ON IDEALS IN LIE-RINEHART ALGEBRAS

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Abstract. Inspired from the notion of infinitesimal ideal system in a Lie algebroid, we describe a possible notion of (weak and strong) ideal in a Lie-Rinehart algebra. We show that the kernel of a morphism of Lie Rinehart algebras carries a weak ideal structure. We prove that a Lie-Rinehart algebra can be quotiented by a strong ideal and we give a factorisation theorem in this context.

1. Introduction

An ideal in a Lie algebroid $(A \to M, \rho, [\cdot\,,\cdot])$ is a vector subbundle $I \subseteq A$ such that $\Gamma(I)$ is a Lie algebra ideal in $\Gamma(A)$. This notion of ideal is not very permissive since the Leibniz condition implies immediately $\rho(I) = 0$ or $I = A$. *Ideal systems* were proposed by Higgins and Mackenzie [4] as the kernels of fibrations of Lie algebroids. Of course, the kernel of a Lie algebroid morphism over the identity on a common base is always an ideal, and the setting of ideal systems is required for Lie algebroid morphisms over two different base manifolds. In [6], the author and Ortiz proposed an infinitesimal version of Higgins and Mackenzie's ideal systems. Such an *infinitesimal ideal system* consists of a subalgebroid $J \subseteq A$, an involutive subbundle $F_M \subseteq TM$ with $\rho(J) \subseteq F_M$ and a flat connection $\nabla\colon \Gamma(F_M) \times \Gamma(A/J) \to \Gamma(A/J)$ such that

(1) If $a \in \Gamma(A)$ is ∇-parallel, then $[a, j] \in \Gamma(J)$ for all $j \in \Gamma(J)$.
(2) If $a, b \in \Gamma(A)$ are ∇-parallel, then $[a, b]$ is also ∇-parallel.
(3) If $a \in \Gamma(A)$ is ∇-parallel, then $\rho(a)$ is ∇^{F_M}-parallel, where

$$\nabla^{F_M} : \Gamma(F_M) \times \Gamma(TM/F_M) \to \Gamma(TM/F_M), \quad \nabla^{F_M}_X \overline{Y} = \overline{[X, Y]}$$

is the Bott connection associated to F_M.

Note that since the connection is flat, one can show that A has local frames consisting of flat sections of A/J together with sections of J. Therefore, the sheaf $\Gamma_\cdot(A)$ is locally finitely generated as a $C^\infty_\cdot(M)$-module by flat sections.

This notion of ideal in a Lie algebroid seems to be the right one since – under the usual topological conditions that are in fact satisfied if and only if the infinitesimal ideal system integrates to an ideal system [6] – $A \to M$ can be reduced to a Lie algebroid $(A/J)/\nabla \to M/F_M$. In addition, an infinitesimal ideal system is equivalent to a subrepresentation up to homotopy of the adjoint representation up to homotopy [2].

The purely algebraic formulation (and generalisation) of the notion of Lie algebroid is the one of *Lie-Rinehart algebra*, first introduced by Herz [3] as *pseudo-Lie algebras*, then studied by Palais [7] under the name *d-Lie ring*; Rinehart then found [8] hat a Poisson structure on an algebra A over a commutative ring R gives rise to a structure of an "(R, A)-Lie algebra; see the introduction of [5] for more details on the historic development of the notion. In this context, we observe the same phenomenom: the usual notion of ideal in a Lie-Rinehart algebra is rather restrictive [1]. Inspired from our knowledge of infinitesimal ideal systems in Lie algebroids, we propose here a more elaborate notion of ideal in a Lie Rinehart-algebra (A, L) over a ring R. We explain morphisms of Lie-Rinehart algebras and their kernels in Section 2. The latter are called *weak kernels*, and become *strong kernels* if the subspace corresponding here to the flat sections above generates L as an A-module. In Section 3 we define then *weak* and *strong ideals*, so that a weak kernel is a weak ideal and a strong kernel is a strong ideal. We explain why we propose to call a morphism of Lie-Rinehart algebras *factorisable* if its kernel is a strong ideal.

2. Lie-Rinehart algebras and kernels of morphisms

2.1. **Definitions.** We begin by recalling the definition of Lie-Rinehart algebras, of Lie-Rinehart subalgebras and of morphisms of Lie-Rinehart algebras.

Definition 2.1. *Let R be a commutative ring with 1. A Lie Rinehart algebra over R is a tuple $(A, L, [\cdot\,,\cdot], \rho)$ where*

1. *A is a commutative R-algebra;*
2. *L is an A-module and $(L, [\cdot\,,\cdot])$ is a Lie algebra over R;*
3. *$\rho\colon L \to \mathrm{Der}(A)$ is a homomorphism of Lie algebras and of A-modules*

such that

$$[l_1, a \cdot l_2] = \rho(l_1)(a) \cdot l_2 + a[l_1, l_2]$$

for all $a, b \in A$ and $l, l_1, l_2 \in L$.

We simply write that (A, L) is a Lie-Rinehart algebra over R.

For instance, consider a Lie algebroid A over a smooth manifold M. Then $(C^\infty(M), \Gamma(A))$ is a Lie-Rinehart algebra over $\mathbb{R}$, with $\rho\colon \Gamma(A) \to \mathfrak{X}(M)$ the composition of sections of A with the anchor, and with the Lie algebra bracket on sections of A. For the sake of our later considerations, note that we get this Lie-Rinehart algebra by considering *global sections* of A and *global smooth functions* on M.

In the literature we find the following notions:

Definition 2.2. *Let R be a commutative ring with 1, let (A, L) be a Lie-Rinehart algebra. Then a Lie-Rinehart subalgebra of (A, L) is a pair (A, S) with*

1. *S a Lie subalgebra of L*
2. *$A \cdot S \subseteq S$ and*
3. *S acts on A via $S \hookrightarrow L \to \mathrm{Der}(A)$.*

The pair (A, S) has then automatically the structure of a Lie-Rinehart algebra over R.

A Lie-Rinehart subalgebra (A, I) of (A, L) is an ideal [1] *in (A, L) if*

1. *$\rho(I)(A)L \subseteq I$ and*
2. *I is an ideal in L.*

In the case of the Lie-Rinehart algebra $(C^\infty(M), \Gamma(A))$ defined by a Lie algebroid, (1) in the definition of an ideal implies that an ideal must consist first of all in a $C^\infty(M)$-submodule of sections of A, which are all sent by the anchor to the zero vector field (unless the submodule is the whole of $\Gamma(A)$). In particular, there are no non-trivial ideals in the prototype $(C^\infty(M), \mathfrak{X}(M))$.

Next we turn to the notion of morphism of Lie-Rinehart algebras. Consider two Lie-Rinehart algebras (A, L) and (A', L') over a commutative ring R with 1. Assume that there is a morphism $\varphi\colon A' \to A$ of R-algebras. Then A is naturally an A'-module and we can build the tensor product $A \otimes_\varphi L'$ of A and L' as A-modules. In particular, we have $(\varphi(a') \cdot a) \otimes_\varphi l' = a \otimes_\varphi (a' \cdot l')$ for all $a \in A$, $a \in A'$ and $l' \in L'$.

Then we can construct the pullback

$$\mathrm{Der}(A) \times_\varphi L' := \left\{ \left(d, \sum_{i=1}^m a_i \otimes_\varphi l_i' \right) \,\middle|\, \begin{array}{c} d \in \mathrm{Der}(A), m \in \mathbb{N}, a_i \in A, l_i \in L' \quad (i = 1, \ldots, m) \\ \text{and } \sum a_i \cdot \varphi \circ \rho'(l_i) = d \circ \varphi \end{array} \right\}$$

It is easy to check that $(A, \mathrm{Der}(A) \times_\varphi L')$ is a Lie-Rinehart algebra over R, with the morphism $\pi_1\colon \mathrm{Der}(A) \times_\varphi L' \to \mathrm{Der}(A)$, $(d, \sum_{i=1}^m a_i \otimes_\varphi l_i) \mapsto d$, and the bracket

$$\left[\left(d_1, \sum_{i=1}^{m_1} a_i^1 \otimes_\varphi l_i^1\right), \left(d_2, \sum_{j=1}^{m_2} a_j^2 \otimes_\varphi l_j^2\right)\right]$$
$$= \left([d_1, d_2], \sum_{i,j} a_i^1 a_j^2 \otimes_\varphi [l_i^1, l_j^2] + \sum_j d_1(a_j^2) \otimes_\varphi l_j^2 - \sum_i d_2(a_i^1) \otimes_\varphi l_j^1\right)$$

Note that if $\varphi\colon A \to A$ is the identity, then $\mathrm{Der}(A) \times_{\mathrm{id}_A} L' = \mathrm{graph}(\rho'\colon L' \to \mathrm{Der}(A)) \simeq L'$ since then $(d, \sum_{i=1}^m a_i \otimes_{\mathrm{id}_A} l_i) = (d, \sum_{i=1}^m a_i \cdot l_i) \in \mathrm{Der}(A) \times_{\mathrm{id}_A} L'$ implies

$$d = d \circ \mathrm{id}_A = \sum_{i=1}^m a_i \cdot \mathrm{id}_A \circ \rho'(l_i) = \rho'\left(\sum_{i=1}^m a_i \cdot l_i\right).$$

Definition 2.3. *Let R be a commutative ring with 1, and let (A, L) and (A', L') be Lie-Rinehart algebras over R.*

A morphism $(\varphi, \Phi)\colon (A, L) \to (A', L')$ of Lie-Rinehart algebras is a unital morphism $\varphi\colon A' \to A$ of commutative R-algebras, together with a morphism $\Phi\colon L \to \mathrm{Der}(A) \times_\varphi L'$ of Lie algebras and of A-modules such that $\pi_1 \circ \Phi = \rho\colon L \to \mathrm{Der}(A)$.

Note that if $A = A'$ and $\varphi\colon A \to A$ is the identity, then we can simply define a morphism $\phi\colon L \to L' \simeq \mathrm{Der}(A) \times_{\mathrm{id}_A} L'$ of Lie-Rinehart algebras over A to be a morphism of Lie algebras and of A-modules with $\rho' \circ \phi = \rho$. We say then that $\phi\colon L \to L'$ is a morphism of Lie-Rinehart algebras over A.

2.2. **The 'kernel' of a morphism of Lie-Rinehart algebras.** Now we are ready to consider the kernel of a morphism of Lie-Rinehart algebras. In the situation of the last definition, we set $B = \varphi(A') \subseteq A$, a commutative R-subalgebra with 1, $J \subseteq L$,

$$J = \{l \in L \mid \Phi(l) = (\rho(l), 0) \in \mathrm{Der}(A) \times_\varphi L'\}.$$

We set finally $\bar{L} \subseteq L$,

$$\bar{L} = \{l \in L \mid \exists l' \in L' :\ \Phi(l) = (\rho(l), 1 \otimes_\varphi l')\}.$$

The triple $(\varphi(A'), J, \bar{L})$ is called the **kernel** of the morphism (φ, Φ).

We define the map $\phi\colon \bar{L} \to L'$ by $\Phi(l) = (\rho(l), 1 \otimes_\varphi \phi(l))$ for all $l \in \bar{L}$. We will see below that ϕ is a morphism of Lie-Rinehart algebras over A'.

Note that for $\phi\colon L \to L'$ a morphism of Lie-Rinehart algebras over A, we have simply $J = \mathrm{kern}(\phi)$ and $\bar{L} = L$. Then J is an ideal in L and the quotient L/J has a unique structure of a Lie-Rinehart algebra over A, such that the projection $\pi\colon L \to L/J$ is a morphism of Lie-Rinehart algebras (over A). The morphism $\phi\colon L \to L'$ factors then as $\bar{\phi}\colon \pi = \phi$, with $\bar{\phi}\colon L/J \to L'$ defined by $\bar{\phi}(l + J) = \bar{\phi}(l)$ for $l \in L$.

In general, the situation is more delicate and we prove the following theorem:

Proposition 2.4. *Let R be a commutative ring with 1, and let (A, L) and (A', L') be Lie-Rinehart algebras over R. Consider a morphism $(\varphi, \Phi)\colon (A, L) \to (A', L')$ of Lie-Rinehart algebras over R. Then the subsets defined above have the following properties:*

1. *$\bar{L}$ is a Lie subalgebra of L and a $\varphi(A')$-module;*
2. *$(\varphi(A'), \bar{L})$ is a Lie-Rinehart algebra over R;*
3. *$\phi\colon \bar{L} \to L'$ is a morphism of Lie-Rinehart algebras over A';*
4. *J is an ideal in $(\varphi(A'), \bar{L})$ with $\rho(J)(\varphi(A')) = 0$;*
5. *J is an A-module.*

Proof. (1) For $l_1, l_2 \in \bar{L}$ we have $\Phi[l_1, l_2] = [\Phi(l_1), \Phi(l_2)] = [(\rho(l_1), 1 \otimes_\varphi l_1'), (\rho(l_2), 1 \otimes_\varphi l_2')] = ([\rho(l_1), \rho(l_2)], 1 \otimes_\varphi [l_1', l_2'])$ which shows that $[l_1, l_2] \in \bar{L}$. We have further for $l \in \bar{L}$ and $a' \in A'$: $\Phi(\varphi(a') \cdot l) = (\rho(\varphi(a') \cdot l), \varphi(a') \otimes_\varphi l') = (\rho(\varphi(a') \cdot l), 1 \otimes_\varphi (a' \cdot l'))$, which shows that $B = \varphi(A')$ acts on $\bar{L}$.

(2) It remains to show that for $l \in \bar{L}$, the derivation $\rho(l)$ restricts to a derivation of B. We have

$$\rho(l) \circ \varphi = 1_A \cdot \varphi \circ (\rho'(l')) = \varphi \circ (1_{A'} \cdot \rho'(l')) = \varphi \circ (\rho'(l'))$$

since $\Phi(l) = (\rho(l), 1 \otimes_\varphi l') \in \mathrm{Der}(A) \times_\varphi L'$, and so $\rho(l)$ sends an element of B to an element of B. The remaining conditions follow from the fact that (A, L) is a Lie-Rinehart algebra.

(3) By the proof of (1), ϕ is a morphism of Lie algebras over R. By definition, ϕ is a morphism of A'-modules, and $\bar{\rho} = \rho' \circ \phi$.

(4) This is now easy to check. Note that for $j \in J$ we have $\rho(j)(B) = 0$ so $\rho(J)(B) \cdot \bar{L} \subseteq J$ is trivially satisfied.

(5) This works as (5).

$\square$

Now we reduce the Lie-Rinehart algebras $(A', \bar{L})$ and $(\varphi(A'), \bar{L})$ by their ideal J.

Theorem 2.5. *In the situation of the last proposition, $(\varphi(A'), \bar{L}/J)$ with the morphism $\bar{\rho}\colon \bar{L}/J \to \mathrm{Der}(\varphi(A'))$, $\bar{\rho}(l + J) = \rho(l)|_{\varphi(A')}$ and the bracket $[l_1 + J, l_2 + J] = [l_1, l_2] + J$ for all $l, l_1, l_2 \in \bar{L}$ is a Lie-Rinehart algebra over R.*

Proof. The proof follows immediately from the preceding proposition. $\square$

Further, if we assume that L is generated as an A-module by $\bar{L}$ we have a canonical projection $(\iota, \pi)\colon (A, L) \to (\varphi(A'), \bar{L}/J)$, where $\iota\colon \varphi(A') \to A$ is the inclusion; a morphism of R-algebras with 1.

Lemma 2.6. *Let R be a commutative ring with 1, and let (A, L) and (A', L') be Lie-Rinehart algebras over R. Consider a morphism $(\varphi, \Phi)\colon (A, L) \to (A', L')$ of Lie-Rinehart algebras over R. We denote by $\iota\colon \varphi(A') \to A$ the inclusion. Then*

$$\mathrm{Der}(A) \times_\varphi \bar{L}/J = \mathrm{Der}(A) \times_\iota \bar{L}/J,$$

where on the left-hand side, $\bar{L}/J$ is considered as an A'-module and on the right-hand side it is considered as a $\varphi(A')$-module.

Proof. The tensor products $A \otimes_\iota \bar{L}/J$ and $A \otimes_\varphi \bar{L}/J$ are equal as sets since in both cases, A is considered as an A'-module via φ.

Take $(d, \sum_{i=1}^m a_i \otimes_\varphi (l_i + J)) \in \mathrm{Der}(A) \times_\varphi \bar{L}/J$. Then for all $b = \varphi(a') \in \varphi(A')$ we have

$$d(\iota(b)) = d(\varphi(a')) = \sum_{i=1}^m a_i \cdot \varphi(\bar{\rho}(l_i + J)(a')) = \sum_{i=1}^m a_i \cdot \rho(l_i)(\varphi(a')) = \sum_{i=1}^m a_i \cdot \rho(l_i)(\iota(b))$$

and so $(d, \sum_{i=1}^m a_i \otimes_\varphi (l_i + J)) \in \mathrm{Der}(A) \times_\iota \bar{L}/J$. $\square$

Theorem 2.7. *Let R be a commutative ring with 1, and let (A, L) and (A', L') be Lie-Rinehart algebras over R. Consider a morphism $(\varphi, \Phi)\colon (A, L) \to (A', L')$ of Lie-Rinehart algebras over R. Assume that L is generated as an A-module by $\bar{L}$. Then we can define $\pi\colon L \to \mathrm{Der}(A) \times_\varphi \bar{L}/J = \mathrm{Der}(A) \times_\iota \bar{L}/J$ by*

$$\pi\left(\sum_{i=1}^r a_i \cdot l_i\right) = \left(\rho(a), \sum_{i=1}^r a_i \otimes_\varphi (l_i + J)\right)$$

for $a_1, \ldots, a_r \in A$ and $l_1, \ldots, l_r \in \bar{L}$. $(\iota, \pi)\colon (A, L) \to (\varphi(A'), \bar{L}/J)$ is a morphism of Lie-Rinehart algebras.

Finally, we immediately get a factorisation theorem in this setting.

Theorem 2.8. *Let R be a commutative ring with 1, and let (A,L) and (A',L') be Lie-Rinehart algebras over R. Consider a morphism $(\varphi,\Phi)\colon (A,L)\to(A',L')$ of Lie-Rinehart algebras over R. Assume that L is generated as an A-module by $\bar{L}$. Then the morphism (φ,Φ) factors as*

$$\begin{tikzcd}
(A,L) \arrow[r, "(\iota,\pi)"] \arrow[dr, "(\varphi,\Phi)"'] & (\varphi(A'),\bar{L}/J) \arrow[d, "(\varphi,\phi)"] \\
& (A',L')
\end{tikzcd}$$

3. Ideals in Lie-Rinehart algebras

The considerations in the last paragraph lead us to the following definition.

Definition 3.1. *Let (A,L) be a Lie-Rinehart algebra over a commutative ring R with 1. A weak ideal in (A,L) is a triple $(B,J,\bar{L})$ with*

1. *B is an R-subalgebra of A with 1;*
2. *$\bar{L}$ is a Lie subalgebra of L and a B-module;*
3. *$(B,\bar{L})$ is a Lie-Rinehart algebra over R;*
4. *J is an ideal in $(B,\bar{L})$ with $\rho(J)(B)=0$;*
5. *J is an A-module.*

A weak ideal $(B,J,\bar{L})$ is a strong ideal if L is generated as an A-module by $\bar{L}$.

We immediately have the following theorem, the proof of which can be taken from the considerations in the previous section.

Theorem 3.2. *Let (A,L) be a Lie-Rinehart algebra over a commutative ring R with 1, and let $(B,J,\bar{L})$ be a weak ideal in (A,L). Then $(B,\bar{L}/J)$ is a Lie-Rinehart algebra over R. If $(B,J,\bar{L})$ is a strong ideal, then there is a canonical surjective morphism*

$$(\iota_B,\pi)\colon (A,L)\to(B,\bar{L}/J)$$

of Lie-Rinehart algebras, with kernel $(B,J,\bar{L})$ in (A,L).

In view of the considerations above, we propose to call a morphism $(\varphi,\Phi)\colon(A,L)\to(A',L')$ of Lie-Rinehart algebras over R **factorisable** if its kernel as in Theorem 2.4 is a strong ideal in (A,L). The morphism (φ,Φ) factors then to $(\varphi,\phi)\circ(\iota,\pi)$.

Example 3.3. Let (F_M,J,∇) be an infinitesimal ideal system in a Lie algebroid $A\to M$. Take $p\in M$ and a chart (U,φ) of M around p, that is adapted to the involutive subbundle $F_M\subseteq TM$ and trivialises J and A. Then $A|_U\to U$ is a Lie algebroid and so $(C^\infty(U),\Gamma_U(A))$ is a Lie-Rinehart algebra. Set $B=C^\infty_U(M)^{F_M}$, $\mathcal{J}=\Gamma_U(J)$ and let $\bar{L}$ be the space of ∇-flat sections of A on U (where sections of J are understood as ∇-flat). Then $(B,\mathcal{J},\bar{L})$ is a strong ideal in $A|_U$.

In general, an infinitesimal ideal system does not come with global flat sections of A/J, so we do not get a strong ideal in $(C^\infty(M),\Gamma_M(A))$, unless we consider the sheaf of Lie-Rinehart algebras over M defined by A, thereby loosing the purely algebraic flavor of the notion of Lie-Rinehart algebra associated to a Lie algebroid.

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