla,R_\nabla)$ defined by the Lagrangian splitting $\Sigma^\nabla\colon E\times_M TM\to TE$ and the VB-algebroid $(TE\to E, TM\to M)$ is then self-dual (see also §4.1.1).

The Poisson [2]-manifold $\mathcal{M}(TE)$ associated to TE is given as follows. The functions of degree 0 are elements of $C^\infty(M)$, the functions of degree 1 are sections of E (E is identified with E^* via the isomorphism $\boldsymbol{\beta}\colon E\to E^*$ defined by the pairing) and the functions of degree 2 are the vector fields $\tilde X$ on E that preserve the pairing, i.e. $\tilde X=\widehat{D}\in\mathfrak{X}(E)$ for a derivation $D_{\tilde X}$ over $X\in\mathfrak{X}(M)$, that preserves the pairing. The Poisson bracket is given by $\{\tilde X,\tilde Y\}=\widetilde{[X,Y]}$, $\{\tilde X,e\}=D_{\tilde X}(e)$ and $\{\tilde X,f\}=X(f)$, $\{e_1,e_2\}=\langle e_1,e_2\rangle$, and $\{e,f\}=\{f_1,f_2\}=0$ for all $e,e_1,e_2\in\Gamma(E)$, $f,f_1,f_2\in C^\infty(M)$ and $\tilde X,\tilde Y\in\mathfrak{X}^{\langle\cdot\,,\cdot\rangle,l}(E)$. The Poisson [2]-manifold $\mathcal{M}(TE)$ splits as the split Poisson [2]-manifold described in §4.1.1. It is hence symplectic. Thus, we have found that the equivalence found in this section restricts to an equivalence of symplectic [2]-manifolds with tangent doubles of Euclidean vector bundles.

4.4.2. *The metric double of a VB-algebroid.* Take a VB-algebroid $(D\to A, B\to M)$ with core C and a linear splitting $\Sigma\colon A\times_M B\to D$. Let $(\partial_A\colon C\to A,\nabla^A,\nabla^C,R\in\Omega^2(B,\mathrm{Hom}(A,C)))$ be the 2-representation of the Lie algebroid B that is induced by Σ and the VB-algebroid $(D \to A, B \to M)$. Recall from Section §2.4.1 that $(D \mathbin{\ast} B \to C^*, B \to M)$ has an induced VB-algebroid structure and from Lemma 2.3 that the splitting Σ induces a splitting $\Sigma^\star \colon B \times_M C^* \to D \mathbin{\ast} B$. The 2-representation that is defined by this splitting and this VB-algebroid is the 2-representation $(\partial_A^* \colon A^* \to C^*, \nabla^{A^*}, \nabla^{C^*}, -R^* \in \Omega^2(B, \operatorname{Hom}(C^*, A^*)))$.

The direct sum $D \oplus_B (D \mathbin{\ast} B)$ over B

$$\begin{tikzcd}
D \oplus_B (D \mathbin{\ast} B) \arrow[r] \arrow[d] & B \arrow[d] \\
A \oplus C^* \arrow[r] & M
\end{tikzcd}$$

has then a VB-algebroid structure $(D \oplus_B (D \mathbin{\ast} B) \to A \oplus C^*, B \to M)$ with core $C \oplus A^*$. It is easy to see that Σ and $\Sigma^\star$ define a linear splitting $\tilde\Sigma \colon B \times_M (A \oplus C^*) \to D \oplus_B (D \mathbin{\ast} B)$, $\tilde\Sigma(b_m, (a_m, \gamma_m)) = (\Sigma(a_m, b_m), \Sigma^\star(b_m, \gamma_m))$. The induced 2-representation is

$$(\partial_A \oplus \partial_A^* \colon C \oplus A^* \to A \oplus C^*, \nabla^A \oplus \nabla^{C^*}, \nabla^C \oplus \nabla^{A^*}, R \oplus (-R^*)),$$

a self-dual 2-representation of the Lie algebroid B. This gives us a new class of examples of (split) Poisson 2-manifolds induced from ordinary 2-representations or VB-algebroids. Note that the splittings of $D \oplus_B (D \mathbin{\ast} B)$ obtained as above are not the only Lagrangian splittings, and that the Example of $(TA \oplus T^*A \to TM \oplus A^*, A \to M)$ discussed in the next example and in [16] is a special case.

4.4.3. *The Pontryagin algebroid over a Lie algebroid.* If A is a Lie algebroid, then since $TA \mathbin{\ast} A = T^*A$, the double vector bundle T^*A has a VB-algebroid structure $(T^*A \to A^*, A \to M)$ with core T^*M. As a consequence, the fibered product $TA \oplus_A T^*A$ has a VB-algebroid structure $(TA \oplus_A T^*A \to TM \oplus A^*, A \to M)$. Recall from Example 3.12 that $(TA \oplus T^*A; TM \oplus A^*, A; M)$ has also a natural linear metric, which is given by (23).

Recall from Example 3.12 that linear splittings of $TA \oplus_A T^*A$ are in bijection with dull brackets on sections of $TM \oplus A^*$, and so also with Dorfman connections $\Delta \colon \Gamma(TM \oplus A^*) \times \Gamma(A \oplus T^*M) \to \Gamma(A \oplus T^*M)$. We give in [16] the 2-representation $((\rho, \rho^*) \colon A \oplus T^*M \to TM \oplus A^*, \nabla^{\text{bas}}, \nabla^{\text{bas}}, R_\Delta^{\text{bas}})$ of A that is defined by the VB-algebroid $(TA \oplus_A T^*A \to TM \oplus A^*, A \to M)$ side and any such Dorfman connection: The connections $\nabla^{\text{bas}} \colon \Gamma(A) \times \Gamma(A \oplus T^*M) \to \Gamma(A \oplus T^*M)$ and $\nabla^{\text{bas}} \colon \Gamma(A) \times \Gamma(TM \oplus A^*) \to \Gamma(TM \oplus A^*)$ are

$$\nabla_a^{\text{bas}}(X, \alpha) = (\rho, \rho^*)(\Omega_{(X,\alpha)} a) + \pounds_a(X, \alpha)$$

and

$$\nabla_a^{\text{bas}}(b, \theta) = \Omega_{(\rho,\rho^*)(b,\theta)} a + \pounds_a(b, \theta),$$

where $\Omega \colon \Gamma(TM \oplus A^*) \times \Gamma(A) \to \Gamma(A \oplus T^*M)$ is defined by

$$\Omega_{(X,\alpha)} a = \Delta_{(X,\alpha)}(a, 0) - (0, \mathbf{d}\langle \alpha, a \rangle)$$

and for $a \in \Gamma(A)$, the derivations $\pounds_a$ over $\rho(a)$ are defined by:

$$\pounds_a \colon \Gamma(A \oplus T^*M) \to \Gamma(A \oplus T^*M), \quad \pounds_a(b, \theta) = ([a, b], \pounds_{\rho(a)} \theta)$$

and

$$\pounds_a \colon \Gamma(TM \oplus A^*) \to \Gamma(TM \oplus A^*), \quad \pounds_a(X, \alpha) = ([\rho(a), X], \pounds_a \alpha).$$

We prove in [16] that the two connections above are dual to each other if and only if the dull bracket dual to Δ is skew-symmetric. Hence, the two connections are dual to each other if and only if the chosen linear splitting is Lagrangian (see Example 3.12). The basic curvature

$$R_\Delta^{\rm bas}\colon \Gamma(A)\times\Gamma(A)\times\Gamma(TM\oplus A^*)\to\Gamma(A\oplus T^*M)$$

is given by

$$\begin{aligned} R_\Delta^{\rm bas}(a,b)(X,\xi) = &-\Omega_{(X,\xi)}[a,b] + \pounds_a\left(\Omega_{(X,\xi)}b\right) - \pounds_b\left(\Omega_{(X,\xi)}a\right)\\ &+\Omega_{\nabla_b^{\rm bas}(X,\xi)}a - \Omega_{\nabla_a^{\rm bas}(X,\xi)}b. \end{aligned}$$

Assume that the linear splitting is Lagrangian. A relatively long but straightforward computation shows that $R_\Delta^{{\rm bas}^*} = -R_\Delta^{\rm bas}$, and so that the 2-representation is self-dual. Hence $(TA\oplus_A T^*A\to TM\oplus A^*, A\to M)$ is a metric VB-algebroid.

5. Split Lie 2-algebroids and Dorfman 2-representations

In this section we recall the notions of Courant algebroids, Dirac structures, dull algebroids and Dorfman connections. Then we discuss (split) Lie 2-algebroids and the dual Dorfman 2-representations. We give several classes of examples of split Lie 2-algebroids.

5.1. **Preliminaries.** We introduce in this section a slights generalisations of the notion of Courant algebroid, namely *degenerate Courant algebroids* and *Courant algebroids with pairing in a vector bundle.* Degenerate Courant algebroids will appear naturally in our study of LA-Courant algebroids, and we will see that the fat bundle associated to a VB-Courant algebroid will carry a natural Courant algebroid structure with pairing in a vector bundle.

In the following, an anchored vector bundle is a vector bundle $Q\to M$ endowed with a vector bundle morphism $\rho_Q\colon Q\to TM$ over the identity. An anchored vector bundle $(Q\to M,\rho_Q)$ and a vector bundle $B\to M$ are said to be paired if there exists a fibrewise pairing $\langle\cdot\,,\cdot\rangle\colon Q\times_M B\to\mathbb{R}$ and a map $\mathbf{d}_B\colon C^\infty(M)\to\Gamma(B)$ such that

$$\langle q,\mathbf{d}_B f\rangle = \rho_Q(q)(f) \tag{27}$$

for all $q\in\Gamma(Q)$ and $f\in C^\infty(M)$. The triple $(B,\mathbf{d}_B,\langle\cdot\,,\cdot\rangle)$ will be called a **pre-dual** of Q and Q and B are said to be **paired by** $\langle\cdot\,,\cdot\rangle$.

Consider an anchored vector bundle $(\mathsf{E}\to M,\rho)$ and a vector bundle V over the same base M together with a map $\tilde\rho\colon\Gamma(\mathsf{E})\to{\rm Der}(V)$, such that the symbol of $\tilde\rho(e)$ is $\rho(e)\in\mathfrak{X}(M)$ for all $e\in\Gamma(\mathsf{E})$. Assume that E is paired with itself via a pairing $\langle\cdot\,,\cdot\rangle\colon\mathsf{E}\times_M\mathsf{E}\to V$ with values in V and that there exists a map $\mathcal{D}\colon\Gamma(V)\to\Gamma(\mathsf{E})$ such that $\langle\mathcal{D}v,e\rangle=\tilde\rho(e)(v)$ for all $v\in\Gamma(V)$.

Then $\mathsf{E}\to M$ is a **degenerate Courant algebroid with pairing in** V over the manifold M if E is in addition equipped with an $\mathbb{R}$-bilinear bracket $[\![\cdot\,,\cdot]\!]$ on the smooth sections $\Gamma(\mathsf{E})$ such that the following conditions are satisfied:

(CA1) $[\![e_1,[\![e_2,e_3]\!]]\!] = [\![[\![e_1,e_2]\!],e_3]\!] + [\![e_2,[\![e_1,e_3]\!]]\!]$,
(CA2) $\tilde\rho(e_1)\langle e_2,e_3\rangle = \langle[\![e_1,e_2]\!],e_3\rangle + \langle e_2,[\![e_1,e_3]\!]\rangle$,
(CA3) $[\![e_1,e_2]\!] + [\![e_2,e_1]\!] = \mathcal{D}\langle e_1,e_2\rangle$,
(CA4) $\rho([\![e_1,e_2]\!]) = [\rho(e_1),\rho(e_2)]$,
(CA5) $[\![e_1,fe_2]\!] = f[\![e_1,e_2]\!] + (\rho(e_1)f)e_2$

for all $e_1, e_2, e_3 \in \Gamma(\mathsf{E})$ and $f \in C^\infty(M)$. If the pairing $\langle\cdot\,,\cdot\rangle$ is nondegenerate, then $(\mathsf{E} \to M, \tilde{\rho}, \langle\cdot\,,\cdot\rangle, [\![\cdot\,,\cdot]\!])$ is a **Courant algebroid with pairing in V**. If $V = \mathbb{R} \times M \to M$ is the trivial bundle and the pairing is nondegenerate, then $\mathcal{D} = \beta^{-1} \circ \rho^* \circ \mathbf{d}\colon C^\infty(M) \to \Gamma(\mathsf{E})$, where β is the isomorphism $\mathsf{E} \to \mathsf{E}^*$ given by $\boldsymbol{\beta}(e) = \langle e, \cdot\rangle$ for all $e \in \mathsf{E}$. The quadruple $(\mathsf{E} \to M, \rho, \langle\cdot\,,\cdot\rangle, [\![\cdot\,,\cdot]\!])$ is then a **Courant algebroid** [22, 34] and Conditions (CA4) and (CA5) follow then from (CA1), (CA2) and (CA3) (see [38] and also [16] for a quicker proof).

In our study of VB-Courant algebroids, we will need the following two lemmas.

Lemma 5.1 ([35])**.** *Let $(\mathsf{E} \to M, \rho, \langle\cdot\,,\cdot\rangle, [\![\cdot\,,\cdot]\!])$ be a Courant algebroid. For all $\theta \in \Omega^1(M)$ and $e \in \Gamma(\mathsf{E})$, we have:*

$$[\![e, \beta^{-1}\rho^*\theta]\!] = \beta^{-1}\rho^*(\pounds_{\rho(e)}\theta), \qquad [\![\beta^{-1}\rho^*\theta, e]\!] = -\beta^{-1}\rho^*(\mathbf{i}_{\rho(e)}\mathbf{d}\theta)$$

and

$$\rho(\beta^{-1}\rho^*\theta) = 0. \tag{28}$$

In particular, it follows from (28) that

$$\rho \circ \mathcal{D} = 0. \tag{29}$$

Lemma 5.2 ([19])**.** *Let $\mathsf{E} \to M$ be a vector bundle, $\rho\colon \mathsf{E} \to TM$ be a bundle map, $\langle\cdot,\cdot\rangle$ a bundle metric on E, and let $\mathcal{S} \subseteq \Gamma(\mathsf{E})$ be a subspace of sections which generates $\Gamma(\mathsf{E})$ as a $C^\infty(M)$-module. Suppose that $[\![\cdot\,,\cdot]\!] : \mathcal{S} \times \mathcal{S} \to \mathcal{S}$ is a bracket which satisfies*

1. $[\![s_1, [\![s_2, s_3]\!]]\!] = [\![[\![s_1, s_2]\!], s_3]\!] + [\![s_2, [\![s_1, s_3]\!]]\!]$,
2. $\rho(s_1)\langle s_2, s_3\rangle = \langle [\![s_1, s_2]\!], s_3\rangle + \langle s_2, [\![s_1, s_3]\!]\rangle$,
3. $[\![s_1, s_2]\!] + [\![s_2, s_1]\!] = \rho^*\mathbf{d}\langle s_1, s_2\rangle$,
4. $\rho[\![s_1, s_2]\!] = [\rho(s_1), \rho(s_2)]$,

for any $s_i \in \mathcal{S}$, and that $\rho \circ \rho^* = 0$. Then there is a unique extension of $[\![\cdot\,,\cdot]\!]$ to a bracket on all of $\Gamma(\mathsf{E})$ such that $(\mathsf{E}, \rho, \langle\cdot\,,\cdot\rangle, [\![\cdot\,,\cdot]\!])$ is a Courant algebroid.

A **Dirac structure** with support in a Courant algebroid $\mathsf{E} \to M$ is a subbundle $D \to S$ over a sub-manifold S of M, such that $D(s)$ is maximal isotropic in $\mathsf{E}(s)$ for all $s \in S$ and

$$e_1|_S \in \Gamma_S(D), e_2 \in \Gamma_S(D) \quad \Rightarrow \quad [\![e_1, e_2]\!]|_S \in \Gamma_S(D)$$

for all $e_1, e_2 \in \Gamma(\mathsf{E})$.

We will use the following lemma in two of our technical proofs involving Dirac structures with support. We leave the proof to the reader.

Lemma 5.3. *Let $\mathsf{E} \to M$ be a Courant algebroid and $D \to S$ a subbundle; with S a sub-manifold of M. Assume that $D \to S$ is spanned by the restrictions to S of a family $\mathcal{S} \subseteq \Gamma(\mathsf{E})$ of sections of E. Then D is a Dirac structure with support S if and only if*

1. $\rho_{\mathsf{E}}(e)(s) \in T_sS$ *for all* $e \in \mathcal{S}$ *and* $s \in S$,
2. D_s *is Lagrangian in* $\mathbb{E}_s$ *for all* $s \in S$ *and*
3. $[\![e_1, e_2]\!]|_S \in \Gamma_S(D)$ *for all* $e_1, e_2 \in \mathcal{S}$.

Next we recall the notion of Dorfman connection [16].

Definition 5.4. *Let $(Q \to M, \rho_Q)$ be an anchored vector bundle and let $(B \to M, \mathbf{d}_B, \langle\cdot\,,\cdot\rangle)$ be paired with (Q, ρ_Q). A* ***Dorfman (Q-)connection on*** *B is an $\mathbb{R}$-linear map*

$$\Delta\colon \Gamma(Q) \to \operatorname{Der}(B)$$

such that

1. *Δ_q is a derivation over $\rho_Q(q) \in \mathfrak{X}(M)$,*
2. *$\Delta_{fq} b = f\Delta_q b + \langle q, b\rangle \cdot \mathbf{d}_B f$ and*
3. *$\Delta_q \mathbf{d}_B f = \mathbf{d}_B(\rho_Q(q) f)$*

for all $f \in C^\infty(M)$, $q, q' \in \Gamma(Q)$, $b \in \Gamma(B)$.

In the last definition, the map $\langle q, \cdot\rangle \mathbf{d}_B f\colon B \to B$ is seen as a section of $\operatorname{Hom}(B, B)$, i.e. a derivation over $0 \in \mathfrak{X}(M)$.

Remark 5.5. Note that if the pairing $\langle\cdot\,,\cdot\rangle : Q \times_M B \to \mathbb{R}$ is nondegenerate, then $B \simeq Q^*$ and the map $\mathbf{d}_B = \mathbf{d}_{Q^*}\colon C^\infty(M) \to \Gamma(Q^*)$ is *defined* by (27): we have then $\mathbf{d}_{Q^*} f = \rho_Q^* \mathbf{d} f$ for all $f \in C^\infty(M)$.

The map $\Delta^* : \Gamma(Q) \times \Gamma(Q) \to \Gamma(Q)$ that is dual to Δ in the sense of dual derivations, i.e. $\langle \Delta^*_{q_1} q_2, \tau\rangle = \rho_Q(q_1)\langle q_2, \tau\rangle - \langle q_2, \Delta_{q_1}\tau\rangle$ for all $q_1, q_2 \in \Gamma(Q)$ and $\tau \in \Gamma(Q^*)$ defines then a *dull bracket* on $\Gamma(Q)$:

$$[\![q_1, q_2]\!]_\Delta = \Delta^*_{q_1} q_2$$

in the sense of the following definition.

Definition 5.6. *A* ***dull algebroid*** *is an anchored vector bundle $(Q \to M, \rho_Q)$ with a bracket $[\![\cdot\,,\cdot]\!]$ on $\Gamma(Q)$ such that*

$$\rho_Q[\![q_1, q_2]\!] = [\rho_Q(q_1), \rho_Q(q_2)] \tag{30}$$

and (the Leibniz identity)

$$[\![f_1 q_1, f_2 q_2]\!] = f_1 f_2 [\![q_1, q_2]\!] + f_1 \rho_Q(q_1)(f_2) q_2 - f_2 \rho_Q(q_2)(f_1) q_1$$

for all $f_1, f_2 \in C^\infty(M)$, $q_1, q_2 \in \Gamma(Q)$.

In other words, a dull algebroid is a **Lie algebroid** if its bracket is in addition skew-symmetric and satisfies the Jacobi identity. Note that a skew symmetric dull bracket can be constructed as follows from an arbitrary dull bracket on Q; the skew-symmetrisation $[\![\cdot\,,\cdot]\!]'$ of $[\![\cdot\,,\cdot]\!]$ is defined by $[\![q_1, q_2]\!]' = \frac{1}{2}\left([\![q_1, q_2]\!] - [\![q_2, q_1]\!]\right)$ for all $q_1, q_2 \in \Gamma(Q)$.

Assume that $\Delta\colon \Gamma(Q) \to \Gamma(Q^*) \to \Gamma(Q^*)$ is a Dorfman connection and let $[\![\cdot\,,\cdot]\!]_\Delta$ be the dual dull bracket. Note that (30) is equivalent to (3) in Definition 5.4 and the Leibniz identity corresponds to (2) and (3).

The *curvature* of a general Dorfman connection $\Delta\colon \Gamma(Q) \times \Gamma(B) \to \Gamma(B)$ is the map

$$R_\Delta\colon \Gamma(Q) \times \Gamma(Q) \to \Gamma(B^* \otimes B),$$

defined on $q, q' \in \Gamma(Q)$ by $R_\Delta(q, q') := \Delta_q \Delta_{q'} - \Delta_{q'} \Delta_q - \Delta_{[q,q']_Q}$. If $B = Q^*$ and the pairing is the natural one, the curvature is equivalent to the Jacobiator of the dull bracket:

$$\langle \tau, \operatorname{Jac}_{[\![\cdot\,,\cdot]\!]_\Delta}(q_1, q_2, q_3)\rangle = \langle R_\Delta(q_1, q_2)\tau, q_3\rangle \tag{31}$$

for $q_1, q_2, q_3 \in \Gamma(Q)$ and $b \in \Gamma(B)$, where

$$\operatorname{Jac}_{[\![\cdot\,,\cdot]\!]_\Delta}(q_1, q_2, q_3) = [\![[\![q_1, q_2]\!]_\Delta, q_3]\!]_\Delta + [\![q_2, [\![q_1, q_3]\!]_\Delta]\!]_\Delta - [\![q_1, [\![q_2, q_3]\!]_\Delta]\!]_\Delta$$

is the Jacobiator of $[\![\cdot\,,\cdot]\!]_\Delta$ in Leibniz form. Hence, the Dorfman connection is flat if and only if the corresponding dull bracket satisfies the Jacobi identity in Leibniz form $[\![q_1, [\![q_2, q_3]\!]_\Delta]\!]_\Delta = [\![[\![q_1, q_2]\!]_\Delta, q_3]\!]_\Delta + [\![q_2, [\![q_1, q_3]\!]_\Delta$ for all $q_1, q_2, q_3 \in \Gamma(Q)$. A flat Dorfman connection is called a **Dorfman representation**, if the dual dull bracket is in addition skew-symmetric. The dull bracket $[\![\cdot\,,\cdot]\!]_\Delta$ and the anchor ρ_Q define then a Lie algebroid structure on $\Gamma(Q)$. Conversely, given a Lie algebroid A, then the Lie derivative $\pounds^A \colon \Gamma(A) \times \Gamma(A^*) \to \Gamma(A^*)$ is a Dorfman representation. Hence, Lie algebroids are dual to Dorfman representations.

Assume that (Q, ρ_Q) is an anchored vector bundle with a Dorfman connection $\Delta : \Gamma(Q) \times \Gamma(Q^*) \to \Gamma(Q^*)$ and let $\nabla : \Gamma(Q) \times \Gamma(B) \to \Gamma(B)$ be a linear Q-connection on a vector bundle[9] B. For each $q \in \Gamma(Q)$, ∇_q and Δ_q define a derivation $\Diamond_q$ of $\Gamma(\mathrm{Hom}(B, Q^*))$: for $\Phi \in \Gamma(\mathrm{Hom}(B, Q^*))$ and $b \in \Gamma(B)$, we have

$$(\Diamond_q \Phi)(b) = \Delta_q(\Phi(b)) - \Phi(\nabla_q b). \tag{32}$$

The map $\Diamond : \Gamma(Q) \times \Gamma(\mathrm{Hom}(B, Q^*)) \to \Gamma(\mathrm{Hom}(B, Q^*))$ satisfies[10]

$$\Diamond_{fq} \Phi = f \Diamond_q \Phi + \Phi^*(q) \cdot \rho_Q^* \mathbf{d} f$$

and $\Diamond_q(\xi \cdot \rho_Q^* \mathbf{d} f) = \nabla_q^* \xi \cdot \rho_Q^* \mathbf{d} f + \xi \cdot \rho_Q^* \mathbf{d}(\pounds_{\rho_Q(q)} f)$ for all $q \in \Gamma(Q)$, $f \in C^\infty(M)$ and $\xi \in \Gamma(B^*)$. If the dull bracket $[\![\cdot\,,\cdot]\!]_\Delta$ dual to Δ is skew-symmetric, then the complex $\Omega(Q, \mathrm{Hom}(B, Q^*))$ has an induced operator $\mathbf{d}_\Diamond \colon \Omega^q(Q, \mathrm{Hom}(B, Q^*)) \to \Omega^{q+1}(Q, \mathrm{Hom}(B, Q^*))$ given by the Koszul formula

$$\begin{aligned} \mathbf{d}_\Diamond \eta(q_1, \ldots, q_{k+1}) = & \sum_{i<j} (-1)^{i+j} \omega([\![q_i, q_i]\!]_\Delta, \ldots, \hat{q}_i, \ldots, \hat{q}_j, \ldots, q_{k+1}) \\ & + \sum_i (-1)^{i+1} \Diamond_{q_i}(\omega(q_1, \ldots, \hat{q}_i, \ldots, q_{k+1})) \end{aligned}$$

for all $\eta \in \Omega^k(Q, \mathrm{Hom}(B, Q^*))$ and $q_1, \ldots, q_{k+1} \in \Gamma(Q)$.

5.2. **Dorfman 2-representations and split Lie 2-algebroids.** We now define the *Dorfman 2-representations* and show that they are the dual derivations to split Lie 2-algebroids.

Recall from §4.1.1 the definition of a graded derivation on an $[n]$-manifold. A **homological** vector field χ on $\mathcal{M}$ is a derivation of degree 1 of $C^\infty(\mathcal{M})$ such that $\chi^2 = \frac{1}{2}[Q, Q]$ vanishes. A homological vector field on a [1]-manifold $\mathcal{M} = E[-1]$ is the Cartan differential $\mathbf{d}_E$ associated to a Lie algebroid structure on E. This result is due to Vaintrob [39] and was explained in our introduction. A **Lie n-algebroid** is an $[n]$-manifold endowed with a homological vector field (an *NQ-manifold* of degree n).

A **split Lie n-algebroid** is a split $[n]$-manifold endowed with a homological vector field. Split Lie n-algebroids were studied by Sheng and Zhu [37] and described as vector bundles endowed with a bracket that satisfies the Jacobi identity up to some correction terms, see also [3]. Let us first give in our own words their definition of a split Lie 2-algebroid.

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[9] Note that the notation slightly changes here. Q is paired with its dual Q^* and B is just a vector bundle over the same base.

[10] One could define this as a Dorfman connection over a vector bundle valued pairing: here the pairing $Q \times_M \mathrm{Hom}(B, Q^*) \to B^*$ would be defined by $(q, \Phi) \mapsto \Phi^*(q)$.

Definition 5.7. *A split Lie 2-algebroid $Q \oplus B^* \to M$ is a pair of an anchored vector bundle* [11] *$(Q \to M, \rho_Q)$ and a vector bundle $B \to M$, together with*

1. *a vector bundle map $l_1 \colon B^* \to Q$,*
2. *a skew-symmetric dull bracket $\llbracket \cdot\,,\cdot \rrbracket \colon \Gamma(Q) \times \Gamma(Q) \to \Gamma(Q)$,*
3. *a linear connection $\nabla \colon \Gamma(Q) \times \Gamma(B) \to \Gamma(B)$ and*
4. *a vector valued 3-form $l_3 \in \Omega^3(Q, B^*)$,*

such that

- (i) $\nabla^*_{l_1(\beta_1)}\beta_2 + \nabla^*_{l_1(\beta_2)}\beta_1 = 0$ *for all* $\beta_1, \beta_2 \in \Gamma(B^*)$,
- (ii) $\llbracket q, l_1(\beta) \rrbracket = l_1(\nabla^*_q \beta)$ *for* $q \in \Gamma(Q)$ *and* $\beta \in \Gamma(B^*)$,
- (iii) $\mathrm{Jac}_{\llbracket \cdot,\cdot \rrbracket} = -l_1 \circ l_3 \in \Omega^3(Q, Q)$,
- (iv) $R_{\nabla^*}(q_1, q_2)\beta = l_3(q_1, q_2, l_1(\beta))$ *for* $q \in \Gamma(Q)$ *and* $\beta \in \Gamma(B^*)$, *and*
- (v) $\mathbf{d}_{\nabla^*} l_3 = 0$.

From (iii) follows the identity $\rho_Q \circ l_1 = 0$. To get the definition that was first given in [37], consider the skew symmetric bracket $l_2 \colon \Gamma(Q \oplus B^*) \times \Gamma(Q \oplus B^*) \to \Gamma(Q \oplus B^*)$,

$$l_2((q_1, \beta_1), (q_2, \beta_2)) = (\llbracket q_1, q_2 \rrbracket, \nabla^*_{q_1}\beta_2 - \nabla^*_{q_2}\beta_1) \tag{33}$$

for $q_1, q_2 \in \Gamma(Q)$ and $\beta_1, \beta_2 \in \Gamma(B^*)$. Note that this bracket satisfies a Leibniz identity with anchor $\rho_Q \circ \mathrm{pr}_Q \colon Q \oplus B^* \to TM$ and that the Jacobiator of this bracket is then given by

$$\mathrm{Jac}_{l_2}((q_1, \beta_1), (q_2, \beta_2), (q_3, \beta_3)) = (-l_1(l_3(q_1, q_2, q_3)), l_3(q_1, q_2, l_1(\beta_3)) + \text{c.p.}$$

Since $\mathbf{d}_{\nabla^*} l_3 = 0$, one could say that a split Lie 2-algebroid is a Lie algebroid "up to homotopy". Conversely, all the geometric objects in the previous definition can easily be constructed from the original definition of split Lie 2-algebroids.

As we have seen above, flat Dorfman connections with skew-symmetric dual dull brackets are in duality with Lie algebroids (or equivalently "split Lie 1-algebroids"). As we will see below, we define in fact Dorfman 2-representations as the Lie derivatives that are dual to split Lie 2-algebroids. The notion of Dorfman 2-representation defined below resembles the notion of 2-representation. The meaning of this analogy will become clearer in our study of VB-Courant algebroids.

Definition 5.8. *Let $(Q \to M, \rho_Q)$ be an anchored vector bundle. A (Q, ρ_Q)-Dorfman 2-representation is a quadruple $(\partial_B, \Delta, \nabla, R)$ where $\partial_B \colon Q^* \to B$ is a vector bundle morphism, Δ is a Dorfman connection*

$$\Delta \colon \Gamma(Q) \times \Gamma(Q^*) \to \Gamma(Q^*),$$

∇ is a linear connection

$$\nabla \colon \Gamma(Q) \times \Gamma(B) \to \Gamma(B),$$

and R is an element of $\Omega^2(Q, \mathrm{Hom}(B, Q^*))$ such that

- (D1) $\partial_B \circ \Delta_q = \nabla_q \circ \partial_B$,
- (D2) $\llbracket q_1, q_2 \rrbracket_\Delta = -\llbracket q_2, q_1 \rrbracket_\Delta$,
- (D3) $\nabla^*_{\partial_B^* \xi_1}\xi_2 + \nabla^*_{\partial_B^* \xi_2}\xi_1 = 0$,
- (D4) $\partial_B \circ R(q_1, q_2) = R_\nabla(q_1, q_2)$ *and* $R(q_1, q_2) \circ \partial_B = R_\Delta(q_1, q_2)$,
- (D5) $R(q_1, q_2)^* q_3 = -R(q_1, q_3)^* q_2$ *and*
- (D6) $\mathbf{d}_\Diamond R(q_1, q_2, q_3) = \nabla^*_{\cdot}\, (R(q_1, q_2)^* q_3)$

for all $\xi_1, \xi_2 \in \Gamma(B^*)$ and $q, q_1, q_2, q_3 \in \Gamma(Q)$ and $f \in C^\infty(M)$.

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[11] The names that we choose for the vector bundles will become natural in a moment.

Note that (D5) is equivalent to R defining an element ω_R of $\Omega^3(Q, B^*)$ by $\omega_R(q_1, q_2, q_3) = R(q_1, q_2)^* q_3$. Axiom (D6) is equivalent to $\mathbf{d}_{\nabla^*}\omega_R = 0$ and (D4) is equivalent to $(R_\nabla(q_1, q_2))^*\xi = -R_{\nabla^*}(q_1, q_2)\xi = \omega_R(q_1, q_2, \partial_B^*\xi)$ for $q_1, q_2 \in \Gamma(Q)$ and $\xi \in \Gamma(B^*)$ and $\mathrm{Jac}_{[\![\cdot\,,\cdot]\!]_\Delta} = \partial_B^*\omega_R$.

One can see quite easily that the definition of a Dorfman 2-representations is just a rephrasing of the definition of a Lie 2-algebroid. Set $\omega_R = l_3$ and $\partial_B^* = -l_1$. Then (D1) is (ii) in Definition 5.7 and the other axioms have already been explained above. Note that the vector bundle $Q \oplus B^*$ is anchored by ρ_Q and paired with $Q^* \oplus B$ by the natural pairing and the map $C^\infty(M) \to \Gamma(Q^* \oplus B)$, $f \mapsto (\rho_Q^*\mathbf{d}f, 0)$. Hence we can define a new Dorfman connection

$$\Delta^2 \colon \Gamma(Q \oplus B^*) \times \Gamma(Q^* \oplus B) \to \Gamma(Q^* \oplus B)$$

by

$$\Delta^2_{(q,\beta)}(\tau, b) = (\Delta_q\tau, \nabla_q b),$$

then Δ^2 is the Dorfman connection that is dual to the bracket l_2 defined in (33). Hence, we can think of Dorfman 2-representations as “Lie 2-derivatives”, or “Lie derivatives up to homotopy”. In other words, as the duals of Lie algebroids are Dorfman representations, the duals of Lie 2-algebroids are Dorfman 2-representations.

5.3. **Split Lie-2-algebroids as split [2]Q-manifolds.** Before we go on with the study of examples, we briefly describe how to construct from the objects in Definitions 5.7 and 5.8 the corresponding homological vector fields on split [2]-manifolds.

Consider a split [2]-manifold $\mathcal{M} = Q[-1] \oplus B^*[-2]$. On an open chart $U \subseteq M$ trivialising both Q and B, we have coordinates $(x_1, \ldots, x_p)$ and we can choose a local frame $(q_1, \ldots, q_{r_1})$ of sections of Q, and a local frame $(\beta_1, \ldots, \beta_{r_2})$ of sections of B^*. We denote by $(\tau_1, \ldots, \tau_{r_1})$ and $(b_1, \ldots, b_{r_2})$ the dual frames for Q^* and B, respectively. As functions on $\mathcal{M}$, the coordinates functions $x_1, \ldots, x_p$ have degree 0, the functions $\tau_1, \ldots, \tau_{r_1}$ have degree 1 and the functions $b_1, \ldots, b_{r_2}$ have degree 2. The vector fields $\partial_{x_1}, \ldots, \partial_{x_p}, \partial_{\tau_1}, \ldots, \partial_{\tau_{r_1}}, \partial_{b_1}, \ldots, \partial_{b_{r_2}}$ have degree $0, -1, -2$, respectively, and generate (locally) the set of vector fields on $\mathcal{M}$ as a $C^\infty_{\mathcal{M}}$-module. Assume that Q is endowed with an anchor ρ_Q and take a Dorfman 2-representation $(\partial_B \colon Q^* \to B, \Delta, \nabla, R)$ of Q on $B \oplus Q^*$. Define a vector field $\mathcal{Q}$ of degree 1 on $\mathcal{M}$ by the following formula in local coordinates:

$$\begin{aligned}\mathcal{Q} = &\sum_{i,j} \rho_Q(q_i)(x_j)\tau_i\partial_{x_j} - \sum_{i<j}\sum_k \langle [\![q_i, q_j]\!]_\Delta, \tau_k\rangle \tau_i\tau_j\partial_{\tau_k} \\ &+ \sum_{r,k}\langle \partial_B^*\beta_r, \tau_k\rangle b_r\partial_{\tau_k} - \sum_{i<j<k}\sum_l \omega_R(q_i, q_j, q_k)(b_l)\tau_i\tau_j\tau_k\partial_{b_l} \\ &- \sum_{ijl}\langle \nabla_{q_i}^*\beta_j, b_l\rangle \tau_i b_j \partial_{b_l}.\end{aligned} \tag{34}$$

This vector field satisfies $[\mathcal{Q}, \mathcal{Q}] = 0$. To see this, we compute $\mathcal{Q} \circ \mathcal{Q}$ in local coordinates.

For $f \in C^\infty(M)$ we have $\mathcal{Q}(f) = \rho_Q^*\mathbf{d}f$ and a relatively easy computation shows that $\mathcal{Q}(\mathcal{Q}(f)) = 0$ for all $f \in C^\infty(M)$ if and only if $\rho_Q \circ \partial_B^* = 0$ and $\rho_Q[\![q_1, q_2]\!]_\Delta = [\rho_Q(q_1), \rho_Q(q_2)]$ for all $q_1, q_2 \in \Gamma(Q)$.

A longer, but straightforward computation shows that $\mathcal{Q}(\mathcal{Q}(\tau_k)) = 0$ for $k = 1, \ldots, r_1$ if and only if $\nabla_{q_i}(\partial_B\tau_k) = \partial_B(\Delta_{q_i}\tau_k)$ for all $i = 1, \ldots, n$ and $[\![[\![q_i, q_j]\!]_\Delta, q_l]\!]_\Delta +$

$[\![q_j, [\![q_i, q_l]\!]_\Delta]\!]_\Delta - [\![q_i, [\![q_j, q_l]\!]_\Delta]\!]_\Delta = \partial_B^* R(q_i, q_j)^* q_l = \partial_B^* \omega_R(q_i, q_j, q_l)$ for all $i, j, l = 1, \ldots, n$.

Finally, a very long but also straightforward computation shows that $\mathcal{Q}^2(b_k) = 0$ for $k = 1, \ldots, l$ if and only if $\nabla_{\partial_B^* \beta_r} \beta_j + \nabla_{\partial_B^* \beta_j} \beta_r = 0$ for $j, r = 1, \ldots, l$, $\partial_B(R(q_i, q_j) b_k) = R_\nabla(q_i, q_j) b_l$ for $i, j = 1, \ldots, n$ and $k = 1, \ldots, l$ and $\mathbf{d}_{\nabla^*} \omega_R = 0$.

Conversely, write locally the homological vector field in coordinates and define ρ_Q, ω_R and $[\![\cdot\,,\cdot]\!]_\Delta$ on basis sections $q_1, \ldots, q_n$ of Q, ∂_B^* on basis sections of B^* and ∇^* on basis sections of B^* and Q. The global objects can then be defined locally using Leibniz identities and tensoriality of these objects. These local definitions will then be compatible with changes of trivialisations and define globally the structure components of a split Lie 2-algebroid.

Hence, we have found an explicit way of writing the homological vector field that is equivalent to a split Lie 2-algebroid. Note that by definition, a [2]-manifolds is always locally split. Hence, a Lie 2-algebroid always defines local Dorfman 2-representation, and the homological vector field can always be locally written as in (34). Going from an open set to an other will involve a change of local basis and a change of splitting, but we will show later that changes of splittings of Lie 2-algebroids are easy to describe

5.4. **Examples of Dorfman 2-representations and split Lie 2-algebroids.**

We describe here five classes of examples of split Lie 2-algebroids. Later we will discuss their geometric meanings. We do not verify in detail the axioms for Dorfman 2-representations of for split Lie 2-algebroids. The computations in order to do this for Examples 5.4.2, 5.4.3 and 5.5 are long, but straightforward. Note that, alternatively, the next section will provide a geometric proof of the fact that the following objects are Dorfman 2-representations (and split Lie 2-algebroids), since we will find them to be equivalent to special classes of VB-Courant algebroids.

5.4.1. *Dorfman 2-representation associated to a Lie algebroid representation.* Let $(Q \to M, \rho, [\cdot\,,\cdot])$ be a Lie algebroid and $\nabla\colon \Gamma(Q) \times \Gamma(B) \to \Gamma(B)$ a representation of Q on a vector bundle B. Then $(\partial_B = 0\colon Q^* \to B, \pounds^Q, \nabla, R = 0)$ is a Dorfman 2-representation.

The corresponding Lie 2-algebroid is a semi-direct extension of the Lie algebroid Q (and a special case of the double Lie 2-algebroids defined later). Here $l_1 = 0$ and l_2 is given by $l_2(q_1 + \beta_1, q_2 + \beta_2) = [q_1, q_2] + (\nabla^*_{q_1} \beta_2 - \nabla^*_{q_2} \beta_1)$ for $q_1, q_2 \in \Gamma(Q)$ and $\beta_1, \beta_2 \in \Gamma(B^*)$, and $l_3 = 0$. Hence $(Q \oplus B^* \to M, \rho = \rho_Q \circ \mathrm{pr}_Q, l_2)$ is simply a Lie algebroid.

5.4.2. *Standard Dorfman 2-representations.* Let $E \to M$ be a vector bundle, set $\partial_E = \mathrm{pr}_E\colon E \oplus T^*M \to E$ and consider a skew-symmetric dull bracket $[\![\cdot\,,\cdot]\!]$ on $\Gamma(TM \oplus E^*)$, with $TM \oplus E^*$ anchored by pr_{TM} and let $\Delta\colon \Gamma(TM \oplus E^*) \times \Gamma(E \oplus T^*M) \to \Gamma(E \oplus T^*M)$ be the dual Dorfman connection. This defines as follows a split Lie 2-algebroid structure on the vector bundles $(TM \oplus E^*, \mathrm{pr}_{TM})$ and E^*, or a Dorfman 2-representation of $(TM \oplus E^*, \mathrm{pr}_{TM})$ on $E \oplus T^*M$ and E^*.

Let $\nabla\colon \Gamma(TM \oplus E^*) \times \Gamma(E) \to \Gamma(E)$ be the ordinary linear connection[12] defined by $\nabla = \mathrm{pr}_E \circ \Delta \circ \iota_E$, where Δ is dual to $[\![\cdot\,,\cdot]\!]$. The vector bundle map $l_1 = -\mathrm{pr}_E^*\colon E^* \to TM \oplus E^*$ is just the opposite of the canonical inclusion. Finally

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[12] To see that $\nabla = \mathrm{pr}_E \circ \Delta \circ \iota_E$ is an ordinary connection, recall that since $TM \oplus E^*$ is anchored by pr_{TM}, the map $\mathbf{d}_{E \oplus T^*M} = \mathrm{pr}_{TM}^* \mathbf{d}\colon C^\infty(M) \to \Gamma(E \oplus T^*M)$ sends $f \to (0, \mathbf{d}f)$.

define l_3 by $l_3(v_1, v_2, v_3) = \mathrm{Jac}_{[\![\cdot\,,\cdot]\!]}(v_1, v_2, v_3)$ and accordingly $R = R_\Delta \circ \iota_E \in \Omega^2(TM \oplus E^*, \mathrm{Hom}(E, E \oplus T^*M))$. (Note that since $TM \oplus E^*$ is anchored by pr_{TM}, the tangent part of the dull bracket must just be the Lie bracket of vector fields. The Jacobiator $\mathrm{Jac}_{[\![\cdot\,,\cdot]\!]}$ can hence really be seen as an element of $\Omega^3(TM \oplus E^*, E^*)$.)

A straightforward verification of the axioms shows that $(\partial_E, \Delta, \nabla, R)$ is a $(TM \oplus E^*, \mathrm{pr}_{TM})$-Dorfman 2-representation, or equivalently, that l_1, $[\![\cdot\,,\cdot]\!]$, ∇^*, l_3 define a split Lie 2-algebroid. For reasons that will become clearer later, we call *standard* this type of split Lie 2-algebroid and Dorfman 2-representation.

5.4.3. *Adjoint Dorfman 2-representations.* Let $\mathsf{E} \to M$ be a Courant algebroid with anchor ρ_{E} and bracket $[\![\cdot\,,\cdot]\!]$ and choose a metric linear connection $\nabla \colon \mathfrak{X}(M) \times \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E})$, i.e. a linear connection that preserves the pairing. Set $\partial_{TM} = \rho_{\mathsf{E}} \colon \mathsf{E} \to TM$, identifying E with its dual via the pairing. The map $\Delta \colon \Gamma(\mathsf{E}) \times \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E})$,

$$\Delta_e e' = [\![e, e']\!] + \nabla_{\rho(e')} e$$

is a Dorfman connection, which we call the *basic Dorfman connection associated to* ∇. The dual skew-symmetric (!) dull bracket is given by $[\![e, e']\!]_\Delta = [\![e, e']\!] - \boldsymbol{\beta}^{-1}\rho^*\langle \nabla_{.} e, e' \rangle$ for all $e, e' \in \Gamma(\mathsf{E})$. The map $\nabla^{\mathrm{bas}} \colon \Gamma(\mathsf{E}) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$,

$$\nabla^{\mathrm{bas}}_e X = [\rho(e), X] + \rho(\nabla_X e)$$

is a linear connection, the *basic connection associated to* ∇.

We now define the *basic curvature* $R^{\mathrm{bas}}_\Delta \in \Omega^2(\mathsf{E}, \mathrm{Hom}(TM, \mathsf{E}))$ by[13]

$$\begin{aligned} R^{\mathrm{bas}}_\Delta(e_1, e_2)X = &- \nabla_X [\![e_1, e_2]\!]_\Delta + [\![\nabla_X e_1, e_2]\!]_\Delta + [\![e_1, \nabla_X e_2]\!]_\Delta \\ &+ \nabla_{\nabla^{\mathrm{bas}}_{e_2} X} e_1 - \nabla_{\nabla^{\mathrm{bas}}_{e_1} X} e_2 - \beta^{-1}\rho^*\langle R_\nabla(X, \cdot)e_1, e_2 \rangle \end{aligned}$$

for all $e_1, e_2 \in \Gamma(\mathsf{E})$ and $X \in \mathfrak{X}(M)$. Then $(\rho, \Delta, \nabla^{\mathrm{bas}}, R^{\mathrm{bas}}_\Delta)$ is a Dorfman 2-representation. Note the similarity of this construction with the one of the adjoint representation up to homotopy in Example 2.9.

The corresponding *adjoint* split Lie 2-algebroid can be described as follows. The map l_1 is $-\boldsymbol{\beta}^{-1} \circ \rho_{\mathsf{E}}^* \colon T^*M \to \mathsf{E}$ and $l_2(e_1 + \theta_1, e_2 + \theta_2)$ is

$$\left([\![e_1, e_2]\!] - \beta^{-1}\rho_{\mathsf{E}}\langle \nabla_{.} e_1, e_2 \rangle\right) + \left(\pounds_{\rho(e_1)}\theta_2 - \pounds_{\rho(e_2)}\theta_1 + \langle \rho^*\theta_1, \nabla_{.} e_2 \rangle - \langle \rho^*\theta_2, \nabla_{.} e_1 \rangle\right)$$

for $e_1, e_2 \in \Gamma(\mathsf{E})$ and $\theta_1, \theta_2 \in \Omega^1(M)$. The form $l_3 \in \Omega^3(\mathsf{E}, T^*M)$ is given by $l_3(e_1, e_2, e_3) = \langle R^{\mathrm{bas}}_\Delta(e_1, e_2), e_3 \rangle$ and corresponds to the tensor Ψ defined in [19, Definition 4.1.2] (the left-hand side of (63)). We will see later that the adjoint split Lie 2-algebroids are exactly the *split symplectic Lie 2-algebroids*, and correspond hence to splittings of the tangent doubles of Courant algebroids. They are usually considered as the super-geometric objects corresponding to Courant algebroids, but we will explain this correspondence in more detail later.

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[13] Alternatively, R^{bas}_Δ can be defined by $R^{\mathrm{bas}}_\Delta(e_1, e_2)X = -\nabla_X [\![e_1, e_2]\!] + [\![\nabla_X e_1, e_2]\!] + [\![e_1, \nabla_X e_2]\!] + \nabla_{\nabla^{\mathrm{bas}}_{e_2} X} e_1 - \nabla_{\nabla^{\mathrm{bas}}_{e_1} X} e_2 - \beta^{-1}\langle \nabla_{\nabla^{\mathrm{bas}}_{.} X} e_1, e_2 \rangle$. Using $-R^*_\nabla = R_{\nabla^*} = R_\nabla$ (where we identify E with its dual using $\langle \cdot\,, \cdot \rangle$), the identity $R^{\mathrm{bas}}_\Delta(e_1, e_2) = -R^{\mathrm{bas}}_\Delta(e_2, e_1)$ is obvious using the first definition. It is an easy computation using the second one.

5.4.4. *Dorfman 2-representation defined by a 2-representation.* Let $(\partial_B\colon C \to B, \nabla, \nabla, R)$ be a representation up to homotopy of a Lie algebroid A on $B \oplus C$. Then the quadruple $(\partial_B \circ \mathrm{pr}_C\colon C \oplus A^* \to B, \Delta, \nabla, R)$ defined by

$$\begin{gathered}\Delta\colon \Gamma(A \oplus C^*) \times \Gamma(C \oplus A^*) \to \Gamma(C \oplus A^*)\\ \Delta_{(a,\gamma)}(c, \alpha) = (\nabla_a c, \pounds_a \alpha + \langle \nabla^*_{\cdot}\gamma, c\rangle),\end{gathered} \tag{35}$$

$$\nabla\colon \Gamma(A \oplus C^*) \times \Gamma(B) \to \Gamma(B), \qquad \nabla_{(a,\gamma)} b = \nabla_a b \tag{36}$$

with $A \oplus C^*$ anchored by $\rho_A \circ \mathrm{pr}_A$, and $R \in \omega^2(A \oplus C^*, \mathrm{Hom}(B, C \oplus A^*))$,

$$R((a_1, \gamma_1), (a_2, \gamma_2)) = (R(a_1, a_2), \langle \gamma_2, R(a_1, \cdot)\rangle + \langle \gamma_1, R(\cdot, a_2)\rangle) \tag{37}$$

is a Dorfman 2-representation.

The vector bundle map l_1 is here $l_1 = \iota_{C^*} \circ \partial_B^*$, where $\iota_{C^*}\colon C^* \to A \oplus C^*$ is the canonical inclusion, and the bracket l_2 is given by

$$l_2((a_1, \gamma_1) + \alpha_1, (a_2, \gamma_2) + \alpha_2) = ([a_1, a_2], \nabla^*_{a_1}\gamma_2 - \nabla^*_{a_2}\gamma_1) + (\nabla^*_{a_1}\alpha_2 - \nabla^*_{a_2}\alpha_1)$$

for $a_1, a_2 \in \Gamma(A)$, $\gamma_1, \gamma_2 \in \Gamma(C^*)$ and $\alpha_1, \alpha_2 \in \Gamma(B^*)$. The tensor l_3 is finally given by

$$l_3((a_1, \gamma_1), (a_2, \gamma_2), (a_3, \gamma_3)) = \langle R(a_1, a_2), \gamma_3\rangle + \text{c.p.}$$

Note that if we work with the dual A-representation up to homotopy $(\partial_B^*\colon B^* \to C^*, \nabla^*, \nabla^*, -R^*)$, then we get the Lie 2-algebroid defined in [37, Proposition 3.5] as the semi-direct product of a 2- representation and a Lie algebroid. This is then also a special case of the bicrossproduct of a matched pair of 2-representations (see the next section). We will explain later the slightly peculiar choice that we make here.

5.5. **The bicrossproduct of a matched pair of 2-representations.** In this section we describe a class of examples, which is of great independent interest: we show that the bicrossproduct of a matched pair of 2-representations is a split Lie 2-algebroid.

We construct a split Lie 2-algebroid $(A \oplus B) \oplus C$ induced by a matched pair of 2-representations as in Definition 2.10. The vector bundle $A \oplus B \to M$ is anchored by $\rho_A \circ \mathrm{pr}_A + \rho_B \circ \mathrm{pr}_B$ and paired with $A^* \oplus B^*$ as follows:

$$\langle (a, b), (\alpha, \beta)\rangle = \alpha(a) - \beta(b)$$

for all $a \in \Gamma(A)$, $b \in \Gamma(B)$, $\alpha \in \Gamma(A^*)$ and $\beta \in \Gamma(B^*)$. The morphism $A^* \oplus B^* \to C^*$ is $\partial_A^* \circ \mathrm{pr}_{A^*} + \partial_B^* \circ \mathrm{pr}_{B^*}$. The $A \oplus B$-Dorfman connection on $A^* \oplus B^*$ is defined by

$$\Delta_{(a,b)}(\alpha, \beta) = (\nabla^*_b \alpha + \pounds_a \alpha - \langle \nabla_{\cdot} b, \beta\rangle, \nabla^*_a \beta + \pounds_b \beta - \langle \nabla_{\cdot} a, \alpha\rangle).$$

The dual dull bracket on $\Gamma(A \oplus B)$ is

$$[\![(a, b), (a', b')]\!] = ([a, a'] + \nabla_b a' - \nabla_{b'} a, [b, b'] + \nabla_a b' - \nabla_{a'} b). \tag{38}$$

The $A \oplus B$-connection on C^* is simply given by $\nabla^*_{(a,b)}\gamma = \nabla^*_a \gamma + \nabla^*_b \gamma$ and the dual connection is $\nabla\colon \Gamma(A \oplus B) \times \Gamma(C) \to \Gamma(C)$,

$$\nabla_{(a,b)} c = \nabla_a c + \nabla_b c. \tag{39}$$

Finally, the form $\omega_R = l_3 \in \Omega^3(A \oplus B, C)$ is given by

$$\begin{aligned}\omega_R((a_1, b_1), (a_2, b_2), (a_3, b_3)) = {} & R(a_1, a_2)b_3 + R(a_2, a_3)b_1 + R(a_3, a_1)b_2\\ & - R(b_1, b_2)a_3 - R(b_2, b_3)a_1 - R(b_3, b_1)a_2.\end{aligned} \tag{40}$$

The quadruple $(\partial_A^*+\partial_B^*\colon A^*\oplus B^*\to C^*,\nabla,\Delta,R)$ is a Dorfman 2-representation. Equivalently the vector bundle $(A\oplus B)\oplus C\to M$ with the anchor $\rho_A\circ\operatorname{pr}_A+\rho_B\circ\operatorname{pr}_B\colon A\oplus B\to TM$, $l_1=(-\partial_A,\partial_B)\colon C\to A\oplus B$, $l_3=\omega_R$ and the skew-symmetric dull bracket (38) define a split Lie 2-algebroid.

Moreover, we prove the following theorem:

Theorem 5.9. *The double of a matched pair of 2-representations is a split Lie 2-algebroid with the structure given above. Conversely if $(A\oplus B)\oplus C$ has a split Lie 2-algebroid structure such that*

1. $[\![(a_1,0),(a_2,0)]\!]=([a_1,a_2],0)$ *with a section* $[a_1,a_2]\in\Gamma(A)$ *for all* $a_1,a_2\in\Gamma(A)$ *and in the same manner* $[\![(0,b_1),(0,b_2)]\!]=(0,[b_1,b_2])$ *with a section* $[b_1,b_2]\in\Gamma(B)$ *for all* $b_1,b_2\in\Gamma(B)$*, and*
2. $l_3((a_1,0),(a_2,0),(a_3,0))=0$ *and* $l_3((0,b_1),(0,b_2),(0,b_3))=0$ *for all* a_1,a_2,a_3 *in* $\Gamma(A)$ *and* b_1,b_2,b_3 *in* $\Gamma(B)$*,*

then A and B are Lie subalgebroids of $(A\oplus B)\oplus C$ and $(A\oplus B)\oplus C$ is the double of a matched pair of 2-representations of A on $B\oplus C$ and of B on $A\oplus C$. The 2-representation of A is given by

$$\begin{aligned}&\partial_B(c)=\operatorname{pr}_B(l_1(c)),\ \nabla_a b=\operatorname{pr}_B[\![(a,0),(0,b)]\!],\ \nabla_a c=\nabla_{(a,0)}c,\\ &R_{AB}(a_1,a_2)b=l_3(a_1,a_2,b)\end{aligned}\tag{41}$$

and the B-representation is given by

$$\begin{aligned}&\partial_A(c)=-\operatorname{pr}_A(l_1(c)),\ \nabla_b a=\operatorname{pr}_A[\![(0,b),(a,0)]\!],\ \nabla_b c=\nabla_{(0,b)}c,\\ &R_{BA}(b_1,b_2)a=-l_3(b_1,b_2,a).\end{aligned}\tag{42}$$

Proof. Assume first that $(A\oplus B)\oplus C$ is a split Lie 2-algebroid with (1) and (2). The bracket $[\cdot\,,\cdot]\colon\Gamma(A)\times\Gamma(A)\to\Gamma(A)$ defined by $[\![(a_1,0),(a_2,0)]\!]=([a_1,a_2],0)$ is obviously skew-symmetric and $\mathbb{R}$-bilinear. Define an anchor ρ_A on A by $\rho_A(a)=\rho_{A\oplus B}(a,0)$. Then we get immediately

$$([a_1,fa_2],0)=[\![(a_1,0),f(a_2,0)]\!]=f([a_1,a_2],0)+\rho_{A\oplus B}(a_1,0)(f)(a_2,0),$$

which shows that $[a_1,fa_2]=f[a_1,a_2]+\rho_A(a_1)(f)a_2$ for all $a_1,a_2\in\Gamma(A)$. Further, we find $\operatorname{Jac}_{[\cdot\,,\cdot]}(a_1,a_2,a_3)=\operatorname{pr}_A(\operatorname{Jac}_{[\![\cdot\,,\cdot]\!]}((a_1,0),(a_2,0),(a_3,0)))=-(\operatorname{pr}_A\circ l_1\circ l_3)((a_1,0),(a_2,0),(a_3,0)))=0$ since l_3 vanishes on sections of A. Hence A is a wide subalgebroid of the split Lie 2-algebroid. In a similar manner, we find a Lie algebroid structure on B. Next we prove that (41) defines a 2-representation of A. Using (ii) in Definition 5.7 we find for $a\in\Gamma(A)$ and $c\in\Gamma(C)$ that

$$\begin{aligned}\partial_B(\nabla_a c)&=(\operatorname{pr}_B\circ l_1)(\nabla_{(a,0)}c)\overset{\text{(ii)}}{=}\operatorname{pr}_B[\![(a,0),l_1(c)]\!]\\&=\operatorname{pr}_B[\![(a,0),(0,\operatorname{pr}_B(l_1(c)))]\!]=\nabla_a(\partial_B c).\end{aligned}$$

In the third equation we have used Condition (1) and in the last equation the definitions of ∂_B and $\nabla_a\colon\Gamma(B)\to\Gamma(B)$. In the following, we will write for simplicity a for $(a,0)\in\Gamma(A\oplus B)$, etc. We get easily

$$R_{AB}(a_1,a_2)\partial_B c=l_3(a_1,a_2,\operatorname{pr}_B(l_1(c)))=l_3(a_1,a_2,l_1(c))\overset{\text{(iv)}}{=}R_\nabla(a_1,a_2)c$$

and

$$\partial_B R_{AB}(a_1,a_2)b=(\operatorname{pr}_B\circ l_1\circ l_3)(a_1,a_2,b)\overset{\text{(iii)}}{=}-\operatorname{pr}_B(\operatorname{Jac}_{[\![\cdot\,,\cdot]\!]}(a_1,a_2,b))$$

for all $a_1, a_2 \in \Gamma(A)$, $b \in \Gamma(B)$ and $c \in \Gamma(C)$. By Condition (1) and the definition of $\nabla_a \colon \Gamma(B) \to \Gamma(B)$, we find

$$\begin{aligned} R_\nabla(a_1, a_2)b &= \operatorname{pr}_B [\![a_1, [\![a_2, b]\!]]\!] - \operatorname{pr}_B [\![a_2 [\![a_1, b]\!]]\!] - \operatorname{pr}_B [\![[\![a_1, a_2]\!], b]\!] \\ &= - \operatorname{pr}_B(\operatorname{Jac}_{[\![\cdot\,,\cdot]\!]}(a_1, a_2, b)). \end{aligned}$$

Hence, $\partial_B R_{AB}(a_1, a_2)b = R_\nabla(a_1, a_2)b$. Finally, an easy computation along the same lines shows that

$$\langle (\mathbf{d}_{\nabla^{\mathrm{Hom}}} R_{AB})(a_1, a_2, a_3), b\rangle = (\mathbf{d}_\nabla l_3)(a_1, a_2, a_3, b) \tag{43}$$

for $a_1, a_2, a_3 \in \Gamma(A)$ and $b \in \Gamma(B)$. Since $\mathbf{d}_\nabla l_3 = 0$, we find $\mathbf{d}_{\nabla^{\mathrm{Hom}}} R_{AB} = 0$. In a similar manner, we prove that (42) defines a 2-representation of B. Further, by construction of the 2-representations, the split Lie 2-algebroid structure on $(A \oplus B) \oplus C)$ must be defined as in (38), (39) and (40), with the anchor $\rho_A \circ \operatorname{pr}_A + \rho_B \circ \operatorname{pr}_B$ and $l_1 = \partial_B \circ \operatorname{pr}_B - \partial_A \circ \operatorname{pr}_A$. Hence, to conclude the proof, it only remains to check that the split Lie 2-algebroid conditions for these objects are equivalent to the seven conditions in Definition 2.10 for the two 2-representations.

First, we find immediately that (M1) is equivalent to (i). Then we find by construction

$$[a, \partial_A c] + \nabla_{\partial_B c} a = -[a, \operatorname{pr}_A(l_1(c))] + \nabla_{\operatorname{pr}_B(l_1(c))} a = \operatorname{pr}_A [\![l_1(c), a]\!] = - \operatorname{pr}_A [\![a, l_1(c)]\!].$$

Hence, we find (M2) if and only if $\operatorname{pr}_A [\![a, l_1(c)]\!] = \operatorname{pr}_A \circ l_1(\nabla_a c)$. But since

$$\begin{aligned} [\![a, l_1 c]\!] &= (\operatorname{pr}_A [\![a, l_1(c)]\!], \nabla_a \operatorname{pr}_B l_1(c)) = (\operatorname{pr}_A [\![a, l_1(c)]\!], \nabla_a \partial_B(c)) \\ &= (\operatorname{pr}_A [\![a, l_1(c)]\!], \partial_B \nabla_a c) = (\operatorname{pr}_A [\![a, l_1(c)]\!], \operatorname{pr}_B(l_1(\nabla_a c))), \end{aligned}$$

we have $\operatorname{pr}_A [\![a, l_1(c)]\!] = \operatorname{pr}_A \circ l_1(\nabla_a c)$ if and only if $[\![a, l_1 c]\!] = l_1(\nabla_a c)$. Hence (M2) is satisfied if and only if $[\![a, l_1 c]\!] = l_1(\nabla_a c)$ for all $a \in \Gamma(A)$ and $c \in \Gamma(C)$. In a similar manner, we find that (M3) is equivalent to $[\![b, l_1 c]\!] = l_1(\nabla_b c)$ for all $b \in \Gamma(B)$ and $c \in \Gamma(C)$. This shows that (M2) and (M3) together are equivalent to (ii).

Next, a simple computation shows that (M4) is equivalent to $R_\nabla(b, a)c = l_3(b, a, l_1(c))$. Since $R_\nabla(a, a')c = R_{AB}(a, a')\partial_B c = l_3(a, a', \operatorname{pr}_B(l_1(c))) = l_3(a, a', l_1(c))$ and $R_\nabla(b, b')c = l_3(b, b', l_1(c))$, we get that (M4) is equivalent to (iv).

Two straightwforward computations show that (M5) is equivalent to

$$\operatorname{pr}_A(\operatorname{Jac}_{[\![\cdot\,,\cdot]\!]}(a_1, a_2, b)) = - \operatorname{pr}_A(l_1 l_3(a_1, a_2, b))$$

and that (M6) is equivalent to

$$\operatorname{pr}_B(\operatorname{Jac}_{[\![\cdot\,,\cdot]\!]}(b_1, b_2, a)) = - \operatorname{pr}_A(l_1 l_3(b_1, b_2, a)).$$

But since $\operatorname{pr}_B(\operatorname{Jac}_{[\![\cdot\,,\cdot]\!]}(a_1, a_2, b)) = -R_\nabla(a_1, a_2)b$ by construction and $R_\nabla(a_1, a_2)b = \partial_B R_{AB}(a_1, a_2)b = \operatorname{pr}_B(l_1 l_3(a_1, a_2, b))$, we find

$$\operatorname{pr}_B(\operatorname{Jac}_{[\![\cdot\,,\cdot]\!]}(a_1, a_2, b)) = - \operatorname{pr}_B(l_1 l_3(a_1, a_2, b)),$$

and in a similar manner

$$\operatorname{pr}_A(\operatorname{Jac}_{[\![\cdot\,,\cdot]\!]}(b_1, b_2, a)) = - \operatorname{pr}_A(l_1 l_3(b_1, b_2, a)).$$

Since $\operatorname{Jac}_{[\![\cdot\,,\cdot]\!]}(a_1, a_2, a_3) = 0$, $\operatorname{Jac}_{[\![\cdot\,,\cdot]\!]}(b_1, b_2, b_3) = 0$, and l_3 vanishes on sections of A, and respectively on sections of B, we conclude that (M5) and (M6) together are equivalent to (iii).

Finally, a slightly longer, but still straightforward computation shows that

$$(\mathbf{d}_{\nabla^A} R_{BA})(a_1, a_2)(b_1, b_2) - (\mathbf{d}_{\nabla^B} R_{AB})(b_1, b_2)(a_1, a_2) = (\mathbf{d}_\nabla l_3)(a_1, a_2, b_1, b_2)$$

for all $a_1, a_2 \in \Gamma(A)$ and $b_1, b_2 \in \Gamma(B)$. This, (43) and a similar identity for R_{BA}, and the vanishing of l_3 on sections of A, and respectively on sections of B, show that (M7) is equivalent to (v). □

If $C = 0$, then $R_{AB} = 0$ and $R_{BA} = 0$ and the matched pair of 2-representations is just a matched pair of Lie algebroids. The double is then concentrated in degree 0, with $l_3 = 0$ and l_2 is the bicrossproduct Lie algebroid structure on $A \oplus B$ with anchor $\rho_A + \rho_B$ [23, 30]. Hence, in that case the split Lie 2-algebroid is just the bicrossproduct of a matched pair of representations and the dual (flat) Dorfman connection is the corresponding Lie derivative. The Lie 2-algebroid is in that case a genuine Lie 1-algebroid.

It would be interesting to define the notion of matched pair of higher representations up to homotopy, and show that the induced doubles are split Lie n-algebroids. This will be studied in a subsequent project.

In the case where B is a trivial Lie algebroid and acts trivially up to homotopy on $\partial_A = 0\colon C \to A$, the double is the semi-direct product Lie 2-algebroid found in [37, Proposition 3.5] (see our preceding example).

5.6. **Morphisms of (split) Lie 2-algebroids.**

In this section we quickly discuss morphisms of split Lie 2-algebroids.

Definition 5.10. *A morphism $\mu\colon (\mathcal{M}_1, \mathcal{Q}_1) \to (\mathcal{M}_2, \mathcal{Q}_2)$ of Lie 2-algebroids is a morphism $\mu\colon \mathcal{M}_1 \to \mathcal{M}_2$ of the underlying $[2]$-manifolds, such that*

$$\mu^\star \circ \mathcal{Q}_2 = \mathcal{Q}_1 \circ \mu^\star \colon C^\infty(\mathcal{M}_2) \to C^\infty(\mathcal{M}_1). \tag{44}$$

Assume that the two [2]-manifolds $\mathcal{M}_1$ and $\mathcal{M}_2$ are split [2]-manifold $\mathcal{M}_1 = Q_1[-1] \oplus B_1^*[-2]$ and $\mathcal{M}_2 = Q_2[-1] \oplus B_2^*[-2]$. Then the homological vector fields $\mathcal{Q}_1$ and $\mathcal{Q}_2$ can be written in coordinates as in (34), with two Dorfman 2-representations; $(\partial_1\colon Q_1^* \to B_1, \Delta^1, \nabla^1, R_1)$ of $(Q_1 \to M_1, \rho_1\colon Q_1 \to TM_1)$ and $(\partial_2\colon Q_2^* \to B_2, \Delta^2, \nabla^2, R_2)$ of $(Q_2 \to M_2, \rho_2\colon Q_2 \to TM_2)$.

Since $\mathcal{M}_1$ and $\mathcal{M}_2$ are split, the morphism $\mu^\star\colon C^\infty(\mathcal{M}_2) \to C^\infty(\mathcal{M}_1)$ over $\mu_0^*\colon C^\infty(M_2) \to C^\infty(M_1)$ decomposes as well as $\mu_Q\colon Q_1 \to Q_2$, $\mu_B\colon B_1^* \to B_2^*$ and $\mu_{12} \in \Omega^2(Q_1, \mu_0^* B_2^*)$ as in (18) and (19).

A study of (44) in coordinates, which we leave to the reader, shows that (44) is equivalent to

(1) $\mu_Q\colon Q_1 \to Q_2$ over $\mu_0\colon M_1 \to M_2$ is compatible with the anchors $\rho_1\colon Q_1 \to TM_1$ and $\rho_2\colon Q_2 \to TM_2$:
$$T_m\mu_0(\rho_1(q_m)) = \rho_2(\mu_Q(q_m))$$
for all $q_m \in Q_1$,

(2) $\partial_1 \circ \mu_Q^\star = \mu_B^\star \circ \partial_2$ as maps from $\Gamma(Q_2^*)$ to $\Gamma(B_1)$, or in other words $\mu_Q \circ \partial_1^* = \partial_2^* \circ \mu_B$.

(3) μ_Q preserves the dull brackets up to $\partial_2^*\mu_{12}$: i.e. if $q^1 \sim_{\mu_Q} r^1$ and $q^2 \sim_{\mu_Q} r^2$, then
$$[\![q^1, r^1]\!]_1 \sim_{\mu_Q} [\![q^2, r^2]\!]_2 - \partial_2^*\mu_{12}(q^1, r^1).$$

(4) μ_B and μ_Q intertwines the connections ∇^1 and ∇^2 up to $\partial_1 \circ \mu_{12}$:
$$\mu_B^\star((\mu_Q^\star\nabla^2)_q b) = \nabla^1_q(\mu_B^\star(b)) - \partial_1 \circ \langle \mu_{12}(q, \cdot), b\rangle \in \Gamma(B_1)$$
for all $q_m \in Q_1$ and $b \in \Gamma(B^2)$, and

(5) $\mu_Q^\star \omega_{R_2} - \mu_B \circ \omega_{R_1} = -\mathbf{d}_{(\mu_Q^\star\nabla^2)}\mu_{12} \in \Omega^3(Q_1, \mu_0^* B_2^*)$.

In the equalities above we have used the following constructions. Recall that μ_{12} is an element of $\Omega^2(Q_1, \mu_0^* B_2^*)$, and $\omega_{R_i} \in \Omega^3(Q_i, B_i^*)$ for $i = 1, 2$. The tensors $\mu_Q^\star \omega_{R_2} \in \Omega^2(Q_1, \mu_0^* B_2^*)$ and $\mu_B \circ \omega_{R_1} \in \Omega^2(Q_1, \mu_0^* B_2^*)$ can be defined as follows:

$$(\mu_Q^\star \omega_{R_2})(q_1(m), q_2(m), q_3(m)) = \omega_{R_2}(\mu_Q(q_1(m)), \mu_Q(q_2(m)), \mu_Q(q_3(m)))$$

in $B_2^*(\mu_0(m))$, and

$$(\mu_B \circ \omega_{R_1})(q_1(m), q_2(m), q_3(m)) = \mu_B(\omega_{R_1})(q_1(m), q_2(m), q_3(m))) \in B_2^*(\mu_0(m))$$

for all $q_1, q_2, q_3 \in \Gamma(Q_1)$. The linear connection

$$\mu_Q^\star \nabla^2 \colon \Gamma(Q_1) \times \Gamma(\mu_0^* B_2^*) \to \Gamma(\mu_0^* B_2^*)$$

is defined by

$$(\mu_Q^\star \nabla^2)_q(\mu_0^! \beta)(m) = \nabla^2_{\mu_Q(q(m))} \beta \in B_2^*(\mu_0(m))$$

for all $q \in \Gamma(Q_1)$ and $\beta \in \Gamma(B_2^*)$.

Definition 5.11. *We call a triple (μ_Q, μ_B, μ_{12}) over μ_0 satisfying the five conditions above a* ***morphism of split Lie 2-algebroids****.*

In particular, if $\mathcal{M}_1 = \mathcal{M}_2$, $\mu_0 = \mathrm{Id}_M \colon M \to M$, $\mu_Q = \mathrm{Id}_Q \colon Q \to Q$ and $\mu_B = \mathrm{Id}_{B^*} \colon B^* \to B^*$, then $\mu_{12} \in \Gamma(Q^* \wedge Q^* \otimes B^*)$ is just a change of splitting The five conditions above simplify to

1. The dull brackets are related by $[\![q, q']\!]_2 = [\![q, q']\!]_1 + \partial_B^* \mu_{12}(q, q')$.
2. The connections are related by $\nabla_q^2 = \nabla_q^1 - \partial_B \circ \mu_{12}(q)$.
3. The curvature terms are related by $\omega_{R^2} - \omega_{R^1} = -\mathbf{d}_{\nabla^{2*}} \mu_{12}$, where the Cartan differential $\mathbf{d}_{\nabla^{2*}}$ is defined with the dull bracket $[\![\cdot\, , \cdot]\!]_1$ on $\Gamma(Q)$.

6. VB-Courant algebroids and Lie 2-algebroids

In this section we describe and prove in detail the equivalence between VB-Courant algebroids and Lie 2-algebroids. In short, a homological vector field on a [2]-manifold defines an anchor and a Courant bracket on the corresponding metric double vector bundle. This Courant bracket and this anchor are automatically compatible with the metric and define so a linear Courant algebroid structure on the double vector bundle. Be aware that a correspondence of Lie 2-algebroids and VB-Courant algebroids has already been discussed by Li-Bland [19]. Our goal is to make this result more constructive, to deduce it from the two preceding sections, and to illustrate it with several (partly new) examples.

6.1. Definition and observations.

We will work with the following definition of a VB-Courant algebroid, which is basically due to Li-Bland [19].

Definition 6.1. *A VB-Courant algebroid is a metric double vector bundle*

$$\begin{tikzcd}
\mathbb{E} \arrow[r, "\pi_B"] \arrow[d, "\pi_Q"'] & B \arrow[d, "q_B"] \\
Q \arrow[r, "q_Q"'] & M
\end{tikzcd}$$

with core Q^* such that $\mathbb{E} \to B$ is a Courant algebroid and the following conditions are satisfied.

(1) *The anchor map* $\Theta\colon \mathbb{E} \to TB$ *is linear. That is,*

$$\begin{tikzcd}
\mathbb{E} \arrow[rr, "\pi_B"] \arrow[dd, "\pi_Q"'] & & B \arrow[dd, "q_B"] & & & TB \arrow[rr, "p_B"] \arrow[dd, "Tq_B"'] & & B \arrow[dd, "q_B"] \\
& C \arrow[dr] & & {} \arrow[r, "\Theta"] & {} & & B \arrow[dr] & \\
Q \arrow[rr, "q_Q"'] & & M & & & TM \arrow[rr, "p_M"'] & & M
\end{tikzcd} \tag{45}$$

is a morphism of double vector bundles.

(2) *The Courant bracket is linear. That is*

$$[\![\Gamma^l_B(\mathbb{E}), \Gamma^l_B(\mathbb{E})]\!] \subseteq \Gamma^l_B(\mathbb{E}), \qquad [\![\Gamma^l_B(\mathbb{E}), \Gamma^c_B(\mathbb{E})]\!] \subseteq \Gamma^c_B(\mathbb{E}),$$
$$[\![\Gamma^c_B(\mathbb{E}), \Gamma^c_B(\mathbb{E})]\!] = 0.$$

We make the following observations. Let $\rho_Q\colon Q \to TM$ be the side map of the anchor, i.e. if $\pi_Q(\chi) = q$ for $\chi \in \mathbb{E}$, then $Tq_B(\Theta(\chi)) = \rho_Q(q)$. Let $\partial_B\colon Q^* \to B$ be the core map defined as follows by the anchor Θ:

$$\Theta(\sigma^\dagger) = (\partial_B \sigma)^\uparrow \tag{46}$$

for all $\sigma \in \Gamma(Q^*)$ (∂_B is sometimes called the "core-anchor"). Then the operator $\mathcal{D} = \Theta^* \mathbf{d}\colon C^\infty(B) \to \Gamma_B(\mathbb{E})$ satisfies $\mathcal{D}(q_B^* f) = (\rho_Q^* \mathbf{d} f)^\dagger$ for all $f \in C^\infty(M)$ and (29) yields immediately

$$\partial_B \circ \rho_Q^* = 0, \quad \text{which is equivalent to} \quad \rho_Q \circ \partial_B^* = 0. \tag{47}$$

Recall finally that if $\chi \in \Gamma^l_B(\mathbb{E})$ is linear over $q \in \Gamma(Q)$, then $\langle \chi, \tau^\dagger \rangle = q_B^* \langle q, \tau \rangle$ for all $\tau \in \Gamma(Q^*)$ and $\rho(\chi)$ is linear over $\rho_Q(q)$.

6.2. **The fat Courant algebroid.** Recall from (7) that there exists a vector bundle $\widehat{\mathbb{E}} \to M$, which sheaf of sections is the sheaf of $C^\infty(M)$-modules $\Gamma^l_B(\mathbb{E})$, the linear sections of $\mathbb{E}$ over B. Gracia-Saz and Mehta show in [13] that if $\mathbb{E}$ is endowed with a linear Lie algebroid structure over B, then $\widehat{\mathbb{E}} \to M$ inherits a Lie algebroid structure, which is called the "fat Lie algebroid". For completeness, we describe here quickly the counterpart of this in the case of a linear Courant algebroid structure on $\mathbb{E} \to B$.

Note that the restriction of the pairing on $\mathbb{E}$ to linear sections of $\mathbb{E}$ defines a nondegenerate pairing on $\widehat{\mathbb{E}}$ with values in B^*. Since the Courant bracket of linear sections is again linear, we get the following theorem.

Theorem 6.2. *The vector bundle* $\widehat{\mathbb{E}}$ *inherits a Courant algebroid structure with pairing in* B^*.

We will come back later to this structure. Recall that for $\phi \in \Gamma(\operatorname{Hom}(B, Q^*))$, the core-linear section $\widetilde{\phi}$ of $\mathbb{E} \to B$ is defined by

$$\widetilde{\phi}(b_m) = 0_{b_m} +_B \overline{\phi(b_m)}.$$

Note that $\widehat{\mathbb{E}}$ is also naturally paired with Q^*: $\langle \chi(m), \sigma(m) \rangle = \langle \pi_Q(\chi(m)), \sigma(m) \rangle$ for all $\chi \in \Gamma^l_B(\mathbb{E}) = \Gamma(\widehat{\mathbb{E}})$ and $\sigma \in \Gamma(Q^*)$. This pairing is degenerate since it restricts to 0 on $\operatorname{Hom}(B, Q^*) \times_M Q^*$. The following proposition can easily be proved.

Proposition 6.3. (1) *The map*

$$\Delta\colon \Gamma(\widehat{\mathbb{E}})\times\Gamma(Q^*)\to\Gamma(Q^*) \quad \textit{defined by} \quad (\Delta_\chi\tau)^\dagger = [\![\chi,\tau^\dagger]\!]$$

is a flat Dorfman connection, where $\widehat{\mathbb{E}}$ is endowed with the anchor $\rho_Q\circ\pi_Q$ and paired with Q^* as above.

(2) *The map*

$$\nabla^*\colon \Gamma(\widehat{\mathbb{E}})\times\Gamma(B^*)\to\Gamma(B^*) \quad \textit{defined by} \quad \ell_{\nabla^*_\chi\beta} = \Theta(\chi)(\ell_\beta)$$

for all $\beta\in\Gamma(B^*)$ is a flat connection.

We dualise the connection ∇^* to a flat connection $\nabla\colon\Gamma(\widehat{\mathbb{E}})\times\Gamma(B)\to\Gamma(B)$.

Proposition 6.4. *The following hold for Δ and ∇:*

1. $\partial_B\circ\Delta=\nabla\circ\partial_B$ *and*
2. $\left[\!\left[\chi,\widetilde{\phi}\right]\!\right]_{\widehat{\mathbb{E}}} = \widetilde{\phi\circ\nabla_\chi-\Delta_\chi\circ\phi}$

for $\chi\in\Gamma(\widehat{\mathbb{E}})$ and $\phi\in\Gamma(\operatorname{Hom}(B,Q^*))$.

Proof. (1) Choose $\chi\in\Gamma^l_B(\mathbb{E})$ and $\tau\in\Gamma(Q^*)$. Then

$$(\partial_B\circ\Delta_\delta\tau)^\uparrow = \Theta(\Delta_\chi\tau^\dagger) = \Theta\left([\![\chi,\tau^\dagger]\!]\right) = \left[\Theta(\chi),(\partial_B\tau)^\uparrow\right] = (\nabla_\chi(\partial_B\tau))^\uparrow.$$

(2) The second equation is easy to check by writing $\widetilde{\phi}=\sum_{i=1}^n\ell_{\beta_i}\cdot\tau_i^\dagger$ with $\beta_i\in\Gamma(B^*)$ and $\tau_i\in\Gamma(Q^*)$.

□

Lemma 6.5. *For $\phi,\psi\in\Gamma(\operatorname{Hom}(B,Q^*))$ and $\tau\in\Gamma(Q^*)$, we have*

1. $\left[\!\left[\tau^\dagger,\widetilde{\phi}\right]\!\right] = (\phi(\partial_B\tau))^\dagger = -\left[\!\left[\widetilde{\phi},\tau^\dagger\right]\!\right]$ *and*
2. $\left[\!\left[\widetilde{\phi},\widetilde{\psi}\right]\!\right] = \widetilde{\psi\circ\partial_B\circ\phi-\phi\circ\partial_B\circ\psi}.$

Remark 6.6. Note that by the second equality, $\operatorname{Hom}(B,Q^*)$ has the structure of a Lie algebra bundle.

Proof. We write $\phi=\sum_{i=1}^n\beta_i\cdot\tau_i$ and $\psi=\sum_{j=1}^n\beta'_j\cdot\tau_j$ with $\beta_1,\dots,\beta_n,\beta'_1,\dots,\beta'_n\in\Gamma(B^*)$ and $\tau_1,\dots,\tau_n\in\Gamma(Q^*)$. Hence, we have $\widetilde{\phi}=\sum_{i=1}^n\ell_{\beta_i}\tau_i^\dagger$ and $\widetilde{\psi}=\sum_{j=1}^n\ell_{\beta'_j}\tau_j^\dagger$. First we compute

$$\begin{aligned}\left[\!\left[\tau^\dagger,\sum_{i=1}^n\ell_{\beta_i}\tau_i^\dagger\right]\!\right] &= \sum_{i=1}^n(\partial_B\tau)^\uparrow(\ell_{\beta_i})\tau_i^\dagger\\ &= \sum_{i=1}^n q_B^*\langle\partial_B\tau,\beta_i\rangle\tau_i^\dagger = \left(\sum_{i=1}^n\langle\partial_B\tau,\beta_i\rangle\tau_i\right)^\dagger\end{aligned}$$

and we get (1). Since $\langle\tau^\dagger,\widetilde{\phi}\rangle=0$, the second equality follows.

Then we have

$$\begin{aligned}\left[\!\left[\sum_{i=1}^n\ell_{\beta_i}\tau_i^\dagger,\sum_{j=1}^n\ell_{\beta'_j}\tau_j^\dagger\right]\!\right] &= \sum_{i=1}^n\sum_{j=1}^n\ell_{\beta_i}(\partial_B\tau_i)^\uparrow(\ell_{\beta'_j})\tau_j^\dagger-\ell_{\beta'_j}(\partial_B\tau_j)^\uparrow(\ell_{\beta_i})\tau_i^\dagger\\ &= \left(\sum_{i=1}^n\sum_{j=1}^n\langle\partial_B\tau_i,\beta'_j\rangle\cdot\beta_i\cdot\tau_j-\langle\partial_B\tau_j,\beta_i\rangle\cdot\beta'_j\cdot\tau_i\right)^\dagger.\end{aligned}$$

Thus, we get (2).

$\square$

6.3. **Dorfman 2-representations and Lagrangian decompositions of VB-algebroids.** In this section, we study in detail the structure of a VB-Courant algebroid, using Lagrangian decompositions of the underlying metric double vector bundle. Our goal is the following theorem. Note the similarity of this result with Theorem 2.6 in the VB-algebroid case.

Theorem 6.7. *Let* $(\mathbb{E}; Q, B; M)$ *be a VB-Courant algebroid and choose a Lagrangian splitting* $\Sigma\colon Q \times_M B \to \mathbb{E}$*. Then there exists a Dorfman 2-representation* (Δ, ∇, R) *of* (Q, ρ_Q) *on the core-anchor* $\partial_B\colon Q^* \to B$ *such that*

$$
\begin{aligned}
&\Theta(\sigma_Q(q)) = \widehat{\nabla_q} \in \mathfrak{X}(B),\\
&[\![\sigma_Q(q_1), \sigma_Q(q_2)]\!] = \sigma_Q([\![q_1, q_2]\!]_\Delta) - \widetilde{R(q_1, q_2)},\\
&[\![\sigma_Q(q), \tau^\dagger]\!] = (\Delta_q \tau)^\dagger
\end{aligned}
\tag{48}
$$

for all $q, q_1, q_2 \in \Gamma(Q)$ *and* $\tau \in \Gamma(Q^*)$*, where* $[\![\cdot\,,\cdot]\!]_\Delta\colon \Gamma(Q) \times \Gamma(Q) \to \Gamma(Q)$ *is the dull bracket that is dual to* Δ*.*

Conversely, a Lagrangian splitting $\Sigma\colon Q \times B^* \to \mathbb{E}$ *of the metric double vector bundle* $\mathbb{E}$ *together with a Dorfman 2-representation define a linear Courant algebroid structure on* $\mathbb{E}$ *by* (48)*.*

First we will construct the objects $[\![\cdot\,,\cdot]\!]_\Delta, \Delta, \nabla, R$ as in the theorem, and then we will prove in the appendix that they satisfy the axioms of a Dorfman 2-representation.

6.3.1. *Construction of the Dorfman 2-representation and outline of the proof.* The objects $[\![\cdot\,,\cdot]\!]_\Delta$, ∇, Δ and R are defined as in the theorem. Let us be more precise.

First recall that, by definition, the Courant bracket of two linear sections of $\mathbb{E} \to B$ is again linear. Hence, we can denote by $[\![q_1, q_2]\!]_\sigma$ the section of Q such that

$$
\pi_Q \circ [\![\sigma_Q(q_1), \sigma_Q(q_2)]\!] = [\![q_1, q_2]\!]_\sigma \circ q_B. \tag{49}
$$

Since for each $q \in \Gamma(Q)$, the anchor $\Theta(\sigma_Q(q))$ is a linear vector field on B over $\rho_Q(q) \in \mathfrak{X}(M)$, there exists a derivation $D_q\colon \Gamma(B^*) \to \Gamma(B^*)$ over $\rho_Q(q)$ such that $\Theta(\sigma_Q(q))(\ell_\beta) = \ell_{D_q\beta}$ for all $\beta \in \Gamma(B^*)$, and $\Theta(\sigma_Q(q))(q_B^* f) = q_B^*(\rho_Q(q)(f))$ for all $f \in C^\infty(M)$. This defines a linear Q-connection $\nabla\colon \Gamma(Q) \times \Gamma(B) \to \Gamma(B)$:

$$
\nabla_q b = D_q^* b
$$

for all $b \in \Gamma(B)$. Then by definition, $\Theta(\sigma_Q(q)) = \widehat{\nabla_q} \in \mathfrak{X}^l(B)$.

For $q \in \Gamma(Q)$ and $\tau \in \Gamma(Q^*)$, the bracket $\left[\!\left[\sigma_Q(q), \tau^\dagger\right]\!\right]$ is a core section. It is easy to check that the map $\Delta\colon \Gamma(Q) \times \Gamma(Q^*) \to \Gamma(Q^*)$ defined by

$$
\left[\!\left[\sigma_Q(q), \tau^\dagger\right]\!\right] = (\Delta_q \tau)^\dagger
$$

is a Dorfman connection.[14]

The difference of the two linear sections $[\![\sigma_Q(q_1), \sigma_Q(q_2)]\!] - \sigma_Q([\![q_1, q_2]\!]_\sigma)$ is again a linear section, which projects to 0 under π_Q. Hence, there exists a vector bundle morphism $R(q_1, q_2)\colon B \to Q^*$ such that $\sigma_Q([\![q_1, q_2]\!]_\sigma) - [\![\sigma_Q(q_1), \sigma_Q(q_2)]\!] = \widetilde{R(q_1, q_2)}$. This defines a map $R\colon \Gamma(Q) \times \Gamma(Q) \to \Gamma(\operatorname{Hom}(B, Q^*))$. We show in the appendix that $(\partial_B, \nabla, \Delta, R)$ is a Dorfman 2-representation, and that $[\![\cdot\,,\cdot]\!]_\Delta = [\![\cdot\,,\cdot]\!]_\sigma$.

---

[14] Note that Condition $(C3)$ then implies that $[\![\tau^\dagger, \sigma_Q(q)]\!] = \left(-\Delta_q \tau + \rho_Q^* \mathbf{d}\langle \tau, q\rangle\right)^\dagger$.

Conversely, choose a Lagrangian splitting $\Sigma\colon Q\times_M B$ of a metric double vector bundle $(\mathbb{E}, Q; B, M)$ with core Q^* and let $\mathcal{S}\subseteq\Gamma_B(\mathbb{E})$ be the subset $\{\tau^\dagger \mid \tau\in\Gamma(Q^*)\}\cup\{\sigma_Q(q)\mid q\in\Gamma(Q)\}\subseteq\Gamma(\mathbb{E})$. Choose a Dorfman 2-representation $(\partial_B\colon Q^*\to B, \nabla, \Delta, R)$ of (Q,ρ_Q). Define then a vector bundle map $\Theta\colon\mathbb{E}\to TB$ over the identity on B by $\Theta(\sigma_Q(q))=\widehat{\nabla_q}$ and $\Theta(\tau^\dagger)=(\partial_B\tau)^\dagger$ and a bracket $[\![\cdot\,,\cdot]\!]$ on $\mathcal{S}$ by

$$[\![\sigma_Q(q_1),\sigma_Q(q_2)]\!]=\sigma_Q([\![q_1,q_2]\!]_\Delta)-\widetilde{R(q_1,q_2)},\quad [\![\sigma_Q(q),\tau^\dagger]\!]=(\Delta_q\tau)^\dagger,$$
$$[\![\tau^\dagger,\sigma_Q(q)]\!]=\left(-\Delta_q\tau+\rho_Q^*\mathbf{d}\langle\tau,q\rangle\right)^\dagger,\quad [\![\tau_1^\dagger,\tau_2^\dagger]\!]=0.$$

We show in the appendix that this bracket, the pairing and the anchor satisfy the conditions of Lemma 5.2, and so that $(\mathbb{E}, B; Q, M)$ with this structure is a VB-Courant algebroid.

6.3.2. *Change of Lagrangian decomposition.* Next we study how the Dorfman 2-representation $(\partial_B\colon Q^*\to B,\nabla,\Delta,R,\Lambda)$ associated to a Lagrangian decomposition of a VB-Courant algebroid changes when one changes the Lagrangian decomposition.

The proof of the following proposition is straightforward and left to the reader. Compare this result with the equations subsequent to Definition 5.11, that describe a change of splittings of split Lie 2-algebroid.

Proposition 6.8. *Let $\Sigma^1,\Sigma^2\colon B\times_M Q\to\mathbb{E}$ be two Lagrangian splittings and let $\phi_{12}\in\Gamma(Q^*\otimes Q^*\otimes B^*)$ be the change of lift.*

1. *The Dorfman connections are related by $\Delta_q^2\sigma=\Delta_q^1\sigma+\phi_{12}(q)(\partial_B\sigma)$*
2. *and the dull brackets consequently by $[\![q,q']\!]_2=[\![q,q']\!]_1-\partial_B^*\phi_{12}(q)^*(q')$.*
3. *The connections are related by $\nabla_q^2=\nabla_q^1+\partial_B\circ\phi_{12}(q)$.*
4. *The curvature terms are related by $\omega_{R^2}-\omega_{R^1}=\mathbf{d}_{\nabla^{2*}}\phi_{12}$, where the Cartan differential $\mathbf{d}_{\nabla^{2*}}$ is defined with the dull bracket $[\![\cdot\,,\cdot]\!]_1$ on $\Gamma(Q)$.*

As an application, we get the following corollary of Theorem 6.7 and Theorem 6.2.

Corollary 6.9. *Let $(Q\oplus B^*\to M,\rho_Q,l_1,l_2,l_3)$ be a split Lie 2-algebroid and $(\partial_B=-l_1^*,\Delta,\nabla,R)$ the dual Dorfman 2-representation. Then the vector bundle $\mathsf{E}:=Q\oplus\operatorname{Hom}(B,Q^*)$ is a Courant algebroid with pairing in B^* given by $\langle(q_1,\phi_1),(q_2,\phi_2)\rangle=\phi_1^*(q_2)+\phi_2^*(q_1)$, with the anchor $\tilde\rho\colon\mathsf{E}\to\widehat{\operatorname{Der}(B)}$, $\tilde\rho(q,\phi)=\nabla_q^*+\phi^*\circ l_1$ over $\rho(q)$ and the bracket given by*

$$[\![(q_1,\phi_1),(q_2,\phi_2)]\!]=\Big([\![q_1,q_2]\!]_\Delta+l_1^*(\phi_1^*(q_2)),\Diamond_{q_1}\phi_2-\Diamond_{q_2}\phi_1+\nabla_\cdot^*(\phi_1^*(q_2))$$
$$+\phi_2\circ l_1^*\circ\phi_1-\phi_1\circ l_1^*\circ\phi_2+R(q_1,q_2)\Big).$$

The map $\mathcal{D}\colon\Gamma(B^*)\to\Gamma(\mathsf{E})$ sends q to $(l_1(q),\nabla_\cdot^*q)$. The bracket does not depend on the choice of splitting.

6.4. **Examples of VB-algebroids and the corresponding Dorfman 2-representations.** We give here some examples of VB-Courant algebroids, and we compute the corresponding classes of Dorfman 2-representations. We find the Dorfman 2-representations described in Section 5.4. In each of the examples below, it is easy to check that the Courant algebroid structure is linear. Hence, it is easy to check geometrically that the objects described in 5.4 are indeed split Lie 2-algebroids. This is why we omitted the detailed computations in that section.

6.4.1. *The standard Courant algebroid over a vector bundle.* We have discussed this example in great detail in [16], but not in the language of Dorfman 2-representations and Lie 2-algebroids. In [16], we worked with general, not necessarily Lagrangian, linear splittings.

Let $q_E\colon E \to M$ be a vector bundle and consider the VB-Courant algebroid

$$\begin{tikzcd}
TE \oplus T^*E \arrow[r, "\Phi_E := (q_{E_*}, r_E)"] \arrow[d, "\pi_E"'] & TM \oplus E^* \arrow[d] \\
E \arrow[r, "q_E"'] & M
\end{tikzcd}$$

with base E and side $TM \oplus E^* \to M$, and with core $E \oplus T^*M \to M$, or in other words the standard (VB-)Courant algebroid over a vector bundle $q_E\colon E \to M$. Recall that $TE \oplus T^*E$ was found in Example 3.12 to have a linear metric. Recall also from this example that linear splittings of $TE \oplus_E T^*E$ are in bijection with dull brackets on sections of $TM \oplus E^*$, and so also with Dorfman connections $\Delta\colon \Gamma(TM \oplus E^*) \times \Gamma(E \oplus T^*M) \to \Gamma(E \oplus T^*M)$, and that Lagrangian splittings of $TE \oplus_E T^*E$ are in bijection with skew-symmetric dull brackets on sections of $TM \oplus E^*$.

The anchor $\Theta = \operatorname{pr}_{TE}\colon TE \oplus T^*E \to TE$ restricts to the map $\partial_E = \operatorname{pr}_E\colon E \oplus T^*M \to E$ on the cores, and defines an anchor $\rho_{TM\oplus E^*} = \operatorname{pr}_{TM}\colon TM \oplus E^* \to TM$ on the side. In other words, the anchor of $(e,\theta)^\dagger$ is $e^\uparrow \in \mathfrak{X}^c(E)$ and if $\widetilde{(X,\epsilon)}$ is a linear section of $TE \oplus T^*E \to E$ over $(X,\epsilon) \in \Gamma(TM \oplus E^*)$, the anchor $\Theta(\widetilde{(X,\epsilon)})$ is linear over X.

Let $\iota_E\colon E \to E \oplus T^*M$ be the canonical inclusion. In [16] we prove the following result (for general linear splittings).

Theorem 6.10. *Choose* $q, q_1, q_2 \in \Gamma(TM \oplus E^*)$ *and* $\tau, \tau_1, \tau_2 \in \Gamma(E \oplus T^*M)$. *The Courant-Dorfman bracket on sections of* $TE \oplus T^*E \to E$ *is given by*

(3) $[\![\sigma(q), \tau^\dagger]\!] = (\Delta_q \tau)^\dagger$,

(4) $[\![\sigma(q_1), \sigma(q_2)]\!] = \sigma([\![q_1, q_2]\!]_\Delta) - R_\Delta\widetilde{(q_1, q_2)} \circ \iota_E$.

The anchor ρ *is described by*

(5) $\Theta(\sigma(q)) = \widehat{\nabla^*_q} \in \mathfrak{X}(E)$,

where $\nabla\colon \Gamma(TM \oplus E^*) \times \Gamma(E) \to \Gamma(E)$ *is defined by* $\nabla_q = \operatorname{pr}_E \circ \Delta_q \circ \iota_E$ *for all* $q \in \Gamma(TM \oplus E^*)$.

Hence, if we choose a Lagrangian splitting of $TE \oplus_E T^*E$, we find the Dorfman 2-representation of Example 5.4.2.

6.4.2. *VB-Courant algebroid defined by a VB-Lie algebroid.* More generally, let

$$\begin{tikzcd}
D \arrow[r, "\pi_B"] \arrow[d, "\pi_A"'] & B \arrow[d, "q_B"] \\
A \arrow[r, "q_A"'] & M
\end{tikzcd}$$

with core C, be endowed with a VB-Lie algebroid structure $(D \to B, A \to M)$. Then the pair $(D, D \mathbin{\ast\!\!\!\mid} B)$ of vector bundles over B is a Lie bialgebroid, with $D \mathbin{\ast\!\!\!\mid} B$

endowed with the trivial Lie algebroid structure. We get a linear Courant algebroid $D \oplus_B (D \mathbin{\ast\!\!\!|} B)$ over B with side $A \oplus C^*$

$$\begin{tikzcd}
D \oplus_B (D \mathbin{\ast\!\!\!|} B) \arrow[r] \arrow[d] & B \arrow[d] \\
A \oplus C^* \arrow[r] & M
\end{tikzcd}$$

and core $C \oplus A^*$. We check that the Courant algebroid structure is linear. Let $\Sigma\colon A \times_M B \to D$ be a linear splitting of D. Recall from Lemma 2.3 that we can define a linear splitting of $D \mathbin{\ast\!\!\!|} B$ by $\Sigma^\star\colon B \times_M C^* \to D \mathbin{\ast\!\!\!|} B$, $\langle \Sigma^\star(b_m, \gamma_m), \Sigma(a_m, b_m)\rangle = 0$ and $\langle \Sigma^\star(b_m, \gamma_m), c^\dagger(b_m)\rangle = \langle \gamma_m, c(m)\rangle$ for all $b_m \in B$, $a_m \in A$, $\gamma_m \in C^*$ and $c \in \Gamma(C)$. The linear splitting $\tilde{\Sigma}\colon B \times_M (A \oplus C^*) \to D \oplus_B (D \mathbin{\ast\!\!\!|} B)$, $\tilde{\Sigma}(b_m, (a_m, \gamma_m)) = (\Sigma(a_m, b_m), \Sigma^\perp(b_m, \gamma_m))$ as in §4.4.2 is then a Lagrangian splitting by Lemma 2.3. A computation shows that the Courant bracket on $\Gamma_B(D \oplus_B (D \mathbin{\ast\!\!\!|} B))$ is given by

$$\begin{aligned}
&[\![\tilde{\sigma}_{A\oplus C^*}(a_1, \gamma_1), \tilde{\sigma}_{A\oplus C^*}(a_2, \gamma_2)]\!] \\
&= ([\sigma_A(a_1), \sigma_A(a_2)], \pounds_{\sigma_A(a_1)}\sigma^\star_{C^*}(\gamma_2) - \mathbf{i}_{\sigma_A(a_2)}\mathbf{d}\sigma^\star_{C^*}(\gamma_1)) \\
&= \Big(\sigma_A([a_1, a_2]) - \widetilde{R(a_1, a_2)}, \sigma^\star_{C^*}(\nabla^*_{a_1}\gamma_2 - \nabla^*_{a_2}\gamma_1) - \langle \gamma_2, \widetilde{R(a_1, \cdot)}\rangle + \langle \gamma_1, \widetilde{R(a_2, \cdot)}\rangle\Big) \\
&\left[\!\!\left[\tilde{\sigma}_{A\oplus C^*}(a, \gamma), (c, \alpha)^\dagger\right]\!\!\right] = \left(\nabla_a c^\dagger, (\pounds_a \alpha + \langle \nabla^*_{\cdot}\gamma, c\rangle)^\dagger\right) \\
&\left[\!\!\left[(c_1, \alpha_1)^\dagger, (c_2, \alpha_2)^\dagger\right]\!\!\right] = 0,
\end{aligned}$$

and the anchor of $D \oplus_B (D \mathbin{\ast\!\!\!|} B)$ is defined by

$$\Theta(\tilde{\sigma}_{A\oplus C^*}(a, \gamma)) = \Theta(\sigma_A(a)) = \widehat{\nabla_a} \in \mathfrak{X}^l(B), \quad \Theta((c, \alpha)^\dagger) = (\partial_B c)^\uparrow \in \mathfrak{X}^c(B),$$

where $(\partial_B\colon C \to B, \nabla\colon \Gamma(A) \times \Gamma(B) \to \Gamma(B), \nabla\colon \Gamma(A) \times \Gamma(C) \to \Gamma(C), R)$ is the 2-representation of A associated to the splitting $\Sigma\colon A \times_M B \to D$ of the VB-algebroid $(D \to B, A \to M)$. Hence, we have found the split Lie 2-algebroid described in Example 5.4.4.

6.4.3. *The tangent Courant algebroid.* We consider here a Courant algebroid $(\mathsf{E}, \rho_{\mathsf{E}}, [\![\cdot\,,\cdot]\!], \langle\cdot\,,\cdot\rangle)$. In this example, E will always be anchored by the Courant algebroid anchor map ρ_{E} and paired with itself by $\langle\cdot\,,\cdot\rangle$ and $\mathcal{D} = \boldsymbol{\beta}^{-1} \circ \rho_{\mathsf{E}}^* \circ \mathbf{d}\colon C^\infty(M) \to \Gamma(\mathsf{E})$. Note that $[\![\cdot\,,\cdot]\!]$ is not a dull bracket.

We show that, after the choice of a metric connection on E and so of a Lagrangian splitting $\Sigma^\nabla\colon TM \times_M \mathsf{E} \to T\mathsf{E}$ (see Example 3.11 and §2.2.2), the VB-Courant algebroid structure on $(T\mathsf{E} \to TM, \mathsf{E} \to M)$ is equivalent to the Dorfman 2-representation defined by ∇ as in Example 5.4.3.

Theorem 6.11. *Choose a linear connection $\nabla\colon \mathfrak{X}(M) \times \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E})$ that preserves the pairing on E. The Courant algebroid structure on $T\mathsf{E} \to TM$ can be described as follows:*

(1) *The pairing is given by*

$$\left\langle e_1^\dagger, e_2^\dagger\right\rangle = 0, \quad \left\langle \sigma_{\mathsf{E}}^\nabla(e_1), e_2^\dagger\right\rangle = p_M^*\langle e_1, e_2\rangle, \qquad \textit{and } \left\langle \sigma_{\mathsf{E}}^\nabla(e_1), \sigma_{\mathsf{E}}^\nabla(e_2)\right\rangle = 0,$$

(2) *the anchor is given by*

$$\Theta(\sigma_{\mathsf{E}}^\nabla(e)) = \widehat{\nabla_e^{\mathrm{bas}}\cdot} \textit{ and } \Theta(e^\dagger) = (\rho_{\mathsf{E}}(e))^\uparrow,$$

(3) *and the bracket is given by*

$$\left\llbracket e_1^\dagger, e_2^\dagger\right\rrbracket = 0, \qquad \left\llbracket \sigma_{\mathsf{E}}^\nabla(e_1), e_2^\dagger\right\rrbracket = (\Delta_{e_1}e_2)^\dagger$$

and

$$\llbracket \sigma_{\mathsf{E}}^\nabla(e_1), \sigma_{\mathsf{E}}^\nabla(e_2)\rrbracket = \sigma_{\mathsf{E}}^\nabla(\llbracket e_1, e_2\rrbracket_\Delta) - \widetilde{R_\Delta^{\mathrm{bas}}(e_1, e_2)}$$

for all $e, e_1, e_2 \in \Gamma(\mathsf{E})$.

Proof. We use the characterisation of the tangent Courant algebroid in [4] (see also [19]): the pairing has already been discussed in Example 3.11 and §4.4.1. It is given by $\langle Te_1, Te_2\rangle = \ell_{\mathbf{d}\langle e_1,e_2\rangle}$ and $\langle Te_1, e_2^\dagger\rangle = p_M^*\langle e_1, e_2\rangle$. The anchor is given by $\Theta(Te) = \widehat{\pounds_{\rho_{\mathsf{E}}(e)}} \in \mathfrak{X}(TM)$ and $\Theta(e^\dagger) = (\rho_{\mathsf{E}}(e))^\uparrow \in \mathfrak{X}(TM)$. The bracket is given by $\llbracket Te_1, Te_2\rrbracket = T\llbracket e_1, e_2\rrbracket$ and $\llbracket Te_1, e_2^\dagger\rrbracket = \llbracket e_1, e_2\rrbracket^\dagger$ for all $e, e_1, e_2 \in \Gamma(\mathsf{E})$.

(1) has already been checked in Example 3.11. We check (2), i.e. that the anchor satisfies $\Theta(\sigma_{\mathsf{E}}^\nabla(e)) = \widehat{\nabla_e^{\mathrm{bas}}}$. We have for $\theta \in \Omega^1(M)$ and $v_m \in TM$: $\Theta(\sigma_{\mathsf{E}}^\nabla(e)(v_m))(\ell_\theta) = \ell_{\pounds_{\rho(e)}\theta}(v_m) - \langle \theta_m, \rho_{\mathsf{E}}(\nabla_{v_m}e)\rangle = \ell_{\nabla^{\mathrm{bas}}{}_e^*\theta}(v_m)$ and for $f \in C^\infty(M)$: $\Theta(\sigma_{\mathsf{E}}^\nabla(e))(p_M^*f) = p_M^*(\rho_{\mathsf{E}}(e)f)$. This proves the equality.

Then we compute the brackets of our linear and core sections. Choose sections ϕ, ϕ' of $\mathrm{Hom}(TM, \mathsf{E})$. Then $\left\llbracket Te, \widetilde{\phi}\right\rrbracket = \widetilde{\pounds_e\phi}$, with $\pounds_e\phi \in \Gamma(\mathrm{Hom}(TM, \mathsf{E}))$ defined by $(\pounds_e\phi)(X) = \llbracket e, \phi(X)\rrbracket - \phi([\rho_{\mathsf{E}}(e), X])$ for all $X \in \mathfrak{X}(M)$. The equality $\left\llbracket \widetilde{\phi}, Te\right\rrbracket = -\widetilde{\pounds_e\phi} + \mathcal{D}\ell_{\langle\phi(\cdot),e\rangle}$ follows. For $\theta \in \Omega^1(M)$, we compute $\left\langle \mathcal{D}\ell_\theta, e^\dagger\right\rangle = \Theta(e^\dagger)(\ell_\theta) = p_M^*\langle\rho_{\mathsf{E}}(e), \theta\rangle$. Thus, $\mathcal{D}\ell_\theta = T(\boldsymbol{\beta}^{-1}\rho_{\mathsf{E}}^*\theta) + \widetilde{\psi}$ for a section $\psi \in \Gamma(\mathrm{Hom}(TM, \mathsf{E}))$ to be determined. Since $\langle\mathcal{D}\ell_\theta, Te\rangle = \Theta(Te)(\ell_\theta) = \ell_{\pounds_{\rho_{\mathsf{E}}(e)}\theta}$, the bracket $\langle T(\boldsymbol{\beta}^{-1}\rho_{\mathsf{E}}^*\theta) + \widetilde{\psi}, Te\rangle = \ell_{\mathbf{d}\langle\theta,\rho_{\mathsf{E}}(e)\rangle + \langle\psi(\cdot),e\rangle}$ must equal $\ell_{\pounds_{\rho_{\mathsf{E}}(e)}\theta}$, and we find $\langle\psi(\cdot), e\rangle = \mathbf{i}_{\rho_{\mathsf{E}}(e)}\mathbf{d}\theta$. Because $e \in \Gamma(\mathsf{E})$ was arbitrary we find $\psi(X) = -\boldsymbol{\beta}^{-1}\rho_{\mathsf{E}}^*\mathbf{i}_X\mathbf{d}\theta$ for $X \in \mathfrak{X}(M)$. We get in particular

$$\left\llbracket \widetilde{\phi}, Te\right\rrbracket = -\widetilde{\pounds_e\phi} + T(\boldsymbol{\beta}^{-1}\rho_{\mathsf{E}}^*\langle\phi(\cdot), e\rangle) - (\boldsymbol{\beta}^{-1}\rho_{\mathsf{E}}^*\widetilde{\mathbf{i}_X\mathbf{d}\langle\phi(\cdot), e\rangle}.$$

The formula $\left\llbracket \widetilde{\phi}, \widetilde{\phi'}\right\rrbracket = \phi' \circ \rho_{\mathsf{E}} \circ \widetilde{\phi - \phi} \circ \rho_{\mathsf{E}} \circ \phi'$ can easily be checked, as well as $\left\llbracket \widetilde{\phi}, e^\dagger\right\rrbracket = -\left\llbracket e^\dagger, \widetilde{\phi}\right\rrbracket = -(\phi(\rho_{\mathsf{E}}(e)))^\dagger$. Using this, we find now easily that

$$\begin{aligned}\llbracket \sigma_{\mathsf{E}}^\nabla(e_1), \sigma_{\mathsf{E}}^\nabla(e_2)\rrbracket &= \left\llbracket Te_1 - \widetilde{\nabla_\cdot e_1}, Te_2 - \widetilde{\nabla_\cdot e_2}\right\rrbracket \\ &= T\llbracket e_1, e_2\rrbracket - \widetilde{\pounds_{e_1}\nabla_\cdot e_2} + \widetilde{\pounds_{e_2}\nabla_\cdot e_1} - T(\boldsymbol{\beta}^{-1}\rho_{\mathsf{E}}^*\langle\nabla_\cdot e_1, e_2\rangle) \\ &\quad + \boldsymbol{\beta}^{-1}\rho_{\mathsf{E}}^*\widetilde{\mathbf{d}\langle\nabla_\cdot e_1, e_2\rangle} + \widetilde{\nabla_{\rho_{\mathsf{E}}(\nabla_\cdot e_1)}e_2} - \widetilde{\nabla_{\rho_{\mathsf{E}}(\nabla_\cdot e_2)}e_1} \\ &= T\llbracket e_1, e_2\rrbracket_\Delta - \widetilde{\pounds_{e_1}\nabla_\cdot e_2} + \widetilde{\pounds_{e_2}\nabla_\cdot e_1} + \boldsymbol{\beta}^{-1}\rho_{\mathsf{E}}^*\widetilde{\mathbf{d}\langle\nabla_\cdot e_1, e_2\rangle} \\ &\quad + \widetilde{\nabla_{\rho_{\mathsf{E}}(\nabla_\cdot e_1)}e_2} - \widetilde{\nabla_{\rho_{\mathsf{E}}(\nabla_\cdot e_2)}e_1}.\end{aligned}$$

Since for all $X \in \mathfrak{X}(M)$, we have

$$\begin{aligned}&-(\pounds_{e_1}\nabla_\cdot e_2)(X) + (\pounds_{e_2}\nabla_\cdot e_1)(X) + \beta^{-1}\rho_{\mathsf{E}}^*\mathbf{i}_X\mathbf{d}\langle\nabla_\cdot e_1, e_2\rangle \\ &= -\llbracket e_1, \nabla_X e_2\rrbracket + \nabla_{[\rho_{\mathsf{E}}(e_1),X]}e_2 + \llbracket e_2, \nabla_X e_1\rrbracket - \nabla_{[\rho_{\mathsf{E}}(e_2),X]}e_1 + \beta^{-1}\rho_{\mathsf{E}}^*\mathbf{i}_X\mathbf{d}\langle\nabla_\cdot e_1, e_2\rangle \\ &= -\llbracket e_1, \nabla_X e_2\rrbracket + \nabla_{[\rho_{\mathsf{E}}(e_1),X]}e_2 - \llbracket \nabla_X e_1, e_2\rrbracket - \nabla_{[\rho_{\mathsf{E}}(e_2),X]}e_1 + \beta^{-1}\rho_{\mathsf{E}}^*\pounds_X\langle\nabla_\cdot e_1, e_2\rangle,\end{aligned}$$

we find that $\left[\!\left[\sigma_{\mathsf{E}}^{\nabla}(e_1), \sigma_{\mathsf{E}}^{\nabla}(e_2)\right]\!\right] = T[\![e_1, e_2]\!]_\Delta - \widetilde{R_\Delta^{\rm bas}(e_1, e_2)}$. Finally we compute $\left[\!\!\left[\sigma_{\mathsf{E}}^{\nabla}(e_1), e_2^\dagger\right]\!\!\right] = \left[\!\!\left[Te_1 - \widetilde{\nabla.e_1}, e_2^\dagger\right]\!\!\right] = [\![e_1, e_2]\!]^\dagger + \nabla_{\rho_{\mathsf{E}}(e_2)} e_1^\dagger = \Delta_{e_1} e_2^\dagger$. □

6.4.4. *The VB-Courant algebroid associated to a double Lie algebroid.* Consider a double vector bundle $(D; A, B; M)$ with core C and a VB-Lie algebroid structure on each of its sides. Recall from §2.4.2 that $(D; A, B, M)$ is a double Lie algebroid if and only if, for any linear splitting of D, the two induced 2-representations (denoted as in §2.4.2) form a matched pair. By definition of a double Lie algebroid, $(D \mathbin{\ast\mkern-11mu|} A, D \mathbin{\ast\mkern-11mu|} B)$ is then a Lie bialgebroid over C^*, and so the double vector bundle

$$\begin{tikzcd}
D \mathbin{\ast\mkern-11mu|} A \oplus D \mathbin{\ast\mkern-11mu|} B \arrow[r] \arrow[d] & C^* \arrow[d] \\
A \oplus B \arrow[r] & M
\end{tikzcd}$$

with core $A^* \oplus B^*$ has the structure of a VB-Courant algebroid with base C^* and side $A \oplus B$.

Consider the splitting $\tilde\Sigma\colon (A \oplus B) \times_M C^* \to D \mathbin{\ast\mkern-11mu|} A \oplus D \mathbin{\ast\mkern-11mu|} B$ given by $\tilde\Sigma((a(m), b(m)), \gamma_m) = (\sigma_A^\star(\gamma_m), \sigma_B^\star(\gamma_m))$, where $\sigma_A^\star\colon \Gamma(A) \to \Gamma_{C^*}^l(D \mathbin{\ast\mkern-11mu|} A)$ and $\sigma_B^\star\colon \Gamma(B) \to \Gamma_{C^*}^l(D \mathbin{\ast\mkern-11mu|} B)$ are defined as in (13) or Lemma 2.3. By (14), this is a Lagrangian splitting. Recall from §2.4.1 that the splitting $\Sigma^\star\colon A \times_M C^* \to D \mathbin{\ast\mkern-11mu|} A$ of the VB-algebroid $(D \mathbin{\ast\mkern-11mu|} A \to C^*, A \to M)$ corresponds to the 2-representation $(\nabla^{C^*}, \nabla^{B^*}, -R^*)$ of A on the complex $\partial_B^*\colon B^* \to C^*$. In the same manner, the splitting $\Sigma^\star\colon B \times_M C^* \to D \mathbin{\ast\mkern-11mu|} B$ of the VB-algebroid $(D \mathbin{\ast\mkern-11mu|} B \to C^*, B \to M)$ corresponds to the 2-representation $(\nabla^{C^*}, \nabla^{A^*}, -R^*)$ of B on the complex $\partial_A^*\colon A^* \to C^*$.

We quickly check that the split Lie 2-algebroid corresponding to the linear splitting $\tilde\Sigma$ of $D \mathbin{\ast\mkern-11mu|} A \oplus D \mathbin{\ast\mkern-11mu|} B$ is the bicrossproduct of the matched pair of 2-representations (see Example 5.5). The equalities in (14) imply that we have to consider $A \oplus B$ as paired with $A^* \oplus B^*$ in the non standard way:

$$\langle (a, b), (\alpha, \beta) \rangle = \alpha(a) - \beta(b)$$

for all $a \in \Gamma(A)$, $b \in \Gamma(B)$, $\alpha \in \Gamma(A^*)$ and $\beta \in \Gamma(B^*)$. The anchor of $\tilde\sigma(a, b) = (\sigma^\star(a), \sigma^\star(b))$ is $\widehat{\nabla_a^*} + \widehat{\nabla_b^*} \in \mathfrak{X}^l(C^*)$, and the anchor of $(\alpha, \beta)^\dagger = (\beta^\dagger, \alpha^\dagger) \in \Gamma_{C^*}^c(D \mathbin{\ast\mkern-11mu|} A \oplus D \mathbin{\ast\mkern-11mu|} B)$ is $(\partial_B^*\beta + \partial_A^*\alpha)^\uparrow \in \mathfrak{X}^c(C^*)$. The Courant bracket $[\![(\sigma_A^\star(a), \sigma_B^\star(b)), (\beta^\dagger, \alpha^\dagger)]\!]$ is

$$\left([\sigma_A^\star(a), \beta^\dagger] + \pounds_{\sigma_B^\star(b)}\beta^\dagger - \mathbf{i}_{\alpha^\dagger}\mathbf{d}_{D \mathbin{\ast\mkern-11mu|} B}\sigma_A^\star(a), [\sigma_B^\star(b), \alpha^\dagger] + \pounds_{\sigma_A^\star(a)}\alpha^\dagger - \mathbf{i}_{\beta^\dagger}\mathbf{d}_{D \mathbin{\ast\mkern-11mu|} A}\sigma_B^\star(b)\right),$$

where $\mathbf{d}_{D \mathbin{\ast\mkern-11mu|} A}\colon \Gamma_{C^*}(\bigwedge^\bullet D \mathbin{\ast\mkern-11mu|} B) \to \Gamma_{C^*}(\bigwedge^{\bullet+1} D \mathbin{\ast\mkern-11mu|} B)$ is defined as usual by the Lie algebroid $D \mathbin{\ast\mkern-11mu|} A$, and similarly for $D \mathbin{\ast\mkern-11mu|} B$ (bear in mind that some non standard signs arise from the signs in (14)). The derivation $\pounds\colon \Gamma(D \mathbin{\ast\mkern-11mu|} A) \times \Gamma(D \mathbin{\ast\mkern-11mu|} B) \to \Gamma(D \mathbin{\ast\mkern-11mu|} B)$ is described by

$$\pounds_{\beta^\dagger}\alpha^\dagger = 0, \quad \pounds_{\beta^\dagger}\sigma_B^\star(b) = -\langle b, \nabla_\cdot^*\beta \rangle^\dagger, \quad \pounds_{\sigma_A^\star(a)}\alpha^\dagger = \pounds_a\alpha^\dagger,$$
$$\pounds_{\sigma_A^\star(a)}\sigma_B^\star(b) = \sigma_B^\star(\nabla_a b) + \widetilde{R(a, \cdot)}b,$$

in [12, Lemma 4.8]. Similar formulae hold for $\pounds\colon \Gamma(D \mathbin{\ast\mkern-11mu|} B) \times \Gamma(D \mathbin{\ast\mkern-11mu|} A) \to \Gamma(D \mathbin{\ast\mkern-11mu|} A)$. We get

$$[\![(\sigma_A^\star(a), \sigma_B^\star(b)), (\beta^\dagger, \alpha^\dagger)]\!] = \left((\nabla_a^*\beta + \pounds_b\beta - \langle \nabla.a, \alpha \rangle)^\dagger, (\nabla_b^*\alpha + \pounds_a\alpha - \langle \nabla.b, \beta \rangle)^\dagger\right).$$

In the same manner, we get

$$\begin{aligned}&[\![(\sigma_A^\star(a_1),\sigma_B^\star(b_1)),(\sigma_A^\star(a_2),\sigma_B^\star(b_2))]\!]\\ &= (\sigma_A^\star([a,a']+\nabla_b a'-\nabla_{b'}a),\sigma_B^\star([b,b']+\nabla_a b'-\nabla_{a'}b))\\ &\qquad+\Big(-\widetilde{R(a_1,a_2)}+\widetilde{R(b_1,\cdot)}a_2-\widetilde{R(b_2,\cdot)}a_1,-\widetilde{R(b_1,b_2)}+\widetilde{R(a_1,\cdot)}b_2-\widetilde{R(a_2,\cdot)}b_1\Big).\end{aligned}$$

Hence we have found the following result.

Theorem 6.12. *There is a bijection between decomposed double Lie algebroids and Lie 2-algebroids that are the bicrossproducts of matched pairs of 2-representations.*

Recall that if the vector bundle C is trivial, the matched pair of 2-representations is just a matched pair of the Lie algebroids A and B. The corresponding double Lie algebroid is the decomposed double Lie algebroid $(A\times_M B, A, B, M)$ found in [28]. The corresponding VB-Courant algebroid is

$$\begin{tikzcd}A\times_M B^*\oplus A^*\times_M B \arrow[r] \arrow[d] & 0 \arrow[d]\\ A\oplus B \arrow[r] & M\end{tikzcd}$$

with core $B^*\oplus A^*$. In that case there is a natural Lagrangian splitting and the corresponding Lie 2-algebroid is just the bicrossproduct Lie algebroid structure defined on $A\oplus B$ by the matched pair, see also the end of §5.5. This shows that the two notions of double of a matched pair of Lie algebroids; the bicrossproduct Lie algebroid in [30] and the double Lie algebroid in [28] are just the N-geometric and the classical descriptions of the same phenomenon, and special cases of Theorem 6.12.

6.5. **Categorical equivalence of Lie 2-algebroids and VB-Courant algebroids.**

In this section we quickly describe morphisms of VB-Courant algebroids. Then we find an equivalence between the category of VB-Courant algebroids and the category of Lie 2-algebroids.

6.5.1. *Morphisms of VB-Courant algebroids.* Recall from §2.1 and §3.2.2 that a morphism $\Omega\colon \mathbb{E}_1 \dashrightarrow \mathbb{E}_2$ of metric double vector bundles is an isotropic relation $\Omega\colon \overline{\mathbb{E}_1}\times\mathbb{E}_2$. Assume that $\mathbb{E}_1$ and $\mathbb{E}_2$ have linear Courant algebroid structures. Then Ω is a morphism of VB-Courant algebroid if it is a Dirac structure (with support) in $\overline{\mathbb{E}_1}\times\mathbb{E}_2$.

Choose two Lagrangian splittings $\Sigma^1\colon Q_1\times B_1\to\mathbb{E}_1$ and $\Sigma^2\colon Q_2\times B_2\to\mathbb{E}_2$. Then, bY §3.2.2 there exists four structure maps $\omega_0\colon M_1\to M_2$, $\omega_Q\colon Q_1\to Q_2$, $\omega_B\colon B_1^*\to B_2^*$ and $\omega_{12}\in\Omega^2(Q_1,\omega_0^*B_2^*)$ that define completely Ω. More precisely, Ω is spanned over $\operatorname{Graph}(\omega_Q\colon Q_1\to Q_2)$ by sections

$$\tilde{b}\colon \operatorname{Graph}(\omega_Q)\to\Omega,$$

$$\tilde{b}(q_m,\omega_Q(q_m))=\Big(\sigma_{B_1}(\omega_B^\star b)(q_m)+\widetilde{\omega_{12}^\star(b)}(q_m),\sigma_{B_2}(b)(\omega_Q(q_m))\Big)$$

for all $b\in\Gamma_{M_2}(B_2)$, and

$$\tau^\times\colon \operatorname{Graph}(\omega_Q)\to\Omega,\qquad \tau^\times(q_m,\omega_Q(q_m))=\big((\omega_Q^\star\tau)^\dagger(q_m),\tau^\dagger(\omega_Q(q_m))\big)$$

for all $\tau\in\Gamma_{M_2}(Q_2^*)$. Note that Ω projects under $\pi_{B_1}\times\pi_{B_2}$ to $R_{\omega_B^*}\subseteq B_1\times B_2$. But if $q\in\Gamma(Q_1)$ then $\omega_Q^! q\in\Gamma_{M_1}(\omega_0^*Q_2)$ can be written as $\sum_i f_i\omega_0^! q_i$ with $f_i\in C^\infty(M_1)$

and $q_i \in \Gamma_{M_2}(Q_2)$. The pair $\Big(\sigma_{B_1}(\omega_B^\star b)(q_m) + \widetilde{\omega_{12}^\star(b)}(q_m), \sigma_{B_2}(b)(\omega_Q(q_m))\Big)$ can be written as

$$\left(\left(\sigma_{Q_1}(q) + \langle \omega_{12}(q,\cdot), b(\omega_0(m))\rangle^\dagger\right)(\omega_B^\star b(m)), \sum_i f_i(m)\sigma_{Q_2}(q_i)(b(\omega_0(m)))\right).$$

Hence, Ω is spanned over $R_{\omega_B^*}$ by sections

$$\left(\sigma_{Q_1}(q)\circ \mathrm{pr}_1 + \langle \omega_{12}(q,\cdot), \mathrm{pr}_2\rangle^\dagger \circ \mathrm{pr}_1, \sum_i (f_i \circ q_{B_1} \circ \mathrm{pr}_1)\cdot(\sigma_{Q_2}(q_i)\circ \mathrm{pr}_2)\right) \tag{50}$$

for all $q \in \Gamma_{M_1}(Q_1)$ and

$$\left((\omega_Q^\star \tau)^\dagger \circ \mathrm{pr}_1, \tau^\dagger \circ \mathrm{pr}_2\right) \tag{51}$$

for all $\tau \in \Gamma(Q_2^*)$. Note also that $\langle \omega_{12}(q,\cdot), \mathrm{pr}_2\rangle^\dagger \circ \mathrm{pr}_1$ can be written

$$\sum_{ijk} ((f_{ijk}\chi_i(q)) \circ q_{B_1} \circ \mathrm{pr}_1) \cdot (\ell_{\beta_k} \circ \mathrm{pr}_2) \cdot (\tau_j^\dagger \circ \mathrm{pr}_1)$$

for some $f_{ijk} \in C^\infty(M_1)$, $\chi_i, \tau_j \in \Gamma(Q_1^*)$ and $\beta_k \in \Gamma(B_2^*)$.

Checking all the conditions in Lemma 5.3 on the two types of sections (50) and (51) yield that $\Omega \to R_{\omega_B^*}$ is a Dirac structure with support if and only if

(1) $\omega_Q \colon Q_1 \to Q_2$ over $\omega_0 \colon M_1 \to M_2$ is compatible with the anchors $\rho_1 \colon Q_1 \to TM_1$ and $\rho_2 \colon Q_2 \to TM_2$:

$$T_m\omega_0(\rho_1(q_m)) = \rho_2(\omega_Q(q_m))$$

for all $q_m \in Q_1$,

(2) $\partial_1 \circ \omega_Q^\star = \omega_B^\star \circ \partial_2$ as maps from $\Gamma(Q_2^*)$ to $\Gamma(B_1)$, or equivalently $\omega_Q \circ \partial_1^* = \partial_2^* \circ \omega_B$,

(3) ω_Q preserves the dull brackets up to $\partial_2^*\omega_{12}$: i.e. if[15] $q^1 \sim_{\omega_Q} r^1$ and $q^2 \sim_{\omega_Q} r^2$, then

$$[\![q^1, r^1]\!]_1 \sim_{\omega_Q} [\![q^2, r^2]\!]_2 - \partial_2^*\omega_{12}(q^1, r^1).$$

(4) ω_B and ω_Q intertwines the connections ∇^1 and ∇^2 up to $\partial_1 \circ \omega_{12}$:

$$\omega_B^\star((\omega_Q^\star\nabla^2)_q b) = \nabla_q^1(\omega_B^\star(b)) - \partial_1 \circ \langle \omega_{12}(q,\cdot), b\rangle \in \Gamma(B_1)$$

for all $q_m \in Q_1$ and $b \in \Gamma(B^2)$, and

(5) $\omega_Q^\star \omega_{R_2} - \omega_B \circ \omega_{R_1} = -\mathbf{d}_{(\omega_Q^\star\nabla^2)}\omega_{12} \in \Omega^3(Q_1, \omega_0^* B_2^*)$.

Hence, Ω is a morphism of VB-Courant algebroid if and only if it induces a morphism of Dorfman 2-representations after any choice of Lagrangian decompositions of $\mathbb{E}_1$ and $\mathbb{E}_2$.

---

[15] For simplicity, we assume that sections $q \sim_{\omega_Q} r$ exist and span Q_1, respectively Q_2. A general discussion without this assumption would be along the same lines, but much more technical (see the general definition of morphisms of Lie algebroids in [27].

6.5.2. *Equivalence of categories.* The functors found in Section 3.3 between the category of metric double vector bundles and the category of [2]-manifolds restrict to functors between the category of VB-Courant algebroids and the category of Lie [2]-algebroids.

Theorem 6.13. *The category of Lie* 2*-algebroids is equivalent to the category of VB-Courant algebroids.*

Proof. Let $(\mathcal{M}, \mathcal{Q})$ be a Lie 2-algebroid and consider the double vector bundle $\mathbb{E}_{\mathcal{M}}$ corresponding to $\mathcal{M}$. Choose a splitting $\mathcal{M} \simeq Q[-1] \oplus B^*[-2]$ of $\mathcal{M}$ and consider the corresponding Lagrangian splitting Σ of $\mathbb{E}_{\mathcal{M}}$.

As we have seen in §5.2, the split Lie 2-algebroid $(Q[-1]\oplus B^*[-2], \mathcal{Q})$ is equivalent to a Dorfman 2-representation. By Theorem 6.7, this Dorfman 2-representation defines a VB-Courant algebroid structure on the decomposition of $\mathbb{E}_{\mathcal{M}}$ and so by isomorphism on $\mathbb{E}_{\mathcal{M}}$. Further, by Proposition 6.8 and §3.3.4, the Courant algebroid structure on $\mathbb{E}_{\mathcal{M}}$ does not depend on the choice of splitting of $\mathcal{M}$, since a different choice of splitting will induce a change of Lagrangian splitting of $\mathbb{E}_{\mathcal{M}}$. This shows that the functor $\mathcal{G}$ restricts to a functor $\mathcal{G}_Q$ from the category of Lie 2-algebroids to the category of VB-Courant algebroids.

Sections 5.6 and 6.5.1 show that morphisms of split Lie 2-algebroids are sent by $\mathcal{G}$ to morphisms of decomposed VB-Courant algebroids.

The functor $\mathcal{F}$ restricts in a similar manner to a functor $\mathcal{F}_{\mathrm{VBC}}$ from the category of VB-Courant algebroids to the category of Lie 2-algebroids. The natural transformations found in the proof of Theorem 3.17 restrict to natural transformations $\mathcal{F}_{\mathrm{VBC}}\mathcal{G}_Q \simeq \mathrm{Id}$ and $\mathcal{G}_Q\mathcal{F}_{\mathrm{VBC}} \simeq \mathrm{Id}$. □

7. LA-Courant algebroids vs Poisson Lie 2-algebroids

In this section, we prove that a split Poisson Lie 2-algebroid is equivalent to the *matched pair* of a Dorfman 2-representation with a self-dual 2-representation.

Take a double vector bundle

$$\begin{tikzcd}
\mathbb{E} \arrow[r, "\pi_B"] \arrow[d, "\pi_Q"'] & B \arrow[d, "q_B"] \\
Q \arrow[r, "q_Q"'] & M
\end{tikzcd}$$

with core Q^*, with a VB-Lie algebroid structure on $(\mathbb{E} \to Q, B \to M)$ and a VB-Courant algebroid structure on $(\mathbb{E} \to B, Q \to M)$. In this section we show that the double vector bundle is an LA-Courant algebroid if and only if the VB-algebroid is metric and the self-dual 2-representation defined by any Lagrangian decomposition of $\mathbb{E}$ and the VB-algebroid side forms a *matched pair* with the Dorfman 2-representation describing the Courant algebroid side.

We conclude by recovering in a constructive manner the equivalence between LA-Courant algebroids and Poisson Lie 2-algebroids [19].

We begin with the following definition.

Definition 7.1. *Let* $(B \to M, \rho_B, [\cdot\,,\cdot])$ *be a Lie algebroid and* $(Q \to M, \rho_Q)$ *an anchored vector bundle. Assume that* B *acts on* $\partial_Q\colon Q^* \to Q$ *up to homotopy* *via a self-dual 2-representation* $(\nabla^Q, \nabla^{Q^*}, R_B)$*, and let* $(\partial_B\colon Q^*\to B, \Delta, \nabla, R_Q)$[16] *be a Q-Dorfman 2-representation. Then we say that the 2-representation and the Dorfman 2-representation form a matched pair if*

(M1) $\partial_Q(\Delta_q\tau) = \nabla_{\partial_B\tau}q + [\![q, \partial_Q\tau]\!] + \partial_B^*\langle\tau, \nabla_\cdot q\rangle$,
(M2) $\partial_B(\nabla_b\tau) = [b, \partial_B\tau] + \nabla_{\partial_Q\tau}b$,
(M3) $\partial_B R(b_1, b_2)q = -\nabla_q[b_1, b_2] + [\nabla_q b_1, b_2] + [b_1, \nabla_q b_2] + \nabla_{\nabla_{b_2}q}b_1 - \nabla_{\nabla_{b_1}q}b_2$,
(M4) $\partial_Q R(q_1, q_2)b = -\nabla_b[\![q_1, q_2]\!] + [\![q_1, \nabla_b q_2]\!] + [\![\nabla_b q_1, q_2]\!] + \nabla_{\nabla_{q_2}b}q_1 - \nabla_{\nabla_{q_1}b}q_2 + \partial_B^*\langle R(\cdot, b)q_1, q_2\rangle$.
(M5) $\mathbf{d}_{\nabla^B}\omega_R = \mathbf{d}_{\nabla^Q}\omega_B \in \Omega^2(B, \wedge^3Q^*) = \Omega^3(Q, \wedge^2B^*)$*, where* ω_R *is seen as an element of* $\Omega^1(B, \wedge^3Q^*)$ *and* $\omega_B \in \Omega^2(Q, \wedge^2B^*)$ *is defined by* $\omega_B(q_1, q_2)(b_1, b_2) = \langle R(b_1, b_2)q_1, q_2\rangle$.

Remark 7.2. (1) (M5) is written out as

$$\begin{aligned}
&\nabla_{b_2}R(q_1, q_2)b_1 - \nabla_{b_1}R(q_1, q_2)b_2 + R(q_1, q_2)[b_1, b_2]\\
&+ R(\nabla_{b_1}q_1, q_2)b_2 + R(q_1, \nabla_{b_1}q_2)b_2 - R(\nabla_{b_2}q_1, q_2)b_1 - R(q_1, \nabla_{b_2}q_2)b_1\\
&+ \Delta_{q_1}R(b_1, b_2)q_2 - \Delta_{q_2}R(b_1, b_2)q_1 - R(b_1, b_2)[\![q_1, q_2]\!]\\
&- R(\nabla_{q_1}b_1, b_2)q_2 - R(b_1, \nabla_{q_1}b_2)q_2 + R(\nabla_{q_2}b_1, b_2)q_1 - R(b_1, \nabla_{q_2}b_2)q_1\\
=&\langle(R(b_1, \nabla_\cdot b_2) + R(\nabla_\cdot b_1, b_2))q_1, q_2\rangle - \rho_Q^*\mathbf{d}\langle R(b_1, b_2)q_1, q_2\rangle
\end{aligned}$$

for all $q_1, q_2 \in \Gamma(Q)$ and $b_1, b_2 \in \Gamma(B)$.

(2) The equality $\rho_Q\circ\partial_Q = \rho_B\circ\partial_B$ follows easily from (M1) if Q has positive rank, and from (M2) if B has positive rank. If both Q and B have rank zero, then $\rho_Q\circ\partial_Q = \rho_B\circ\partial_B$ is trivially satisfied.

(3) The equation $[\rho_Q(q), \rho_B(b)] = \rho_B(\nabla_q b) - \rho_Q(\nabla_b q)$ follows easily from (M3) if B has positive rank, and from (M4) if Q has positive rank. If both Q and B have rank zero, then it is trivially satisfied.

(4) If $\rho_Q\circ\partial_Q = \rho_B\circ\partial_B$, then (M1) is equivalent to

$$\left(\Delta_{\partial_Q\sigma_1}\sigma_2 - \nabla_{\partial_B\sigma_2}\sigma_1\right) + \left(\Delta_{\partial_Q\sigma_2}\sigma_1 - \nabla_{\partial_B\sigma_1}\sigma_2\right) = \rho_Q^*\mathbf{d}\langle\sigma_1, \partial_Q\sigma_2\rangle \tag{52}$$

for all $\sigma_1, \sigma_2 \in \Gamma(Q^*)$.

(5) If $[\rho_Q(q), \rho_B(b)] = \rho_B(\nabla_q b) - \rho_Q(\nabla_b q)$, then (M4) is equivalent to

$$R(q, \partial_Q\tau)b - R(b, \partial_B\tau)q = \Delta_q\nabla_b\tau - \nabla_b\Delta_q\tau + \Delta_{\nabla_b q}\tau - \nabla_{\nabla_q b}\tau - \langle\nabla_{\nabla_\cdot b}q, \tau\rangle \tag{53}$$

for all $b \in \Gamma(B)$, $q \in \Gamma(Q)$ and $\tau \in \Gamma(Q^*)$.

7.1. Poisson Lie 2-algebroids via matched pairs.

We begin this subsection with the definition of a Poisson Lie 2-algebroid.

Definition 7.3. *Let* $\mathcal{M}$ *be an Poisson* $[2]$*-manifold with algebra of functions* $\mathcal{A} = C^\infty(\mathcal{M})$ *and Poisson bracket* $\{\cdot\,,\cdot\}$*. Assume that* $\mathcal{M}$ *has in addition a Lie 2-algebroid structure, i.e. it is endowed with a homological vector field* $\mathcal{Q} \in \operatorname{Der}^1\mathcal{A}$*. Then* $(\mathcal{M}, \mathcal{Q}, \{\cdot\,,\cdot\})$ *is a* ***Poisson Lie 2-algebroid*** *if the homological vector field preserves the Poisson structure, i.e. if*

$$\mathcal{Q}\{\xi_1, \xi_2\} = \{\mathcal{Q}(\xi_1), \xi_2\} + (-1)^{\deg\xi_1}\{\xi_1, \mathcal{Q}(\xi_2)\} \tag{54}$$

for all $\xi_1, \xi_2 \in \mathcal{A}$.

---

[16] For the sake of simplicity, we write in this definition ∇ for three different connections, unless it is not clear from the indexes which connection is meant.

A morphism of Poisson Lie 2-algebroids is a morphism of the underlying [2]*-manifold that is a morphism of Poisson* [2]*-manifolds* and *a morphism of Lie 2-algebroids.*

The main theorem of this section shows that matched pairs as in Definition 7.1 are equivalent to split Poisson Lie 2-algebroids.

Theorem 7.4. *Let $\mathcal{M} = Q[-1] \oplus B^*[-2]$ be a split* [2]*-manifold endowed with a homological vector field $\mathcal{Q}$ and a Poisson bracket $\{\cdot\,,\cdot\}$ of degree -2. Let $(\partial_B\colon Q^* \to B, \Delta, \nabla, R_Q)$ be the Dorfman 2-representation of (Q, ρ_Q) that encodes $\mathcal{Q}$ in coordinates, and let $(\partial_Q\colon Q^* \to Q, \nabla^*, \nabla, R_B)$ be the self-dual* 2*-representation that is equivalent to the Poisson bracket.*

Then $(\mathcal{M}, \mathcal{Q}, \{\cdot\,,\cdot\})$ is a Poisson Lie 2-algebroid if and only if the self dual 2-representation and the Dorfman 2-representation form a matched pair.

Proof. The idea of this proof is very simple, but requires rather long computations in coordinates. We will leave some of the detailed verifications to the reader.

We check (54) in coordinates, by using the formulae found in (34) and Theorem 4.3 for χ and $\{\cdot\,,\cdot\}$, respectively. On an open chart $U \subseteq M$ trivialising both Q and B, we have coordinates $(x_1, \ldots, x_p)$ and we can choose a local frame $(q_1, \ldots, q_{r_1})$ of sections of Q, and a local frame $(\beta_1, \ldots, \beta_{r_2})$ of sections of B^*. We denote by $(\tau_1, \ldots, \tau_{r_1})$ and $(b_1, \ldots, b_{r_2})$ the dual frames for Q^* and B, respectively. As functions on $\mathcal{M}$, the coordinates functions $x_1, \ldots, x_p$ have degree 0, the functions $\tau_1, \ldots, \tau_{r_1}$ have degree 1 and the functions $b_1, \ldots, b_{r_2}$ have degree 2.

First we have $\mathcal{Q}(f) = \sum_i \rho_Q(q_i)(x_k)\sigma_i\partial_{x_k}(f) = \rho_Q^*\mathbf{d}f \in \Gamma(Q^*)$ and $\{f, g\} = 0$ for $f, g \in C^\infty(M)$. This yields

$$\{\mathcal{Q}(f), g\} + \{f, \mathcal{Q}(g)\} = \left\{\rho_Q^*\mathbf{d}f, g\right\} + \left\{f, \rho_Q^*\mathbf{d}g\right\} = 0 = \mathcal{Q}\{f, g\}$$

by the graded skew-symmetry and $\{\tau, f\} = 0$ for $\tau \in \Gamma(Q^*)$ and $f \in C^\infty(M)$. Then we have

$$\begin{aligned}
&\{\mathcal{Q}(\tau_k), f\} - \{\tau_k, \mathcal{Q}(f)\}\\
&= \left\{-\sum_{i<j}\langle [\![q_i, q_j]\!], \tau_k\rangle\tau_i\tau_j + \sum_r \langle \partial_B^*\beta_r, \tau_k\rangle b_r, f\right\} - \{\tau_k, \rho_Q^*\mathbf{d}f\}\\
&= \sum_r \langle \partial_B^*\beta_r, \tau_k\rangle \rho_B(b_r)(f) - \langle \partial_Q\tau_k, \rho_Q^*\mathbf{d}f\rangle\\
&= \rho_B(\partial_B\tau_k)(f) - \rho_Q(\partial_Q\tau_k)(f).
\end{aligned}$$

But we also have $\mathcal{Q}\{\tau_k, f\} = \mathcal{Q}(0) = 0$. Hence, $\{\mathcal{Q}(\tau_k), f\} - \{\tau_k, \mathcal{Q}(f)\} = \mathcal{Q}\{\tau_k, f\}$ is equivalent to $\rho_B(\partial_B\tau_k)(f) = \rho_Q(\partial_Q\tau_k)(f)$.

In a similar manner, we have $\mathcal{Q}\{b_l, f\} = \rho_Q^*\mathbf{d}(\rho_B(b_l)(f))$ and $\{\mathcal{Q}(b_l), f\} + \{b_l, \mathcal{Q}(f)\}$ is

$$\begin{aligned}
&-\left\{\sum_{i<j<k} \omega_R(q_i, q_j, q_k)(b_l)\tau_i\tau_j\tau_k + \sum_{ij}\langle \nabla^*_{q_i}\beta_j, b_l\rangle\tau_i b_j, f\right\} + \{b_l, \rho_Q^*\mathbf{d}f\}\\
&= -\sum_{ij}\langle \nabla^*_{q_i}\beta_j, b_l\rangle\tau_i\rho_B(b_j)(f) + \nabla^*_{b_l}(\rho_Q^*\mathbf{d}f) = \sum_{ij}\langle \rho_B^*\mathbf{d}f, \nabla_{q_i}b_l\rangle\tau_i + \nabla^*_{b_l}(\rho_Q^*\mathbf{d}f).
\end{aligned}$$

Hence, $\mathcal{Q}\{b_l, f\} = \{\mathcal{Q}(b_l), f\} + \{b_l, \mathcal{Q}(f)\}$ if and only if

$$\rho_Q(q)\rho_B(b_l)(f) = \langle \rho_B^*\mathbf{d}f, \nabla_q b_l\rangle + \rho_B(b_l)\rho_Q(q)(f) - \rho_Q(\nabla_{b_l}q)(f)$$

for all $q \in \Gamma(Q)$. This is

$$[\rho_Q(q), \rho_B(b)] = \rho_B(\nabla_q b_l)(f) - \rho_Q(\nabla_{b_l} q)(f).$$

Then we have $\mathcal{Q}\{b_l, \tau_k\} = \mathcal{Q}(\nabla_{b_l}\tau_k)$, which is

$$\sum_{i,j} \rho_Q(q_i)\langle \nabla_{b_l}\tau_k, q_j\rangle \tau_i\tau_j - \sum_{i<j}\langle [\![q_i, q_j]\!], \nabla_b \tau_k\rangle \tau_i\tau_j + \sum_r \langle \nabla_{b_l}\tau_k, \partial_B^*\beta_r\rangle b_r$$

in coordinates. The Poisson bracket $\{\mathcal{Q}(b_l), \tau_k\}$ is computed to be

$$-\sum_{i<j}\langle R(\partial_Q\tau_k, q_i)b_l, q_j\rangle \tau_i\tau_j - \sum_r \langle \nabla^*_{\partial_Q\tau_k}\beta_r, b_l\rangle b_r + \sum_{i,j}\langle \nabla_{\nabla_{q_i}b_l}\tau_k, q_j\rangle \tau_i\tau_j$$

in coordinates, and the Poisson bracket $\{b_l, \mathcal{Q}(\tau_k)\}$ is

$$\begin{aligned}
&-\sum_{i<j}\rho_B(b_l)\langle [\![q_i, q_j]\!], \tau_k\rangle \tau_i\tau_j + \sum_{i,j}\sum_r \langle [\![q_r, q_j]\!], \tau_k\rangle \langle \tau_r, \nabla_{b_l} q_i\rangle \tau_i\tau_j \\
&+ \sum_r \langle [b_l, \partial_B\tau_k], \beta_r\rangle b_r - \sum_{i<j}\langle R(b_l, \partial_B\tau_k)q_i, q_j\rangle \tau_i\tau_j.
\end{aligned}$$

By comparing the coefficients in these three equations, we find that

$$\mathcal{Q}\{b_l, \tau_k\} = \{\mathcal{Q}(b_l), \tau_k\} + \{b_l, \mathcal{Q}(\tau_k)\}$$

if and only if

$$\partial_B \nabla_{b_l}\tau_k = \nabla_{\partial_Q\tau_k} b_l + [b_l, \partial_B\tau_k],$$

which is (M2) and

$$\begin{aligned}
&\cancel{\rho_Q(q_i)\langle \nabla_{b_l}\tau_k, q_j\rangle} - \cancel{\rho_Q(q_j)\langle \nabla_{b_l}\tau_k, q_i\rangle} - \langle [\![q_i, q_j]\!], \nabla_b\tau_k\rangle \\
= &- \langle R(\partial_Q\tau_k, q_i)b_l, q_j\rangle + \langle \nabla_{\nabla_{q_i}b_l}\tau_k, q_j\rangle - \langle \nabla_{\nabla_{q_j}b_l}\tau_k, q_i\rangle \\
&- \rho_B(b_l)\langle [\![q_i, q_j]\!], \tau_k\rangle + \langle [\![\nabla_{b_l}q_i, q_j]\!], \tau_k\rangle + \cancel{\rho_Q(q_j)\langle \tau_k, \nabla_{b_l}q_i\rangle} \\
&- \langle [\![\nabla_{b_l}q_j, q_i]\!], \tau_k\rangle - \cancel{\rho_Q(q_i)\langle \tau_k, \nabla_{b_l}q_j\rangle} - \langle R(b_l, \partial_B\tau_k)q_i, q_j\rangle
\end{aligned}$$

which simplifies to (M4). Next we study the condition $\mathcal{Q}\{\tau_k, \tau_l\} = \{\mathcal{Q}(\tau_k), \tau_l\} - \{\tau_k, \mathcal{Q}(\tau_l)\}$. The left hand side is $\rho_Q^*\mathbf{d}\langle \tau_k, \partial_Q\tau_l\rangle = \sum_i \rho_Q(q_i)\langle \tau_k, \partial_Q\tau_l\rangle \tau_i$. To get the right hand side we note that $-\{\tau_k, \mathcal{Q}(\tau_l)\} = \{\mathcal{Q}(\tau_l), \tau_k\}$. A short computation yields

$$\{\mathcal{Q}(\tau_k), \tau_l\} = \sum_{i,j}\langle [\![q_i, q_j]\!], \tau_k\rangle \langle \tau_l, \partial_Q\tau_i\rangle \tau_j + \sum_r\sum_j \langle \partial_B^*\beta_r, \tau_k\rangle \langle \nabla_{b_r}\tau_l, q_j\rangle \tau_j.$$

We get easily that $\mathcal{Q}\{\tau_k, \tau_l\} = \{\mathcal{Q}(\tau_k), \tau_l\} - \{\tau_k, \mathcal{Q}(\tau_l)\}$ is equivalent to (52). Recall from Remark 7.2 that together with $\rho_B \circ \partial_B = \rho_Q \circ \partial_Q$, this is equivalent to (M1).

The study of the equation $\mathcal{Q}\{b_l, b_k\} = \{\mathcal{Q}(b_l), b_k\} + \{b_l, \mathcal{Q}(b_k)\}$ is long, but relatively straightforward. We get $C^\infty(M)$-linear combinations of $\tau_i\tau_j\tau_k$ and of $\tau_i b_r$. Comparing the factors of $\tau_i\tau_j\tau_k$ yields (M5) and comparing the factors of $\tau_i b_r$ yields (M3). □

7.2. **LA-Courant algebroids and equivalence of categories.** Li-Bland's definition of an LA-Courant algebroid is quite technical and requires the consideration of triple vector bundles. We give it and study it in Appendix B, where we prove the following theorem.

Theorem 7.5. *Let* $(\mathbb{E}; Q, B; M)$ *be a double vector bundle with a VB-Courant algebroid structure on* $(\mathbb{E} \to B, Q \to M)$ *and a VB-Lie algebroid structure on* $(\mathbb{E} \to Q, B \to M)$*. Then in particular,* $\mathbb{E}$ *is a metric double vector bundle with the linear metric underlying the linear Courant algebroid structure on* $\mathbb{E} \to B$*. Choose a Lagrangian decomposition* $\Sigma\colon B \times_M Q \to \mathbb{E}$ *of* $\mathbb{E}$*. Then* $(\mathbb{E}; Q, B; M)$ *is an LA-Courant algebroid if and only if*

1. *the linear Lie algebroid structure on* $\mathbb{E} \to Q$ *is compatible in the sense of Definition 4.4 with the linear metric, and*
2. *the self-dual 2-representation and the Dorfman 2-representation obtained from the Lagrangian splitting form a matched pair as in Definition 7.1.*

The proof of this theorem is very long and technical, showing that the definition of an LA-Courant algebroid is hard to handle. Hence our result provides a new definition of LA-Courant algebroids, that is much simpler to articulate and probably also easier to use.

Further, we now explain how this theorem shows that LA-Courant algebroid are equivalent to Poisson Lie 2-algebroids. (Note that this has already been found by Li-Bland in [19].) First, morphisms of LA-Courant algebroids are morphisms of metric double vector bundles that preserve the Courant algebroid structure and the Lie algebroid structure [19]. Hence, the category of LA-Courant algebroids is a full subcategory of the intersection of the category of metric VB-algebroids and the category of VB-Courant algebroids.

On the other hand, Definition 7.3 shows that the category of Poisson Lie 2-algebroids is a full subcategory of the intersection of the categories of Poisson [2]-manifolds and of Lie 2-algebroids.

This, Theorem 7.4 and Theorem 7.5 show that the equivalences of the categories of metric VB-algebroids and of Poisson [2]-manifolds and of the categories of VB-Courant algebroids and Lie 2-algebroids restrict to an equivalence of the category of LA-Courant algebroids with the category of Poisson Lie 2-algebroids.

7.3. **Examples of LA-Courant algebroids and Poisson Lie 2-algebroids.** Next we discuss some classes of Examples of LA-Courant algebroids, and the corresponding Poisson Lie 2-algebroids.

7.3.1. *The tangent double of a Courant algebroid.* Let $\mathsf{E} \to M$ be a Courant algebroid and choose a metric connection $\nabla\colon \mathfrak{X}(M) \times \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E})$. We have seen in Examples 2.8, 3.11 and §4.4.1 that the triple $(\nabla, \nabla, R_\nabla)$ is then the TM-representation up to homotopy describing $(T\mathsf{E} \to \mathsf{E}, TM \to M)$ after the choice of the splitting $\Sigma^\nabla\colon \mathsf{E} \times_M M \to T\mathsf{E}$. We have also seen in Section 6.4.3 that the E-Dorfman 2-representation encoding the Courant algebroid side $(T\mathsf{E} \to TM, \mathsf{E} \to TM)$ is $(\rho_\mathsf{E}\colon \mathsf{E} \to TM, \Delta, \nabla^{\rm bas}, R_\Delta^{\rm bas})$. A straightforward computation resembling the one in [18, Section 3.2] for the tangent double of a Lie algebroid shows that this 2-representation and this Dorfman 2-representation are compatible, and so that $T\mathsf{E}$ is an LA-Courant algebroid (see also [19]).

Example 4.4.1 and §4.1.1 show that this class of LA-Courant algebroids is equivalent to the *symplectic* Lie 2-algebroids.

7.3.2. *The standard Courant algebroid over a Lie algebroid.* Let A be a Lie algebroid. Then $TA\oplus T^*A$ has a VB-Courant algebroid structure $(TA\oplus_A T^*A; TM\oplus A^*, A; M)$ by §6.4.1 and a metric VB-algebroid structure $(TA\oplus_A T^*A \to TM\oplus A^*, A \to M)$ by Example 3.12. Recall from §6.4.1 and Example 5.4.2 the Dorfman 2-representation given by a Lagrangian splitting and the VB-Courant algebroid structure, and recall from Example 3.12 the self-dual 2-representation defined by the same Lagrangian splitting and the VB-algebroid structure. A straightforward computation, that also resembles the one in [18, Section 3.2] for the tangent double of a Lie algebroid, shows that the Dorfman 2-representation and the self-dual 2-representation form a matched pair. Hence, $TA\oplus_A T^*A$ is an LA-Courant algebroid.

7.3.3. *The LA-Courant algebroid defined by a double Lie algebroid.* More generally, take a double Lie algebroid (D, A, B, M) and consider the metric VB-algebroid

$$\begin{tikzcd}
D \oplus_B (D \ast B) \arrow[r] \arrow[d] & B \arrow[d] \\
A \oplus C^* \arrow[r] & M
\end{tikzcd}$$

defined as in §4.4.2 by the VB-algebroid $(D \to A, B \to M)$. Since $(D \to B, A \to M)$ is a VB-algebroid as well, we get a linear Courant algebroid structure on $D \oplus_B (D \ast B) \to B$ as in §6.4.2. Given a linear splitting $\Sigma\colon A\times_M B \to D$, we get a matched pair of 2-representations as in Definition 2.10, see §2.4.2.

The induced linear splitting $\tilde{\Sigma}\colon B\times_M (A\oplus C^*) \to D\oplus_B (D \ast B)$ defines hence a 2-representation

$$(\partial_A \oplus \partial_A^*\colon C\oplus A^* \to A\oplus C^*, \nabla^A \oplus \nabla^{C^*}, \nabla^C \oplus \nabla^{A^*}, R\oplus(-R^*)), \tag{55}$$

as in §4.4.2 and a Dorfman 2-representation $(\partial_B \circ \mathrm{pr}_C\colon C\oplus A^* \to B, \Delta, \nabla, R)$ defined by

$$\begin{gathered}\Delta\colon \Gamma(A\oplus C^*)\times\Gamma(C\oplus A^*) \to \Gamma(C\oplus A^*)\\ \Delta_{(a,\gamma)}(c,\alpha) = (\nabla_a c, \pounds_a\alpha + \langle \nabla_\cdot^*\gamma, c\rangle),\end{gathered} \tag{56}$$

$$\nabla\colon \Gamma(A\oplus C^*)\times\Gamma(B) \to \Gamma(B), \qquad \nabla_{(a,\gamma)}b = \nabla_a b \tag{57}$$

with $A\oplus C^*$ anchored by ρ_A, and $R\in\omega^2(A\oplus C^*, \mathrm{Hom}(B, C\oplus A^*))$,

$$R((a_1,\gamma_1),(a_2,\gamma_2)) = (R(a_1,a_2), \langle\gamma_2, R(a_1,\cdot)\rangle + \langle\gamma_1, R(\cdot,a_2)\rangle) \tag{58}$$

as in Example 5.4.4, see §6.4.2. A straightforward computation shows that the matched pair conditions for the 2-representations describing the sides of D imply that the 2-representation (55) and the Dorfman 2-representation (56)–(58) form a matched pair. Hence, $(D\oplus_B (D \ast B), A\oplus C^*, B, M)$ has a natural LA-Courant algebroid structure. In the same manner, $(D\oplus_A (D \ast A), B\oplus C^*, A, M)$ has a natural LA-Courant algebroid structure. Hence, we get the following theorem.

Theorem 7.6. *Consider a matched pair of* 2*-representations with the usual notation. Then the split* [2]*-manifold* $(A\oplus C^*)[-1]\oplus B^*[-2]$ *endowed with the semi-direct Lie* 2*-algebroid structure in §5.4.4 and the Poisson bracket defined by* (55) *and §4.4.2, is a split Poisson Lie* 2*-algebroid.*

By symmetry, the split [2]*-manifold* $(B\oplus C^*)[-1]\oplus A^*[-2]$ *also inherits a split Poisson Lie* 2*-algebroid structure.*

7.4. **The degenerate Courant algebroid structure on the core.** We prove in this section that the core of an LA-Courant algebroid inherits a natural structure of degenerate Courant algebroid. We discuss some examples and we deduce a new way of describing the equivalence between Courant algebroids and symplectic Lie 2-algebroids.

Theorem 7.7. *Let* $(\mathbb{E}; B, Q; M)$ *be an LA-Courant algebroid and choose a Lagrangian splitting. Then the core* Q^* *inherits the structure of a degenerate Courant algebroid over* M*, with the anchor* $\rho_Q\partial_Q$*, the map* $\mathcal{D} = \rho_Q^*\mathbf{d}\colon C^\infty(M) \to \Gamma(Q^*)$*, the pairing defined by* $\langle \tau_1, \tau_2\rangle_{Q^*} = \langle \tau_1, \partial_Q\tau_2\rangle$ *and the bracket defined by* $[\![\tau_1, \tau_2]\!]_{Q^*} = \Delta_{\partial_Q\tau_1}\tau_2 - \nabla_{\partial_B\tau_2}\tau_1$ *for all* $\tau_1, \tau_2 \in \Gamma(Q^*)$*. This structure does not depend on the choice of the Lagrangian splitting, and the map* $\partial_B\colon Q^* \to B$ *preserves the brackets and the anchors.*

Proof. Theorem 7.5 states that the 2-representation and the Dorfman 2-representation defined by a Lagrangian splitting form a matched pair and that the 2-representation is self-dual. Hence, by (1) of Definition 4.2, the pairing $\langle\cdot\,,\cdot\rangle_{Q^*}$ is symmetric. The map $\rho_Q^*\mathbf{d}\colon C^\infty(M) \to \Gamma(Q^*)$ satisfies $\langle \tau, \rho_Q^*\mathbf{d}\varphi\rangle_{Q^*} = \langle \partial_Q\tau, \rho_Q^*\mathbf{d}\varphi\rangle = \langle \rho_Q\partial_Q\tau, \mathbf{d}\varphi\rangle = (\rho_Q \circ \partial_Q)(\tau)(\varphi)$ for all $\tau \in \Gamma(Q^*)$ and $\varphi \in C^\infty(M)$. We check (CA1)–(CA5) in Definition 5.1. Condition (CA5) is immediate by definition of the bracket. Condition (CA3) is exactly (52). Note that (M5) and (D1) imply

$$\begin{aligned}\partial_B[\![\tau_1, \tau_2]\!]_{Q^*} &= \partial_B(\Delta_{\partial_Q\tau_1}\tau_2 - \nabla_{\partial_B\tau_2}\tau_1)\\ &= \nabla_{\partial_Q\tau_1}\partial_B\tau_2 - [\partial_B\tau_2, \partial_B\tau_1] - \nabla_{\partial_Q\tau_1}\partial_B\tau_2 = [\partial_B\tau_1, \partial_B\tau_2].\end{aligned} \tag{59}$$

This and $\rho_Q \circ \partial_Q = \rho_B \circ \partial_B$ (see Remark 7.2) imply the last assertion of the theorem. In the same manner (M1) and $\nabla^Q \circ \partial_Q = \partial_Q \circ \nabla^{Q^*}$ (by Definition of a 2-representation) imply the equation

$$\partial_Q[\![\tau_1, \tau_2]\!]_{Q^*} = [\![\partial_Q\tau_1, \partial_Q\tau_2]\!]_\Delta + \partial_B^*\langle \tau_2, \nabla_\cdot\partial_Q\tau_1\rangle. \tag{60}$$

The compatibility of the bracket with the anchor (CA4) follows then immediately from (60) with (47), or from (59) with $\rho_Q \circ \partial_Q = \rho_B \circ \partial_B$. Next we check (CA2) using (M1) and $\nabla^Q \circ \partial_Q = \partial_Q \circ \nabla^{Q^*}$. We have

$$\begin{aligned}&\rho_Q\partial_Q(\tau_1)\langle \tau_2, \tau_3\rangle_{Q^*} - \langle [\![\tau_1, \tau_2]\!]_{Q^*}, \tau_3\rangle_{Q^*} - \langle \tau_2, [\![\tau_1, \tau_3]\!]_{Q^*}\rangle_{Q^*}\\ &= \langle \tau_2, [\![\partial_Q\tau_1, \partial_Q\tau_3]\!]_\Delta\rangle + \langle \nabla_{\partial_B\tau_2}\tau_1, \partial_Q\tau_3\rangle - \langle \tau_2, \partial_Q\Delta_{\partial_Q\tau_1}\tau_3 - \partial_Q\nabla_{\partial_B\tau_3}\tau_1\rangle\\ &= \langle \tau_2, -\partial_Q\Delta_{\partial_Q\tau_1}\tau_3 + \nabla_{\partial_B\tau_3}\partial_Q\tau_1 + [\![\partial_Q\tau_1, \partial_Q\tau_3]\!]_\Delta + \partial_B^*\langle \nabla_\cdot\partial_Q\tau_1, \tau_3\rangle\rangle = 0.\end{aligned}$$

Finally we check the Jacobi identity (CA1). Using (59) and (60), we have for $\tau_1, \tau_2, \tau_3 \in \Gamma(Q^*)$:

$$\begin{aligned}&[\![[\![\tau_1, \tau_2]\!]_{Q^*}, \tau_3]\!]_{Q^*} + [\![\tau_2, [\![\tau_1, \tau_3]\!]_{Q^*}]\!]_{Q^*} - [\![\tau_1, [\![\tau_2, \tau_3]\!]_{Q^*}]\!]_{Q^*}\\ &= [\![\Delta_{\partial_Q\tau_1}\tau_2 - \nabla_{\partial_B\tau_2}\tau_1, \tau_3]\!]_{Q^*} + [\![\tau_2, \Delta_{\partial_Q\tau_1}\tau_3 - \nabla_{\partial_B\tau_3}\tau_1]\!]_{Q^*}\\ &- [\![\tau_1, \Delta_{\partial_Q\tau_2}\tau_3 - \nabla_{\partial_B\tau_3}\tau_2]\!]_{Q^*}\\ &= \Delta_{[\![\partial_Q\tau_1, \partial_Q\tau_2]\!] + \partial_B^*\langle \tau_2, \nabla_\cdot\partial_Q\tau_1\rangle}\tau_3 - \nabla_{\partial_B\tau_3}(\Delta_{\partial_Q\tau_1}\tau_2 - \nabla_{\partial_B\tau_2}\tau_1)\\ &\quad + \Delta_{\partial_Q\tau_2}(\Delta_{\partial_Q\tau_1}\tau_3 - \nabla_{\partial_B\tau_3}\tau_1) - \nabla_{[\partial_B\tau_1, \partial_B\tau_3]}\tau_2\\ &\quad - \Delta_{\partial_Q\tau_1}(\Delta_{\partial_Q\tau_2}\tau_3 - \nabla_{\partial_B\tau_3}\tau_2) + \nabla_{[\partial_B\tau_2, \partial_B\tau_3]}\tau_1\\ &= R_\nabla(\partial_B\tau_3, \partial_B\tau_2)\tau_1 + \nabla_{\partial_B\tau_2}\nabla_{\partial_B\tau_3}\tau_1 - R_\Delta(\partial_Q\tau_1, \partial_Q\tau_2)\tau_3 + \Delta_{\partial_B^*\langle \tau_2, \nabla_\cdot\partial_Q\tau_1\rangle}\tau_3\\ &\quad - \nabla_{\partial_B\tau_3}\Delta_{\partial_Q\tau_1}\tau_2 + \Delta_{\partial_Q\tau_1}\nabla_{\partial_B\tau_3}\tau_2 - \Delta_{\partial_Q\tau_2}\nabla_{\partial_B\tau_3}\tau_1 - \nabla_{[\partial_B\tau_1, \partial_B\tau_3]}\tau_2.\end{aligned}$$

Using the equalities

$$R_\Delta(\partial_Q\tau_1,\partial_Q\tau_2)\tau_3 = R(\partial_Q\tau_1,\partial_Q\tau_2)\partial_B\tau_3 \quad \text{by (D4)},$$

$$R_\nabla(\partial_B\tau_3,\partial_B\tau_2)\tau_1 = R(\partial_B\tau_3,\partial_B\tau_2)\partial_Q\tau_1 \quad \text{by the Def. of a 2-representation},$$

and (53), this is

$$\begin{aligned}&-\Delta_{\nabla_{\partial_B\tau_3}\partial_Q\tau_1}\tau_2 + \nabla_{\nabla_{\partial_Q\tau_1}\partial_B\tau_3}\tau_2 + \langle \nabla_{\nabla_{\cdot}\partial_B\tau_3}\partial_Q\tau_1,\tau_2\rangle \\ &\quad + \nabla_{\partial_B\tau_2}\nabla_{\partial_B\tau_3}\tau_1 + \Delta_{\partial_B^*\langle\tau_2,\nabla_{\cdot}\partial_Q\tau_1\rangle}\tau_3 - \nabla_{[\partial_B\tau_1,\partial_B\tau_3]}\tau_2 - \Delta_{\partial_Q\tau_2}\nabla_{\partial_B\tau_3}\tau_1.\end{aligned}$$

By (52), we can replace

$$\begin{aligned}&-\Delta_{\nabla_{\partial_B\tau_3}\partial_Q\tau_1}\tau_2 + \nabla_{\partial_B\tau_2}\nabla_{\partial_B\tau_3}\tau_1 - \Delta_{\partial_Q\tau_2}\nabla_{\partial_B\tau_3}\tau_1 \\ =&-\Delta_{\partial_Q(\nabla_{\partial_B\tau_3}\tau_1)}\tau_2 + \nabla_{\partial_B\tau_2}(\nabla_{\partial_B\tau_3}\tau_1) - \Delta_{\partial_Q\tau_2}(\nabla_{\partial_B\tau_3}\tau_1)\end{aligned}$$

by

$$-\nabla_{\partial_B(\nabla_{\partial_B\tau_3}\tau_1)}\tau_2 - \rho_Q^*\mathbf{d}\langle\tau_2,\partial_Q\nabla_{\partial_B\tau_3}\tau_1\rangle$$

and we get

$$\begin{aligned}&\nabla_{\nabla_{\partial_Q\tau_1}\partial_B\tau_3-[\partial_B\tau_1,\partial_B\tau_3]-\partial_B(\nabla_{\partial_B\tau_3}\tau_1)}\tau_2 + \langle \nabla_{\nabla_{\cdot}\partial_B\tau_3}\partial_Q\tau_1,\tau_2\rangle \\ &\quad + \Delta_{\partial_B^*\langle\tau_2,\nabla_{\cdot}\partial_Q\tau_1\rangle}\tau_3 - \rho_Q^*\mathbf{d}\langle\tau_2,\partial_Q\nabla_{\partial_B\tau_3}\tau_1\rangle\end{aligned}$$

Since $\nabla_{\partial_Q\tau_1}\partial_B\tau_3 - [\partial_B\tau_1,\partial_B\tau_3] - \partial_B(\nabla_{\partial_B\tau_3}\tau_1) = 0$ by (M2), we finally get

$$\langle \nabla_{\nabla_{\cdot}\partial_B\tau_3}\partial_Q\tau_1,\tau_2\rangle + \Delta_{\partial_B^*\langle\tau_2,\nabla_{\cdot}\partial_Q\tau_1\rangle}\tau_3 - \rho_Q^*\mathbf{d}\langle\tau_2,\partial_Q\nabla_{\partial_B\tau_3}\tau_1\rangle. \tag{61}$$

We write $\beta := \langle\nabla_{\cdot}\partial_Q\tau_1,\tau_2\rangle \in \Gamma(B^*)$. Since $\rho_Q\circ\partial_Q = \rho_B\circ\partial_B$ and $\nabla^Q\circ\partial_Q = \partial_Q\circ\nabla^{Q^*}$, we find $\beta = \langle\partial_Q\nabla_{\cdot}\tau_1,\tau_2\rangle = \langle\nabla_{\cdot}\tau_1,\partial_Q\tau_2\rangle \in \Gamma(B^*)$. To see that (61), a section of Q^*, vanishes, we evaluate it on an arbitrary $q\in\Gamma(Q)$. We use (D1) and the definition of a 2-representation and we get

$$\begin{aligned}&\langle\nabla_{\nabla_q\partial_B\tau_3}\partial_Q\tau_1,\tau_2\rangle + \langle\Delta_{\partial_B^*\beta}\tau_3,q\rangle - \rho_Q(q)\langle\tau_2,\partial_Q\nabla_{\partial_B\tau_3}\tau_1\rangle \\ =&\langle\nabla_{\partial_B\Delta_q\tau_3}\partial_Q\tau_1,\tau_2\rangle + \langle\Delta_{\partial_B^*\beta}\tau_3,q\rangle - \rho_Q(q)\langle\tau_2,\partial_Q\nabla_{\partial_B\tau_3}\tau_1\rangle \\ =&\langle\beta,\partial_B\Delta_q\tau_3\rangle + \langle\Delta_{\partial_B^*\beta}\tau_3,q\rangle - \rho_Q(q)\langle\beta,\partial_B\tau_3\rangle = -\langle[\![q,\partial_B^*\beta]\!]_\Delta,\tau_3\rangle + \langle\Delta_{\partial_B^*\beta}\tau_3,q\rangle.\end{aligned}$$

Since the Dorfman connection Δ is dual to the skew-symmetric dull bracket $[\![\cdot\,,\cdot]\!]_\Delta$, this is

$$\rho_Q(\partial_B^*\beta)\langle q,\tau_3\rangle.$$

Since $\rho_Q\circ\partial_B^* = 0$ by (47), we can conclude.

We finally prove that the degenerate Courant algebroid structure does not depend on the choice of the Lagrangian splitting. Clearly the pairing and anchor are independent of the splitting, so we only need to check that the bracket remains the same if we choose a different Lagrangian splitting. Assume that $\Sigma_1,\Sigma_2\colon B\times_M Q\to\mathbb{E}$ are two Lagrangian splittings. Then $\Delta^2_{\partial_Q\tau_1}\tau_2 - \nabla^2_{\partial_B\tau_2}\tau_1 = \Delta^1_{\partial_Q\tau_1}\tau_2 + \phi_{12}(\partial_Q\tau_1,\partial_B\tau_2) - \nabla^2_{\partial_B\tau_2}\tau_1 - \phi_{12}(\partial_Q\tau_1,\partial_B\tau_2) = \Delta^1_{\partial_Q\tau_1}\tau_2 - \nabla^1_{\partial_B\tau_2}\tau_1$ by Proposition 6.8 and the considerations following Theorem 2.6. □

We will study in a future work the N-geometric description of degenerate Courant algebroids.

Example 7.8 (Tangent Courant algebroid)**.** Consider the example described in §4.4.1, §5.4.3 and §6.4.3. The degenerate Courant algebroid structure on the core E of $T\mathsf{E}$ is just the initial Courant algebroid structure on E since $\Delta_{e_1}e_2 = [\![e_1, e_2]\!] + \nabla_{\rho(e_2)}e_1$ by definition and so

$$\Delta_{e_1}e_2 - \nabla_{\rho(e_2)}e_1 = [\![e_1, e_2]\!].$$

We have hence proved that the Courant algebroid associated to a symplectic Lie 2-algebroid can be defined directly from any of the splittings of the Lie 2-algebroid, and so does not need to be obtained as a derived bracket. Recall that we have characterised split symplectic [2]-manifolds in §4.1.1.

Theorem 7.9. *Let $\mathcal{M}$ be a symplectic Lie 2-algebroid over a base manifold M. Then the corresponding Courant algebroid is defined as follows. Choose any splitting $\mathcal{M} \simeq Q[-1] \oplus T^*M[-2]$ of the underlying symplectic [2]-manifold. Then $Q \simeq Q^*$ via ∂_Q and Q inherits a nondegenerate pairing given by $\langle \tau_1, \partial_Q\tau_2\rangle$. The split Lie 2-algebroid structure on $Q \oplus T^*M$ is dual to a Dorfman 2-representation of Q on $-l_1^* = \partial_{TM} \colon Q^* \to TM$. The map $-l_1^* = \rho_Q\partial_Q$ defines an anchor on Q and the bracket $[\![\cdot\,,\cdot]\!]_{Q^*}$ defined on $\Gamma(Q^*)$ by $[\![\tau_1, \tau_2]\!]_Q = \Delta_{\partial_Q\tau_1}\tau_2 + \{l_1^*\tau_2, \tau_1\}$ does not depend on the choice of the splitting. This anchor, pairing and bracket define a Courant algebroid structure on Q^*. The isomorphism $\partial_Q \colon Q^* \to Q$ transports this Courant algebroid structure to Q.*

Note that the Courant algebroid structure is transported to Q by $\boldsymbol{\beta} = \partial_Q \colon Q^* \to Q$ for our result to be consistent with the constructions in §4.1.1, §5.4.3 and §7.3.1.

Example 7.10 (Core of the standard Courant algebroid over a Lie algebroid)**.** Consider now the Example discussed in Example 5.4.2, §6.4.1 and §7.3.2; namely the standard LA-Courant algebroid

$$\begin{tikzcd} TA \oplus_A T^*A \arrow[r] \arrow[d] & TM \oplus A^* \arrow[d] \\ A \arrow[r] & M \end{tikzcd}$$

over a Lie algebroid A. The degenerate Courant algebroid structure on the core $A \oplus T^*M$ of $TA \oplus T^*A$ is here given by $\rho_{A\oplus T^*M}(a, \theta) = \rho_A(a)$,

$$\langle (a_1, \theta_1), (a_2, \theta_2)\rangle_{A\oplus T^*M} = \langle (a_1, \theta_1), (\rho_A, \rho_A^*)(a_2, \theta_2)\rangle$$

and the bracket defined by

$$[\![(a_1, \theta_1), (a_2, \theta_2)]\!]_{A\oplus T^*M} = ([a_1, a_2], \pounds_{\rho_A(a_1)}\theta_2 - \mathbf{i}_{\rho_A(a_2)}\mathbf{d}\theta_1)$$

for all $a, a_1, a_2 \in \Gamma(A)$ and $\theta, \theta_1, \theta_2 \in \Omega^1(M)$. To see this, use Lemma 5.16 in [16] or the next example; this degenerate Courant algebroid plays a crucial role in the infinitesimal description of Dirac groupoids [17], i.e. in the definition of *Dirac bialgebroids*.

Example 7.11 (LA-Courant algebroid associated to a double Lie algebroid)**.** More generally, the LA-Courant algebroids (and the corresponding Poisson Lie 2-algebroids) constructed in §4.4.2, §5.4.4, §6.4.2 and §7.3.3 and Theorem 7.7 yield the following application.

A matched pair of 2-representations as in Definition 2.10 defines two degenerate Courant algebroids. The first one is $C \oplus A^* \to M$ with the anchor $\rho_{C\oplus A^*} \colon C \oplus A^* \to TM$ defined by $\rho_A \circ \operatorname{pr}_A \circ(\partial_A \oplus \partial_A^*) = \rho_B \circ \partial_B \circ \operatorname{pr}_C$. The pairing is defined by

$$\langle (c_1, \alpha_1), (c_2, \alpha_2)\rangle_{C\oplus A^*} = \langle \alpha_1, \partial_A c_2\rangle + \langle \alpha_2, \partial_A c_1\rangle$$

for all $\alpha_1, \alpha_2 \in \Gamma(A^*)$ and $c_1, c_2 \in \Gamma(C)$, and the bracket by

$$\begin{aligned}
[\![(c_1, \alpha_1), (c_2, \alpha_2)]\!]_{C\oplus A^*} &= \Delta_{(\partial_A c_1, \partial_A^* \alpha_1)}(c_2, \alpha_2) - \nabla_{\partial_B c_2}(c_1, \alpha_1)\\
&= (\nabla_{\partial_A c_1} c_2 - \nabla_{\partial_B c_2} c_1, \pounds_{\partial_A c_1}\alpha_2 + \langle \nabla^*_{\cdot}\partial_A^*\alpha_1, c_2\rangle - \nabla^*_{\partial_B c_2}\alpha_1)\\
&= (\nabla_{\partial_A c_1} c_2 - \nabla_{\partial_B c_2} c_1, \pounds_{\partial_A c_1}\alpha_2 - \mathbf{i}_{\partial_A c_2}\mathbf{d}_A\alpha_1).
\end{aligned}$$

The second degenerate Courant algebroid is $C \oplus B^* \to M$ with the anchor $\rho_{C\oplus B^*} \colon C \oplus B^* \to TM$ defined by $\rho_B \circ \operatorname{pr}_B \circ(\partial_B \oplus \partial_B^*) = \rho_B \circ \partial_B \circ \operatorname{pr}_C$, the pairing defined by

$$\langle (c_1, \beta_1), (c_2, \beta_2)\rangle_{C\oplus B^*} = \langle \beta_1, \partial_B c_2\rangle + \langle \beta_2, \partial_B c_1\rangle$$

for all $\beta_1, \beta_2 \in \Gamma(B^*)$ and $c_1, c_2 \in \Gamma(C)$, and the bracket

$$[\![(c_1, \beta_1), (c_2, \beta_2)]\!]_{C\oplus B^*} = (\nabla_{\partial_B c_1} c_2 - \nabla_{\partial_A c_2} c_1, \pounds_{\partial_B c_1}\beta_2 - \mathbf{i}_{\partial_B c_2}\mathbf{d}_B\beta_1).$$

Note that in both cases, the restriction to $\Gamma(C)$ of the Courant bracket is the Lie algebroid bracket induced on C by the matched pair, see §2.11.

8. VB-Dirac structures, LA-Dirac structures and pseudo Dirac structures

In this section, we study isotropic subalgebroids of metric VB-algebroids and Dirac structures in VB- and LA-Courant algebroids. While we paid attention in the preceding sections to bridge [2]-geometric objects to geometric structures on metric double vector bundles, we are here more interested in classifications of Dirac structures via the simple geometric descriptions that we found before for VB-Courant algebroids and LA-Courant algebroids.

8.1. **VB-Dirac structures.** Let $(\mathbb{E}; B, Q; M)$ be a VB-Courant algebroid with core Q^* and anchor $\Theta \colon \mathbb{E} \to TB$. Let D be a double vector subbundle structure over $B' \subseteq B$ and $U \subseteq Q$ and with core K. Choose a linear splitting $\Sigma \colon B \times_M Q \to \mathbb{E}$ that is adapted[17] to D, i.e. such that $\Sigma(B' \times_M U) \subseteq D$. Then D is spanned as a vector bundle over B' by the sections $\sigma_Q(u)|_{B'}$ for all $u \in \Gamma(U)$ and $\tau^\dagger|_{B'}$ for all $\tau \in \Gamma(K)$.

The following two propositions follow immediately from the study of linear metrics in §3.2.1 and the definition in (22).

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[17] To see that such a splitting exist, we work with decompositions. Since D and $\mathbb{E}$ are both double vector bundles, there exist two decompositions $\mathbb{I}_D \colon B' \times_M U \times_M K \to D$ and $\mathbb{I} \colon B \times_M Q \times_M Q^* \to \mathbb{E}$. Let $\iota \colon D \to \mathbb{E}$ be the double vector bundle inclusion, over $\iota_U \colon U \to Q$ and $\iota_{B'} \colon B' \to B$, and with core $\iota_K \colon K \to Q^*$. Then there exists $\phi \in \Gamma(B'^* \otimes U^* \otimes Q^*)$ such that the map $\mathbb{I}^{-1} \circ \iota \circ \mathbb{I}_D \colon B' \times_M U \times_M K \to B \times_M Q \times_M Q^*$ sends (b_m, u_m, k_m) to $(\iota_B(b_m), \iota_U(u_m), \iota_K(k_m) + \phi(b_m, u_m))$. Using local basis sections of B and Q adapted to B' and U and a partition of unity on M, extend ϕ to $\hat{\phi} \in \Gamma(B^* \otimes Q \otimes Q^*)$. Then define a new decomposition $\tilde{\mathbb{I}}^{-1} \colon \mathbb{E} \to B \times_M Q \times_M Q^*$ by $\tilde{\mathbb{I}}^{-1}(e) = \mathbb{I}^{-1}(e) +_B (b_m, 0^Q_m, -\hat{\phi}(b_m, q_m)) = \mathbb{I}^{-1}(e) +_Q (0^B_m, q_m, -\hat{\phi}(b_m, q_m))$ for $e \in \mathbb{E}$ with $\pi_B(e) = b_m$ and $\pi_Q(e) = q_m$. Then $(\tilde{\mathbb{I}} \circ \iota \circ \mathbb{I}_D)(b_m, u_m, k_m) = (\iota_B(b_m), \iota_U(u_m), \iota_K(k_m))$ for all $(b_m, u_m, k_m) \in B' \times_M U \times_M K$. The corresponding linear splitting $\tilde{\Sigma} \colon B \times_M Q \to \mathbb{E}$, $\tilde{\Sigma}(b_m, q_m) = \tilde{\mathbb{I}}(b_m, q_m, 0^{Q^*}_m)$ sends $(\iota_{B'}(b_m), \iota_U(u_m))$ to $\iota(\mathbb{I}_D(b_m, u_m, 0^K_m) \in \iota(D)$.

Proposition 8.1. *In the situation described above, the double subbundle $D \subseteq \mathbb{E}$ over B' is isotropic if and only if $K \subseteq U^\circ$ and Λ as in* (22) *sends $U \otimes U$ to B'°.*

Proposition 8.2. *In the situation described above, D is maximal isotropic if and only if $U = K^\circ$ and Λ sends $U \otimes U$ to B'°.*

Now we can prove that if D is maximal isotropic, then there exists a *Lagrangian* splitting of $\mathbb{E}$ that is adapted to D.

Corollary 8.3. *Let $(\mathbb{E}, B; Q, M)$ be a metric double vector bundle and $D \subseteq \mathbb{E}$ a maximal isotropic double subbundle. Then there exists a Lagrangian splitting that is adapted to D.*

Proof. As before, let $U \subseteq Q$ and $B' \subseteq B$ be the sides of D. Then by Proposition 8.2 the core of D is the vector bundle $U^\circ \subseteq Q^*$. Choose a linear splitting $\Sigma\colon Q \times_M B \to \mathbb{E}$ that is adapted to D. Then D is spanned as a vector bundle over B' by the sections $\sigma_Q(u)|_{B'}$ for all $u \in \Gamma(U)$ and $\tau^\dagger|_{B'}$ for all $\tau \in \Gamma(U^\circ)$. As in the proof of Theorem 3.8, transform Σ into a new Lagrangian linear splitting Σ'. We need to show that $\sigma'_Q(u)|_{B'} - \sigma_Q(u)|_{B'}$ is equivalent to a section of $B'^* \otimes U^\circ$ for all $u \in \Gamma(U)$. But $\sigma'_Q(u) - \sigma_Q(u) = \frac{1}{2}\widetilde{\Lambda(u, \cdot)}$ by construction and, since D is isotropic, we have $\Lambda(u, u')|_{B'} = 0$ for all $u, u' \in \Gamma(U)$. □

Remark 8.4. Consider a Courant algebroid $\mathsf{E} \to M$ and its tangent double $T\mathsf{E}$. Recall from Example 3.11 that Lagrangian splittings of $T\mathsf{E}$ are equivalent to metric connections $\mathfrak{X}(M) \times \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E})$. Let ∇ be such a metric connection, that is adapted to a maximally isotropic double subbundle D over the sides TM and $U \subseteq \mathsf{E}$. Define $[\nabla]\colon \mathfrak{X}(M) \times \Gamma(U) \to \Gamma(\mathsf{E}/U^\perp)$ by $[\nabla]_X u = \overline{\nabla_X u} \in \Gamma(\mathsf{E}/U^\perp)$. A second metric connection $\nabla'\colon \mathfrak{X}(M) \times \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E})$ is adapted to D if and only if $\nabla_X u - \nabla'_X u \in \Gamma(U^\circ)$ for all $X \in \mathfrak{X}(M)$ and for all $u \in \Gamma(U)$. Hence, if and only if $[\nabla] = [\nabla']$. We call $[\nabla]$ the **invariant part of the metric connection adapted to D**.

The existence of Lagrangian splittings of $\mathbb{E}$ adapted to maximal isotropic double subbundles D will now be used to study the involutivity of D. Note that in a very early version of this work, we studied VB-Courant algebroids via general (not necessarily Lagrangian) linear splittings. We found some more general objects than Dorfman 2-representations; basically Dorfman 2-connections with the additional structure object Λ appearing in (D1) to (D5) in Definition 5.8. The study of the involutivity of general (not necessarily isotropic) double subbundles D of $\mathbb{E}$ was then possible, and yielded very similar results.

Proposition 8.5. *Let $(\mathbb{E}, B; Q, M)$ be a VB-Courant algebroid and $D \subseteq \mathbb{E}$ a maximal isotropic double subbundle. Choose a Lagrangian splitting of $\mathbb{E}$ that is adapted to D and consider the corresponding Dorfman 2-representation, denoted as usual. Then D is a Dirac structure in $\mathbb{E}$ with support B' if and only if*

1. $\partial_B(U^\circ) \subseteq B'$,
2. $\nabla_u b \in \Gamma(B')$ *for all* $u \in \Gamma(U)$ *and* $b \in \Gamma(B')$,
3. $[\![u_1, u_2]\!] \in \Gamma(U)$ *for all* $u_1, u_2 \in \Gamma(U)$,
4. $R(u_1, u_2)$ *restricts to a section of* $\Gamma(\operatorname{Hom}(B', U^\circ))$ *for all* $u_1, u_2 \in \Gamma(U)$.

A Dirac double subbundle D of a VB-Courant algebroid $\mathbb{E}$ as in the proposition is called a **VB-Dirac structure**.

Proof. This is easy to prove using Lemma 5.3 on sections $\sigma_Q(u)$ and $\tau^\dagger$, for $u \in \Gamma(U)$ and $\tau \in \Gamma(U^\circ)$. Their anchors and Courant brackets can be described by

$$\begin{aligned} &\Theta(\sigma_Q(u)) = \widehat{\nabla_u} \in \mathfrak{X}^l(B), \qquad \Theta(\tau^\dagger) = (\partial_B\tau)^\uparrow \in \mathfrak{X}^c(B), \\ &\llbracket \sigma_Q(u_1), \sigma_Q(u_2) \rrbracket = \sigma_Q(\llbracket u_1, u_2 \rrbracket) - \widetilde{R(u_1, u_2)}, \\ &\llbracket \sigma_Q(u), \tau^\dagger \rrbracket = (\Delta_u \tau)^\dagger, \qquad \llbracket \tau_1^\dagger, \tau_2^\dagger \rrbracket = 0 \end{aligned} \tag{62}$$

for all $u, u_1, u_2 \in \Gamma(U)$ and $\tau, \tau_1, \tau_2 \in \Gamma(U^\circ)$. The vector field $\widehat{\nabla_u}$ is tangent to B' on B' if and only if for all $\beta \in \Gamma((B')^\circ)$, $\widehat{\nabla_u}(\ell_\beta) = \ell_{\nabla^*_u\beta}$ vanishes on B'. That is, if and only if, for all $\beta \in \Gamma((B')^\circ)$, $\nabla^*_u\beta$ is again a section of $(B')^\circ$. This yields (2). The vector field $(\partial_B\tau)^\uparrow$ is tangent to B' if and only if $\partial_B\tau \in \Gamma(B')$. This yields (1). Next, $\llbracket \sigma_Q(u), \tau^\dagger \rrbracket = (\Delta_u\tau)^\dagger$ is a section of D over B' if and only if $\Delta_u\tau \in \Gamma(U^\circ)$. Since $\Delta_u\tau \in \Gamma(U^\circ)$ for all $u \in \Gamma(U)$ and $\tau \in \Gamma(U^\circ)$ if and only if $\llbracket u_1, u_2 \rrbracket \in \Gamma(U)$ for all $u_1, u_2 \in \Gamma(U)$, this is (3). Further, $\sigma_Q(\llbracket u_1, u_2 \rrbracket)$ takes then values in D over B', and so $\llbracket \sigma_Q(u_1), \sigma_Q(u_2) \rrbracket$ takes values in D over B' if and only if $R(u_1, u_2)$ restricts to a morphism $B' \to U^\circ$. This is (4). □

We get the following result for ordinary VB-Dirac structures (with support B) in $\mathbb{E}$.

Corollary 8.6. *Let $(\mathbb{E}, B; Q, M)$ be a VB-Courant algebroid and $(D, B, U, M) \subseteq \mathbb{E}$ a maximal isotropic double subbundle. Choose a Lagrangian splitting of $\mathbb{E}$ that is adapted to D and consider the corresponding Dorfman 2-representation, denoted as usual. If D is a Dirac structure in $\mathbb{E} \to B$, then U inherits a Lie algebroid structure with bracket $\llbracket \cdot\,, \cdot \rrbracket|_{\Gamma(U)\times\Gamma(U)}$ and anchor $\rho_Q|_U$. This Lie algebroid structure does not depend on the choice of Lagrangian splitting.*

Proof. By (3) in Proposition 8.5, $\llbracket \cdot\,, \cdot \rrbracket$ restricts to a bracket on sections of U. For u_1, u_2, u_3, $\mathrm{Jac}_{\llbracket \cdot\,,\cdot \rrbracket}(u_1, u_2, u_3) = \partial_B^*\omega_R(u_1, u_2, u_3) = 0$ since $\omega_R(u_1, u_2, u_3) = \langle R(u_1, u_2), u_3 \rangle = 0 \in \Gamma(B^*)$ by (4) in Proposition 8.5. Hence, U with the bracket $\llbracket \cdot\,, \cdot \rrbracket|_{\Gamma(U)\times\Gamma(U)}$ and the anchor $\rho_Q|_U$ is a Lie algebroid.

If $\phi_{12} \in \Gamma(Q^* \wedge Q^* \otimes B^*)$ is the tensor defined as in §2.2.1 by a change of Lagrangian splitting adapted to D, then, by Proposition 6.8,

$$\llbracket u, u' \rrbracket_1 = \llbracket u, u' \rrbracket_2 + \partial_B^*\phi_{12}(u, u')$$

for all u, u'. But since both splittings $\Sigma^1, \Sigma^2 \colon B \times_M Q \to \mathbb{E}$ are adapted to D, we know that $\sigma_Q^1(u)$ and $\sigma_Q^2(u)$ have values in D, and their difference $\sigma_Q^1(u) - \sigma_Q^2(u) = \widetilde{\phi_{12}(u)}$ is a core-linear section of $D \to B$. Hence it must takes values in U°, and $\phi_{12}(u, u')$ must so vanish for all $u, u' \in \Gamma(U)$. As a consequence, $\llbracket u, u' \rrbracket_1 = \llbracket u, u' \rrbracket_2$. □

The following two corollaries are now easy to prove. (The first one was already found in [19].)

Corollary 8.7. *Let $(\mathcal{M}, \mathcal{Q})$ be a Lie 2-algebroid, and $(\mathbb{E} \to B, Q \to M)$ the corresponding VB-Courant algebroid. Then VB-Dirac structures in $\mathbb{E}$ are equivalent to wide Lie 1-subalgebroids of $(\mathcal{M}, \mathcal{Q})$.*

Proof. A wide Lie subalgebroid of $(\mathcal{M}, \mathcal{Q})$ is a wide [1]-submanifold $U[-1]$ of $\mathcal{M}$ such that $\mathcal{Q}_U(\mu^\star\xi) = \mu^\star(\mathcal{Q}(\xi))$, $\xi \in C^\infty(\mathcal{M})$, defines a Lie algebroid structure $\mathcal{Q}_U$ on U. Here, $\mu \colon U[-1] \to \mathcal{M}$ is the submanifold inclusion as in Definition 3.4.

In a splitting $Q[-1] \oplus B^*[-2]$ of $\mathcal{M}$, the homological vector field $\mathcal{Q}$ is given by (34). Assume that the choice of local basis vector fields is as in the discussion after Definition 3.4. Then we have $\mathcal{Q}_U(f) = \mu^\star(\rho_Q^* \mathbf{d}f) = \rho_Q^* \mathbf{d}f + U^\circ$ for all $f \in C^\infty(M)$. This translates easily to $\rho_U = \rho_Q|_U$. Then we have $\mathcal{Q}_U(\tau_i + U^\circ) = \mathcal{Q}_U(\mu^\star \tau_i) = \mu^\star(\mathcal{Q}(\tau_k)) = -\sum_{i<j}^r \langle [\![u_i, u_j]\!], \tau_k \rangle \bar{\tau}_i \bar{\tau}_j$ for $k = 1, \ldots, r$. This shows that the bracket on U must be the restriction to $\Gamma(U)$ of the dull bracket on $\Gamma(Q)$. Finally $0 = \mathcal{Q}_U(\mu^\star b_l) = \mu^\star(\mathcal{Q}(b_l)) = -\sum_{i<j<k<r} \omega_R(u_i, u_j, u_k)(b_l) \bar{\tau}_i \bar{\tau}_j \bar{\tau}_k$ for all l shows that $\omega_R(u_1, u_2, u_3)$ must be zero for all $u_1, u_2, u_3 \in \Gamma(U)$. This is equivalent to (3) in Proposition 8.5 (with $B' = B$). Note that since $B' = B$, (1) and (2) in Proposition 8.5 are trivially satisfied. Hence we can conclude. ☐

Corollary 8.8. *A VB-Dirac structure $(D, B; U, M)$ in a VB-Courant algebroid inherits a linear Lie algebroid structure $(D \to B, U \to M)$.*

Proof. ☐

8.2. **LA-Dirac structures.** Assume now that $(\mathbb{E} \to Q; B \to M)$ is a metric VB-algebroid, and take a maximal isotropic double subbundle D of $\mathbb{E}$ over the sides $U \subseteq Q$ and $B' \subseteq B$. We will study conditions on the self-dual 2-representation defined by a Lagrangian splitting and the linear Lie algebroid structure on $\mathbb{E} \to Q$, and on Q and on B', for D to be an isotropic subalgebroid of $\mathbb{E} \to Q$ over U.

Note the similarity of the following result with Proposition 8.5.

Proposition 8.9. *Let $(\mathbb{E}, B; Q, M)$ be a metric VB-algebroid and $(D, B'; U, M) \subseteq \mathbb{E}$ a maximal isotropic double subbundle. Choose a Lagrangian splitting of $\mathbb{E}$ that is adapted to D and consider the corresponding self-dual 2-representation, denoted as usual. Then $D \to U$ is a subalgebroid of $\mathbb{E} \to Q$ if and only if*

(1) $\partial_Q(U^\circ) \subseteq U$,
(2) $\nabla_b u \in \Gamma(U)$ *for all* $u \in \Gamma(U)$ *and* $b \in \Gamma(B')$,
(3) $[b_1, b_2] \in \Gamma(B')$ *for all* $b_1, b_2 \in \Gamma(B')$,
(4) $R(b_1, b_2)$ *restricts to a section of* $\Gamma(\operatorname{Hom}(U, U^\circ))$ *for all* $b_1, b_2 \in \Gamma(B')$.

Proof. This proof is very similar to the proof of Proposition 8.5, and left to the reader. ☐

Now let $(\mathbb{E}; Q, B; M)$ be an LA-Courant algebroid. A VB-Dirac structure $(D; U, B'; M)$ in $\mathbb{E}$ is an **LA-Dirac structure** if $(D \to U, B' \to M)$ is also a subalgebroid of $(\mathbb{E} \to Q; B \to M)$. We deduce from Propositions 8.5 and 8.9 a characterisation of LA-Dirac structures.

Proposition 8.10. *Let $(\mathbb{E}, B; Q, M)$ be an LA-Courant algebroid and $(D, B'; U, M)$ a maximal isotropic double subbundle of $\mathbb{E}$. Choose a Lagrangian splitting of $\mathbb{E}$ that is adapted to D and consider the corresponding matched self-dual 2-representation and Dorfman 2-representation. Then $D \to U$ is an LA-Dirac structure in $\mathbb{E}$ if and only if*

(1) $\partial_B(U^\circ) \subseteq B'$ *and* $\partial_Q(U^\circ) \subseteq U$,
(2) $\nabla_u b \in \Gamma(B')$ *for all* $u \in \Gamma(U)$ *and* $b \in \Gamma(B')$,
(3) $\nabla_b u \in \Gamma(U)$ *for all* $u \in \Gamma(U)$ *and* $b \in \Gamma(B')$,
(4) $[\![u_1, u_2]\!] \in \Gamma(U)$ *for all* $u_1, u_2 \in \Gamma(U)$,
(5) $[b_1, b_2] \in \Gamma(B')$ *for all* $b_1, b_2 \in \Gamma(B')$,
(6) $R(u_1, u_2)$ *restricts to a section of* $\Gamma(\operatorname{Hom}(B', U^\circ))$ *for all* $u_1, u_2 \in \Gamma(U)$,
(7) $\nabla_b u \in \Gamma(U)$ *for all* $u \in \Gamma(U)$ *and* $b \in \Gamma(B')$,

(8) *$[b_1, b_2] \in \Gamma(B')$ for all $b_1, b_2 \in \Gamma(B')$,*

(9) *$R(b_1, b_2)$ restricts to a section of $\Gamma(\mathrm{Hom}(U, U^\circ))$ for all $b_1, b_2 \in \Gamma(B')$.*

Hence, we also have the following result.

Corollary 8.11. *VB-subalgebroids $(D \to U, B \to M)$ of a metric VB-algebroid $(\mathbb{E} \to Q, B \to M)$ are equivalent to wide coisotropic $[1]$-submanifolds of the corresponding Poisson $[2]$-manifold.*

LA-Dirac structures $(D \to U, B \to M)$ in an LA-Courant algebroid $(\mathbb{E} \to Q, B \to M)$ are equivalent to wide coisotropic Lie subalgebroids of the corresponding Poisson Lie 2-algebroid.

Proof. Let $U[-1]$ be a [1]-submanifold of a Poisson [2]-manifold $(\mathcal{M}, \{\cdot\,,\cdot\})$. Then $U[-1]$ is coisotropic if and only if $\mu^\star(\xi) = \mu^\star(\eta) = 0$ imply $\mu^\star(\{\xi, \eta\}) = 0$ for all $\xi, \eta \in C^\infty(\mathcal{M})$, where $\mu\colon Q[-1] \to \mathcal{M}$ is the inclusion as in Definition 3.4. In a local splitting, we find easily that this implies $\partial_Q(U^\circ) \subseteq U$, $\nabla_b^* \tau \in \Gamma(U^\circ)$ for all $b \in \Gamma(B)$ and $\tau \in \Gamma(U^\circ)$, and the restriction to U of $R(b_1, b_2)$ has image in U°. By Proposition 8.9, we can conclude.

The second claim follows with Corollary 8.7. ☐

As a corollary of Theorem 7.5, Proposition 8.5 and Proposition 8.9, we get the following theorem.

Theorem 8.12. *Let $(\mathbb{E}, B; Q, M)$ be an LA-Courant algebroid and $(D, U; B, M) \subseteq \mathbb{E}$ a (wide) LA-Dirac structure in $\mathbb{E}$.*

Then D is a double Lie algebroid with the VB-algebroid structure in Corollary 8.8 and the VB-algebroid structure $(D \to U, B \to M)$.

Proof. Let us study the two linear Lie algebroid structures on D. Choose as before a linear splitting $\Sigma\colon B \times_M Q \to \mathbb{E}$ that restricts to a linear splitting $\Sigma_D\colon U \times_M B \to D$ of D. The LA-Courant algebroid structure of $\mathbb{E}$ is then encoded as in Theorems 6.7 and 2.6, respectively, by a Dorfman 2-representation (Δ, ∇, R) of (Q, ρ_Q) on $\partial_B\colon Q^* \to B$ and by a self-dual 2-representation (∇, ∇^*, R) of the Lie algebroid B on $\partial_Q = \partial_Q^*\colon Q^* \to Q$. By Theorem 7.5, the Dorfman 2-representation and the 2-representation form a matched pair as in Definition 7.1.

By Proposition 8.5 and Corollary 8.6, the restriction to $\Gamma(U)$ of the dull bracket on $\Gamma(Q)$ that is dual to Δ defines a Lie algebroid structure on U, $R|_{U \otimes U}$ can be seen as an element of $\Omega^2(U, \mathrm{Hom}(B, U^\circ))$ and since $\Delta_u \tau \in \Gamma(U^\circ)$ for all $u \in \Gamma(U)$ and $\tau \in \Gamma(U^\circ)$, the Dorfman connection Δ restricts to a map $\Delta^D\colon \Gamma(U) \times \Gamma(U^\circ) \to \Gamma(U^\circ)$. Since $\Delta_u^D(f\tau) = f\Delta_u^D \tau + \rho_Q(u)(f)\tau$ and $\Delta_{fu}^D \tau = f\Delta_u^D \tau + \langle u, \tau\rangle \rho_Q^* \mathbf{d} f = f\Delta_u \tau$ for $f \in C^\infty(M)$, we find that this restriction is in fact an ordinary connection. Since $R(u_1, u_2)^* u_3$ then vanishes for all $u_1, u_2, u_3 \in \Gamma(U)$, it is then easy to see that the restrictions of (D1), (D4) and (D6) to sections of U and U° define a ordinary 2-representation. By (62), this 2-representation $(\partial_B\colon U^\circ \to B, \nabla, \Delta^D, R|_{\Gamma(U) \times \Gamma(U)})$ of the Lie algebroid U on $\partial_B\colon U^\circ \to B$ encodes the VB-algebroid structure that $D \to B$ inherits from the Courant algebroid $\mathbb{E} \to B$.

In a similar manner, we find using Proposition 8.9 that the self-dual 2-representation $(\partial_Q\colon Q^* \to Q, \nabla, \nabla^*, R \in \Omega^2(B, Q^* \wedge Q^*))$ restricts to a 2-representation $(\partial_B\colon U^\circ \to U, \nabla^U\colon \Gamma(B) \times \Gamma(U) \to \Gamma(U), \nabla^{U^\circ}\colon \Gamma(B) \times \Gamma(U^\circ) \to \Gamma(U^\circ), R \in \Omega^2(B, \mathrm{Hom}(U, U^\circ)))$ of B.

A study of the restrictions to sections of U and U° of the equations in Definition 7.1 shows then that (M1) restricts to (2) in Definition 2.10 since $\partial_B^* \langle \tau, \nabla^U \cdot u\rangle = 0$

for all $u \in \Gamma(U)$ and $\tau \in \Gamma(U^\circ)$. The equations (M2), and (M3) and immediately (3) and (6), respectively. (M4) restricts to (5) since $\langle R(\cdot, b)u_1, u_2\rangle = 0$ for all $u_1, u_2 \in \Gamma(U)$ and $b \in \Gamma(B)$. (M5) restricts to (7) since the right-hand side of (M5) in (1) of Remark 7.2 vanishes. Finally, (52) restricts to (1) and (53) restricts to (4) since $\langle \nabla_{\nabla_{\cdot}b}u, \tau\rangle = 0$ for all $b \in \Gamma(B)$, $u \in \Gamma(U)$ and $\tau \in \Gamma(U^\circ)$. Thus, the two 2-representations describing the sides of D given the splitting Σ_D form a matched pair, which implies that D is a double Lie algebroid (see [12] or §2.4.2 for a quick summary of this paper). ☐

Note finally that with a different approach as the one adopted in this paper, we could deduce the main result in [12] from our Theorem 7.5. If we had shown without the use of these results that for each double Lie algebroid (D, A, B, M) with core C, the direct sum over B of D and $D \ast B$ defines an LA-Courant algebroid $(D \oplus_B (D \ast B), A \oplus C^*, B, M)$ as in §7.3.3, then we could use the last Theorem to deduce the equations in Definition 2.10 from the ones in Definition 7.1 and in Remark 7.2: by construction, the double vector subbundle D of $D \oplus_B (D \ast B)$ is a VB-Dirac structure in $D \oplus_B (D \ast B) \to B$ and a linear Lie subalgebroid in $D \oplus_B (D \ast B) \to A \oplus C^*$. We have chosen to use the main theorem in [12] to prove that $(D \oplus_B (D \ast B), A \oplus C^*, B, M)$ is an LA-Courant algebroid, see §7.3.3.

8.3. **Pseudo-Dirac structures.** We explain here the notion of pseudo-Dirac structures that was introduced in [19, 20] and we compare it with our approach to VB- and LA-Dirac structures in the tangent of a Courant algebroid. Consider a VB-Courant algebroid

$$\begin{tikzcd} \mathbb{E} \arrow[r] \arrow[d] & Q \arrow[d] \\ B \arrow[r] & M \end{tikzcd}$$

with core Q^*, and a double vector subbundle

$$\begin{tikzcd} D \arrow[r] \arrow[d] & U \arrow[d] \\ B \arrow[r] & M \end{tikzcd}$$

in $\mathbb{E}$ with core K. Consider the restriction $\mathbb{E}|_U$ of $\mathbb{E}$ to U; i.e. $\mathbb{E}|_U = \pi_Q^{-1}(U)$. This is a double vector bundle

$$\begin{tikzcd} \mathbb{E}|_U \arrow[r] \arrow[d] & U \arrow[d] \\ B \arrow[r] & M \end{tikzcd}$$

with core Q^*. The **total quotient** of $\mathbb{E}|_U$ by D is the map q from

$$\begin{tikzcd} \mathbb{E}|_U \arrow[r] \arrow[d] & Q \arrow[d] \\ B \arrow[r] & M \end{tikzcd} \quad \text{to} \quad \begin{tikzcd} Q^*/K \arrow[r] \arrow[d] & 0^M \arrow[d] \\ 0^M \arrow[r] & M, \end{tikzcd}$$

defined by

$$\mathsf{q}(e) = \bar{\tau} \Leftrightarrow e - \tau^\dagger \in D.$$

After the choice of a linear splitting of $\mathbb{E}$ that is adapted to D, we know that each element of $\mathbb{E}|_U$ can be written $\sigma_Q(u)(b_m) + \tau^\dagger(b_m)$ for some $u \in \Gamma(U)$, $\tau \in \Gamma(Q^*)$ and $b_m \in B$. The image of $\sigma_Q(u)(b_m) + \tau^\dagger(b_m)$ under q is then simply $\overline{\tau}(m)$. It is easy to see that D can be recovered from q. Recall that if $e_1 = \sigma_Q(u_1)(b_m) + \tau_1^\dagger(b_m)$ and $e_2 = \sigma_Q(u_2)(b_m) + \tau_2^\dagger(b_m) \in \mathbb{E}$, then

$$\begin{aligned}\langle e_1, e_2\rangle &= \langle \sigma_Q(u_1)(b_m) + \tau_1^\dagger(b_m), \sigma_Q(u_2)(b_m) + \tau_2^\dagger(b_m)\rangle \\ &= \ell_{\Lambda(u_1,u_2)}(b_m) + \langle u_1(m), \tau_2(m)\rangle + \langle u_2(m), \tau_1(m)\rangle.\end{aligned}$$

In particular,

$$\langle e_1, e_2\rangle = \langle \pi_Q(e_1), \mathsf{q}(e_2)\rangle + \langle \pi_Q(e_2), \mathsf{q}(e_1)\rangle$$

for all $e_1, e_2 \in \mathbb{E}|_U$ if and only if $\Lambda|_{U\otimes U}$ vanishes and $K = U^\circ$, i.e. if and only if D is maximal isotropic (Proposition 8.2).

Now we recall Li-Bland's definition of a pseudo-Dirac structure [20].

Definition 8.13. *Let $\mathsf{E} \to M$ be a Courant algebroid. A pseudo-Dirac structure is a pair (U, ∇^p) consisting of a subbundle $U \subseteq \mathsf{E}$ together with a map $\nabla^p\colon \Gamma(U) \to \Omega^1(M, U^*)$ satisfying*

1. $\nabla^p(fu) = f\nabla^p u + \mathbf{d}f \otimes \langle u, \cdot\rangle$,
2. $\mathbf{d}\langle u_1, u_2\rangle = \langle \nabla^p u_1, u_2\rangle + \langle u_1, \nabla^p u_2\rangle$,
3. $[\![u_1, u_2]\!]_p := [\![u_1, u_2]\!]_{\mathsf{E}} - \rho^*\langle \nabla^p u_1, u_2\rangle$ *defines a bracket* $\Gamma(U)\times\Gamma(U) \to \Gamma(U)$,
4. *and*

$$(\langle [\![u_1, u_2]\!]_p, \nabla^p u_3\rangle + \mathbf{i}_{\rho(u_1)}\mathbf{d}\langle \nabla^p u_2, u_3\rangle) + \text{c.p.} \\ + \mathbf{d}\left(\left\langle \nabla^p_{\rho(u_1)} u_2 - \nabla^p_{\rho(u_2)} u_1, u_3\right\rangle - \langle [\![u_1, u_2]\!]_p, u_3\rangle\right) = 0 \tag{63}$$

for all $u_1, u_2, u_3 \in \Gamma(U)$ and $f \in C^\infty(M)$.

Consider the tangent double $(T\mathsf{E}, TM, \mathsf{E}, M)$ where E is a Courant algebroid over M. Choose a linear (wide) Dirac structure D in $T\mathsf{E}$, over the side $U \subseteq \mathsf{E}$ and a metric connection $\nabla\colon \mathfrak{X}(M)\times\Gamma(\mathsf{E}) \to \Gamma(\mathsf{E})$ that is adapted to D. Li-Bland define the pseudo-Dirac structure associated to D [20] as the map $\nabla^p\colon \Gamma(U) \to \Gamma(\mathrm{Hom}(TM, U^*))$ that is defined by $\nabla^p u = \mathsf{q} \circ Tu$ for all $u \in \Gamma(U)$. By definition of $\sigma_{\mathsf{E}}^\nabla$, we have $Tu = \sigma_{\mathsf{E}}^\nabla(u) + \widetilde{\nabla_. u}$ and we find that $\nabla^p u(v_m) = \overline{\nabla_{v_m} u} = [\nabla]_{v_m} u$. The pseudo-Dirac structure is nothing else than the invariant part of the metric connection that is adapted to D (Remark 8.4). Condition (2) in Definition 8.13 is then

$$\mathbf{d}\langle u_1, u_2\rangle = \langle Tu_1, Tu_2\rangle_{T\mathsf{E}} = \langle u_1, \nabla^p u_2\rangle + \langle u_2, \nabla^p u_1\rangle \tag{64}$$

for all $u_1, u_2 \in \Gamma(U)$ and Condition (1) is

$$\nabla^p(\varphi\cdot u) = \overline{\nabla_.(\varphi\cdot u)} = \varphi\cdot\overline{\nabla_. u} + \mathbf{d}\varphi\otimes\overline{u} = \varphi\cdot\nabla^p u + \mathbf{d}\varphi\otimes\overline{u}. \tag{65}$$

The bracket $[\![\cdot\,,\cdot]\!]_p$ is then

$$[\![u_1, u_2]\!]_p = [\![u_1, u_2]\!]_{\mathsf{E}} - \rho^*\langle \nabla^p u_1, u_2\rangle = [\![u_1, u_2]\!]_{\mathsf{E}} - \rho^*\langle \nabla_. u_1, u_2\rangle = [\![u_1, u_2]\!]_\nabla,$$

the bracket defined in §5.4.3. Finally, a straightforward computation shows that the left-hand side of (63) equals $R_\Delta^{\mathrm{bas}}(u_1, u_2)^* u_3 \in \Gamma(B^*)$, which is zero by Proposition 8.5. Li-Bland proves that the bracket $[\![\cdot\,,\cdot]\!]_p$ defines a Lie algebroid structure on U. More explicitly, he finds that the left-hand side $\Psi(u_1, u_2, u_3)$ of (63) defines a tensor $\Psi \in \Omega^3(U, T^*M)$ that is related as follows to the Jacobiator of $[\![\cdot\,,\cdot]\!]_p$:

$\mathrm{Jac}_{[\![\cdot\,,\cdot]\!]_p} = (\boldsymbol{\beta}^{-1}\circ\rho_{\mathsf{E}}^*)\Psi$. He proves so that (wide) linear Dirac structures in $T\mathsf{E}$ are in bijection with pseudo-Dirac structures on E. Hence, our result in Proposition 8.5 is a generalisation of Li-Bland's result to linear Dirac structures in general VB-Courant algebroids.

Further, our Theorem 8.9 can be formulated as follows in Li-Bland's setting.

Theorem 8.14. *In the correspondence of linear Dirac structures with pseudo-Dirac connections in* [20]*, LA-Dirac structures correspond to pseudo-Dirac connections* (U,∇^p) *such that*

1. $U\subseteq\mathsf{E}$ *is an isotropic (or 'quadratic') subbundle, i.e.* $U^\perp\subseteq U$*,*
2. ∇^p *sends* $U^\perp$ *to zero and so, by Condition (2) in Definition 8.13, has image in* $U/U^\perp\subseteq\mathsf{E}/U^\perp\simeq U^*$*,*
3. *the induced ordinary connection* $\overline{\nabla^p}\colon\Gamma(U/U^\perp)\to\Omega^1(M,U/U^\perp)$ *is flat.*

We propose to call these pseudo-Dirac connections **quadratic pseudo-Dirac connections**. Note that $\overline{\nabla^p}$ equals $\bar\nabla\colon\mathfrak{X}(M)\times\Gamma(U/U^\perp)\to\Gamma(U/U^\perp)$ $\bar\nabla_X\bar u=\overline{\nabla_X u}$, $u\in\Gamma(U)$ and $X\in\mathfrak{X}(M)$, for any metric connection $\nabla\colon\mathfrak{X}(M)\times\Gamma(U)\to\Gamma(U)$ such that $[\nabla]=\nabla^p$. Such a connection must preserve U by Condition (2) in Proposition 8.9, and so also $U^\perp$ since it is metric. The condition $R_\nabla(X_1,X_2)u\in\Gamma(U^\perp)$ for all $X_1,X_2\in\mathfrak{X}(M)$ and $u\in\Gamma(U)$ in Proposition 8.9 is then equivalent to $R_{\bar\nabla}=0$.

8.4. **The Manin pair associated to an LA-Dirac structure.** Consider an LA-Courant algebroid

$$\begin{tikzcd} \mathbb{E} \arrow[r] \arrow[d] & Q \arrow[d] \\ B \arrow[r] & M \end{tikzcd}$$

with core Q^*, and an LA-Dirac structure

$$\begin{tikzcd} D \arrow[r] \arrow[d] & U \arrow[d] \\ B \arrow[r] & M \end{tikzcd}$$

in $\mathbb{E}$ with core U°. Since ∂_Q restricts to a map from U° to U, we can define the vector bundle

$$\mathbb{B}=\frac{U\oplus Q^*}{\mathrm{graph}(-\partial_Q|_{U^\circ})}\to M.$$

This vector bundle is anchored by the map

$$\rho_{\mathbb{B}}\colon\mathbb{B}\to TM,\qquad \rho_{\mathbb{B}}(u\oplus\tau)=\rho_Q(u+\partial_Q\tau)=\rho_Q(u)+\rho_B(\partial_B\tau).$$

Note that this map is well-defined because

$$\rho_{\mathbb{B}}(-\partial_Q\tau\oplus\tau)=\rho_Q(-\partial_Q\tau+\partial_Q\tau)=0$$

for all $\tau\in U^\circ$. We will show that there is a natural symmetric non-degenerate pairing $\langle\cdot\,,\cdot\rangle_{\mathbb{B}}$ on $\mathbb{B}\times_M\mathbb{B}$ and a natural bracket $[\![\cdot\,,\cdot]\!]_{\mathbb{B}}$ on $\Gamma(\mathbb{B})$ such that

$$(\mathbb{B}\to M,\rho_{\mathbb{B}},\langle\cdot\,,\cdot\rangle_{\mathbb{B}},[\![\cdot\,,\cdot]\!]_{\mathbb{B}})$$

is a Courant-algebroid.

We define the pairing on $\mathbb{B}$ by

$$\langle u_1\oplus\tau_1,u_2\oplus\tau_2\rangle_{\mathbb{B}}=\langle u_1,\tau_1\rangle+\langle u_2,\tau_2\rangle+\langle\tau_1,\partial_Q\tau_2\rangle.$$

It is easy to check that this pairing is well-defined and non-degenerate and that the induced map $\mathcal{D}_{\mathbb{B}} : C^\infty(M) \to \Gamma(\mathbb{B})$ given by

$$\langle \mathcal{D}_{\mathbb{B}} f, u \oplus \tau \rangle_{\mathbb{F}} = \rho_{\mathbb{B}}(u \oplus \tau)(f)$$

can alternatively be defined by $\mathcal{D}_{\mathbb{B}} f = 0 \oplus \rho_Q^* \mathbf{d} f$.

Choose as before a Lagrangian splitting of $\mathbb{E}$ that is adapted to D, and recall that the linear Courant algebroid structure and the linear Lie algebroid structure on $\mathbb{E}$ are then encoded by a Dorfman 2-representation and a self-dual 2-representation, respectively, both denoted as usual. We define the bracket on $\Gamma(\mathbb{B})$ by

(66)

$$\begin{aligned}&[\![u_1 \oplus \tau_1, u_2 \oplus \tau_2]\!]_{\mathbb{B}}\\ =&([\![u_1, u_2]\!]_U + \nabla_{\partial_B \tau_1} u_2 - \nabla_{\partial_B \tau_2} u_1) \oplus ([\![\tau_1, \tau_2]\!]_{Q^*} + \Delta_{u_1}\tau_2 - \Delta_{u_2}\tau_1 + \rho_Q^* \mathbf{d}\langle \tau_1, u_2 \rangle).\end{aligned}$$

A quick computation using Remark 2.7 and Proposition 6.8 show that this bracket does not depend on the choice of Lagrangian splitting.

Theorem 8.15. *Let $(D; U, B; M)$ be an LA-Dirac structure in a LA-Courant algebroid $(\mathbb{E}; Q, B; M)$. Then the vector bundle*

$$\mathbb{B} = \frac{U \oplus Q^*}{\operatorname{graph}(-\partial_Q|_{U^\circ})} \to M,$$

with the anchor $\rho_{\mathbb{B}}$, the pairing $\langle \cdot\,, \cdot \rangle_{\mathbb{B}}$ and the bracket $[\![\cdot\,, \cdot]\!]_{\mathbb{B}}$, is a Courant algebroid. Further, U is a Dirac structure in $\mathbb{E}$, via the inclusion $U \hookrightarrow \mathbb{B}$, $u \mapsto u \oplus 0$.

The proof of Theorem 8.15 can be found in Appendix C.

Corollary 8.16. *Let $(D; U, B; M)$ be an LA-Dirac structure in an LA-Courant algebroid $(\mathsf{E}, B; Q, M)$ (with core Q^*). The Manin pair $(\mathbb{B}, U)$ defined in Theorem 8.15 and the degenerate Courant algebroid Q^* satisfy the following conditions:*

1. *There is a morphism $\psi\colon Q^* \to C$ of degenerate Courant algebroids and an embedding $\iota\colon U \to Q$ over the identity on M*
2. *ι is compatible with the anchors: $\rho_Q \circ \iota = \rho_C|_U$,*
3. *$\psi(Q^*) + U = C$ and*
4. *$\langle \psi(\tau), u \rangle_C = \langle \iota(u), tau \rangle$ for all $\tau \in Q^*$ and $u \in U$.*

Proof. Take an LA-Dirac structure $(D; U, B; M)$ in an LA-Courant algebroid $(\mathbb{E}; Q, B; M)$. The morphism $\psi\colon Q^* \to \mathbb{E}$ defined by $\psi(\tau) \mapsto 0 \oplus \tau$ is obviously a morphism of degenerate Courant algebroids. Conditions (1)–(4) are then immediate. □

Conversely take a Manin pair (C, U) over M satisfying with Q^* the conditions in Corollary 8.16 and identify U with a subbundle of Q. If $\tau \in U^\circ \subseteq Q^*$, then $\psi(\tau)$ satisfies

$$\langle u, \psi(\tau) \rangle_C = \langle u, \tau \rangle = 0$$

for all $u \in U$. Since U is a Dirac structure, we find that ψ restricts to a map $U^\circ \to U$. Conversely, we find easily that $\psi(\tau) \in U$ if and only if $\tau \in U^\circ$. Next choose $\tau_1 \in U^\circ$ and $\tau_2 \in Q^*$. Then since $\psi(\tau_1) \in U$,

$$\langle \psi(\tau_1), \tau_2 \rangle = \langle \psi(\tau_1), \psi(\tau_2) \rangle_C = \langle \tau_1, \tau_2 \rangle_{Q^*} = \langle \partial_Q \tau_1, \tau_2 \rangle,$$

which shows that $\psi|_{U^\circ} = \partial_Q|_{U^\circ}$. In particular, ∂_Q sends U° to U, and U° is isotropic in Q^*. Consider the vector bundle map $U \oplus Q^* \to C$, $(u, \tau) \mapsto u + \psi(\tau)$.

By assumption, this map is surjective. Its kernel is the set of pairs (u, τ) with $u = -\psi(\tau)$, i.e. the graph of $-\partial_Q|_{U^\circ} \colon U^\circ \to U$. It follows that

$$C \simeq \frac{U \oplus Q^*}{\operatorname{graph}(-\partial_Q|_{U^\circ} \colon U^\circ \to U)}. \tag{67}$$

Hence, we can use the notation $u \oplus \tau$ for $\overline{u + \psi(\tau)} \in C$.

In the case of an LA-Courant algebroid $(TA \oplus_A T^*A, TM \oplus A^*, A, M)$ as in §7.3.2, for a Lie algebroid A, we could show in [17] that Manin pairs as in Corollary 8.16 are *in bijection* with LA-Dirac structures on A. That is, given a Manin pair (C, U) with an inclusion $U \hookrightarrow TM \oplus A^*$ and a degenerate Courant algebroid morphism $A \oplus T^*M \to C$ satisfying (1)–(4), then via the identification (67), there exists a Lagrangian splitting of $TA \oplus_A T^*A$ such that the Courant bracket on C is given by (66).

In a future project we will study how this result generalises to LA-Dirac structures in general LA-Courant algebroids, and we will compare the data contained in the Manin pair with the infinitesimal description of Dirac groupoids that was found by Li-Bland and Severa in [21].

Appendix A. Proof of Theorem 6.7

Let $(\mathbb{E}; Q, B; M)$ be a VB-Courant algebroid and choose a Lagrangian splitting $\Sigma \colon Q \times_M B$. We prove here that the split linear Courant algebroid is equivalent to a Dorfman 2-representation. Recall that $\mathcal{S} \subseteq \Gamma_B(\mathbb{E})$ is the subset $\{\tau^\dagger \mid \tau \in \Gamma(Q^*)\} \cup \{\sigma_Q(q) \mid q \in \Gamma(Q)\} \subseteq \Gamma(\mathbb{E})$.

Recall also that the tangent double $(TB \to B; TM \to M)$ has a VB-Lie algebroid structure, which is described in §2.2.2.

We start with the proofs of two useful lemmas.

Lemma A.1. *For $\beta \in \Gamma(B^*)$, we have*

$$\mathcal{D}(\ell_\beta) = \sigma_Q(\partial_B^* \beta) + \widetilde{\nabla_\cdot^* \beta},$$

where $\nabla_\cdot^* \beta$ is seen as follows as a section of $\Gamma(\operatorname{Hom}(B, Q^*))$: $(\nabla_\cdot^* \beta)(b) = \langle \nabla_\cdot^* \beta, b\rangle \in \Gamma(Q^*)$ for all $b \in \Gamma(B)$.

Proof. For $\beta \in \Gamma(B^*)$, the section $\mathbf{d}\ell_\beta$ is a linear section of $T^*B \to B$. Since the anchor Θ is linear, the section $\mathcal{D}\ell_\beta = \Theta^* \mathbf{d}\ell_\beta$ is linear. Since for any $\tau \in \Gamma(Q^*)$,

$$\langle \mathcal{D}(\ell_\beta), \tau^\dagger\rangle = \Theta(\tau^\dagger)(\ell_\beta) = q_B^* \langle \partial_B \tau, \beta\rangle,$$

we find that $\mathcal{D}(\ell_\beta) - \sigma_Q(\partial_B^* \beta) \in \Gamma(\ker \pi_Q)$. Hence, $\mathcal{D}(\ell_\beta) - \sigma_Q(\partial_B^* \beta)$ is a core-linear section of $\mathbb{E} \to B$ and there exists a section ϕ of $\operatorname{Hom}(B, Q^*)$ such that $\mathcal{D}(\ell_\beta) - \sigma_Q(\partial_B^* \beta) = \widetilde{\phi}$. We have

$$\ell_{\langle \phi, q\rangle} = \langle \widetilde{\phi}, \sigma_Q(q)\rangle = \langle \mathcal{D}(\ell_\beta) - \sigma_Q(\partial_B^* \beta), \sigma_Q(q)\rangle = \Theta(\sigma_Q(q))(\ell_\beta) = \ell_{\nabla_q^* \beta}$$

and so $\phi(b) = \langle \nabla_\cdot^* \beta, b\rangle \in \Gamma(Q^*)$ for all $b \in \Gamma(B)$. $\square$

Lemma A.2. *For $q \in \Gamma(Q)$ and $\phi \in \Gamma(\operatorname{Hom}(B, Q^*))$, we have*

$$\left[\!\!\left[\sigma_Q(q), \widetilde{\phi} \right]\!\!\right] = \widetilde{\Diamond_q \phi}.$$

Proof. Write $\phi = \sum_{i=1}^n \beta_i \otimes \tau_i$ with $\beta_1, \ldots, \beta_n \in \Gamma(B^*)$ and $\tau_1, \ldots, \tau_n \in \Gamma(Q^*)$. Then $\widetilde{\phi} = \sum_{i=1}^n \ell_{\beta_i} \cdot \tau_i^\dagger$ and we can compute

$$\left[\!\!\left[\sigma_Q(q), \widetilde{\phi}\right]\!\!\right] = \sum_{i=1}^n \left(\ell_{\nabla_q^*\beta_i} \cdot \tau_i^\dagger + \ell_{\beta_i} \cdot (\Delta_q \tau_i)^\dagger\right) = \sum_{i=1}^n \widetilde{\nabla_q^*\beta_i \otimes \tau_i + \beta_i \otimes \Delta_q \tau_i}$$

on the one hand, and on the other hand

$$(\Diamond_q \phi)(b) = \Delta_q \left(\sum_{i=1}^n \langle \beta_i, b\rangle \tau_i\right) - \sum_{i=1}^n \langle \beta_i, \nabla_q b\rangle \tau_i = \left(\sum_{i=1}^n \beta_i \otimes \Delta_q \tau_i + \nabla_q^* \beta_i \otimes \tau_i\right)(b)$$

for any $b \in \Gamma(B)$. □

Now we can express all the conditions of Lemma 5.2 in terms of the data $\partial_B, \Delta, \nabla, [\![\cdot\,,\cdot]\!]_\sigma, R$ found in §6.3.1.

Proposition A.3. *The anchor satisfies* $\Theta \circ \Theta^* = 0$ *if and only if*

1. $\rho_Q \circ \partial_B^* = 0$ *and*
2. $\nabla^*_{\partial_B^*\beta_1}\beta_2 + \nabla^*_{\partial_B^*\beta_2}\beta_1 = 0$ *for all* $\beta_1, \beta_2 \in \Gamma(B^*)$.

Proof. The composition $\Theta \circ \Theta^*$ vanishes if and only if $(\Theta \circ \Theta^*)\mathbf{d}F = 0$ for all linear and pullback functions $F \in C^\infty(B)$. For $f \in C^\infty(M)$, $\Theta(\Theta^*\mathbf{d}(q_B^* f)) = ((\partial_B \circ \rho_Q^*)\mathbf{d}f)^\uparrow$. For $\beta \in \Gamma(B^*)$, we find using Lemma A.1 $\Theta(\Theta^*\mathbf{d}\ell_\beta) = \Theta(\mathcal{D}\ell_\beta) = \Theta(\sigma_Q(\partial_B^*\beta) + \widetilde{\nabla^*_\cdot\beta}) = \widehat{\nabla_{\partial_B^*\beta}} + \partial_B \circ \widetilde{\langle\nabla^*_\cdot\beta, \cdot\rangle}$. Here, $\partial_B \circ \langle\nabla^*_\cdot\beta, \cdot\rangle$ is as follows a morphism $B \to B$; $b \mapsto \partial_B(\langle\nabla^*_\cdot\beta, b\rangle)$. On a linear function $\ell_{\beta'}$, $\beta' \in \Gamma(B^*)$, $\Theta(\Theta^*\mathbf{d}\ell_\beta)(\ell_{\beta'}) = \ell_{\nabla^*_{\partial_B^*\beta}\beta'} + \ell_{\nabla^*_{\partial_B^*\beta'}\beta}$. On a pullback $q_B^* f$, $f \in C^\infty(M)$, this is $q_B^*(\pounds_{(\rho_Q \circ \partial_B^*)(\beta)} f)$. □

Proposition A.4. *The compatibility of* Θ *with the Courant algebroid bracket* $[\![\cdot\,,\cdot]\!]$ *implies*

1. $\partial_B \circ R(q_1, q_2) = R_\nabla(q_1, q_2)$,
2. $\rho_Q \circ [\![\cdot\,,\cdot]\!]_\sigma = [\cdot\,,\cdot] \circ (\rho_Q, \rho_Q)$, *that is* $\Delta_q(\rho_Q^*\mathbf{d}f) = \rho_Q^*\mathbf{d}(\rho_Q(q)(f))$ *for all* $q \in \Gamma(Q)$ *and* $f \in C^\infty(M)$, *and*
3. $\partial_B \circ \Delta = \nabla \circ \partial_B$.

Proof. We have

$$\Theta\,[\![\sigma_Q(q_1), \sigma_Q(q_2)]\!] = [\Theta(\sigma_Q(q_1)), \Theta(\sigma_Q(q_2))] = \left[\widehat{\nabla_{q_1}}, \widehat{\nabla_{q_2}}\right]$$

and

$$\Theta\left(\sigma_Q([\![q_1, q_2]\!]_\sigma) - \widetilde{R(q_1, q_2)}\right) = \widehat{\nabla_{[\![q_1,q_2]\!]_\sigma}} - \partial_B \circ \widetilde{R(q_1, q_2)}.$$

Applying both derivations to a pullback function $q_B^* f$ for $f \in C^\infty(M)$ yields

$$\left[\widehat{\nabla_{q_1}}, \widehat{\nabla_{q_2}}\right](q_B^* f) = q_B^*([\rho_Q(q_1), \rho_Q(q_2)]f).$$

and

$$\left(\widehat{\nabla_{[\![q_1,q_2]\!]_\sigma}} - \partial_B \circ \widetilde{R(q_1, q_2)}\right)(q_B^* f) = q_B^*(\rho_Q[\![q_1, q_2]\!]_\sigma(f))$$

Applying both vector fields to a linear function $\ell_\beta \in C^\infty(B)$, $\beta \in \Gamma(B^*)$, we get

$$\left[\widehat{\nabla_{q_1}}, \widehat{\nabla_{q_2}}\right](\ell_\beta) = \ell_{\nabla^*_{q_1}\nabla^*_{q_2}\beta - \nabla^*_{q_2}\nabla^*_{q_1}\beta}$$

and

$$\left(\widehat{\nabla_{[\![q_1,q_2]\!]_\sigma}} - \partial_B \circ \widetilde{R(q_1,q_2)}\right)(\ell_\beta) = \ell_{\nabla^*_{[\![q_1,q_2]\!]_\sigma}\beta - R(q_1,q_2)^*\partial_B^*\beta}.$$

Since $R_{\nabla^*}(q_1,q_2) = -(R_\nabla(q_1,q_2))^*$, we find that

$$\Theta\,[\![\sigma_Q(q_1),\sigma_Q(q_2)]\!] = [\Theta(\sigma_Q(q_1)),\Theta(\sigma_Q(q_2))]$$

for all $q_1,q_2 \in \Gamma(Q)$ if and only if (1) and (2) are satisfied.

In the same manner we compute for $q \in \Gamma(Q)$ and $\tau \in \Gamma(Q^*)$:

$$\Theta\left([\![\sigma_Q(q),\tau^\dagger]\!]\right) = (\partial_B\Delta_q\tau)^\uparrow$$

and

$$\left[\Theta(\sigma_Q(q)),\Theta(\tau^\dagger)\right] = \left[\widehat{\nabla_q},(\partial_B\tau)^\uparrow\right] = (\nabla_q(\partial_B\tau))^\uparrow.$$

Hence, $\Theta\left([\![\sigma_Q(q),\tau^\dagger]\!]\right) = \left[\Theta(\sigma_Q(q)),\Theta(\tau^\dagger)\right]$ if and only if $\partial_B \circ \Delta = \nabla \circ \partial_B$. □

Proposition A.5. *The condition (3) of Lemma 5.2 is equivalent to*

1. $R(q_1,q_2) = -R(q_2,q_1)$ *and*
2. $[\![q_1,q_2]\!]_\sigma + [\![q_2,q_1]\!]_\sigma = 0$

for $q_1,q_2 \in \Gamma(Q)$.

Proof. Choose q_1,q_2 in $\Gamma(Q)$. Then we have

$$[\![\sigma_Q(q_1),\sigma_Q(q_2)]\!]+[\![\sigma_Q(q_2),\sigma_Q(q_1)]\!] = \sigma_Q([\![q_1,q_2]\!]_\sigma+[\![q_2,q_1]\!]_\sigma)-\widetilde{R(q_1,q_2)}-\widetilde{R(q_2,q_1)}.$$

By the choice of the splitting, we have $\mathcal{D}\langle\sigma_Q(q_1),\sigma_Q(q_2)\rangle = \mathcal{D}(0) = 0$. Hence, (3) of Lemma 5.2 is true for linear sections if and only if $R(q_1,q_2) = -R(q_2,q_1)$ and $[\![q_1,q_2]\!] + [\![q_2,q_1]\!] = 0$. On one linear and one core section, we have $[\![\sigma_Q(q),\tau^\dagger]\!] = (\Delta_q\tau)^\dagger$ and $[\![\tau^\dagger,\sigma_Q(q)]\!] = (-\Delta_q\tau + \rho_Q^*\mathbf{d}\langle\tau,q\rangle)^\dagger$ by definition. On core sections (3) is trivially satisfied since both the pairing and the bracket of two core sections vanish. □

Proposition A.6. *The derivation formula (2) in Lemma 5.2 is equivalent to*

1. Δ *is dual to* $[\![\cdot\,,\cdot]\!]_\sigma$*, that is* $[\![\cdot\,,\cdot]\!]_\sigma = [\![\cdot\,,\cdot]\!]_\Delta$*,*
2. $[\![q_1,q_2]\!]_\sigma + [\![q_2,q_1]\!]_\sigma = 0$ *for all* $q_1,q_2 \in \Gamma(Q)$ *and*
3. $R(q_1,q_2)^*q_3 = -R(q_1,q_3)^*q_2$ *for all* $q_1,q_2,q_3 \in \Gamma(Q)$.

Proof. We compute (CA2) for linear and core sections. First of all, the equations

$$\Theta(\tau_1^\dagger)\langle\tau_2^\dagger,\tau_3^\dagger\rangle = \langle[\![\tau_1^\dagger,\tau_2^\dagger]\!],\tau_3^\dagger\rangle + \langle\tau_2^\dagger,[\![\tau_1^\dagger,\tau_3^\dagger]\!]\rangle,$$

$$\Theta(\tau_1^\dagger)\langle\tau_2^\dagger,\sigma_Q(q)\rangle = \langle[\![\tau_1^\dagger,\tau_2^\dagger]\!],\sigma_Q(q)\rangle + \langle\tau_2^\dagger,[\![\tau_1^\dagger,\sigma_Q(q)]\!]\rangle$$

and

$$\Theta(\sigma_Q(q))\langle\tau_1^\dagger,\tau_2^\dagger\rangle = \langle[\![\sigma_Q(q),\tau_1^\dagger]\!],\tau_2^\dagger\rangle + \langle\tau_1^\dagger,[\![\sigma_Q(q),\tau_2^\dagger]\!]\rangle$$

are trivially satisfied for all $\tau_1,\tau_2,\tau_3 \in \Gamma(Q^*)$ and $q \in \Gamma(Q)$. Next we have for $q_1,q_2 \in \Gamma(Q)$ and $\tau \in \Gamma(Q^*)$:

$$\begin{aligned}
&\Theta(\sigma_Q(q_1))\langle\sigma_Q(q_2),\tau^\dagger\rangle - \langle[\![\sigma_Q(q_1),\sigma_Q(q_2)]\!],\tau^\dagger\rangle - \langle\sigma_Q(q_2),[\![\sigma_Q(q_1),\tau^\dagger]\!]\rangle\\
&= \widehat{\nabla_{q_1}}(q_B^*\langle q_2,\tau\rangle) - q_B^*\langle[\![q_1,q_2]\!]_\sigma,\tau\rangle - q_B^*\langle q_2,\Delta_{q_1}\tau\rangle\\
&= q_B^*\Big(\rho_Q(q_1)\langle q_2,\tau\rangle - \langle[\![q_1,q_2]\!]_\sigma,\tau\rangle - \langle q_2,\Delta_{q_1}\tau\rangle\Big)
\end{aligned}$$

Hence

$$\Theta(\sigma_Q(q_1))\langle\sigma_Q(q_2),\tau^\dagger\rangle = \langle[\![\sigma_Q(q_1),\sigma_Q(q_2)]\!],\tau^\dagger\rangle + \langle\sigma_Q(q_2),[\![\sigma_Q(q_1),\tau^\dagger]\!]\rangle$$

for all $q_1, q_2 \in \Gamma(Q)$ and $\tau \in \Gamma(Q^*)$ if and only if Δ and $[\![\cdot\,,\cdot]\!]_\sigma$ are dual to each other.

In the same manner, we compute

$$\begin{aligned}
&\Theta(\tau^\dagger)\langle \sigma_Q(q_1), \sigma_Q(q_2)\rangle - \langle [\![\tau^\dagger, \sigma_Q(q_1)]\!], \sigma_Q(q_2)\rangle - \langle \sigma_Q(q_1), [\![\tau^\dagger, \sigma_Q(q_2)]\!]\rangle \\
&= 0 - \langle -(\Delta_{q_1}\tau)^\dagger + (\rho_Q^*\mathbf{d}\langle q_1, \tau\rangle)^\dagger, \sigma_Q(q_2)\rangle - \langle \sigma_Q(q_1), -(\Delta_{q_2}\tau)^\dagger + (\rho_Q^*\mathbf{d}\langle q_2, \tau\rangle)^\dagger\rangle \\
&= \ - q_B^*\langle [\![q_1, q_2]\!]_\sigma + [\![q_2, q_1]\!]_\sigma, \tau\rangle.
\end{aligned}$$

Thus,

$$\Theta(\tau^\dagger)\langle \sigma_Q(q_1), \sigma_Q(q_2)\rangle = \langle [\![\tau^\dagger, \sigma_Q(q_1)]\!], \sigma_Q(q_2)\rangle + \langle \sigma_Q(q_1), [\![\tau^\dagger, \sigma_Q(q_2)]\!]\rangle$$

for all $q_1, q_2 \in \Gamma(Q)$ and $\tau \in \Gamma(Q^*)$ if and only if

$$0 = [\![q_1, q_2]\!]_\sigma + [\![q_2, q_1]\!]_\sigma.$$

Finally we have $\Theta(\sigma_Q(q_1))\langle \sigma_Q(q_2), \sigma_Q(q_3)\rangle = 0$ for all $q_1, q_2, q_3 \in \Gamma(Q)$, and $\langle [\![\sigma_Q(q_1), \sigma_Q(q_2)]\!], \sigma_Q(q_3)\rangle = \ell_{-R(q_1,q_2)^* q_3}$. This shows that

$$\Theta(\sigma_Q(q_1))\langle \sigma_Q(q_2), \sigma_Q(q_3)\rangle = \langle [\![\sigma_Q(q_1), \sigma_Q(q_2)]\!], \sigma_Q(q_3)\rangle + \langle \sigma_Q(q_2), [\![\sigma_Q(q_1), \sigma_Q(q_3)]\!]\rangle$$

if and only if

$$0 = -R(q_1, q_2)^* q_3 - R(q_1, q_3)^* q_2.$$

□

Proposition A.7. *Assume that Δ and $[\![\cdot\,,\cdot]\!]_\sigma$ are dual to each other. The Jacobi identity in Leibniz form for sections in $\mathcal{S}$ is equivalent to*

(1) $R(q_1, q_2) \circ \partial_B = R_\Delta(q_1, q_2)$ *and*

(2)

$$\begin{aligned}
&R(q_1, [\![q_2, q_3]\!]_\Delta) - R(q_2, [\![q_1, q_3]\!]_\Delta) - R([\![q_1, q_2]\!]_\Delta), q_3) \\
&+ \lozenge_{q_1}(R(q_2, q_3)) - \lozenge_{q_2}(R(q_1, q_3)) + \lozenge_{q_3}(R(q_1, q_2)) = \nabla^*_{\cdot}\left(R(q_1, q_2)^* q_3\right)
\end{aligned}$$

for all $q_1, q_2, q_3 \in \Gamma(Q)$.

If R is skew-symmetric as in (1) of Proposition A.5, then the second equation is the same as (D6) in Definition 5.8.

Proof. The Jacobi identity is trivially satisfied on core sections since the bracket of two core sections is 0. Similarly, for $\tau_1, \tau_2 \in \Gamma(Q^*)$ and $q \in \Gamma(Q)$, we find $[\![\sigma_Q(q), [\![\tau_1^\dagger, \tau_2^\dagger]\!]]\!] = 0$ and $[\![[\![\sigma_Q(q), \tau_1^\dagger]\!], \tau_2^\dagger]\!] + [\![\tau_1^\dagger, [\![\sigma_Q(q), \tau_2^\dagger]\!]]\!] = 0$. We have

$$\begin{aligned}
&\left[\!\!\left[\sigma_Q(q_1), \left[\!\!\left[\sigma_Q(q_2), \tau^\dagger\right]\!\!\right]\right]\!\!\right] - \left[\!\!\left[\sigma_Q(q_2), \left[\!\!\left[\sigma_Q(q_1), \tau^\dagger\right]\!\!\right]\right]\!\!\right] \\
&= \left[\!\!\left[\sigma_Q(q_1), (\Delta_{q_2}\tau)^\dagger\right]\!\!\right] - \left[\!\!\left[\sigma_Q(q_2), (\Delta_{q_1}\tau)^\dagger\right]\!\!\right] \\
&= (\Delta_{q_1}\Delta_{q_2}\tau)^\dagger - (\Delta_{q_2}\Delta_{q_1}\tau)^\dagger,
\end{aligned}$$

and

$$\begin{aligned}
\left[\!\!\left[[\![\sigma_Q(q_1), \sigma_Q(q_2)]\!]\,, \tau^\dagger\right]\!\!\right] &= \left[\!\!\left[\sigma_Q([\![q_1, q_2]\!]_\Delta) - \widetilde{R(q_1, q_2)}, \tau^\dagger\right]\!\!\right] \\
&= (\Delta_{[\![q_1, q_2]\!]_\Delta}\tau)^\dagger + (R(q_1, q_2)(\partial_B\tau))^\dagger
\end{aligned}$$

by Lemma 6.5. We now choose $q_1, q_2, q_3 \in \Gamma(Q)$ and compute

$$\begin{aligned}
&\llbracket \llbracket \sigma_Q(q_1), \sigma_Q(q_2) \rrbracket, \sigma_Q(q_3) \rrbracket \\
&= \left\llbracket \sigma_Q(\llbracket q_1, q_2 \rrbracket_\Delta) - \widetilde{R(q_1, q_2)}, \sigma_Q(q_3) \right\rrbracket \\
&= \sigma_Q(\llbracket \llbracket q_1, q_2 \rrbracket_\Delta, q_3 \rrbracket_\Delta) - \widetilde{R(\llbracket q_1, q_2 \rrbracket_\Delta, q_3)} - \mathcal{D}\ell_{\langle R(q_1,q_2)\cdot, q_3 \rangle} + \Diamond_{q_3} \widetilde{R(q_1, q_2)} \\
&= \sigma_Q(\llbracket \llbracket q_1, q_2 \rrbracket_\Delta, q_3 \rrbracket_\Delta) - \widetilde{R(\llbracket q_1, q_2 \rrbracket_\Delta, q_3)} \\
&\qquad - \sigma_Q(\partial_B^* \langle R(q_1, q_2)\cdot, q_3 \rangle) - \nabla^*_\cdot \widetilde{\langle R(q_1, q_2)\cdot, q_3 \rangle} + \Diamond_{q_3} \widetilde{R(q_1, q_2)}
\end{aligned}$$

and

$$\begin{aligned}
\llbracket \sigma_Q(q_2), \llbracket \sigma_Q(q_1), \sigma_Q(q_3) \rrbracket \rrbracket &= \left\llbracket \sigma_Q(q_2), \sigma_Q(\llbracket q_1, q_3 \rrbracket_\Delta) - \widetilde{R(q_1, q_3)} \right\rrbracket \\
&= \sigma_Q(\llbracket q_2, \llbracket q_1, q_3 \rrbracket_\Delta \rrbracket_\Delta) - \widetilde{R(q_2, \llbracket q_1, q_3 \rrbracket_\Delta)} - \Diamond_{q_2} \widetilde{R(q_1, q_3)}.
\end{aligned}$$

We hence find that

$$\llbracket \llbracket \sigma_Q(q_1), \sigma_Q(q_2) \rrbracket, \sigma_Q(q_3) \rrbracket + \llbracket \sigma_Q(q_2), \llbracket \sigma_Q(q_1), \sigma_Q(q_3) \rrbracket \rrbracket = \llbracket \sigma_Q(q_1), \llbracket \sigma_Q(q_2), \sigma_Q(q_3) \rrbracket \rrbracket$$

if and only if

$$\llbracket \llbracket q_1, q_2 \rrbracket_\Delta, q_3 \rrbracket_\Delta + \llbracket q_2, \llbracket q_1, q_3 \rrbracket_\Delta \rrbracket_\Delta = \llbracket q_1, \llbracket q_2, q_3 \rrbracket_\Delta \rrbracket_\Delta + \partial_B^* \langle R(q_1, q_2)\cdot, q_3 \rangle$$

and

$$\begin{aligned}
&R(\llbracket q_1, q_2 \rrbracket_\Delta, q_3) + \nabla^*_\cdot \langle R(q_1, q_2)\cdot, q_3 \rangle - \Diamond_{q_3} R(q_1, q_2) + (q_2, \llbracket q_1, q_3 \rrbracket_\Delta) + \Diamond_{q_2} R(q_1, q_3) \\
&= R(q_1, \llbracket q_2, q_3 \rrbracket_\Delta) + \Diamond_{q_1} R(q_2, q_3).
\end{aligned}$$

We conclude using (31). $\square$

A combination of Proposition A.3, A.4, A.5, A.6, A.7 and Lemma 5.2 proves Theorem 6.7.

APPENDIX B. ON THE LA-COURANT ALGEBROID CONDITION

We give here Li-Bland's definition of an LA-Courant algebroid [19], and we sketch the proof of Theorem 7.5.

First consider a Lie algebroid $(q_A \colon A \to M, \rho_A, [\cdot\,,\cdot])$. Then the dual A^* is endowed with a linear Poisson structure π_A, which defines a vector bundle morphism $\pi_A^\sharp \colon T^*A^* \to TA^*$. We begin by studying this morphism in decompositions of T^*A^* and TA^*. Choose a connection $\nabla \colon \mathfrak{X}(M) \times \Gamma(A) \to \Gamma(A)$. Then ∇ defines a decomposition $\mathbb{I}_\nabla \colon A \times_M TM \times_M A \to TA$ by $\mathbb{I}_\nabla(a_m, v_m, a'_m) = T_m a v_m - \frac{d}{dt}\big|_{t=0} a_m + t(\nabla_{v_m} a - a'_m)$ for any $a \in \Gamma(A)$ such that $a(m) = a_m$. In particular, the dual connection $\nabla^* \colon \mathfrak{X}(M) \times \Gamma(A^*) \to \Gamma(A^*)$ defines a decomposition of TA^*:

$$\begin{tikzcd}
A^* \times_M TM \times_M A^* \arrow[rr] \arrow[dd] & & TM \arrow[dd] & & TA^* \arrow[rr] \arrow[dd] & & TM, \arrow[dd] \\
& A^* \arrow[dr] & & \xrightarrow{\mathbb{I}_{\nabla^*}} & & A^* \arrow[dr] & \\
A^* \arrow[rr] & & M & & A^* \arrow[rr] & & M
\end{tikzcd}$$

$$\mathbb{I}_{\nabla^*}(\alpha(m), v_m, \alpha'_m) = T_m \alpha v_m - \frac{d}{dt}\bigg|_{t=0} \alpha(m) + t(\nabla^*_{v_m} \alpha - \alpha'_m).$$

Further, ∇ defines a decomposition

$$\begin{tikzcd}
A^*\times_M A\times_M T^*M \arrow[rr] \arrow[dd] & & A \arrow[dd] & & T^*A^* \arrow[rr] \arrow[dd] & & A \arrow[dd] \\
& T^*M \arrow[dr] & & {} \arrow[r, "\mathbb{J}_\nabla"] & {} & T^*M \arrow[dr] & \\
A^* \arrow[rr] & & M & & A^* \arrow[rr] & & M
\end{tikzcd}$$

of T^*A^* by

$$\mathbb{J}_\nabla(\alpha_m, a(m), \theta_m) = \mathbf{d}\ell_a(\alpha_m) - (T_{\alpha_m} q_{A^*})^*(\langle \nabla_{.} a, \alpha_m\rangle - \theta_m),$$

for all $\alpha_m \in A^*$, $\theta_m \in T^*M$ and $a \in \Gamma(A)$, and ∇^* defines a decomposition $\mathbb{J}_{\nabla^*}\colon A\times_M A^*\times_M T^*M \to T^*A$ by $\mathbb{J}_\nabla(a_m, \alpha(m), \theta_m) = \mathbf{d}\ell_\alpha(a_m) - (T_{a_m} q_A)^*(\langle \nabla^*_{.}\alpha, a_m\rangle - \theta_m)$.

The map $\mathbb{I}_{\nabla^*}^{-1}\circ \pi_A^\sharp \circ \mathbb{J}_\nabla \colon A^*\times_M A\times_M TM \to A^*\times_M TM\times_M A^*$ is given by

$$(\alpha_m, a_m, \theta_m) \mapsto (\alpha_m, \rho_A(a_m), -\langle \alpha_m, \nabla^{\rm bas}_{.} a\rangle - \rho_A^*\theta_m + \rho_A^*\langle \nabla_{.} a, \alpha_m\rangle)$$

(see [16]), where $\nabla^{\rm bas}$ is defined as in Example 2.9. Recall that T^*A^* is isomorphic to T^*A [29]; in the decompositions $\mathbb{J}_\nabla$ and $\mathbb{J}_{\nabla^*}$, of T^*A^* and T^*A, the isomorphism is given by

$$(\alpha_m, a_m, \theta_m) \in T^*A^* \leftrightarrow (\alpha_m, a_m, -\theta_m) \in T^*A.$$

Note further that, via this isomorphism, $(T^*A^*) \mathbin{\ast\!\!\!\!\mid} A \simeq (T^*A) \mathbin{\ast\!\!\!\!\mid} A = TA$ and also $TA^* \mathbin{\ast\!\!\!\!\mid} TM \simeq TA$. In the decompositions, the pairing of T^*A^* with its dual TA is given by

$$\langle (\alpha_m, a_m, \theta_m), (a_m, v_m, a'_m)\rangle = \langle \alpha_m, a_m\rangle - \langle \theta_m, v_m\rangle$$

and the pairing of TA^* with TA is given by

$$\langle (\alpha_m, v_m, \alpha'_m), (a_m, v_m, a'_m)\rangle = \langle \alpha_m, a'_m\rangle + \langle \alpha'_m, a_m\rangle.$$

In [19] Li-Bland defines a relation $\Pi_A \subseteq TA\times TA \simeq (T^*A^*) \mathbin{\ast\!\!\!\!\mid} A \times TA^* \mathbin{\ast\!\!\!\!\mid} TM$ by

$$(V, W) \in \Pi_A \qquad \Leftrightarrow \qquad \langle V, \Phi\rangle_A = \langle W, \pi^\sharp(\Phi)\rangle_{TM} \ \text{ for all } \ \Phi \in T^*A^*.$$

In the decomposition $\mathbb{I}_\nabla$, the relation Π_A is hence given by

$$(v_m, a_m, b_m) \sim_{\Pi_A} (c_m, w_m, d_m)$$

if and only if

$$\begin{aligned} &\langle (v_m, a_m, b_m), (\alpha_m, a_m, \theta_m)\rangle \\ &= \langle (c_m, w_m, d_m), (\alpha_m, \rho_A(a_m), -\langle \alpha_m, \nabla^{\rm bas}_{.} a\rangle - \rho_A^*\theta_m + \rho_A^*\langle \nabla_{.} a, \alpha_m\rangle)\rangle \end{aligned}$$

for all $(\alpha_m, a_m, \theta_m) \in A_m^*\times A_m\times T_m^*M$. That is, $(v_m, a_m, b_m) \sim_{\Pi_A} (c_m, w_m, d_m)$ if and only if $w_m = \rho_A(a_m)$, $b_m = d_m - \nabla^{\rm bas}_{c_m} a + \nabla_{\rho_A(c_m)} a$ and $v_m = \rho_A(c_m)$. In other words, we have

$$(\rho_A(a_m), b_m, c_m - \nabla^{\rm bas}_{a_m} b + \nabla_{\rho_A(a_m)} b) \sim_{\Pi_A} (a_m, \rho_A(b_m), c_m)$$

for all $a_m, b_m, c_m \in A_m$. This leads to

$$\begin{aligned} &(\rho_A(a)(m), b(m), c(m) - [a,b](m) + \nabla_{\rho_A(a)} b(m)) \\ &\qquad\qquad \sim_{\Pi_A} (a(m), \rho_A(b)(m), c(m) + \nabla_{\rho_A(b)} a(m)) \end{aligned}$$

for all $a, b, c \in \Gamma(A)$. But this is just

$$T_m b \rho_A(a)(m) +_A \left.\frac{d}{dt}\right|_{t=0} b(m) + t([b,a]+c)(m)$$
$$\sim_{\Pi_A} T_m a \rho_A(b)(m) +_A \left.\frac{d}{dt}\right|_{t=0} a(m) + tc(m)$$

for all $a, b, c \in \Gamma(A)$.

Given $a, b, c \in \Gamma(A)$, we write $(a,b,c)(m)$ for the pair

$$\left(T_m b \rho_A(a)(m) +_A ([b,a]+c)^{\uparrow}(b_m), T_m a \rho_A(b)(m) +_A c^{\uparrow}(a_m)\right)$$

in Π_A and we get

$$\Pi_A = \{(a,b,c)(m) \mid a,b,c \in \Gamma(A),\, m \in M\}.$$

Note that for $f \in C^\infty(M)$,

$$(a, b, fc)(m) = (a, b, f(m)c)(m),$$
$$(fa, b, c)(m) = (f(m)a, b, c + \rho_A(b)(f)(m)a)(m),$$

and

$$(a, fb, c)(m) = (a, f(m)b, c - \rho_A(a)(f)(m)b)(m)$$

for all $m \in M$. As a consequence, one can easily check that given a family of sections $\mathcal{S} \subseteq \Gamma(A)$ that spans point-wise A, we find

(68)

$$\Pi_A = \left\{ \left(\sum_i x_i a_i, \sum_j y_j b_j, \sum_k z_k c_k \right)(m) \,\middle|\, a_i, b_j, c_k \in \mathcal{S}, x_i, y_j, z_k \in \mathbb{R}, m \in M \right\}.$$

Note further that $\Pi_A \subseteq TA \times TA$ can be described as follows.

Proposition B.1. *A point $P \in TA \times TA$ is an element of Π_A if and only if*

(1) $(q_A \circ p_A \circ \mathrm{pr}_1)(P) = (q_A \circ p_A \circ \mathrm{pr}_2)(P)$,

i.e. $P \in TA \times_{q_A \circ p_A} TA$, and all the following elements of $C^\infty(TA \times_{q_A \circ p_A} TA)$ vanish on P:

(2) $\ell_{\mathbf{d}\ell_\alpha} \circ \mathrm{pr}_1 - \ell_{\mathbf{d}\ell_\alpha} \circ \mathrm{pr}_2 + \mathbf{d}_A\alpha \circ (p_A, p_A)$, *for all $\alpha \in \Gamma(A^*)$,*
(3) $\ell_{q_A^*\mathbf{d}f} \circ \mathrm{pr}_1 - (p_A^* \ell_{\rho_A^*\mathbf{d}f}) \circ \mathrm{pr}_2$ *and*
(4) $\ell_{q_A^*\mathbf{d}f} \circ \mathrm{pr}_2 - (p_A^* \ell_{\rho_A^*\mathbf{d}f}) \circ \mathrm{pr}_1$ *for all $f \in C^\infty(M)$.*

Proof. Any point $P \in TA \times TA$ can be written $P = (T_m a v_m + c^\dagger(a_m), T_n b w_n + d^\dagger(b_n))$ with $v_m, w_n \in TM$ and $a, b, c, d \in \Gamma(A)$. Condition (1) is $m = n$. Condition (4) is equivalent to $w_m = \rho(a(m))$, and Condition (3) to $v_m = \rho(b(m))$. Condition (2) is satisfied for $P = (T_m a \rho(b(m)) + c^\dagger(a(m)), T_m b \rho(a(m)) + d^\dagger(b_n))$ if and only if $c = d + [a, b]$. $\square$

Now consider a double vector bundle

$$\begin{tikzcd}
\mathbb{E} \arrow[r] \arrow[d] & B \arrow[d] \\
Q \arrow[r] & M
\end{tikzcd}$$

endowed with a VB-Lie algebroid structure $(\mathbf{b}, [\cdot\,,\cdot])$ on $\mathbb{E} \to Q$ and a linear metric $\langle\cdot\,,\cdot\rangle$ on $\mathbb{E} \to B$ (hence, $\mathbb{E}$ has core Q^*). Let $\rho_B \colon B \to TM$ be the anchor of the induced Lie algebroid structure on B.

The relation $\Pi_{\mathbb{E}}$ defined as above by the Lie algebroid structure on $\mathbb{E}$ over Q is then a relation $\Pi_{\mathbb{E}}$ of triple vector bundles [19]

$$\begin{tikzcd}
T\mathbb{E} \arrow[rr] \arrow[dr] \arrow[dd] & & TQ \arrow[dr] \arrow[dd] & \\
& \mathbb{E} \arrow[rr] \arrow[dd] & & Q \arrow[dd] \\
TA \arrow[rr] \arrow[dr] & & TM \arrow[dr] & \\
& A \arrow[rr] & & M
\end{tikzcd}
\xrightarrow{\Pi_{\mathbb{E}}}
\begin{tikzcd}
T\mathbb{E} \arrow[rr] \arrow[dr] \arrow[dd] & & \mathbb{E} \arrow[dr] \arrow[dd] & \\
& TQ \arrow[rr] \arrow[dd] & & Q \arrow[dd] \\
TA \arrow[rr] \arrow[dr] & & A \arrow[dr] & \\
& TM \arrow[rr] & & M
\end{tikzcd}$$

Li-Bland's definition [19] is the following.

Definition B.2. *Let* $(\mathbb{E}; Q, B; M)$ *be a double vector bundle with a VB-Courant algebroid structure (over* B*) and a VB-algebroid structure* $(\mathbb{E} \to Q, B \to M)$*. Then* $(\mathbb{E}, B, Q, M)$ *is an LA-Courant algebroid if* $\Pi_{\mathbb{E}}$ *is a Dirac structure with support in* $\overline{T\mathbb{E}} \times T\mathbb{E}$*.*

We choose a Lagrangian splitting $\Sigma \colon B \times_M Q \to \mathbb{E}$. We denote as usual by $(\partial_Q \colon Q^* \to Q, \nabla^Q, \nabla^{Q^*}, R \in \Omega^2(B, \mathrm{Hom}(Q, Q^*)))$ the induced 2-representation[18] of the Lie algebroid B, and by $(\partial_B \colon Q^* \to B, \nabla, \Delta, R \in \Omega^2(Q, \mathrm{Hom}(B, Q^*)))$ the induced Dorfman 2-representation of the anchored vector bundle (Q, ρ_Q). We begin with studying the vector bundle structure of $\Pi_{\mathbb{E}}$.

Proposition B.3. *The image of* $\Pi_{\mathbb{E}}$ *under* $(T\pi_B \times T\pi_B) \colon T\mathbb{E} \times T\mathbb{E} \to TB \times TB$ *is* Π_B*.*

Proof. By (68), it is sufficient to show that the set of points

$$(\sigma_B(b_1) + \tau_1^\dagger, \sigma_B(b_2) + \tau_2^\dagger, \sigma_B(b_3) + \tau_3^\dagger)(q_m)$$

for $b_1, b_2, b_3 \in \Gamma(B)$, $\tau_1, \tau_2, \tau_3 \in \Gamma(Q^*)$ and $q_m \in Q$ projects under $(T\pi_B, T\pi_B)$ to Π_B. It is easy to see that $(T\pi_B, T\pi_B)((\sigma_B(b_1) + \tau_1^\dagger, \sigma_B(b_2) + \tau_2^\dagger, \sigma_B(b_3) + \tau_3^\dagger)(q_m)) = (b_1, b_2, b_3)(m)$. □

It seems at this point useful to enumerate the sections of $\mathbb{E} \to B$, of $\mathbb{E} \to Q$, and of $T\mathbb{E} \to TB$ that we are working with here. Recall that after the choice of a Lagrangian splitting $\Sigma \colon Q \times_M B \to \mathbb{E}$, the vector bundle $\mathbb{E} \to Q$ is spanned by sections $\sigma_B(b)$ for all $b \in \Gamma(B)$ and $\tau^\dagger$ for all $\tau \in \Gamma(Q^*)$. We have used those sections to define $\Pi_{\mathbb{E}}$, as the relation that the Lie algebroid $\mathbb{E} \to Q$ induces.

In the same manner (and with a slight abuse of notation), the vector bundle $\mathbb{E} \to B$ is spanned by sections $\sigma_Q(q)$ for all $q \in \Gamma(Q)$ and $\tau^\dagger$ for all $\tau \in \Gamma(Q^*)$.

As a consequence, the vector bundle $T\mathbb{E} \to TB$ is spanned by the sections

$$T\sigma_Q(q), \qquad T\tau^\dagger, \qquad (\sigma_Q(q))^\times \quad \text{and} \quad (\tau^\dagger)^\times$$

---

[18] As always, we omit the upper indices of the two linear connections when which one is used is clear from its index and term.

for all $q \in \Gamma(Q)$ and $\tau \in \Gamma(Q^*)$. (Recall that for a vector bundle $E \to M$, and a section $e \in \Gamma_M(E)$, the section $e^\times$ is defined in (10).) Since $\mathbb{E} \to B$ has a Courant algebroid structure, its tangent double has a natural Courant algebroid structure, see the proof of Theorem 6.11.

Proposition B.4. *For $b_1, b_2, b_3 \in \Gamma(B)$, $\tau_1, \tau_2, \tau_3 \in \Gamma(Q^*)$ and $q \in \Gamma(Q)$, the pair*

$$\begin{aligned} &T_{q(m)}\left(\sigma_B(b_1) + \tau_1^\dagger\right) \mathsf{b}\left(\left(\sigma_B(b_2) + \tau_2^\dagger\right)(q(m))\right) \\ &\qquad + \left(\left[\sigma_B(b_1) + \tau_1^\dagger, \sigma_B(b_2) + \tau_2^\dagger\right] + \sigma_B(b_3) + \tau_3^\dagger\right)^\times (\sigma_B(b_1) + \tau_1^\dagger)(q(m)) \\ &\sim_{\Pi_{\mathbb{E}}} T_{q(m)}\left(\sigma_B(b_2) + \tau_2^\dagger\right) \mathsf{b}\left(\left(\sigma_B(b_1) + \tau_1^\dagger\right)(q(m))\right) \\ &\qquad\qquad + \left(\sigma_B(b_3) + \tau_3^\dagger\right)^\times \left(\left(\sigma_B(b_2) + \tau_2^\dagger\right)(q(m))\right) \end{aligned}$$

is equal to

$$\begin{aligned} &T_{b_1(m)}\left(\sigma_Q(q) + \tau_1^\dagger\right)\left(T_m b_1 \rho_B(b_2)(m) + [b_1, b_2]^\uparrow(b_1(m)) + b_3^\uparrow(b_1(m))\right) \\ &+\left(-\sigma_Q(\nabla_{b_2} q) - R(b_1, b_2)q^\dagger + \tau_3^\dagger + \nabla_{b_1}\tau_2^\dagger - \nabla_{b_2}\tau_1^\dagger + \sigma_Q(\partial_Q \tau_2)\right)^\times \left(\sigma_Q(q) + \tau_1^\dagger\right)(b_1(m)) \\ &\sim_{\Pi_{\mathbb{E}}} T_{b_2(m)}\left(\sigma_Q(q) + \tau_2^\dagger\right)(T_m b_2 \rho(b_1)(m) + b_3^\uparrow(b_2(m))) \\ &\qquad + \left(-\sigma_Q(\nabla_{b_1} q) + \tau_3^\dagger + \sigma_Q(\partial_Q \tau_1)\right)^\times (\sigma_Q(q) + \tau_2^\dagger)(b_2(m)). \end{aligned}$$

Proof. This proof is tedious, but straightforward. To show that two vectors on $\mathbb{E}$ are equal, we evaluate them on linear and pullback functions in $C^\infty(\mathbb{E})$. We identify $\mathbb{E}$ with $\mathbb{E} \,{\ast\!\!\!\mid}\, B$ using the linear metric. We consider the four types of functions on $\mathbb{E}$: $\ell_{\sigma_Q(q)}$, $\ell_{\tau^\dagger}$, $\pi_B^*\ell_\beta$ and $\pi_B^* q_B^* f$ for all $q \in \Gamma(Q)$, $\tau \in \Gamma(Q^*)$, $\beta \in \Gamma(B^*)$ and $f \in C^\infty(M)$. Recall that the horizontal lift $\sigma_{Q^{**}}^\star : \Gamma(Q^{**}) \to \Gamma_Q^l(\mathbb{E} \,{\ast\!\!\!\mid}\, Q)$ can be defined by $\langle \sigma_{Q^{**}}(q), \sigma_B(b)\rangle = 0$ and $\langle \sigma_{Q^{**}}(q), \tau^\dagger\rangle = q_Q^*\langle q, \tau\rangle$ for all $q \in \Gamma(Q)$, $\tau \in \Gamma(Q^*)$ and $b \in \Gamma(B)$, where we identify Q^{**} with Q via the canonical isomorphism. The sections of $\mathbb{E} \,{\ast\!\!\!\mid}\, Q$ over Q define linear functions on $\mathbb{E}$ and we have $\ell_{\sigma_{Q^{**}}^\star(q)} = \ell_{\sigma_Q(q)}$, $\ell_{\beta^\dagger} = \pi_B^*\ell_\beta$, $\pi_Q^*\ell_\tau = \ell_{\tau^\dagger}$ and $\pi_Q^* q_Q^* f = \pi_B^* q_B^* f$ for all $q \in \Gamma(Q)$, $\tau \in \Gamma(Q^*)$, $\beta \in \Gamma(B^*)$ and $f \in C^\infty(M)$. We use these equalities of functions to prove the equalities of vectors in Proposition B.4. The details are left to the reader. □

The last proposition was quite technical, but it yields useful and relatively simple sections of $\Pi_{\mathbb{E}} \to \Pi_B$:

$$\left((\tau^\dagger)^\times \circ \mathrm{pr}_1, (\tau^\dagger)^\times \circ \mathrm{pr}_2\right), \tag{69}$$

$$\left(T\tau^\dagger \circ \mathrm{pr}_1 - (\nabla_{p_B \circ \mathrm{pr}_2}\tau^\dagger)^\times \circ \mathrm{pr}_1, \sigma_Q(\partial_Q \tau)^\times \circ \mathrm{pr}_2\right), \tag{70}$$

$$\left((\nabla_{p_B \circ \mathrm{pr}_1}\tau^\dagger + \sigma_Q(\partial_Q \tau))^\times \circ \mathrm{pr}_1, T\tau^\dagger \circ \mathrm{pr}_2\right) \tag{71}$$

and

$$\begin{aligned} \Big(&T\sigma_Q(q) \circ \mathrm{pr}_1 - (\sigma_Q(\nabla_{p_B \circ \mathrm{pr}_2} q) + R(p_B \circ \widetilde{\mathrm{pr}_1, p_B} \circ \mathrm{pr}_2)q)^\times \circ \mathrm{pr}_1, \\ &\qquad\qquad T\sigma_Q(q) \circ \mathrm{pr}_2 - (\sigma_Q(\nabla_{p_B \circ \mathrm{pr}_1} q))^\times \circ \mathrm{pr}_2\Big) \end{aligned} \tag{72}$$

for all $q \in \Gamma(Q)$ and $\tau \in \Gamma(Q^*)$. Now we study the compatibility of $\Pi_{\mathbb{E}}$ with the metric on $\mathbb{E} \to B$.

Theorem B.5. *The vector bundle* $\Pi_{\mathbb{E}} \to \Pi_B$ *is maximal isotropic in* $\overline{T\mathbb{E}} \times T\mathbb{E}|_{\Pi_B} \to \Pi_B$ *if and only if*

(1) $\partial_Q = \partial_Q^*$,
(2) $(\nabla^Q)^*_b \tau = \nabla^{Q^*}_b \tau$ *for all* $b \in \Gamma(B)$ *and* $\tau \in \Gamma(Q^*)$, *and*
(3) $\langle R(b_1, b_2)q_1, q_2\rangle + \langle R(b_1, b_2)q_2, q_1\rangle = 0$ *for all* $b_1, b_2 \in \Gamma(B)$ *and* $q_1, q_2 \in \Gamma(Q)$.

In other words, $\Pi_{\mathbb{E}} \to \Pi_B$ is maximal isotropic in $\overline{T\mathbb{E}} \times T\mathbb{E}|_{\Pi_B} \to \Pi_B$ if and only if the 2-representation $(\partial_Q \colon Q^* \to Q, \nabla^Q, \nabla^{Q^*}, R)$ is self-adjoint. This yields the following result:

Corollary B.6. *The VB-algebroid structure on* $\mathbb{E} \to Q$ *is compatible with the linear metric if and only if* $\Pi_{\mathbb{E}} \to \Pi_B$ *is maximal isotropic in* $\overline{T\mathbb{E}} \times T\mathbb{E}|_{\Pi_B} \to \Pi_B$.

Proof of Theorem B.5. The vector bundle $T\mathbb{E} \to TB$ has rank twice the rank of $\mathbb{E} \to B$. Hence, we have $\operatorname{rank}(T\mathbb{E} \times T\mathbb{E} \to TB \times TB) = 2 \cdot 2 \cdot (\operatorname{rank}(Q) + \operatorname{rank}(Q^*)) = 8 \operatorname{rank}(Q)$. It is easy to see that $\Pi_{\mathbb{E}} \to \Pi_B$ has rank $\operatorname{rank}(Q) + 3 \operatorname{rank}(Q^*) = 4 \operatorname{rank}(Q)$. Hence, it is sufficient to find when $\Pi_{\mathbb{E}} \to \Pi_B$ is isotropic in $\overline{T\mathbb{E}} \times T\mathbb{E}|_{\Pi_B} \to \Pi_B$.

The pairing

$$\Big\langle T\left(\sigma_Q(q_1) + \chi_1^\dagger\right) + \left(-\sigma_Q(\nabla_{b_2} q_1) - R(b_1, b_2)q_1^\dagger + \sigma_1^\dagger + \nabla_{b_1}\tau_1^\dagger - \nabla_{b_2}\chi_1^\dagger + \sigma_Q(\partial_Q \tau_1)\right)^\times,$$
$$T\left(\sigma_Q(q_2) + \chi_2^\dagger\right) + \left(-\sigma_Q(\nabla_{b_2} q_2) - R(b_1, b_2)q_2^\dagger + \sigma_2^\dagger + \nabla_{b_1}\tau_2^\dagger - \nabla_{b_2}\chi_2^\dagger + \sigma_Q(\partial_Q \tau_2)\right)^\times \Big\rangle$$
$$\left(Tb_1 + ([b_1, b_2] + b_3)^\times\right)(\rho_B(b_2))$$
$$-\Big\langle T\left(\sigma_Q(q_1) + \tau_1^\dagger\right) + \left(-\sigma_Q(\nabla_{b_1} q_1) + \sigma_1^\dagger + \sigma_Q(\partial_Q \chi_1)\right)^\times,$$
$$T\left(\sigma_Q(q_2) + \tau_2^\dagger\right) + \left(-\sigma_Q(\nabla_{b_1} q_2) + \sigma_2^\dagger + \sigma_Q(\partial_Q \chi_2)\right)^\times \Big\rangle (Tb_2 + b_3^\times)(\rho_B(b_1))$$

equals

(73)
$$\begin{aligned}
&\rho_B(b_2)\langle \chi_1, q_2\rangle + \rho_B(b_2)\langle \chi_2, q_1\rangle - \rho_B(b_1)\langle \tau_1, q_2\rangle - \rho_B(b_1)\langle \tau_2, q_1\rangle \\
&-\langle \nabla_{b_2} q_1, \chi_2\rangle - \langle R(b_1, b_2)q_1, q_2\rangle + \langle \nabla_{b_1}\tau_1, q_2\rangle - \langle \nabla_{b_2}\chi_1, q_2\rangle \\
&+\langle \partial_Q \tau_1, \chi_2\rangle - \langle \nabla_{b_2} q_2, \chi_1\rangle - \langle R(b_1, b_2)q_2, q_1\rangle + \langle \nabla_{b_1}\tau_2, q_1\rangle - \langle \nabla_{b_2}\chi_2, q_1\rangle + \langle \partial_Q \tau_2, \chi_1\rangle \\
&+\langle \nabla_{b_1} q_1, \tau_2\rangle - \langle \partial_Q \chi_1, \tau_2\rangle + \langle \nabla_{b_1} q_2, \tau_1\rangle - \langle \partial_Q \chi_2, \tau_1\rangle.
\end{aligned}$$

This vanishes for all $b_1, b_2 \in \Gamma(B)$, $q_1, q_2 \in \Gamma(Q)$, $\chi_1, \chi_2, \tau_1, \tau_2 \in \Gamma(Q^*)$ if and only if

- (setting $b_1 = b_2 = 0$, $q_1 = q_2 = 0$, $\chi_2 = \tau_1 = 0$)
$$\langle \partial_Q \tau_2, \chi_1\rangle - \langle \partial_Q \chi_1, \tau_2\rangle = 0$$
for all $\chi_1, \tau_2 \in \Gamma(Q^*)$, which is equivalent to $\partial_Q = \partial_Q^*$.
- (setting $b_2 = 0$, $q_2 = 0$, $\chi_1 = \chi_2 = \tau_1 = 0$)
$$-\rho_B(b_1)\langle \tau_2, q_1\rangle + \langle \nabla_{b_1}\tau_2, q_1\rangle + \langle \nabla_{b_1} q_1, \tau_2\rangle = 0$$
for all $\tau_2 \in \Gamma(Q^*)$, $q_1 \in \Gamma(Q)$ and $a \in \Gamma(B)$. This is equivalent to $\nabla^*_{b_1}\tau = \nabla_{b_1}\tau$ for all $\tau \in \Gamma(Q^*)$ and $b_1 \in \Gamma(B)$.

- Using the two equations found above, several terms in (73) cancel. This yields $-\langle R(b_1, b_2)q_1, q_2\rangle - \langle R(b_1, b_2)q_2, q_1\rangle = 0$ for all $b_1, b_2 \in \Gamma(B)$ and $q_1, q_2 \in \Gamma(Q)$.

□

Hence we have found that the second condition in Lemma 5.3 is satisfied if and only if $(\mathbb{E} \to Q; B \to M)$ is a metric VB-algebroid. We study the remaining two conditions in Lemma 5.3 on the sections (69), (70), (71), (72).

Theorem B.7. *The anchor $\rho_{\overline{T\mathbb{E}}\times T\mathbb{E}}$ sends $\Pi_{\mathbb{E}}$ to $T\Pi_B$ if and only if*

(1) $\rho_B \circ \partial_B = \rho_Q \circ \partial_Q$,
(2) $\partial_B(\nabla_b \tau) = [b, \partial_B\tau] + \nabla_{\partial_Q\tau} b$,
(3) $[\rho_B(b), \rho_Q(q)] = \rho_Q(\nabla_b q) - \rho_B(\nabla_q b)$ *and*
(4) $\partial_B R(b_1, b_2)q = -\nabla_q[b_1, b_2] + [\nabla_q b_1, b_2] + [b_1, \nabla_q b_2] + \nabla_{\nabla_{b_2} q} b_1 - \nabla_{\nabla_{b_1} q} b_2$

for all $b, b_1, b_2 \in \Gamma(B)$, $q \in \Gamma(Q)$ and $\tau \in \Gamma(Q^*)$.

Proof. First note that, by construction, the map

$$T(q_B \circ p_B \circ \mathrm{pr}_1) - T(q_B \circ p_B \circ \mathrm{pr}_2) \colon T(TB \times TB) \to TM$$

sends the images under $\rho_{\overline{T\mathbb{E}}\times T\mathbb{E}}$ of our four types of sections to zero. That is, we have $\rho_{\overline{T\mathbb{E}}\times T\mathbb{E}}(\Pi_{\mathbb{E}}) \subseteq T(TB \times_{q_B \circ p_B} TB)$ by construction. Hence, the anchor $\rho_{\overline{T\mathbb{E}}\times T\mathbb{E}}$ sends $\Pi_{\mathbb{E}}$ to $T\Pi_B$ if and only if $\rho_{\overline{T\mathbb{E}}\times T\mathbb{E}}$ applied to our four special types of sections annihilate all the $\mathbb{R}$-valued functions in Proposition B.1.

The anchor of $\big((\tau^\dagger)^\times \circ \mathrm{pr}_1, (\tau^\dagger)^\times \circ \mathrm{pr}_2\big)$ is

$$\big((\Theta(\tau^\dagger))^\uparrow \circ \mathrm{pr}_1, (\Theta(\tau^\dagger))^\uparrow \circ \mathrm{pr}_2\big) = \big((\partial_B\tau^\dagger)^\uparrow \circ \mathrm{pr}_1, (\partial_B\tau^\dagger)^\uparrow \circ \mathrm{pr}_2\big)$$

for $\tau \in \Gamma(Q^*)$. This vanishes on all the functions defining Π_B. The anchor of

$$\big(T\tau^\dagger - (\nabla_{b_2}\tau^\dagger)^\times, \sigma_Q(\partial_Q\tau)^\times\big)\,((b_1, b_2, b_3)(m))$$

is

$$\left(\widehat{[\partial_B\tau^\uparrow, \cdot]} - (\partial_B\nabla_{b_2}\tau^\uparrow)^\uparrow, \widehat{\nabla_{\partial_Q\tau}\cdot}^{\uparrow}\right)((b_1, b_2, b_3)(m)).$$

This derivation, which we call here X sends $\ell_{q_B^*\mathbf{d}f} \circ \mathrm{pr}_1 - (p_B^*\ell_{\rho_B^*\mathbf{d}f}) \circ \mathrm{pr}_2$ to 0. We have further

$$\begin{aligned} X(\ell_{q_B^*\mathbf{d}f} \circ \mathrm{pr}_2 - (p_B^*\ell_{\rho_B^*\mathbf{d}f}) \circ \mathrm{pr}_1) &= \widehat{\nabla_{\partial_Q\tau}\cdot}(q_B^* f)(b_2(m)) - (\partial_B\tau)^\uparrow(\ell_{\rho_B^*\mathbf{d}f})(b_1(m)) \\ &= (\rho_Q\partial_Q\tau)(f)(m) - (\rho_B\partial_B\tau)(f)(m). \end{aligned}$$

This vanishes for all $f \in C^\infty(M)$ and all $m \in M$ if and only if $\rho_B\partial_B(\tau) = \rho_Q\partial_Q(\tau)$. Finally, a computation yields

$$\begin{aligned} &X(\ell_{\mathbf{d}\ell_\beta} \circ \mathrm{pr}_1 - \ell_{\mathbf{d}\ell_\beta} \circ \mathrm{pr}_2 + \mathbf{d}_B\beta \circ (p_B, p_B)) \\ &= \langle \beta, -\partial_B\nabla_{b_2}\tau + \nabla_{\partial_Q\tau} b_2 + [b_2, \partial_B\tau]\rangle + (\rho_B\partial_B\tau - \rho_Q\partial_Q\tau)\langle \beta, b_2\rangle(m). \end{aligned}$$

We find hence that X vanishes on all the functions defining Π_B as in Proposition B.1 if and only if $[b_2, \partial_B\tau] = \partial_B(\nabla_{b_2}\tau) - \nabla_{\partial_Q\tau} b_2$ and $\rho_B\partial_B(\tau) = \rho_Q\partial_Q(\tau)$.

In a similar manner, the anchors of the elements of the type

$$\big((\nabla_a\tau^\dagger + \sigma_Q(\partial_Q\tau))^\times, T\tau^\dagger\big)\,((a, b, c)(m))$$

annihilate all the functions in Proposition B.1 if and only if $\rho_Q\partial_Q = \rho_B\partial_B$ and $\partial_B\nabla_b\tau = [b, \partial_B\tau] + \nabla_{\partial_Q\tau} b$ for all $b \in \Gamma(B)$ and $\tau \in \Gamma(Q^*)$.

The anchor of

$$\left(T\sigma_Q(q) - \left(\sigma_Q(\nabla_{b_2}q) + R(b_1,b_2)q^\dagger\right)^\times, T\tilde{q} - \sigma_Q(\nabla_{b_1}q)^\times\right)((b_1,b_2,b_3)(m))$$

is

$$\left(\widehat{\left[\widehat{\nabla_q\cdot},\cdot\right]} - \left(\widehat{\nabla_{\nabla_{b_2}q}\cdot} + \partial_B R(b_1,b_2)q^\uparrow\right)^\uparrow, \widehat{\left[\widehat{\nabla_q\cdot},\cdot\right]} - \widehat{\nabla_{\nabla_{b_1}q}\cdot}^\uparrow\right)((b_1,b_2,b_3)(m)).$$

We call this vector Y. Applying Y to $\ell_{q_B^*\mathbf{d}f} \circ \mathrm{pr}_1 - (p_B^*\ell_{\rho_B^*\mathbf{d}f}) \circ \mathrm{pr}_2$ yields

$$\begin{aligned}
&\ell_{\pounds_{\widehat{\nabla_q\cdot}} q_B^*\mathbf{d}f}\left((Tb_1 + ([b_1,b_2]+c)^\times)\rho_B(b_2)(m)\right) \\
&\quad - \langle q_B^*\mathbf{d}f, \widehat{\nabla_{\nabla_{b_2}q}\cdot} + \partial_B R(b_1,b_2)q^\dagger\rangle(b_1(m)) - \widehat{\nabla_q\cdot}(\ell_{\rho_B^*\mathbf{d}f})(b_2(m)) \\
&= \ell_{q_B^*\mathbf{d}(\rho_Q(q)(f))}\left((Tb_1 + ([b_1,b_2]+c)^\times)\rho_B(b_2)(m)\right) - \rho_Q(\nabla_{b_2}q)(f)(m) - \langle\nabla_q^*(\rho_B^*\mathbf{d}f), b_m\rangle \\
&= \rho_B(b_2(m))\rho_Q(q)(f) - \rho_Q(\nabla_{b_2}q)(f)(m) - \rho_Q(q(m))\langle\rho_B^*\mathbf{d}f, b_2\rangle + \langle\rho_B^*\mathbf{d}f, \nabla_q b_2\rangle \\
&= \langle\mathbf{d}f, [\rho_B(b_2), \rho_Q(q)] - \rho_Q(\nabla_{b_2}q) + \rho_B(\nabla_q b_2)\rangle(m)
\end{aligned}$$

and applying Y to $\ell_{q_B^*\mathbf{d}f} \circ \mathrm{pr}_2 - (p_B^*\ell_{\rho_B^*\mathbf{d}f}) \circ \mathrm{pr}_1$ yields

$$\langle\mathbf{d}f, [\rho_B(b_1), \rho_Q(q)] - \rho_Q(\nabla_{b_1}q) + \rho_B(\nabla_q b_1)\rangle(m).$$

Now we apply Y to $\ell_{\mathbf{d}\ell_\beta} \circ \mathrm{pr}_1 - \ell_{\mathbf{d}\ell_\beta} \circ \mathrm{pr}_2 + \mathbf{d}_B\beta \circ (p_B, p_B)$. Let Φ_t be the flow of $\widehat{\nabla_q\cdot} \in \mathfrak{X}(B)$, and ϕ_t the flow of $\rho_Q(q) \in \mathfrak{X}(M)$. Since for each t, Φ_t is a vector bundle morphism $B \to B$ over ϕ_t, we can define for each section $b \in \Gamma(B)$ a new section $b^t \in \Gamma(B)$ by $b^t(m) = \Phi_t(b(\phi_{-t}(m)))$ for all $m \in M$. We find that $Y(\ell_{\mathbf{d}\ell_\beta} \circ \mathrm{pr}_1 - \ell_{\mathbf{d}\ell_\beta} \circ \mathrm{pr}_2 + \mathbf{d}_B\beta \circ (p_B, p_B))$ equals

$$\begin{aligned}
&\ell_{\pounds_{\widehat{\nabla_q\cdot}}\mathbf{d}\ell_\beta}\left((Tb_1 + ([b_1,b_2]+b_3)^\times)\rho_B(b_2)(m)\right) - \langle\mathbf{d}\ell_\beta, \widehat{\nabla_{\nabla_{b_2}q}\cdot} + \partial_B R(b_1,b_2)q^\uparrow\rangle(b_1(m)) \\
&- \ell_{\pounds_{\widehat{\nabla_q\cdot}}\mathbf{d}\ell_\beta}\left((Tb_2 + b_3^\times)\rho_B(b_1)(m)\right) + \langle\mathbf{d}\ell_\beta, \widehat{\nabla_{\nabla_{b_1}q}\cdot}\rangle(b_2(m)) \\
&+ \left.\frac{d}{dt}\right|_{t=0} \mathbf{d}_B\beta(b_1^t(m), b_2^t(m)).
\end{aligned}$$

The first term is

$$\ell_{\mathbf{d}\ell_{\nabla_q^*\beta}}\left((Tb_1 + ([b_1,b_2]+b_3)^\times)\rho_B(b_2)(m)\right) = \rho_B(b_2(m))\langle\nabla_q^*\beta, b_1\rangle + \langle\nabla_q^*\beta, [b_1,b_2]+b_3\rangle.$$

The second term is

$$-(\ell_{\nabla^*_{\nabla_{b_2}q}\beta} + q_B^*\langle\beta, \partial_B R(b_1,b_2)q\rangle)(b_1(m)) = -\langle\nabla^*_{\nabla_{b_2}q}\beta, b_1(m)\rangle - \langle\beta, \partial_B R(b_1,b_2)q\rangle(m).$$

The third and fourth terms are

$$-\ell_{\mathbf{d}\ell_{\nabla_q^*\beta}}\left((Tb_2 + b_3^\times)\rho_B(b_1)(m)\right) = -\rho_B(b_1(m))\langle\nabla_q^*\beta, b_2\rangle - \langle\nabla_q^*\beta, b_3\rangle(m)$$

and

$$\ell_{\nabla^*_{\nabla_{b_1}q}\beta}(b_2(m)) = \langle\nabla^*_{\nabla_{b_1}q}\beta, b_2\rangle(m)$$

and the fifth term is

$$\begin{aligned}
&+ \left.\frac{d}{dt}\right|_{t=0}\left(\rho_B(b_1^t)\langle\beta, b_2^t\rangle - \rho_B(b_2^t)\langle\beta, b_1^t\rangle - \langle\beta, [b_1^t, b_2^t]\rangle\right)(\phi_t(m)) \\
=& - \rho_B(\nabla_q b_1(m))\langle\beta, b_2\rangle - \rho_B(b_1(m))\langle\beta, \nabla_q b_2\rangle \\
&+ \rho_B(\nabla_q b_2(m))\langle\beta, b_1\rangle + \rho_B(b_2(m))\langle\beta, \nabla_q b_1\rangle + \langle\beta(m), [\nabla_q b_1, b_2] + [b_1, \nabla_q b_2]\rangle \\
&+ \rho_Q(q(m))\left(\rho_B(b_1)\langle\beta, b_2\rangle - \rho_B(b_2)\langle\beta, b_1\rangle - \langle\beta, [b_1, b_2]\rangle\right).
\end{aligned}$$

Hence, we get

$$
\begin{aligned}
&\left\langle \beta, -\nabla_q[b_1,b_2] + \nabla_{\nabla_{b_2}q}b_1 - \partial_B R(b_1,b_2)q - \nabla_{\nabla_{b_1}q}b_2 + [\nabla_q b_1, b_2] + [b_1, \nabla_q b_2]\right\rangle(m)\\
&+ \left([\rho_B(b_2),\rho_Q(q)](m) - \rho_B(\nabla_{b_2}q(m)) + \rho_B(\nabla_q b_2(m))\right)\langle \beta, b_1\rangle\\
&+ \left(-[\rho_B(b_1),\rho_Q(q)](m) + \rho_Q(\nabla_{b_1}q(m)) - \rho_B(\nabla_q b_1(m))\right)\langle \beta, b_2\rangle
\end{aligned}
$$

Hence, we find that the anchors of sections of type (72) vanish on all the functions defining Π_B as in Proposition B.1 if and only if

$$[\rho_B(b),\rho_Q(q)] = \rho_Q(\nabla_b q) - \rho_B(\nabla_q b)$$

and

$$\partial_B R(b_1,b_2)q = -\nabla_q[b_1,b_2] + \nabla_{\nabla_{b_2}q}b_1 - \nabla_{\nabla_{b_1}q}b_2 + [\nabla_q b_1, b_2] + [b_1, \nabla_q b_2]$$

for all $b, b_1, b_2 \in \Gamma(B)$ and $q \in \Gamma(Q)$. □

Proposition B.8. *Assume that* $\partial_Q = \partial_Q^*$. *The Courant brackets of extensions of any two of the special sections of* $\Pi_{\mathbb{E}} \to \Pi_B$ *restrict to a section of* $\Pi_{\mathbb{E}} \to \Pi_B$ *if and only if*

1. $(\Delta_{\partial_Q\tau_1}\tau_2 - \nabla_{\partial_B\tau_2}\tau_1) + (\Delta_{\partial_Q\tau_2}\tau_1 - \nabla_{\partial_B\tau_1}\tau_2) = \rho_Q^*\mathbf{d}\langle\tau_1, \partial_Q\tau_2\rangle$,
2. $\partial_Q(\Delta_q\tau) = \nabla_{\partial_B\tau}q + [\![q, \partial_Q\tau]\!] + \partial_B^*\langle\tau, \nabla_{\cdot}q\rangle$,
3. $\partial_B R(b_1,b_2)q = -\nabla_q[b_1,b_2] + [\nabla_q b_1, b_2] + [b_1, \nabla_q b_2] + \nabla_{\nabla_{b_2}q}b_1 - \nabla_{\nabla_{b_1}q}b_2$,
4. $R(q,\partial_Q\tau)b - R(b,\partial_B\tau)q = \Delta_q\nabla_b\tau - \nabla_b\Delta_q\tau + \Delta_{\nabla_b q}\tau - \nabla_{\nabla_q b}\tau - \langle\nabla_{\nabla_{\cdot}b}q, \tau\rangle$,
5. $\partial_Q R(q_1,q_2)b = -\nabla_b[\![q_1,q_2]\!] + [\![q_1, \nabla_b q_2]\!] + [\![\nabla_b q_1, q_2]\!] + \nabla_{\nabla_{q_2}b}q_1 - \nabla_{\nabla_{q_1}b}q_2 + \partial_B^*\langle R(\cdot, b)q_1, q_2\rangle$.
6. $$
\begin{aligned}
&\nabla_{b_2}R(q_1,q_2)b_1 - \nabla_{b_1}R(q_1,q_2)b_2 + R(q_1,q_2)[b_1,b_2]\\
&+ R(\nabla_{b_1}q_1,q_2)b_2 + R(q_1,\nabla_{b_1}q_2)b_2 - R(\nabla_{b_2}q_1,q_2)b_1 - R(q_1,\nabla_{b_2}q_2)b_1\\
&+ \Delta_{q_1}R(b_1,b_2)q_2 - \Delta_{q_2}R(b_1,b_2)q_1 - R(b_1,b_2)[\![q_1,q_2]\!]\\
&- R(\nabla_{q_1}b_1,b_2)q_2 - R(b_1,\nabla_{q_1}b_2)q_2 + R(\nabla_{q_2}b_1,b_2)q_1 - R(b_1,\nabla_{q_2}b_2)q_1\\
=&\langle R(b_1,\nabla_{\cdot}b_2)q_1,q_2\rangle + \langle R(\nabla_{\cdot}b_1,b_2)q_1,q_2\rangle - \rho_Q^*\mathbf{d}\langle R(b_1,b_2)q_1,q_2\rangle.
\end{aligned}
$$

for all $b, b_1, b_2 \in \Gamma(B)$, $q, q_1, q_2 \in \Gamma(Q)$, $\tau, \tau_1, \tau_2 \in \Gamma(Q^*)$.

In order to prove this, we need to extend the four types of sections (69),(70),(71),(72) to sections of $T\mathbb{E} \times T\mathbb{E} \to TB \times TB$.

Lemma B.9.
1. *The sections of type* (69) *are already restrictions to* Π_B *of sections* $\left((\tau^\dagger)^\times, (\tau^\dagger)^\times\right) \in \Gamma_{TB\times TB}(T\mathbb{E}\times T\mathbb{E})$, $\tau \in \Gamma(Q^*)$.
2. *For* $\chi \in \Gamma(Q^*)$, *the vector bundle morphism* $\nabla_{\cdot}\chi \colon B \to Q^*$ *can be written*

   $$\nabla_{\cdot}\chi = \sum_{i=1}^n \ell_{\xi_i}\cdot\chi_i$$

   with some $\xi_1, \ldots, \xi_n \in \Gamma(B^*)$ *and* $\chi_1, \ldots, \chi_n \in \Gamma(Q^*)$. *The section*

   $$\left(T\chi^\dagger, 0\right) - \sum_{i=1}^n (p_B \circ \mathrm{pr}_2)^*\ell_{\xi_i}\cdot\left((\chi_i^\dagger)^\times, 0\right) + \left(0, \sigma_Q(\partial_Q\chi)^\times\right)$$

   of $T\mathbb{E}\times T\mathbb{E} \to TB \times TB$ *restricts to* $\left(T\chi^\dagger - (\nabla_{p_B\circ\mathrm{pr}_2}\chi^\dagger)^\times, \sigma_Q(\partial_Q\chi)^\times\right)$ *on* Π_B.

(3) *In the same manner, for* $\tau \in \Gamma(Q^*)$*, the morphism* $\nabla.\tau \colon B \to Q^*$ *can be written*

$$\nabla.\tau = \sum_{i=1}^{k} \ell_{\eta_i} \cdot \tau_i$$

with some $\eta_1, \ldots, \eta_k \in \Gamma(B^*)$ *and* $\tau_1, \ldots, \tau_n \in \Gamma(Q^*)$*. The section*

$$\big(\sigma_Q(\partial_Q \tau)^\times, 0\big) + \sum_{i=1}^{k} (p_B \circ \mathrm{pr}_1)^* \ell_{\eta_i} \cdot \Big((\tau_i^\dagger)^\times, 0\Big) + \big(0, T\tau^\dagger\big)$$

of $T\mathbb{E} \times T\mathbb{E} \to TB \times TB$ *restricts to* $\big((\nabla_{p_B \circ \mathrm{pr}_1} \tau^\dagger + \sigma_Q(\partial_Q \tau))^\times, T\tau^\dagger\big)$ *on* Π_B*.*

(4) *Finally, for* $q \in \Gamma(Q)$*, the morphism* $\nabla.q \colon B \to Q$ *can be written*

$$\nabla.q = \sum_{i=1}^{l} \ell_{\gamma_i} \cdot q_i$$

with some $\gamma_1, \ldots, \gamma_l \in \Gamma(B^*)$ *and* $q_1, \ldots, q_n \in \Gamma(Q)$*, and the tensor* $R(\cdot,\cdot)q \colon B \times B \to Q^*$ *can be written*

$$R(b_1, b_2)q = \sum_{i=1}^{m} \sum_{j=1}^{p} \ell_{\alpha_i}(b_1) \ell_{\beta_j}(b_2) \tau_{ij}$$

with some $\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_p \in \Gamma(B^*)$ *and* $\tau_{ij} \in \Gamma(Q^*)$*,* $i = 1, \ldots, m$*,* $j = 1, \ldots, p$*. The section*

$$(T\sigma_Q(q), T\sigma_Q(q)) - \sum_{i=1}^{l} (p_B \circ \mathrm{pr}_2)^* \ell_{\gamma_i} \cdot \big((\sigma_Q(q_i))^\times, 0\big) - \sum_{i=1}^{l} (p_B \circ \mathrm{pr}_1)^* \ell_{\gamma_i} \cdot \big(0, (\sigma_Q(q_i))^\times\big)$$
$$- \sum_{i=1}^{m} \sum_{j=1}^{p} (p_B \circ \mathrm{pr}_1)^* \ell_{\alpha_i} (p_B \circ \mathrm{pr}_2)^* \ell_{\beta_j} \cdot \Big((\tau_{ij}^\dagger)^\times, 0\Big)$$

restricts to

$$\big(T\sigma_Q(q) - (\sigma_Q(\nabla_{p_B \circ \mathrm{pr}_2} q) + R(p_B \circ \mathrm{pr}_1, p_B \circ \mathrm{pr}_2) q^\dagger)^\times, T\sigma_Q(q) - (\sigma_Q(\nabla_{p_B \circ \mathrm{pr}_1} q))^\times\big)$$

on Π_B*.*

Proof of Theorem B.8. We compute successively the Courant brackets of the extensions of our four special types of sections. The Courant bracket of the extension S of

$$\Bigg(T\left(\sigma_Q(q_1) + \chi_1^\dagger\right) + \left(-\sigma_Q(\nabla_{p_B \circ \mathrm{pr}_2} q_1) - R(p_B \circ \mathrm{pr}_1, p_B \circ \mathrm{pr}_2) q_1^\dagger + \sigma_1^\dagger\right)^\times$$
$$+ \left(\nabla_{p_B \circ \mathrm{pr}_1} \tau_1^\dagger - \nabla_{p_B \circ \mathrm{pr}_2} \chi_1^\dagger + \sigma_Q(\partial_Q \tau_1)\right)^\times,$$
$$T\left(\sigma_Q(q_1) + \tau_1^\dagger\right) + \left(-\sigma_Q(\nabla_{p_B \circ \mathrm{pr}_1} q_1) + \sigma_1^\dagger + \sigma_Q(\partial_Q \chi_1)\right)^\times\Bigg)$$

with $\left((\sigma_2^\dagger)^\times, (\sigma_2^\dagger)^\times\right)$ equals $\left((\Delta_{q_1} \sigma_2^\dagger)^\times, (\Delta_{q_1} \sigma_2^\dagger)^\times\right)$, which restricts to a section of type (69) of $\Pi_{\mathbb{E}}$ on Π_B. Since $\left\langle S, \left((\sigma_2^\dagger)^\times, (\sigma_2^\dagger)^\times\right)\right\rangle = p_B^* \left\langle \sigma_Q(q_1) + \chi_1^\dagger, \sigma_2^\dagger \right\rangle - p_B^* \left\langle \sigma_Q(q_1) + \tau_1^\dagger, \sigma_2^\dagger \right\rangle = p_B^* q_B^*(\langle q_1, \sigma_2 \rangle - \langle q_1, \sigma_2 \rangle) = 0$, the Courant bracket of

$\left((\sigma_2^\dagger)^\times,(\sigma_2^\dagger)^\times\right)$ with S equals $-\left((\Delta_{q_1}\sigma_2^\dagger)^\times,(\Delta_{q_1}\sigma_2^\dagger)^\times\right)$. Hence the bracket of sections of type (69) with the four types of sections have values in $\Pi_{\mathbb{E}}$ on Π_B.

Choose $\chi_1,\chi_2,\tau_1,\tau_2\in\Gamma(Q^*)$. We write

$$\left(T\chi_1^\dagger,0\right)-\sum_{i=1}^n(p_B\circ\mathrm{pr}_2)^*\ell_{\xi_i^1}\cdot\left(({\chi_i^1}^\dagger)^\times,0\right)+\left(0,\sigma_Q(\partial_Q\chi_1)^\times\right)$$

for the extension of $\left(T\chi_1^\dagger-(\nabla_{p_B\circ\mathrm{pr}_2}\chi_1^\dagger)^\times,\sigma_Q(\partial_Q\chi_1)^\times\right)$ as in Lemma B.9,

$$\left(T\chi_2^\dagger,0\right)-\sum_{j=1}^m(p_B\circ\mathrm{pr}_2)^*\ell_{\xi_j^2}\cdot\left(({\chi_j^2}^\dagger)^\times,0\right)+\left(0,\sigma_Q(\partial_Q\chi_2)^\times\right)$$

for the extension of $\left(T\chi_2^\dagger-(\nabla_{p_B\circ\mathrm{pr}_2}\chi_2^\dagger)^\times,\sigma_Q(\partial_Q\chi_2)^\times\right)$,

$$\left(\sigma_Q(\partial_Q\tau_1)^\times,0\right)+\sum_{i=1}^k(p_B\circ\mathrm{pr}_1)^*\ell_{\eta_i^1}\cdot\left(({\tau_i^1}^\dagger)^\times,0\right)+\left(0,T\tau_1^\dagger\right)$$

for the extension of $\left((\nabla_{p_B\circ\mathrm{pr}_1}\tau_1^\dagger+\sigma_Q(\partial_Q\tau_1))^\times,T\tau_1^\dagger\right)$ and

$$\left(\sigma_Q(\partial_Q\tau_2)^\times,0\right)+\sum_{j=1}^l(p_B\circ\mathrm{pr}_1)^*\ell_{\eta_i^2}\cdot\left(({\tau_i^2}^\dagger)^\times,0\right)+\left(0,T\tau_2^\dagger\right)$$

for the extension of $\left((\nabla_{p_B\circ\mathrm{pr}_1}\tau_2^\dagger+\sigma_Q(\partial_Q\tau_2))^\times,T\tau_2^\dagger\right)$. We find

$$\Bigg[\!\!\Bigg[\left(T\chi_1^\dagger,0\right)-\sum_{i=1}^n(p_B\circ\mathrm{pr}_2)^*\ell_{\xi_i^1}\cdot\left(({\chi_i^1}^\dagger)^\times,0\right)+\left(0,\sigma_Q(\partial_Q\chi_1)^\times\right),$$
$$\left(T\chi_2^\dagger,0\right)-\sum_{j=1}^m(p_B\circ\mathrm{pr}_2)^*\ell_{\xi_j^2}\cdot\left(({\chi_j^2}^\dagger)^\times,0\right)+\left(0,\sigma_Q(\partial_Q\chi_2)^\times\right)\Bigg]\!\!\Bigg]=0,$$

$$\Bigg[\!\!\Bigg[\left(\sigma_Q(\partial_Q\tau_2)^\times,0\right)+\sum_{j=1}^l(p_B\circ\mathrm{pr}_1)^*\ell_{\eta_i^2}\cdot\left(({\tau_i^2}^\dagger)^\times,0\right)+\left(0,T\tau_2^\dagger\right),$$
$$\left(\sigma_Q(\partial_Q\tau_1)^\times,0\right)+\sum_{i=1}^k(p_B\circ\mathrm{pr}_1)^*\ell_{\eta_i^1}\cdot\left(({\tau_i^1}^\dagger)^\times,0\right)+\left(0,T\tau_1^\dagger\right)\Bigg]\!\!\Bigg]=0$$

and

$$\Bigg[\!\!\Bigg[\left(T\chi^\dagger,0\right)-\sum_{i=1}^n(p_B\circ\mathrm{pr}_2)^*\ell_{\xi_i}\cdot\left((\chi_i^\dagger)^\times,0\right)+\left(0,\sigma_Q(\partial_Q\chi)^\times\right),$$
$$\left(\sigma_Q(\partial_Q\tau)^\times,0\right)+\sum_{i=1}^k(p_B\circ\mathrm{pr}_1)^*\ell_{\eta_i}\cdot\left((\tau_i^\dagger)^\times,0\right)+\left(0,T\tau^\dagger\right)\Bigg]\!\!\Bigg]$$
$$=\left((-\Delta_{\partial_Q\tau}\chi^\dagger+\mathcal{D}_{\mathbb{E}}q_B^*\langle\chi,\partial_Q\tau\rangle)^\times),0\right)+\sum_{i=1}^k(q_B\circ p_B\circ\mathrm{pr}_1)^*\langle\partial_B\chi,\eta_i\rangle\left((\tau_i^\dagger)^\times,0\right)$$

$$
\begin{aligned}
&+\sum_{i=1}^{n}(q_B\circ p_B\circ \mathrm{pr}_2)^*\langle \xi_i,\chi\rangle\cdot\Big((\chi_i^\dagger)^\times,0\Big)+\big(0,(\Delta_{\partial_Q\chi}\tau^\dagger)^\times\big)\\
=&\big((-\Delta_{\partial_Q\tau}\chi+\rho_Q^*\mathbf{d}\langle\chi,\partial_Q\tau\rangle+\nabla_{\partial_B\chi}\tau+\nabla_{\partial_B\tau}\chi)^\dagger)^\times,(\Delta_{\partial_Q\chi}\tau^\dagger)^\times\big)\,.
\end{aligned}
$$

The restriction of this to Π_B is a section (of type (69)) of $\Pi_{\mathbb{E}}$ if and only if

$$
-\Delta_{\partial_Q\tau}\chi+\rho_Q^*\mathbf{d}\langle\chi,\partial_Q\tau\rangle+\nabla_{\partial_B\chi}\tau+\nabla_{\partial_B\tau}\chi=\Delta_{\partial_Q\chi}\tau.
$$

Since using $\partial_Q=\partial_Q^*$

$$
\begin{aligned}
&\Bigg\langle (T\chi^\dagger,0)-\sum_{i=1}^{n}(p_B\circ\mathrm{pr}_2)^*\ell_{\xi_i}\cdot\Big((\chi_i^\dagger)^\times,0\Big)+\big(0,\sigma_Q(\partial_Q\chi)^\times\big)\,,\\
&\qquad\qquad\big(\sigma_Q(\partial_Q\tau)^\times,0\big)+\sum_{i=1}^{k}(p_B\circ\mathrm{pr}_1)^*\ell_{\eta_i}\cdot\Big((\tau_i^\dagger)^\times,0\Big)+\big(0,T\tau^\dagger\big)\Bigg\rangle\\
=&p_B^*q_B^*\left(\langle\partial_Q\tau,\chi\rangle-\langle\tau,\partial_Q\chi\rangle\right)=0,
\end{aligned}
$$

the bracket

$$
\begin{aligned}
&\Bigg[\!\!\Bigg[\big(\sigma_Q(\partial_Q\tau)^\times,0\big)+\sum_{i=1}^{k}(p_B\circ\mathrm{pr}_1)^*\ell_{\eta_i}\cdot\Big((\tau_i^\dagger)^\times,0\Big)+\big(0,T\tau^\dagger\big)\,,\\
&\qquad\qquad(T\chi^\dagger,0)-\sum_{i=1}^{n}(p_B\circ\mathrm{pr}_2)^*\ell_{\xi_i}\cdot\Big((\chi_i^\dagger)^\times,0\Big)+\big(0,\sigma_Q(\partial_Q\chi)^\times\big)\Bigg]\!\!\Bigg]
\end{aligned}
$$

then also restricts to a section of $\Pi_{\mathbb{E}}$ on Π_B.

We compute

$$\begin{aligned}
&\Bigg\llbracket (T\sigma_Q(q), T\sigma_Q(q)) - \sum_{i=1}^{l}(p_B\circ \mathrm{pr}_2)^*\ell_{\gamma_i}\cdot \big((\sigma_Q(q_i))^\times, 0\big)\\
&\qquad - \sum_{i=1}^{l}(p_B\circ \mathrm{pr}_1)^*\ell_{\gamma_i}\cdot \big(0, (\sigma_Q(q_i))^\times\big)\\
&\qquad - \sum_{i=1}^{m}\sum_{j=1}^{p}(p_B\circ \mathrm{pr}_1)^*\ell_{\alpha_i}(p_B\circ \mathrm{pr}_2)^*\ell_{\beta_j}\cdot \Big((\sigma_{ij}^\dagger)^\times, 0\Big),\\
&\qquad\qquad \big(\sigma_Q(\partial_Q\tau)^\times, 0\big) + \sum_{i=1}^{k}(p_B\circ \mathrm{pr}_1)^*\ell_{\eta_i}\cdot \Big((\tau_i^\dagger)^\times, 0\Big) + \big(0, T\tau^\dagger\big)\Bigg\rrbracket\\
&= \big((\sigma_Q(\llbracket q, \partial_Q\tau\rrbracket) - R(q, \partial_Q\tau)^\dagger)^\times, 0\big)\\
&\quad + \sum_{i=1}^{k}\Big((p_B\circ \mathrm{pr}_1)^*\ell_{\nabla_q^*\eta_i}\cdot \Big((\tau_i^\dagger)^\times, 0\Big) + (p_B\circ \mathrm{pr}_1)^*\ell_{\eta_i}\cdot \Big((\Delta_q\tau_i^\dagger)^\times, 0\Big)\Big)\\
&\quad + \big(0, T\Delta_q\tau^\dagger\big) + \sum_{i=1}^{l}(q_B\circ p_B\circ \mathrm{pr}_2)^*\langle\gamma_i, \partial_B\tau\rangle\cdot \big((\sigma_Q(q_i))^\times, 0\big)\\
&\quad + \sum_{i=1}^{l}(p_B\circ \mathrm{pr}_1)^*\ell_{\gamma_i}\cdot \big(0, ((-\Delta_{q_i}\tau + \rho_Q^*\mathbf{d}\langle q_i, \tau\rangle)^\dagger)^\times\big)\\
&\quad - \mathcal{D}_{\overline{T\mathbb{E}}\times T\mathbb{E}}\sum_{i=1}^{l}(p_B\circ \mathrm{pr}_1)^*\ell_{\gamma_i}(q_B\circ p_B\circ \mathrm{pr}_2)^*\langle q_i, \tau\rangle\\
&\quad + \sum_{i=1}^{m}\sum_{j=1}^{p}(p_B\circ \mathrm{pr}_1)^*\ell_{\alpha_i}(q_B\circ p_B\circ \mathrm{pr}_2)^*\langle\partial_B\tau, \beta_j\rangle\cdot \Big((\sigma_{ij}^\dagger)^\times, 0\Big)
\end{aligned}$$

We have

$$\begin{aligned}
&\mathcal{D}_{\overline{T\mathbb{E}}\times T\mathbb{E}}\sum_{i=1}^{l}(p_B\circ \mathrm{pr}_1)^*\ell_{\gamma_i}(q_B\circ p_B\circ \mathrm{pr}_2)^*\langle q_i, \tau\rangle\\
&= \sum_{i=1}^{l}(p_B\circ \mathrm{pr}_1)^*\ell_{\gamma_i}(0, \mathcal{D}_{T\mathbb{E}}p_B^*q_B^*\langle q_i, \tau\rangle) - \sum_{i=1}^{l}(q_B\circ p_B\circ \mathrm{pr}_2)^*\langle q_i, \tau\rangle(\mathcal{D}_{T\mathbb{E}}p_B^*\ell_{\gamma_i}, 0)\\
&= \sum_{i=1}^{l}(p_B\circ \mathrm{pr}_1)^*\ell_{\gamma_i}(0, (\Theta^*\mathbf{d}q_B^*\langle q_i, \tau\rangle)^\times) - \sum_{i=1}^{l}(q_B\circ p_B\circ \mathrm{pr}_2)^*\langle q_i, \tau\rangle\,\big((\Theta^*\mathbf{d}\ell_{\gamma_i})^\times, 0\big)\\
&= \sum_{i=1}^{l}(p_B\circ \mathrm{pr}_1)^*\ell_{\gamma_i}(0, ((\rho_Q^*\mathbf{d}\langle q_i, \tau\rangle)^\dagger)^\times)\\
&\quad - \sum_{i=1}^{l}(q_B\circ p_B\circ \mathrm{pr}_2)^*\langle q_i, \tau\rangle\,\Big((\sigma_Q(\partial^*\gamma_i) + \widetilde{\nabla_{\cdot}^*\gamma_i})^\times, 0\Big).
\end{aligned}$$

Hence, the bracket equals

$$\begin{aligned}
&\bigl((\sigma_Q([\![q,\partial_Q\tau]\!]) - R(q,\partial_Q\tau)^\dagger)^\times, 0\bigr) + \sum_{i=1}^{k}(p_B\circ \mathrm{pr}_1)^*\ell_{\nabla^*_q\eta_i}\cdot\Bigl((\tau_i^\dagger)^\times, 0\Bigr)\\
&\qquad + \sum_{i=1}^{k}(p_B\circ \mathrm{pr}_1)^*\ell_{\eta_i}\cdot\Bigl((\Delta_q\tau_i^\dagger)^\times, 0\Bigr) + \bigl(0, T\Delta_q\tau^\dagger\bigr)\\
&+ \sum_{i=1}^{l}(q_B\circ p_B\circ \mathrm{pr}_2)^*\langle\gamma_i,\partial_B\tau\rangle\cdot\bigl((\sigma_Q(q_i))^\times, 0\bigr) + \sum_{i=1}^{l}(p_B\circ \mathrm{pr}_1)^*\ell_{\gamma_i}\cdot\bigl(0,(-\Delta_{q_i}\tau^\dagger)^\times\bigr)\\
&\qquad + \sum_{i=1}^{l}(q_B\circ p_B\circ \mathrm{pr}_2)^*\langle q_i,\tau\rangle\Bigl((\sigma_Q(\partial^*\gamma_i) + \widetilde{\nabla^*_{\cdot}\gamma_i})^\times, 0\Bigr)\\
&\qquad\quad + \sum_{i=1}^{m}\sum_{j=1}^{p}(p_B\circ \mathrm{pr}_1)^*\ell_{\alpha_i}(q_B\circ p_B\circ \mathrm{pr}_2)^*\langle\partial_B\tau,\beta_j\rangle\cdot\Bigl((\sigma_{ij}^\dagger)^\times, 0\Bigr).
\end{aligned}\tag{74}$$

On $TB\times_{q_B\circ p_B} TB$ we have

$$\sum_{i=1}^{l}(q_B\circ p_B\circ \mathrm{pr}_2)^*\langle\gamma_i,\partial_B\tau\rangle\cdot\bigl((\sigma_Q(q_i))^\times, 0\bigr) = \bigl(\sigma_Q(\nabla_{\partial_B\tau}q)^\times, 0\bigr)$$

and

$$\sum_{i=1}^{l}(q_B\circ p_B\circ \mathrm{pr}_2)^*\langle q_i,\tau\rangle\Bigl(\left(\sigma_Q(\partial^*\gamma_i)\right)^\times, 0\Bigr) = \Bigl(\left(\sigma_Q(\partial_B^*\langle\nabla_{\cdot}q,\tau\rangle)\right)^\times, 0)\Bigr).$$

(Note that since $\langle\nabla_{\cdot}q,\tau\rangle$ is a section of B^*, $\partial_B^*\langle\nabla_{\cdot}q,\tau\rangle$ is a section of Q.) Since

$$\begin{aligned}
\sum_{i=1}^{k}\langle\nabla^*_q\eta_i, b_1\rangle\cdot\tau_i + \langle\eta_i,b_1\rangle\cdot\Delta_q\tau_i &= \sum_{i=1}^{k} -\langle\eta_i,\nabla_q b_1\rangle\cdot\tau_i + \Delta_q\left(\langle\eta_i,b_1\rangle\cdot\tau_i\right)\\
&= -\nabla_{\nabla_q b_1}\tau + \Delta_q\nabla_{b_1}\tau,
\end{aligned}$$

we find

$$\begin{aligned}
&\sum_{i=1}^{k}\Bigl((p_B\circ \mathrm{pr}_1)^*\ell_{\nabla^*_q\eta_i}\cdot\Bigl((\tau_i^\dagger)^\times, 0\Bigr) + (p_B\circ \mathrm{pr}_1)^*\ell_{\eta_i}\cdot\Bigl((\Delta_q\tau_i^\dagger)^\times, 0\Bigr)\Bigr)(b_1,b_2,b_3)(m)\\
&= \Bigl(\left((-\nabla_{\nabla_q b_1}\tau + \Delta_q\nabla_{b_1}\tau)^\dagger\right)^\times, 0\Bigr)(b_1,b_2,b_3)(m)
\end{aligned}$$

at $(b_1,b_2,b_3)(m) = ((Tb_1{+}([b_1,b_2]{+}b_3)^\times)(\rho_B(b_2)(m)), (Tb_2{+}b_3^\times)(\rho_B(b_1)(m))) \in \Pi_B$.
Note also that

$$\sum_{i=1}^{l} -\langle\gamma_i,b_1\rangle\cdot\Delta_{q_i}\tau = -\Delta_{\nabla_{b_1}q}\tau + \sum_{i=1}^{l}\langle q_i,\tau\rangle\rho_Q^*\mathbf{d}\langle\gamma_i,b_1\rangle$$

and

$$\sum_{i=1}^{l}\langle q_i,\tau\rangle\cdot\langle\nabla^*_{\cdot}\gamma_i,b_1\rangle = -\langle\nabla_{\nabla_{\cdot}b_1}q,\tau\rangle + \sum_{i=1}^{l}\langle q_i,\tau\rangle\rho_Q^*\mathbf{d}\langle\gamma_i,b_1\rangle.$$

The bracket is hence at $(b_1,b_2,b_3)(m)\in\Pi_B$:

$$\Biggl(\sigma_Q([\![q,\partial_Q\tau]\!])^\times{+}\sigma_Q(\nabla_{\partial_B\tau}q)^\times{+}\sigma_Q(\partial_B^*\langle\tau,\nabla_{\cdot}q\rangle)^\times{+}\Bigl(-R(q,\partial_Q\tau)b_1^\dagger + R(b_1,\partial_B\tau)q^\dagger + \Delta_q\nabla_{b_1}\tau^\dagger\Bigr)^\times$$

$$-\left(\nabla_{\nabla_q b_1}\tau^\dagger + \langle\tau, \nabla_{\nabla_{.} b_1} q\rangle^\dagger - \sum_{i=1}^{l} \langle q_i,\tau\rangle \rho_Q^*\mathbf{d}\langle\gamma_i, b_1\rangle^\dagger\right)^\times,$$

$$T\Delta_q\tau^\dagger + \left(-\Delta_{\nabla_{b_1}q}\tau^\dagger + \sum_{i=1}^{l} \langle q_i,\tau\rangle\rho_Q^*\mathbf{d}\langle\gamma_i,b_1\rangle^\dagger\right)^\times\Bigg)\big((b_1,b_2,b_3)(m)\big).$$

This is an element of $\Pi_{\mathbb{E}}$ (of type (69)+(71)) if and only if

$$\partial_Q\Delta_q\tau = [\![q,\partial_Q\tau]\!] + \nabla_{\partial_B\tau}q + \partial_B^*\langle\tau,\nabla_{.}q\rangle$$

and

$$-R(q,\partial_Q\tau)b_1 + R(b_1,\partial_B\tau)q + \Delta_q\nabla_{b_1}\tau - \nabla_{\nabla_q b_1}\tau - \langle\tau,\nabla_{\nabla_{.} b_1}q\rangle = -\Delta_{\nabla_{b_1}q}\tau + \nabla_{b_1}\Delta_q\tau.$$

Choose finally $q_1, q_2 \in \Gamma(Q)$. We write

$$(75)\quad (T\sigma_Q(q_1), T\sigma_Q(q_1)) - \sum_i (p_B\circ \mathrm{pr}_2)^*\ell_{\xi_i}\left(\sigma_Q(q_i)^\times, 0\right)$$
$$-\sum_k\sum_l (p_B\circ\mathrm{pr}_1)^*\ell_{\alpha_k}(p_B\circ\mathrm{pr}_2)^*\ell_{\beta_l}\cdot\left(\tau_{kl}^{\dagger\,\times},0\right) - \sum_i(p_B\circ\mathrm{pr}_1)^*\ell_{\xi_i}\left(0,\sigma_Q(q_i)^\times\right)$$

for the extension of

$$\Big(T\sigma_Q(q_1) - (\sigma_Q(\nabla_{p_B\circ\mathrm{pr}_2}q_1) + R(p_B\circ\mathrm{pr}_1, p_B\circ\mathrm{pr}_2)q_1^\dagger)^\times, T\sigma_Q(q_1) - (\sigma_Q(\nabla_{p_B\circ\mathrm{pr}_1}q_1))^\times\Big)$$

and

$$(76)\quad (T\sigma_Q(q_2), T\sigma_Q(q_2)) - \sum_j (p_B\circ\mathrm{pr}_2)^*\ell_{\eta_j}\left(\sigma_Q(r_j)^\times, 0\right)$$
$$-\sum_s\sum_t (p_B\circ\mathrm{pr}_1)^*\ell_{\gamma_s}(p_B\circ\mathrm{pr}_2)^*\ell_{\delta_t}\left(\chi_{st}^{\dagger\,\times},0\right) - \sum_j(p_B\circ\mathrm{pr}_1)^*\ell_{\eta_j}\left(0,\sigma_Q(r_j)^\times\right)$$

for the extension of

$$\Big(T\sigma_Q(q_2) - (\sigma_Q(\nabla_{p_B\circ\mathrm{pr}_2}q_2) + R(p_B\circ\mathrm{pr}_1, p_B\circ\mathrm{pr}_2)q_2^\dagger)^\times, T\sigma_Q(q_2) - \sigma_Q(\nabla_{p_B\circ\mathrm{pr}_1}q_2)^\times\Big)$$

The bracket of these two extensions equals

$$\Big[\!\!\Big[(T\sigma_Q(q_1), T\sigma_Q(q_1)), (T\sigma_Q(q_2), T\sigma_Q(q_2))\Big]\!\!\Big]$$
$$-\Big[\!\!\Big[(T\sigma_Q(q_1), T\sigma_Q(q_1)), \sum_j (p_B\circ\mathrm{pr}_2)^*\ell_{\eta_j}\left(\sigma_Q(r_j)^\times,0\right)\Big]\!\!\Big]$$
$$-\Big[\!\!\Big[(T\sigma_Q(q_1), T\sigma_Q(q_1)), \sum_s\sum_t (p_B\circ\mathrm{pr}_1)^*\ell_{\gamma_s}(p_B\circ\mathrm{pr}_2)^*\ell_{\delta_t}\left(\chi_{st}^{\dagger\,\times},0\right)\Big]\!\!\Big]$$
$$-\Big[\!\!\Big[(T\sigma_Q(q_1), T\sigma_Q(q_1)), \sum_j (p_B\circ\mathrm{pr}_1)^*\ell_{\eta_j}\left(0,\sigma_Q(r_j)^\times\right)\Big]\!\!\Big]$$
$$-\Big[\!\!\Big[\sum_i (p_B\circ\mathrm{pr}_2)^*\ell_{\xi_i}\left(\sigma_Q(q_i)^\times,0\right), (T\sigma_Q(q_2), T\sigma_Q(q_2))\Big]\!\!\Big]$$
$$-\Big[\!\!\Big[\sum_k\sum_l (p_B\circ\mathrm{pr}_1)^*\ell_{\alpha_k}(p_B\circ\mathrm{pr}_2)^*\ell_{\beta_l}\cdot\left(\tau_{kl}^{\dagger\,\times},0\right), (T\sigma_Q(q_2), T\sigma_Q(q_2))\Big]\!\!\Big]$$
$$-\Big[\!\!\Big[\sum_i (p_B\circ\mathrm{pr}_1)^*\ell_{\xi_i}\left(0,\sigma_Q(q_i)^\times\right), (T\sigma_Q(q_2), T\sigma_Q(q_2))\Big]\!\!\Big].$$

(The remaining terms all vanish.) This is

$$
\begin{aligned}
&\Big(T(\sigma_Q(\llbracket q_1,q_2\rrbracket)-R\widetilde{(q_1,q_2)}),T(\sigma_Q(\llbracket q_1,q_2\rrbracket)-R\widetilde{(q_1,q_2)})\Big)\\
&\qquad\qquad -\sum_j(p_B\circ\mathrm{pr}_2)^*\ell_{\eta_j}\left(\Big(\sigma_Q(\llbracket q_1,r_j\rrbracket)-R\widetilde{(q_1,r_j)}\Big)^{\times},0\right)\\
&\qquad\qquad\qquad -\sum_j(p_B\circ\mathrm{pr}_2)^*\ell_{\nabla^*_{q_1}\eta_j}\left(\sigma_Q(r_j)^{\times},0\right)\\
&-\sum_s\sum_t(p_B\circ\mathrm{pr}_1)^*\ell_{\nabla^*_{q_1}\gamma_s}(p_B\circ\mathrm{pr}_2)^*\ell_{\delta_t}\left({\chi_{st}^{\dagger}}^{\times},0\right)-\sum_s\sum_t(p_B\circ\mathrm{pr}_1)^*\ell_{\gamma_s}(p_B\circ\mathrm{pr}_2)^*\ell_{\nabla^*_{q_1}\delta_t}\left({\chi_{st}^{\dagger}}^{\times},0\right)\\
&\qquad\qquad -\sum_s\sum_t(p_B\circ\mathrm{pr}_1)^*\ell_{\gamma_s}(p_B\circ\mathrm{pr}_2)^*\ell_{\delta_t}\left(\Delta_{q_1}{\chi_{st}^{\dagger}}^{\times},0\right)\\
&-\sum_j(p_B\circ\mathrm{pr}_1)^*\ell_{\nabla^*_{q_1}\eta_j}\left(0,\sigma_Q(r_j)^{\times}\right)-\sum_j(p_B\circ\mathrm{pr}_1)^*\ell_{\eta_j}\left(0,\Big(\sigma_Q(\llbracket q_1,r_j\rrbracket)-R\widetilde{(q_1,r_j)}\Big)^{\times}\right)\\
&+\sum_i(p_B\circ\mathrm{pr}_2)^*\ell_{\nabla^*_{q_2}\xi_i}\left(\sigma_Q(q_i)^{\times},0\right)+\sum_i(p_B\circ\mathrm{pr}_2)^*\ell_{\xi_i}\left(\Big(\sigma_Q(\llbracket q_2,q_i\rrbracket)-R\widetilde{(q_2,q_i)}\Big)^{\times},0\right)\\
&\qquad -\mathcal{D}_{\overline{T\mathbb{E}}\times T\mathbb{E}}\left\langle\sum_i(p_B\circ\mathrm{pr}_2)^*\ell_{\xi_i}\left(\sigma_Q(q_i)^{\times},0\right),(T\sigma_Q(q_2),T\sigma_Q(q_2))\right\rangle\\
&+\sum_k\sum_l(p_B\circ\mathrm{pr}_1)^*\ell_{\alpha_k}(p_B\circ\mathrm{pr}_2)^*\ell_{\nabla^*_{q_2}\beta_l}\cdot\left({\sigma_{kl}^{\dagger}}^{\times},0\right)+\sum_k\sum_l(p_B\circ\mathrm{pr}_1)^*\ell_{\nabla^*_{q_2}\alpha_k}(p_B\circ\mathrm{pr}_2)^*\ell_{\beta_l}\cdot\left({\sigma_{kl}^{\dagger}}^{\times},0\right)\\
&\qquad\qquad +\sum_k\sum_l(p_B\circ\mathrm{pr}_1)^*\ell_{\alpha_k}(p_B\circ\mathrm{pr}_2)^*\ell_{\beta_l}\cdot\left(\Delta_{q_2}{\sigma_{kl}^{\dagger}}^{\times},0\right)\\
&-\mathcal{D}_{\overline{T\mathbb{E}}\times T\mathbb{E}}\left\langle\sum_k\sum_l(p_B\circ\mathrm{pr}_1)^*\ell_{\alpha_k}(p_B\circ\mathrm{pr}_2)^*\ell_{\beta_l}\cdot\left({\sigma_{kl}^{\dagger}}^{\times},0\right),(T\sigma_Q(q_2),T\sigma_Q(q_2))\right\rangle\\
&+\sum_i(p_B\circ\mathrm{pr}_2)^*\ell_{\nabla^*_{q_2}\xi_i}\left(0,\sigma_Q(q_i)^{\times}\right)+\sum_i(p_B\circ\mathrm{pr}_2)^*\ell_{\xi_i}\left(0,\Big(\sigma_Q(\llbracket q_2,q_i\rrbracket)-R\widetilde{(q_2,q_i)}\Big)^{\times}\right)\\
&\qquad\qquad -\mathcal{D}_{\overline{T\mathbb{E}}\times T\mathbb{E}}\left\langle\sum_i(p_B\circ\mathrm{pr}_1)^*\ell_{\xi_i}\left(0,\sigma_Q(q_i)^{\times}\right),(T\sigma_Q(q_2),T\sigma_Q(q_2))\right\rangle.
\end{aligned}
$$

We evaluate this at $(b_1,b_2,b_3)(m)=((Tb_1+([b_1,b_2]+b_3)^{\times})(\rho_B(b_2)(m)),(Tb_2+b_3^{\times})(\rho_B(b_1)(m)))$. We have

$$
\begin{aligned}
-\sum_j\langle b_2,\eta_j\rangle\llbracket q_1,r_j\rrbracket-\sum_j\langle b_2,\nabla^*_{q_1}\eta_j\rangle r_j&=\sum_j(-\llbracket q_1,\langle b_2,\eta_j\rangle\cdot r_j\rrbracket+(\rho_Q(q_1)\langle b_2,\eta_j\rangle-\langle b_2,\nabla^*_{q_1}\eta_j\rangle)\cdot r_j)\\
&=-\llbracket q_1,\nabla_{b_2}q_2\rrbracket+\nabla_{\nabla_{q_1}b_2}q_2
\end{aligned}
$$

and in the same manner

$$
\begin{aligned}
&\sum_s\sum_t\left(\langle\nabla^*_{q_1}\gamma_s,b_1\rangle\langle\delta_t,b_2\rangle\chi_{st}+\langle b_1,\gamma_s\rangle\langle b_2,\nabla^*_{q_1}\delta_t\rangle\chi_{st}+\langle b_1,\gamma_s\rangle\langle b_2,\delta_t\rangle\Delta_{q_1}\chi_{st}\right)\\
=&\sum_s\sum_t\left(-\langle\gamma_s,\nabla_{q_1}b_1\rangle\langle\delta_t,b_2\rangle\chi_{st}-\langle b_1,\gamma_s\rangle\langle\delta_t,\nabla_{q_1}b_2\rangle\chi_{st}+\rho_Q(q_1)(\langle\gamma_s,b_1\rangle\cdot\langle\delta_t,b_2\rangle)+\langle b_1,\gamma_s\rangle\langle b_2,\delta_t\rangle\Delta_{q_1}\chi_{st}\right)\\
=&-R(\nabla_{q_1}b_1,b_2)q_2-R(b_1,\nabla_{q_1}b_2)q_2+\Delta_{q_1}(R(b_1,b_2)q_2).
\end{aligned}
$$

We also have $\mathcal{D}_{\overline{T\mathbb{E}}\times T\mathbb{E}} \left\langle \sum_i (p_B\circ \mathrm{pr}_2)^*\ell_{\xi_i}\,(\sigma_Q(q_i)^\times, 0)\,, (T\sigma_Q(q_2), T\sigma_Q(q_2))\right\rangle = 0$ and

$$\begin{aligned}
&\mathcal{D}_{\overline{T\mathbb{E}}\times T\mathbb{E}} \left\langle \sum_k\sum_l (p_B\circ \mathrm{pr}_1)^*\ell_{\alpha_k}(p_B\circ \mathrm{pr}_2)^*\ell_{\beta_l}\cdot \left(\tau_{kl}^{\dagger\,\times}, 0\right), (T\sigma_Q(q_2), T\sigma_Q(q_2))\right\rangle\\
&= -\mathcal{D}_{\overline{T\mathbb{E}}\times T\mathbb{E}}\left(\sum_k\sum_l (p_B\circ \mathrm{pr}_1)^*\ell_{\alpha_k}\cdot (p_B\circ \mathrm{pr}_2)^*\ell_{\beta_l}\cdot (q_B\circ p_B\circ \mathrm{pr}_1)^*\langle \tau_{kl}, q_2\rangle\right)\\
&= \sum_k\sum_l (p_B\circ \mathrm{pr}_1)^*\ell_{\alpha_k}\cdot (p_B\circ \mathrm{pr}_2)^*\ell_{\beta_l}\cdot \left(\left(\rho_Q^*\mathbf{d}\langle \tau_{kl}, q_2\rangle^\dagger\right)^\times, 0\right)\\
&+ \sum_k\sum_l (p_B\circ \mathrm{pr}_2)^*\ell_{\beta_l}\cdot (q_B\circ p_B\circ \mathrm{pr}_1)^*\langle \tau_{kl}, q_2\rangle\cdot \left(\left(\sigma_Q(\partial_B^*\alpha_k) + \widetilde{\nabla^*_{\cdot}\alpha_k}\right)^\times, 0\right)\\
&- \sum_k\sum_l (p_B\circ \mathrm{pr}_1)^*\ell_{\alpha_k}\cdot (q_B\circ p_B\circ \mathrm{pr}_1)^*\langle \tau_{kl}, q_2\rangle\cdot \left(0, \left(\sigma_Q(\partial_B^*\beta_l) + \nabla^*_{\cdot}\beta_l^\dagger\right)^\times\right)
\end{aligned}$$

which is

$$\begin{aligned}
&\sum_k\sum_l \langle\alpha_k, b_1\rangle\cdot\langle\beta_l, b_2\rangle\cdot\left(\left(\rho_Q^*\mathbf{d}\langle\tau_{kl}, q_2\rangle^\dagger\right)^\times, 0\right) + \left(\left(\sigma_Q(\partial_B^*\langle R(\cdot, b_2)q_1, q_2\rangle) - \langle R(\nabla_{\cdot} b_1, b_2)q_1, q_2\rangle^\dagger\right)^\times, 0\right)\\
&+\sum_k\sum_l \langle\beta_l, b_2\rangle\cdot\langle\tau_{kl}, q_2\rangle\cdot\left(\left(\rho_Q^*\mathbf{d}\langle\alpha_k, b_1\rangle^\dagger\right)^\times, 0\right) - \left(0, \left(\sigma_Q(\partial_B^*\langle R(b_1, \cdot)q_1, q_2\rangle) - \langle R(b_1, \nabla_{\cdot} b_2)q_1, q_2\rangle^\dagger\right)^\times\right)\\
&\qquad\qquad - \sum_k\sum_l \langle\alpha_k, b_1\rangle\cdot\langle\tau_{kl}, q_2\rangle\cdot\left(0, \left(\rho_Q^*\mathbf{d}\langle\beta_l, b_2\rangle^\dagger\right)^\times\right)
\end{aligned}$$

at $(b_1, b_2, b_3)(m)$. We evaluate the bracket at $(b_1, b_2, b_3)(m)$ and reorganize the terms to get

$$\begin{aligned}
&\left(T\sigma_Q(\llbracket q_1, q_2\rrbracket) - TR(q_1, q_2){b_1}^\dagger - \left(R(q_1, q_2)([b_1, b_2] + c)^\dagger\right)^\times, T\sigma_Q(\llbracket q_1, q_2\rrbracket) - TR(q_1, q_2){b_2}^\dagger - \left(R(q_1, q_2)c^\dagger\right)^\times\right)\\
&+\Big(-\sigma_Q(\llbracket q_1, \nabla_{b_2}q_2\rrbracket)^\times + \sigma_Q(\nabla_{\nabla_{q_1}b_2}q_2)^\times + \sigma_Q(\llbracket q_2, \nabla_{b_2}q_1\rrbracket)^\times - \sigma_Q(\nabla_{\nabla_{q_2}b_2}q_1)^\times - \sigma_Q(\partial_B^*\langle R(\cdot, b_2)q_1, q_2\rangle)^\times,\\
&\quad -\sigma_Q(\llbracket q_1, \nabla_{b_1}q_2\rrbracket)^\times + \sigma_Q(\nabla_{\nabla_{q_1}b_1}q_2)^\times + \sigma_Q(\partial_B^*\langle R(b_1, \cdot)q_1, q_2\rangle)^\times + \sigma_Q(\llbracket q_2, \nabla_{b_1}q_1\rrbracket)^\times - \sigma_Q(\nabla_{\nabla_{q_2}b_1}q_1)^\times\Big)\\
&+\left((R(q_1, \nabla_{b_2}q_2){b_1}^\dagger)^\times + \left(R(\nabla_{q_1}b_1, b_2)q_2^\dagger + R(b_1, \nabla_{q_1}b_2)q_2^\dagger - \Delta_{q_1}(R(b_1, b_2)q_2)^\dagger\right)^\times - (R(q_2, \nabla_{b_2}q_1){b_1}^\dagger)^\times, 0\right)\\
&+\left(0, (R(q_1, \nabla_{b_1}q_2){b_2}^\dagger)^\times\right) + \left(\left(-R(\nabla_{q_2}b_1, b_2)q_1^\dagger - R(b_1, \nabla_{q_2}b_2)q_1^\dagger + \Delta_{q_2}(R(b_1, b_2)q_1)^\dagger\right)^\times, 0\right)\\
&-\sum_k\sum_l \langle\alpha_k, b_1\rangle\cdot\langle\beta_l, b_2\rangle\cdot\left(\left(\rho_Q^*\mathbf{d}\langle\tau_{kl}, q_2\rangle^\dagger\right)^\times, 0\right) - \sum_k\sum_l \langle\beta_l, b_2\rangle\cdot\langle\tau_{kl}, q_2\rangle\cdot\left(\left(\rho_Q^*\mathbf{d}\langle\alpha_k, b_1\rangle^\dagger\right)^\times, 0\right)\\
&+\left(\left(\langle R(\nabla_{\cdot} b_1, b_2)q_1, q_2\rangle^\dagger\right)^\times, 0\right) - \left(0, \left(\langle R(b_1, \nabla_{\cdot} b_2)q_1, q_2\rangle^\dagger\right)^\times\right) - \left(0, \left(R(q_2, \nabla_{b_1}q_1){b_2}^\dagger\right)^\times\right)\\
&\qquad\qquad + \sum_k\sum_l \langle\alpha_k, b_1\rangle\cdot\langle\tau_{kl}, q_2\rangle\cdot\left(0, \left(\rho_Q^*\mathbf{d}\langle\beta_l, b_2\rangle^\dagger\right)^\times\right)
\end{aligned}$$

This is an element of $\Pi_{\mathbb{E}}$ if and only if

$$\begin{aligned}
&-\nabla_{b_1}[\![q_1,q_2]\!] - \partial_Q R(q_1,q_2)b_1\\
&= -[\![q_1,\nabla_{b_1}q_2]\!] + \nabla_{\nabla_{q_1}b_1}q_2 + \partial_B^*\langle R(b_1,\cdot)q_1,q_2\rangle + [\![q_2,\nabla_{b_1}q_1]\!] - \nabla_{\nabla_{q_2}b_1}q_1\\
&= -[\![q_1,\nabla_{b_1}q_2]\!] + \nabla_{\nabla_{q_1}b_1}q_2 + \partial_B^*\langle R(b_1,\cdot)q_1,q_2\rangle - [\![\nabla_{b_1}q_1,q_2]\!] - \nabla_{\nabla_{q_2}b_1}q_1
\end{aligned}$$

and

$$\begin{aligned}
&-\nabla_{b_1}R(q_1,q_2)b_2 + \nabla_{b_2}R(q_1,q_2)b_1 - R(b_1,b_2)[\![q_1,q_2]\!] - R(q_1,q_2)b_3\\
&\qquad - \langle R(b_1,\nabla_{\cdot}b_2)q_1,q_2\rangle - R(q_2,\nabla_{b_1}q_1)b_2\\
&\qquad + \sum_k\sum_l\langle\alpha_k,b_1\rangle\cdot\langle\tau_{kl},q_2\rangle\cdot\rho_Q^*\mathbf{d}\langle\beta_l,b_2\rangle + R(q_1,\nabla_{b_1}q_2)b_2\\
&= -R(q_1,q_2)([b_1,b_2]+b_3) + R(q_1,\nabla_{b_2}q_2)b_1 + R(\nabla_{q_1}b_1,b_2)q_2 + R(b_1,\nabla_{q_1}b_2)q_2\\
&\qquad - \Delta_{q_1}(R(b_1,b_2)q_2) - R(q_2,\nabla_{b_2}q_1)b_1\\
&\qquad - R(\nabla_{q_2}b_1,b_2)q_1 - R(b_1,\nabla_{q_2}b_2)q_1 + \Delta_{q_2}(R(b_1,b_2)q_1)\\
&\qquad - \sum_k\sum_l\langle\alpha_k,b_1\rangle\cdot\langle\beta_l,b_2\rangle\cdot\rho_Q^*\mathbf{d}\langle\tau_{kl},q_2\rangle\\
&\qquad - \sum_k\sum_l\langle\beta_l,b_2\rangle\cdot\langle\tau_{kl},q_2\rangle\cdot\rho_Q^*\mathbf{d}\langle\alpha_k,b_1\rangle + \langle R(\nabla_{\cdot}b_1,b_2)q_1,q_2\rangle.
\end{aligned}$$

The last equation can be rewritten

$$\begin{aligned}
&-\nabla_{b_1}R(q_1,q_2)b_2 + \nabla_{b_2}R(q_1,q_2)b_1 - R(b_1,b_2)[\![q_1,q_2]\!] - \langle R(b_1,\nabla_{\cdot}b_2)q_1,q_2\rangle\\
&- R(q_2,\nabla_{b_1}q_1)b_2 + R(q_1,\nabla_{b_1}q_2)b_2\\
&= -R(q_1,q_2)[b_1,b_2] + R(q_1,\nabla_{b_2}q_2)b_1 + R(\nabla_{q_1}b_1,b_2)q_2 + R(b_1,\nabla_{q_1}b_2)q_2\\
&- \Delta_{q_1}(R(b_1,b_2)q_2) - R(q_2,\nabla_{b_2}q_1)b_1 - R(\nabla_{q_2}b_1,b_2)q_1 - R(b_1,\nabla_{q_2}b_2)q_1\\
&+ \Delta_{q_2}(R(b_1,b_2)q_1) - \rho_Q^*\mathbf{d}\langle R(b_1,b_2)q_1,q_2\rangle + \langle R(\nabla_{\cdot}b_1,b_2)q_1,q_2\rangle.
\end{aligned}$$

With (DS6), this is

$$\begin{aligned}
&R(q_1,q_2)[b_1,b_2] - R(b_1,b_2)[\![q_1,q_2]\!]\\
&- \nabla_{b_1}R(q_1,q_2)b_2 + R(\nabla_{b_1}q_1,q_2)b_2 + R(q_1,\nabla_{b_1}q_2)b_2\\
&+ \nabla_{b_2}R(q_1,q_2)b_1 - R(\nabla_{b_2}q_1,q_2)b_1 - R(q_1,\nabla_{b_2}q_2)b_1\\
&+ \Delta_{q_1}(R(b_1,b_2)q_2) - R(\nabla_{q_1}b_1,b_2)q_2 - R(b_1,\nabla_{q_1}b_2)q_2\\
&- \Delta_{q_2}(R(b_1,b_2)q_1) + R(\nabla_{q_2}b_1,b_2)q_1 + R(b_1,\nabla_{q_2}b_2)q_1\\
&= -\rho_Q^*\mathbf{d}\langle R(b_1,b_2)q_1,q_2\rangle + \langle R(\nabla_{\cdot}b_1,b_2)q_1,q_2\rangle + \langle R(b_1,\nabla_{\cdot}b_2)q_1,q_2\rangle.
\end{aligned}$$

$\square$

Appendix C. Proof of Theorem 8.15

Note that in the following computations, we will make extensive use of the identity $\partial_Q = \partial_Q^*$ without always mentioning it. We begin by proving the two following Lemmas.

Lemma C.1. *Consider an LA-Courant algebroid $(\mathbb{E}, Q, B, M)$. The bracket $[\![\cdot\,,\cdot]\!]_{Q^*}$ on sections of the core Q^* satisfies the following equation:*

$$\begin{aligned}
R(\partial_B\tau_1,\partial_B\tau_2)q =& -\Delta_q[\![\tau_1,\tau_2]\!]_{Q^*} + [\![\Delta_q\tau_1,\tau_2]\!]_{Q^*} + [\![\tau_1,\Delta_q\tau_2]\!]_{Q^*}\\
&+ \Delta_{\nabla_{\partial_B\tau_2}q}\tau_1 - \Delta_{\nabla_{\partial_B\tau_1}q}\tau_2 - \rho_Q^*\mathbf{d}\langle\tau_1,\nabla_{\partial_B\tau_2}q\rangle
\end{aligned}\tag{77}$$

for all $q \in \Gamma(Q)$ *and* $\tau_1, \tau_2 \in \Gamma(Q^*)$.

Proof. The proof is just a computation using (M1) and (53). We have

$$\begin{aligned}
&\Delta_q[\![\tau_1, \tau_2]\!]_{Q^*} - [\![\Delta_q\tau_1, \tau_2]\!]_{Q^*} - [\![\tau_1, \Delta_q\tau_2]\!]_{Q^*} + \Delta_{\nabla_{\partial_B\tau_1}q}\tau_2 \\
&- \Delta_{\nabla_{\partial_B\tau_2}q}\tau_1 + \rho_Q^*\mathbf{d}\langle\tau_1, \nabla_{\partial_B\tau_2}q\rangle \\
=&\Delta_q\Delta_{\partial_Q\tau_1}\tau_2 - \Delta_q\nabla_{\partial_B\tau_2}\tau_1 - \Delta_{\partial_Q(\Delta_q\tau_1)}\tau_2 + \nabla_{\partial_B\tau_2}\Delta_q\tau_1 - \Delta_{\partial_Q\tau_1}\Delta_q\tau_2 \\
&+ \nabla_{\partial_B(\Delta_q\tau_2)}\tau_1 + \Delta_{\nabla_{\partial_B\tau_1}q}\tau_2 - \Delta_{\nabla_{\partial_B\tau_2}q}\tau_1 + \rho_Q^*\mathbf{d}\langle\tau_1, \nabla_{\partial_B\tau_2}q\rangle
\end{aligned}$$

Replacing $\Delta_q\Delta_{\partial_Q\tau_1}\tau_2 - \Delta_{\partial_Q\tau_1}\Delta_q\tau_2$ by $R_\Delta(q, \partial_Q\tau_1)\tau_2 + \Delta_{[\![q,\partial_Q\tau_1]\!]}\tau_2$ and reordering the terms yields

$$\begin{aligned}
&R_\Delta(q, \partial_Q\tau_1)\tau_2 + \Delta_{[\![q,\partial_Q\tau_1]\!]-\partial_Q(\Delta_q\tau_1)+\nabla_{\partial_B\tau_1}q}\tau_2 - \Delta_q\nabla_{\partial_B\tau_2}\tau_1 + \nabla_{\partial_B\tau_2}\Delta_q\tau_1 \\
&+ \nabla_{\partial_B(\Delta_q\tau_2)}\tau_1 - \Delta_{\nabla_{\partial_B\tau_2}q}\tau_1 + \rho_Q^*\mathbf{d}\langle\tau_1, \nabla_{\partial_B\tau_2}q\rangle.
\end{aligned}$$

Since $R_\Delta(q, \partial_Q\tau_1)\tau_2 = R(q, \partial_Q\tau_1)\partial_B\tau_2$ by (D4), we can now use (53) and $\nabla_q \circ \partial_B = \partial_B \circ \Delta_q$ to replace

$$R_\Delta(q, \partial_Q\tau_1)\tau_2 - \Delta_q\nabla_{\partial_B\tau_2}\tau_1 + \nabla_{\partial_B\tau_2}\Delta_q\tau_1 - \Delta_{\nabla_{\partial_B\tau_2}q}\tau_1 + \nabla_{\nabla_q\partial_B\tau_2}\tau_1$$

by

$$-\langle\nabla_{\nabla_\cdot\partial_B\tau_2}q, \tau_1\rangle + R(\partial_B\tau_2, \partial_B\tau_1)q.$$

We use (M1) to replace $\Delta_{[\![q,\partial_Q\tau_1]\!]-\partial_Q(\Delta_q\tau_1)+\nabla_{\partial_B\tau_1}q}\tau_2$ by $-\Delta_{\partial_B^*\langle\tau_1,\nabla_\cdot q\rangle}\tau_2$. These two steps yield that the right hand side of our equation is

$$-\langle\nabla_{\nabla_\cdot\partial_B\tau_2}q, \tau_1\rangle + R(\partial_B\tau_2, \partial_B\tau_1)q - \Delta_{\partial_B^*\langle\tau_1,\nabla_\cdot q\rangle}\tau_2 + \rho_Q^*\mathbf{d}\langle\tau_1, \nabla_{\partial_B\tau_2}q\rangle.$$

To conclude, let us show that

$$-\langle\nabla_{\nabla_\cdot\partial_B\tau_2}q, \tau_1\rangle - \Delta_{\partial_B^*\langle\tau_1,\nabla_\cdot q\rangle}\tau_2 + \rho_Q^*\mathbf{d}\langle\tau_1, \nabla_{\partial_B\tau_2}q\rangle \in \Gamma(Q^*)$$

vanishes. On $q' \in \Gamma(Q)$, this is

$$\begin{aligned}
&-\langle\nabla_{\nabla_{q'}(\partial_B\tau_2)}q, \tau_1\rangle + \langle[\![\partial_B^*\langle\nabla_\cdot q, \tau_1\rangle, q']\!], \tau_2\rangle + \rho_Q(q')\langle\tau_1, \nabla_{\partial_B\tau_2}q\rangle \\
=&-\langle\nabla_{\nabla_{q'}(\partial_B\tau_2)}q, \tau_1\rangle + \langle\Delta_{q'}\tau_2, \partial_B^*(\langle\nabla_\cdot q, \tau_1\rangle)\rangle \\
=&-\langle\nabla_{\nabla_{q'}(\partial_B\tau_2)}q, \tau_1\rangle + \langle\nabla_{\partial_B(\Delta_{q'}\tau_2)}q, \tau_1\rangle = 0.
\end{aligned}$$

We have used (47)$\rho_Q \circ \partial_B^* = 0$ and the duality of the Dorfman connection with the dull bracket in the first line, as well as for the first equality. To conclude, we have used $\partial_B \circ \Delta_{q'} = \nabla_{q'} \circ \partial_B$ by (D1). □

Lemma C.2. *The bracket on* Q^* *satisfies*

$$[\![\rho_Q^*\mathbf{d}f, \tau]\!]_{Q^*} = 0 \tag{78}$$

for all $f \in C^\infty(M)$ *and* $\tau \in \Gamma(Q^*)$.

Proof. We have $\Delta_q(\rho_Q^*\mathbf{d}f) = \rho_Q^*\mathbf{d}(\rho_Q(q)f)$ by the definition of a Dorfman connection (Definition 5.4).

By (52), we have $[\![\rho_Q^*\mathbf{d}f, \tau]\!]_{Q^*} = \nabla_{\partial_B\rho_Q^*\mathbf{d}f}\tau - \Delta_{\partial_Q\tau}(\rho_Q^*\mathbf{d}f) + \rho_Q^*\mathbf{d}((\rho_Q\partial_Q\tau)f)$. Since $\partial_B\rho_Q^* = 0$ by (47), and $\Delta_{\partial_Q\tau}(\rho_Q^*\mathbf{d}f) = \rho_Q^*\mathbf{d}(\rho_Q(\partial_Q\tau)(f))$, the bracket $[\![\rho_Q^*\mathbf{d}f, \tau]\!]_{Q^*}$ is 0. □

Now we check that the bracket $[\![\cdot\,,\cdot]\!]_{\mathbb{B}}$ in Theorem 8.15 is well-defined. We have for all $\upsilon \in \Gamma(U^\circ)$, $\tau \in \Gamma(Q^*)$ and $u \in \Gamma(U)$:

$$\begin{aligned}[\![u \oplus \tau, (-\partial_Q \upsilon) \oplus \upsilon]\!] = (&-[\![u, \partial_Q \upsilon]\!]_U + \nabla_{\partial_B \tau}(-\partial_Q \upsilon) - \nabla_{\partial_B \upsilon} u) \\ &\oplus \left([\![\tau, \upsilon]\!]_{Q^*} + \Delta_u \upsilon - \Delta_{-\partial_Q \upsilon} \tau + \rho_Q^* \mathbf{d}\langle \tau, -\partial_Q \upsilon\rangle\right).\end{aligned}$$

By (M1), the properties of 2-representations and (52), this is

$$\begin{aligned}&(-\partial_Q(\Delta_u \upsilon) + \partial_B^*\langle \upsilon, \nabla_{.} u\rangle + \cancel{\nabla_{\partial_B \upsilon} u} - \partial_Q \nabla_{\partial_B \tau} \upsilon - \cancel{\nabla_{\partial_B \upsilon} u}) \\ &\quad \oplus \left(-\cancel{\Delta_{\partial_Q \upsilon} \tau} + \nabla_{\partial_B \tau} \upsilon + \cancel{\rho_Q^* \mathbf{d}\langle \tau, \partial_Q \upsilon\rangle} + \Delta_u \upsilon + \cancel{\Delta_{\partial_Q \upsilon} \tau} - \cancel{\rho_Q^* \mathbf{d}\langle \tau, \partial_Q \upsilon\rangle}\right) \\ &= (-\partial_Q(\Delta_u \upsilon) + \partial_B^*\langle \upsilon, \nabla_{.} u\rangle - \partial_Q \nabla_{\partial_B \tau} \upsilon) \oplus (\nabla_{\partial_B \tau} \upsilon + \Delta_u \upsilon).\end{aligned}$$

Since $\upsilon \in \Gamma(U^\circ)$ and ∇_b preserves $\Gamma(U)$ for all $b \in \Gamma(B)$, the section $\langle \upsilon, \nabla_{.} u\rangle$ of B^* vanishes and we get

$$[\![u \oplus \tau, (-\partial_Q \upsilon) \oplus \upsilon]\!] = (-\partial_Q(\Delta_u \upsilon + \nabla_{\partial_B \tau} \upsilon)) \oplus (\nabla_{\partial_B \tau} \upsilon + \Delta_u \upsilon).$$

Because Δ_u preserves as well $\Gamma(U^\circ)$, the sum $\nabla_{\partial_B \tau} \upsilon + \Delta_u \upsilon$ is a section of U°, and so $[\![u \oplus \tau, (-\partial_Q \upsilon) \oplus \upsilon]\!]$ is zero in $\mathbb{B}$.

We now check the Courant algebroid axioms (CA1), (CA2) and (CA3). The last one, (CA3), is immediate:

$$\begin{aligned}&[\![u_1 \oplus \tau_1, u_2 \oplus \tau_2]\!]_{\mathbb{B}} + [\![u_2 \oplus \tau_2, u_1 \oplus \tau_1]\!]_{\mathbb{B}} \\ &= 0 \oplus \left(\rho_Q^* \mathbf{d}\langle \tau_1, \partial_Q \tau_2\rangle + \rho_Q^* \mathbf{d}\langle \tau_1, u_2\rangle + \rho_Q^* \mathbf{d}\langle \tau_2, u_1\rangle\right) = 0 \oplus \rho_Q^* \mathbf{d}\langle u_1 \oplus \tau_1, u_2 \oplus \tau_2\rangle_{\mathbb{B}} \\ &= \mathcal{D}_{\mathbb{B}}\langle u_1 \oplus \tau_1, u_2 \oplus \tau_2\rangle_{\mathbb{B}}.\end{aligned}$$

Next we prove (CA2). We have, using (52) to replace $[\![\tau_1, \tau_2]\!]_{Q^*}$ by $-\Delta_{\partial_Q \tau_2} \tau_1 + \nabla_{\partial_B \tau_1} \tau_2 + \rho_Q^* \mathbf{d}\langle \tau_1, \partial_Q \tau_2\rangle$:

$$\begin{aligned}&\langle [\![u_1 \oplus \tau_1, u_2 \oplus \tau_2]\!], u_3 \oplus \tau_3\rangle_{\mathbb{B}} \\ &= \langle [\![u_1, u_2]\!]_U + \nabla_{\partial_B \tau_1} u_2 - \nabla_{\partial_B \tau_2} u_1, \tau_3\rangle \\ &+ \langle -\Delta_{\partial_Q \tau_2} \tau_1 + \nabla_{\partial_B \tau_1} \tau_2 + \rho_Q^* \mathbf{d}\langle \tau_1, \partial_Q \tau_2\rangle + \Delta_{u_1} \tau_2 - \Delta_{u_2} \tau_1 + \rho_Q^* \mathbf{d}\langle \tau_1, u_2\rangle, u_3 + \partial_Q \tau_3\rangle \\ &= \rho_Q(u_1)\langle u_2, \tau_3\rangle - \langle u_2, \Delta_{u_1} \tau_3\rangle + \langle \nabla_{\partial_B \tau_1} u_2 - \nabla_{\partial_B \tau_2} u_1, \tau_3\rangle \\ &- \langle \Delta_{\partial_Q \tau_2} \tau_1, u_3\rangle + \langle \nabla_{\partial_B \tau_1} \tau_2, u_3\rangle + \rho_Q(u_3)\langle \tau_1, \partial_Q \tau_2\rangle + \langle \Delta_{u_1} \tau_2, u_3\rangle - \langle \Delta_{u_2} \tau_1, u_3\rangle \\ &+ \rho_Q(u_3)\langle \tau_1, u_2\rangle - \langle \Delta_{\partial_Q \tau_2} \tau_1, \partial_Q \tau_3\rangle + \langle \nabla_{\partial_B \tau_1} \tau_2, \partial_Q \tau_3\rangle + \rho_Q(\partial_Q \tau_3)\langle \tau_1, \partial_Q \tau_2\rangle \\ &+ \langle \Delta_{u_1} \tau_2, \partial_Q \tau_3\rangle - \langle \Delta_{u_2} \tau_1, \partial_Q \tau_3\rangle + \rho_Q(\partial_Q \tau_3)\langle \tau_1, u_2\rangle.\end{aligned}$$

We sum $\langle [\![u_1 \oplus \tau_1, u_2 \oplus \tau_2]\!], u_3 \oplus \tau_3\rangle_{\mathbb{B}}$ with $\langle u_2 \oplus \tau_2, [\![u_1 \oplus \tau_1, u_3 \oplus \tau_3]\!]\rangle_{\mathbb{B}}$, and replace only in the first summand the term $\langle \Delta_{u_1} \tau_2, \partial_Q \tau_3\rangle$ by $\rho_Q(u_1)\langle \tau_2, \partial_Q \tau_3\rangle - \langle \tau_2, [\![u_1, \partial_Q \tau_3]\!]\rangle$.

This yields

$$\begin{aligned}
&\langle [\![u_1 \oplus \tau_1, u_2 \oplus \tau_2]\!], u_3 \oplus \tau_3\rangle_{\mathbb{B}} + \langle u_2 \oplus \tau_2, [\![u_1 \oplus \tau_1, u_3 \oplus \tau_3]\!]\rangle_{\mathbb{B}}\\
&= \rho_Q(u_1)\langle u_2, \tau_3\rangle - \cancel{\langle u_2, \Delta_{u_1}\tau_3\rangle} + \langle \nabla_{\partial_B\tau_1}u_2 - \nabla_{\partial_B\tau_2}u_1, \tau_3\rangle\\
&- \langle \Delta_{\partial_Q\tau_2}\tau_1, u_3\rangle + \langle \nabla_{\partial_B\tau_1}\tau_2, u_3\rangle + \rho_Q(u_3)\langle \tau_1, \partial_Q\tau_2\rangle + \cancel{\langle \Delta_{u_1}\tau_2, u_3\rangle} - \langle \Delta_{u_2}\tau_1, u_3\rangle\\
&+ \rho_Q(u_3)\langle \tau_1, u_2\rangle - \langle \Delta_{\partial_Q\tau_2}\tau_1, \partial_Q\tau_3\rangle + \langle \nabla_{\partial_B\tau_1}\tau_2, \partial_Q\tau_3\rangle + \rho_Q(\partial_Q\tau_3)\langle \tau_1, \partial_Q\tau_2\rangle\\
&+ \rho_Q(u_1)\langle \tau_2, \partial_Q\tau_3\rangle - \langle \tau_2, [\![u_1, \partial_Q\tau_3]\!]\rangle - \langle \Delta_{u_2}\tau_1, \partial_Q\tau_3\rangle + \rho_Q(\partial_Q\tau_3)\langle \tau_1, u_2\rangle\\
&+ \rho_Q(u_1)\langle u_3, \tau_2\rangle - \cancel{\langle u_3, \Delta_{u_1}\tau_2\rangle} + \langle \nabla_{\partial_B\tau_1}u_3 - \nabla_{\partial_B\tau_3}u_1, \tau_2\rangle\\
&- \langle \Delta_{\partial_Q\tau_3}\tau_1, u_2\rangle + \langle \nabla_{\partial_B\tau_1}\tau_3, u_2\rangle + \rho_Q(u_2)\langle \tau_1, \partial_Q\tau_3\rangle + \cancel{\langle \Delta_{u_1}\tau_3, u_2\rangle} - \langle \Delta_{u_3}\tau_1, u_2\rangle\\
&+ \rho_Q(u_2)\langle \tau_1, u_3\rangle - \langle \Delta_{\partial_Q\tau_3}\tau_1, \partial_Q\tau_2\rangle + \langle \nabla_{\partial_B\tau_1}\tau_3, \partial_Q\tau_2\rangle + \rho_Q(\partial_Q\tau_2)\langle \tau_1, \partial_Q\tau_3\rangle\\
&+ \langle \Delta_{u_1}\tau_3, \partial_Q\tau_2\rangle - \langle \Delta_{u_3}\tau_1, \partial_Q\tau_2\rangle + \rho_Q(\partial_Q\tau_2)\langle \tau_1, u_3\rangle.
\end{aligned}$$

We reorder the remaining terms and replace eight times sums like $\rho_Q(\partial_Q\tau_2)\langle \tau_1, u_3\rangle - \langle \Delta_{\partial_Q\tau_2}\tau_1, u_3\rangle$ by $\langle [\![\partial_Q\tau_2, u_3]\!], \tau_1\rangle$, and, using $\nabla^{Q^*} = \nabla^{Q^*}$, three times sums like $\langle \nabla_{\partial_B\tau_1}u_2, \tau_3\rangle + \langle u_2, \nabla_{\partial_B\tau_1}\tau_3\rangle$ by $\rho_B(\partial_B\tau_1)\langle u_2, \tau_3\rangle$. This leads to

$$\begin{aligned}
&\langle [\![u_1 \oplus \tau_1, u_2 \oplus \tau_2]\!], u_3 \oplus \tau_3\rangle_{\mathbb{B}} + \langle u_2 \oplus \tau_2, [\![u_1 \oplus \tau_1, u_3 \oplus \tau_3]\!]\rangle_{\mathbb{B}}\\
&= \rho_Q(u_1)\langle u_2 \oplus \tau_2, u_3 \oplus \tau_3\rangle_{\mathbb{B}}\\
&+ \cancel{\langle [\![\partial_Q\tau_2, \partial_Q\tau_3]\!], \tau_1\rangle} + \cancel{\langle [\![\partial_Q\tau_2, u_3]\!], \tau_1\rangle} + \cancel{\langle [\![u_3, \partial_Q\tau_2]\!], \tau_1\rangle} + \cancel{\langle [\![u_2, u_3]\!], \tau_1\rangle}\\
&+ \cancel{\langle [\![u_3, u_2]\!], \tau_1\rangle} + \cancel{\langle [\![\partial_Q\tau_3, u_2]\!], \tau_1\rangle} + \cancel{\langle [\![u_2, \partial_Q\tau_3]\!], \tau_1\rangle} + \cancel{\langle [\![\partial_Q\tau_3, \partial_Q\tau_2]\!], \tau_1\rangle}\\
&+ \rho_B(\partial_B\tau_1)\langle u_2, \tau_3\rangle + \rho_B(\partial_B\tau_1)\langle \tau_3, \partial_Q\tau_2\rangle + \rho_B(\partial_B\tau_1)\langle u_3, \tau_2\rangle\\
&- \langle \nabla_{\partial_B\tau_2}u_1, \tau_3\rangle - \langle \tau_2, [\![u_1, \partial_Q\tau_3]\!]\rangle - \langle \nabla_{\partial_B\tau_3}u_1, \tau_2\rangle + \langle \Delta_{u_1}\tau_3, \partial_Q\tau_2\rangle.
\end{aligned}$$

Because $[\![\cdot\,,\cdot]\!]$ is skew-symmetric on $\Gamma(U)$, $\langle [\![u_2, u_3]\!], \tau_1\rangle$ and $\langle [\![u_3, u_2]\!], \tau_1\rangle$ cancel each other, etc. The four last terms cancel each other by (M1) and $\partial_Q = \partial_Q^*$. This yields

$$\begin{aligned}
&\langle [\![u_1 \oplus \tau_1, u_2 \oplus \tau_2]\!], u_3 \oplus \tau_3\rangle_{\mathbb{B}} + \langle u_2 \oplus \tau_2, [\![u_1 \oplus \tau_1, u_3 \oplus \tau_3]\!]\rangle_{\mathbb{B}}\\
&= (\rho_Q(u_1) + \rho_B\partial_B\tau_1)\langle u_2 \oplus \tau_2, u_3 \oplus \tau_3\rangle_{\mathbb{B}} = \rho_{\mathbb{B}}(u_1 \oplus \tau_1)\langle u_2 \oplus \tau_2, u_3 \oplus \tau_3\rangle_{\mathbb{B}}.
\end{aligned}$$

Finally we check the Jacobi identity in Leibniz form (CA1). We will check that

$$\mathrm{Jac}_{[\![\cdot\,,\cdot]\!]}(u_1 \oplus \tau_1, u_2 \oplus \tau_2, u_3 \oplus \tau_3) = (-\partial_Q\upsilon) \oplus \upsilon$$

with

$$\upsilon = (R(\partial_B\tau_1, \partial_B\tau_2)u_3 - R(u_1, u_2)\partial_B\tau_3) + \text{cyclic permutations.}$$

Since by Proposition 8.9 $R(u_1, u_2)$ has image in U° for all $u_1, u_2 \in \Gamma(U)$ and $R(b_1, b_2)$ restricts to a morphism $U \to U^\circ$ for all $b_1, b_2 \in \Gamma(B)$ by Proposition 8.5, υ is a section of U° and so $\mathrm{Jac}_{[\![\cdot\,,\cdot]\!]}(u_1 \oplus \tau_1, u_2 \oplus \tau_2, u_3 \oplus \tau_3)$ will be zero in $\mathbb{B}$.

We compute

$$\begin{aligned}
&\llbracket \llbracket u_1 \oplus \tau_1, u_2 \oplus \tau_2 \rrbracket, u_3 \oplus \tau_3 \rrbracket \\
&= \llbracket (\llbracket u_1, u_2 \rrbracket_U + \nabla_{\partial_B \tau_1} u_2 - \nabla_{\partial_B \tau_2} u_1) \oplus (\llbracket \tau_1, \tau_2 \rrbracket_{Q^*} + \Delta_{u_1}\tau_2 - \Delta_{u_2}\tau_1 + \rho_Q^* \mathbf{d}\langle \tau_1, u_2\rangle), u_3 \oplus \tau_3 \rrbracket \\
&= \Big(\llbracket \llbracket u_1, u_2 \rrbracket_U, u_3 \rrbracket_U + \llbracket \nabla_{\partial_B \tau_1} u_2 - \nabla_{\partial_B \tau_2} u_1, u_3 \rrbracket_U \\
&\qquad + \nabla_{\partial_B(\llbracket \tau_1, \tau_2 \rrbracket_{Q^*} + \Delta_{u_1}\tau_2 - \Delta_{u_2}\tau_1 + \rho_Q^*\mathbf{d}\langle \tau_1, u_2\rangle)} u_3 - \nabla_{\partial_B \tau_3}(\llbracket u_1, u_2 \rrbracket_U + \nabla_{\partial_B \tau_1} u_2 - \nabla_{\partial_B \tau_2} u_1) \Big) \\
&\oplus \Big(\llbracket \llbracket \tau_1, \tau_2 \rrbracket_{Q^*}, \tau_3 \rrbracket_{Q^*} + \llbracket \Delta_{u_1}\tau_2 - \Delta_{u_2}\tau_1 + \rho_Q^*\mathbf{d}\langle \tau_1, u_2 \rangle, \tau_3 \rrbracket_{Q^*} \\
&\qquad + \Delta_{\llbracket u_1, u_2 \rrbracket_U + \nabla_{\partial_B \tau_1} u_2 - \nabla_{\partial_B \tau_2} u_1} \tau_3 - \Delta_{u_3}(\llbracket \tau_1, \tau_2 \rrbracket_{Q^*} + \Delta_{u_1}\tau_2 - \Delta_{u_2}\tau_1 + \rho_Q^*\mathbf{d}\langle \tau_1, u_2 \rangle) \\
&\qquad + \rho_Q^*\mathbf{d}\left\langle \llbracket \tau_1, \tau_2 \rrbracket_{Q^*} + \Delta_{u_1}\tau_2 - \Delta_{u_2}\tau_1 + \rho_Q^*\mathbf{d}\langle \tau_1, u_2\rangle, u_3 \right\rangle \Big).
\end{aligned}$$

Using $\partial_B \llbracket \tau_1, \tau_2 \rrbracket_{Q^*} = [\partial_B \tau_1, \partial_B \tau_2]$, $\Delta_q(\rho_Q^*\mathbf{d}f) = \rho_Q^*\mathbf{d}(\rho_Q(q)(f))$, (78) and (47), this is

$$\begin{aligned}
&\Big(\llbracket \llbracket u_1, u_2 \rrbracket_U, u_3 \rrbracket_U + \llbracket \nabla_{\partial_B \tau_1} u_2 - \nabla_{\partial_B \tau_2} u_1, u_3 \rrbracket_U \\
&\qquad + \nabla_{[\partial_B \tau_1, \partial_B \tau_2] + \partial_B(\Delta_{u_1}\tau_2 - \Delta_{u_2}\tau_1)} u_3 - \nabla_{\partial_B \tau_3}(\llbracket u_1, u_2 \rrbracket_U + \nabla_{\partial_B \tau_1} u_2 - \nabla_{\partial_B \tau_2} u_1) \Big) \\
&\oplus \Big(\llbracket \llbracket \tau_1, \tau_2 \rrbracket_{Q^*}, \tau_3 \rrbracket_{Q^*} + \llbracket \Delta_{u_1}\tau_2 - \Delta_{u_2}\tau_1, \tau_3 \rrbracket_{Q^*} \\
&\qquad + \Delta_{\llbracket u_1, u_2 \rrbracket_U + \nabla_{\partial_B \tau_1} u_2 - \nabla_{\partial_B \tau_2} u_1} \tau_3 - \Delta_{u_3}(\llbracket \tau_1, \tau_2 \rrbracket_{Q^*} + \Delta_{u_1}\tau_2 - \Delta_{u_2}\tau_1) \\
&\qquad + \rho_Q^*\mathbf{d}\left\langle \llbracket \tau_1, \tau_2 \rrbracket_{Q^*} + \Delta_{u_1}\tau_2 - \Delta_{u_2}\tau_1, u_3 \right\rangle \Big).
\end{aligned}$$

In the same manner, we compute

$$\begin{aligned}
&\llbracket u_1 \oplus \tau_1, \llbracket u_2 \oplus \tau_2, u_3 \oplus \tau_3 \rrbracket \rrbracket \\
&= \llbracket u_1 \oplus \tau_1, (\llbracket u_2, u_3 \rrbracket_U + \nabla_{\partial_B \tau_2} u_3 - \nabla_{\partial_B \tau_3} u_2) \oplus (\llbracket \tau_2, \tau_3 \rrbracket_{Q^*} + \Delta_{u_2}\tau_3 - \Delta_{u_3}\tau_2 + \rho_Q^*\mathbf{d}\langle \tau_2, u_3 \rangle) \rrbracket \\
&= \Big(\llbracket u_1, \llbracket u_2, u_3 \rrbracket_U \rrbracket_U + \llbracket u_1, \nabla_{\partial_B \tau_2} u_3 - \nabla_{\partial_B \tau_3} u_2 \rrbracket_U \\
&\qquad + \nabla_{\partial_B \tau_1}(\llbracket u_2, u_3 \rrbracket_U + \nabla_{\partial_B \tau_2} u_3 - \nabla_{\partial_B \tau_3} u_2) - \nabla_{[\partial_B \tau_2, \partial_B \tau_3] + \partial_B(\Delta_{u_2}\tau_3 - \Delta_{u_3}\tau_2)} u_1 \Big) \\
&\oplus \Big(\llbracket \tau_1, \llbracket \tau_2, \tau_3 \rrbracket_{Q^*} \rrbracket_{Q^*} + \llbracket \tau_1, \Delta_{u_2}\tau_3 - \Delta_{u_3}\tau_2 \rrbracket_{Q^*} + \rho_Q^*\mathbf{d}(\rho_Q(\partial_Q \tau_1)\langle \tau_2, u_3 \rangle) \\
&\qquad + \Delta_{u_1}(\llbracket \tau_2, \tau_3 \rrbracket_{Q^*} + \Delta_{u_2}\tau_3 - \Delta_{u_3}\tau_2) + \rho_Q^*\mathbf{d}(\rho_Q(u_1)\langle \tau_2, u_3 \rangle) - \Delta_{\llbracket u_2, u_3 \rrbracket_U + \nabla_{\partial_B \tau_2} u_3 - \nabla_{\partial_B \tau_3} u_2} \tau_1 \\
&\qquad + \rho_Q^*\mathbf{d}\langle \tau_1, \llbracket u_2, u_3 \rrbracket_U + \nabla_{\partial_B \tau_2} u_3 - \nabla_{\partial_B \tau_3} u_2 \rangle \Big)
\end{aligned}$$

Since $\mathrm{Jac}_{\llbracket \cdot\,,\cdot \rrbracket_U}(u_1, u_2, u_3) = 0$ and $\partial_B \circ \Delta_q = \nabla_q \circ \partial_B$ for all $q \in \Gamma(Q)$, the U-term of $\mathrm{Jac}_{\llbracket \cdot\,,\cdot \rrbracket}(u_1 \oplus \tau_1, u_2 \oplus \tau_2, u_3 \oplus \tau_3)$ equals

$$\begin{aligned}
&\llbracket \nabla_{\partial_B \tau_1} u_2 - \nabla_{\partial_B \tau_2} u_1, u_3 \rrbracket_U \\
&+ \nabla_{[\partial_B \tau_1, \partial_B \tau_2] + \nabla_{u_1}\partial_B \tau_2 - \nabla_{u_2}\partial_B \tau_1} u_3 - \nabla_{\partial_B \tau_3}(\llbracket u_1, u_2 \rrbracket_U + \nabla_{\partial_B \tau_1} u_2 - \nabla_{\partial_B \tau_2} u_1) \\
&+ \llbracket u_2, \nabla_{\partial_B \tau_1} u_3 - \nabla_{\partial_B \tau_3} u_1 \rrbracket_U \\
&+ \nabla_{\partial_B \tau_2}(\llbracket u_1, u_3 \rrbracket_U + \nabla_{\partial_B \tau_1} u_3 - \nabla_{\partial_B \tau_3} u_1) - \nabla_{[\partial_B \tau_1, \partial_B \tau_3] + \nabla_{u_1}\partial_B \tau_3 - \nabla_{u_3}\partial_B \tau_1} u_2 \\
&- \llbracket u_1, \nabla_{\partial_B \tau_2} u_3 - \nabla_{\partial_B \tau_3} u_2 \rrbracket_U \\
&- \nabla_{\partial_B \tau_1}(\llbracket u_2, u_3 \rrbracket_U + \nabla_{\partial_B \tau_2} u_3 - \nabla_{\partial_B \tau_3} u_2) + \nabla_{[\partial_B \tau_2, \partial_B \tau_3] + \nabla_{u_2}\partial_B \tau_3 - \nabla_{u_3}\partial_B \tau_2} u_1.
\end{aligned}$$

Note that since for any $b_1, b_2 \in \Gamma(B)$, $R(b_1, b_2)$ restricts to a section of $\mathrm{Hom}(U, U^\circ)$ (see §8.2), the last summand on the right hand side of (M4) vanishes on sections of

U. By sorting out the terms and using (M4) on sections of U, we get

$$\begin{aligned}&-R_\nabla(\partial_B\tau_3,\partial_B\tau_1)u_2-R_\nabla(\partial_B\tau_1,\partial_B\tau_2)u_3-R_\nabla(\partial_B\tau_2,\partial_B\tau_3)u_1\\&+\partial_QR(u_1,u_2)\partial_B\tau_3+\partial_QR(u_2,u_3)\partial_B\tau_1+\partial_QR(u_3,u_1)\partial_B\tau_2.\end{aligned}$$

Since $R_\nabla=\partial_Q\circ R$, this is $-\partial_Q\upsilon$. We conclude by computing the Q^*-part of $\mathrm{Jac}_{\llbracket\cdot\,,\cdot\rrbracket}(u_1\oplus\tau_1,u_2\oplus\tau_2,u_3\oplus\tau_3)$. Again, because $\mathrm{Jac}_{\llbracket\cdot\,,\cdot\rrbracket_{Q^*}}(\tau_1,\tau_2,\tau_3)=0$, we get

$$\begin{aligned}&\llbracket\Delta_{u_1}\tau_2-\Delta_{u_2}\tau_1,\tau_3\rrbracket_{Q^*}+\Delta_{\llbracket u_1,u_2\rrbracket_U+\nabla_{\partial_B\tau_1}u_2-\nabla_{\partial_B\tau_2}u_1}\tau_3-\Delta_{u_3}(\llbracket\tau_1,\tau_2\rrbracket_{Q^*}+\Delta_{u_1}\tau_2-\Delta_{u_2}\tau_1)\\&+\rho_Q^*\mathbf{d}\left\langle\llbracket\tau_1,\tau_2\rrbracket_{Q^*}+\cancel{\Delta_{u_1}\tau_2}-\cancel{\Delta_{u_2}\tau_1},u_3\right\rangle+\llbracket\tau_2,\Delta_{u_1}\tau_3-\Delta_{u_3}\tau_1\rrbracket_{Q^*}+\rho_Q^*\mathbf{d}(\rho_Q(\partial_Q\tau_2)\langle\tau_1,u_3\rangle)\\&+\Delta_{u_2}(\llbracket\tau_1,\tau_3\rrbracket_{Q^*}+\Delta_{u_1}\tau_3-\Delta_{u_3}\tau_1)+\cancel{\rho_Q^*\mathbf{d}(\rho_Q(u_2)\langle\tau_1,u_3\rangle)}-\Delta_{\llbracket u_1,u_3\rrbracket_U+\nabla_{\partial_B\tau_1}u_3-\nabla_{\partial_B\tau_3}u_1}\tau_2\\&+\rho_Q^*\mathbf{d}\langle\tau_2,\cancel{\llbracket u_1,u_3\rrbracket_U}+\nabla_{\partial_B\tau_1}u_3-\nabla_{\partial_B\tau_3}u_1\rangle-\llbracket\tau_1,\Delta_{u_2}\tau_3-\Delta_{u_3}\tau_2\rrbracket_{Q^*}-\rho_Q^*\mathbf{d}(\rho_Q(\partial_Q\tau_1)\langle\tau_2,u_3\rangle)\\&-\Delta_{u_1}(\llbracket\tau_2,\tau_3\rrbracket_{Q^*}+\Delta_{u_2}\tau_3-\Delta_{u_3}\tau_2)-\cancel{\rho_Q^*\mathbf{d}(\rho_Q(u_1)\langle\tau_2,u_3\rangle)}+\Delta_{\llbracket u_2,u_3\rrbracket_U+\nabla_{\partial_B\tau_2}u_3-\nabla_{\partial_B\tau_3}u_2}\tau_1\\&-\rho_Q^*\mathbf{d}\langle\tau_1,\cancel{\llbracket u_2,u_3\rrbracket_U}+\nabla_{\partial_B\tau_2}u_3-\nabla_{\partial_B\tau_3}u_2\rangle\end{aligned}$$

The six cancelling terms cancel by the duality of the dull bracket and the Dorfman connection. Reordering the terms, we get using Lemma C.1:

$$\begin{aligned}&-R_\Delta(u_3,u_1)\tau_2-R_\Delta(u_2,u_3)\tau_1-R_\Delta(u_1,u_2)\tau_3\\&+R(\partial_B\tau_2,\partial_B\tau_3)u_1+R(\partial_B\tau_1,\partial_B\tau_2)u_3-R(\partial_B\tau_1,\partial_B\tau_3)u_2\\&+\cancel{\rho_Q^*\mathbf{d}\langle\tau_2,\nabla_{\partial_B\tau_3}u_1\rangle}+\cancel{\rho_Q^*\mathbf{d}\langle\tau_1,\nabla_{\partial_B\tau_2}u_3\rangle}-\cancel{\rho_Q^*\mathbf{d}\langle\tau_1,\nabla_{\partial_B\tau_3}u_2\rangle}\\&-\rho_Q^*\mathbf{d}\langle\tau_2,\partial_Q\Delta_{u_3}\tau_1\rangle+\rho_Q^*\mathbf{d}\left\langle\llbracket\tau_1,\tau_2\rrbracket_{Q^*},u_3\right\rangle+\rho_Q^*\mathbf{d}(\rho_Q(\partial_Q\tau_2)\langle\tau_1,u_3\rangle)\\&+\rho_Q^*\mathbf{d}\langle\tau_2,\nabla_{\partial_B\tau_1}u_3-\cancel{\nabla_{\partial_B\tau_3}u_1}\rangle-\rho_Q^*\mathbf{d}(\rho_Q(\partial_Q\tau_1)\langle\tau_2,u_3\rangle)-\rho_Q^*\mathbf{d}\langle\tau_1,\cancel{\nabla_{\partial_B\tau_2}u_3}-\cancel{\nabla_{\partial_B\tau_3}u_2}\rangle.\end{aligned}$$

For the third use of Lemma C.1, we had to replace $-\llbracket\tau_2,\Delta_{u_3}\tau_1\rrbracket_{Q^*}$ by $\llbracket\Delta_{u_3}\tau_1,\tau_2\rrbracket_{Q^*}-\rho_Q^*\mathbf{d}\langle\tau_2,\partial_Q\Delta_{u_3}\tau_1\rangle$. This is why we get the first term on the fourth line. Six terms cancel immediately pairwise. Using $R_\Delta=R\circ\partial_B$, we get

$$\upsilon+\rho_Q^*\mathbf{d}f$$

with $f\in C^\infty(M)$ defined by

$$\begin{aligned}f=&-\langle\tau_2,\partial_Q\Delta_{u_3}\tau_1\rangle+\left\langle\Delta_{\partial_Q\tau_1}\tau_2-\nabla_{\partial_B\tau_2}\tau_1,u_3\right\rangle+\rho_Q(\partial_Q\tau_2)\langle\tau_1,u_3\rangle\\&+\langle\tau_2,\nabla_{\partial_B\tau_1}u_3\rangle-\rho_Q(\partial_Q\tau_1)\langle\tau_2,u_3\rangle\\=&\langle\tau_2,-\partial_Q\Delta_{u_3}\tau_1+\llbracket\partial_Q\tau_1,u_3\rrbracket+\nabla_{\partial_B\tau_1}u_3\rangle+\rho_Q(\partial_Q\tau_2)\langle\tau_1,u_3\rangle-\langle\nabla_{\partial_B\tau_2}\tau_1,u_3\rangle.\end{aligned}$$

By (M1), the first pairing equals

$$-\langle\tau_1,\nabla_{\partial_B\tau_2}u_3\rangle.$$

Hence, we find using $\nabla^{Q^*}=\nabla^{Q^*}$ and $\rho_B\circ\partial_B=\rho_Q\circ\partial_Q$ that $f=0$, and so

$$\mathrm{Jac}_{\llbracket\cdot\,,\cdot\rrbracket}(u_1\oplus\tau_1,u_2\oplus\tau_2,u_3\oplus\tau_3)=(-\partial_Q\upsilon)\oplus\upsilon.$$

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