BOUND STATES FOR NANO-TUBES WITH A DISLOCATION

RAINER HEMPEL, MARTIN KOHLMANN, MARKO STAUTZ, AND JÜRGEN VOIGT

ABSTRACT. As a model for an interface in solid state physics, we consider two real-valued potentials $V^{(1)}$ and $V^{(2)}$ on the cylinder or tube $S = \mathbb{R} \times (\mathbb{R}/\mathbb{Z})$ where we assume that there exists an interval (a_0, b_0) which is free of spectrum of $-\Delta + V^{(k)}$ for $k = 1, 2$. We are then interested in the spectrum of $H_t = -\Delta + V_t$, for $t \in \mathbb{R}$, where $V_t(x, y) = V^{(1)}(x, y)$, for $x > 0$, and $V_t(x, y) = V^{(2)}(x + t, y)$, for $x < 0$. While the essential spectrum of H_t is independent of t, we show that discrete spectrum, related to the interface at $x = 0$, is created in the interval (a_0, b_0) at suitable values of the parameter t, provided $-\Delta + V^{(2)}$ has some essential spectrum in $(-\infty, a_0]$. We do not require $V^{(1)}$ or $V^{(2)}$ to be periodic. We furthermore show that the discrete eigenvalues of H_t are Lipschitz continuous functions of t if the potential $V^{(2)}$ is locally of bounded variation.

1. INTRODUCTION

In the quantum mechanical theory of solids, the spectrum $\sigma(H)$ of self-adjoint Schrödinger operators $H = -\Delta + V$, acting in $\mathsf{L}_2(\mathbb{R}^3)$, yields basic information on the electronic energy levels in a crystal. In the most simple cases the potential V is periodic and then $\sigma(H)$ has *band structure,* i.e., $\sigma(H)$ is the locally finite union of compact intervals $I_k \subset \mathbb{R}$ with $k \in \mathbb{N}$. The bands may be separated by (non-empty) open intervals, the *spectral gaps.* These well-known results are a part of Floquet theory; cf., e.g., [35], [12]. In reality, however, a solid body occupying all of $\mathbb{R}^3$ in a periodic way is an idealization and one has to deal with various deviations from periodicity which may correspond to almost periodic or random potentials. Notice that there is considerable interest in the properties of ordered, but non-periodic, solid materials (cf. e.g. Baake et al. [3], Baake and Grimm [4], Penrose [30], Radin [32]). In addition, one would like to understand the effect of *interfaces* occuring in certain types of alloys where two different structures meet, or of *surfaces* where a solid only fills a half-space.

In order to facilitate the mathematical analysis of electronic levels associated with interfaces or surfaces, *dislocation potentials* have been introduced some time ago in the 1-dimensional case; cf. the introduction in [26] for the history of this problem. Here one starts from a periodic, real-valued potential $V = V(x)$ on the real line and considers the Schrödinger operators $h_t = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_t$, for $t \in \mathbb{R}$, where $V_t(x) = V(x)$, for $x > 0$, and $V_t(x) = V(x + t)$, for $x < 0$. Assuming that the

spectrum of the periodic operator h_0 has a non-trivial gap (a_0, b_0), located above the infimum of the (essential) spectrum of h_0, one can show that h_t has some (discrete) spectrum in (a_0, b_0) for suitable t.

In the present paper, we will study dislocation problems on an infinite cylinder $S := \mathbb{R} \times (\mathbb{R}/\mathbb{Z})$ without a periodicity assumption. Given two (bounded and measurable) potentials $V^{(k)} \colon S \to \mathbb{R}$, $k = 1, 2$, the family of *dislocation potentials* is defined by

$$(1.1) \qquad V_t(x, y) = \begin{cases} V^{(1)}(x, y), & x \geq 0, \\ V^{(2)}(x + t, y), & x < 0, \end{cases}$$

for $(x, y) \in S$ and $t \in \mathbb{R}$. In the Hilbert space $\mathcal{H} := \mathsf{L}_2(S)$, we let L denote the (unique) self-adjoint extension of $-\Delta$ defined on $\mathsf{C}_c^\infty(S)$. For each $t \in \mathbb{R}$ the Schrödinger operator $H_t = L + V_t$ describes the energy of an electron on a tube made of the same or two different materials to the left and to the right of the interface $\{0\} \times \mathbb{R}/\mathbb{Z}$. We are interested in the bound states produced by and at this junction where we focus on energies in a spectral gap of H_0. In our main theorem, given below, we will also need the Dirichlet Laplacian $L_{(0,\infty)}$ of $S^+ := (0, \infty) \times \mathbb{R}/\mathbb{Z}$, defined as the Friedrichs extension of $-\Delta$ on $\mathsf{C}_c^\infty(S^+)$.

Theorem 1.1. *Let $V^{(1)}, V^{(2)} \colon S \to \mathbb{R}$ be bounded and measurable, and let V_t be as in (1.1). Suppose $E \in \mathbb{R}$ is such that*

$$(1.2) \qquad E \notin \sigma(L + V^{(k)}), \qquad k = 1, 2,$$

and

$$(1.3) \qquad \inf \sigma_{\mathrm{ess}}(L_{(0,\infty)} + V^{(2)} \restriction S^+) < E.$$

Then there exists a sequence $(\tau_j)_{j \in \mathbb{N}} \subset [0, \infty)$ of dislocation parameters such that $E \in \sigma(L + V_{\tau_j})$, and $\tau_j \to \infty$ as $j \to \infty$.

Remarks.

(a) For the case of periodic potentials $V^{(k)}$ on the real line or on $\mathbb{R}^2$ corresponding results can be found in the paper [18]. The assumptions in Theorem 1.1 are purely spectral and do not involve any further features of the potentials. In this sense, the occurrence of eigenvalues in gaps for dislocation problems is not an exception, but it is the rule; to convey this message is the main objective of the present investigations.

(b) In many applications of Theorem 1.1 both $L + V^{(1)}$ and $L + V^{(2)}$ have some essential spectrum below the common gap (a_0, b_0). In the case of half-space problems, however, only one of the operators has essential spectrum below the gap; cf. Example 4.5.

(c) We expect the statement of Theorem 1.1 to be true for all $E \in \mathbb{R}$ that satisfy condition (1.3) and $E \notin \sigma_{\mathrm{ess}}(L + V^{(k)})$ for $k = 1, 2$.

(d) Our proof of Theorem 1.1 is based on an approximation on large sections $(-n - t, n) \times \mathbb{R}/\mathbb{Z}$ of the tube, much as in [18] where periodic boundary conditions at the ends $-n - t$ and n have been used. Since the potentials $V^{(k)}$ need not be periodic in x, there is no natural boundary condition at the ends, and we simply take Dirichlet boundary conditions. Of course, this may introduce spurious eigenvalues

into the gap and we adapt a technique of Deift and Hempel [1, 8] to remove those eigenvalues. This rather technical construction is at the heart of Section 3.

The Laplacian L of Thm. 1.1 is unitarily equivalent to the operator L_{per}, defined as the self-adjoint realization of $-\Delta$ on the strip $\mathbb{R} \times (0,1)$ with periodic boundary conditions in y. We may extend the potentials $V^{(k)}$, $k = 1,2$, and V_t periodically with respect to the y-variable to all of $\mathbb{R}^2$, and consider the dislocation problem in $\mathbb{R}^2$ with the operators $H_t = -\Delta + V_t$. Then Thm. 1.1 can be used to obtain lower bounds for the integrated density of states inside a gap (a_0, b_0) for certain values of the parameter t. A precise description of the result requires quite a bit of notation and we refer to Section 4 for details.

We finally address the question of continuity of the (discrete) eigenvalues of the family of operators H_t as functions of t. For periodic potentials in one dimension continuity is easy as $V_0 - V_t$ tends to zero in $L_{1,\mathrm{loc,unif}}(S)$, as $t \to 0$; cf. the appendix in [18]. Without periodicity, we now have to face the problem that, no matter how small $t > 0$ might be, $V_0 - V_t$ need not be small on the global scale. Here we use a change of variables to the effect that, in the new coordinates, the potential is altered only in a compact set. This leads to the following basic result.

Theorem 1.2. *Let $V^{(1)}, V^{(2)} \in \mathsf{L}_\infty(S)$ be real-valued, and let $H_t := L + V_t$ with V_t as in (1.1). Then the discrete eigenvalues of H_t depend continuously on $t \in \mathbb{R}$. If, in addition, the distributional derivative $\partial_1 V^{(2)}$ is a (signed) Borel measure, the discrete eigenvalues of H_t are (locally) Lipschitz continuous functions of $t \in \mathbb{R}$.*

The second part of the theorem applies in particular if $V^{(2)}$ is of locally bounded variation; cf. [15] for details on this class of functions. Note that Thm. 1.2 also applies to discrete eigenvalues below the essential spectrum of H_t. Corresponding results for the case of periodic potentials on the real line are given in [18].

The paper is organized as follows. Section 2 introduces basic notation and describes the main technical tools used later on. There are several lemmas that will allow us to control the decoupling via additional Dirichlet boundary conditions. The finer results depend on exponential decay estimates, which we recall. Another line of argument deals with coordinate transformations on S.

In Section 3 we construct operators $\tilde{H}_{n,t}$, acting in $\mathsf{L}_2((-n-t,n) \times \mathbb{R}/\mathbb{Z})$, that serve as an approximation to H_t. We show that there exists a bounded sequence of parameters $t_n \geq 0$ such that E is an eigenvalue of $\tilde{H}_{n,t_n}$. Taking limits then leads to a $\bar{t} \geq 0$ such that $E \in \sigma(H_{\bar{t}})$ and we may take $\bar{t}$ as the first parameter τ_1 in Thm. 1.1.

Section 4 contains examples of various kinds. The first two examples show that there exist potentials $V = V(x,y)$ on S that satisfy the assumptions of Thm. 1.1 without requiring periodicity. The third example explains in some detail how to use periodic extension in y in order to go from Thm. 1.1 and the tube S to problems in $\mathbb{R}^2$ with potentials that are periodic in the y-variable only. The fourth example is an application of Thm. 1.1 and Thm. 4.4 to half-space problems and the existence of surface states.

In Section 5 we analyze continuity properties of the discrete eigenvalues of H_t as functions of the parameter t. We obtain, in particular, Lipschitz continuity if the distributional derivative $\partial_1 V^{(2)}$ is a measure.

Finally, the appendix gives a detailed account of some basic Hilbert-Schmidt-estimates on resolvent differences which are then used to control the decoupling by additional Dirichlet boundary conditions; this material is fairly standard and has been included chiefly for the convenience of the reader. In addition, we recall a basic method for obtaining exponential decay estimates for resolvent kernels, and we also give a proof of Lemma 2.5 which is based on exponential decay.

We conclude the introduction with some remarks on the literature. For results concerning the existence of eigenvalues in gaps for dislocation problems in one dimension, we refer to the work of Korotyaev [24, 25], Korotyaev and Schmidt [26], Hempel and Kohlmann [18], Dohnal, Plum and Reichel [10], and the references given therein. While many of the deeper results in 1-d clearly depend on the periodicity assumption, the basic question of *existence* of surface states in a gap can be treated in great generality using limit-point/limit-circle theory (cf. Stautz [40]).

Several basic questions concerning surface states and the associated integrated density of states in higher dimensions are discussed in Davies and Simon [6], Englisch et al. [13], and Kostrykin and Schrader [27]. However, there are only few results concerning the *existence* of eigenstates in a gap for higher-dimensional dislocations; cf. Hempel and Kohlmann [19]. Dohnal, Plum and Reichel [11] study eigenstates of a non-linear problem at an interface with energies below the spectrum of the linear operator. As for the general history of surface states, going back to Rayleigh, Kelvin, and Landau, we recommend the paper [23] by Khoruzhenko and Pastur.

2. Preliminaries

In this section we introduce notation and collect some basic results related to the dislocation problem.

2.1. **Notation and Basic Assumptions.** For basic notation and definitions concerning self-adjoint operators in Hilbert space, we refer to [22] and [35]. The spectral projection associated with a self-adjoint operator T and an interval $I \subset \mathbb{R}$ is denoted as $\mathbb{E}_I(T)$. If T has purely discrete spectrum in I, the number of eigenvalues (counting multiplicities) in I is given by the trace of $\mathbb{E}_I(T)$, denoted as $\operatorname{tr} \mathbb{E}_I(T)$. The Schatten-von Neumann classes will be denoted by $\mathcal{B}_p$, for $1 \le p < \infty$.

Our basic coordinate space is the tube $S = \mathbb{R} \times (\mathbb{R}/\mathbb{Z})$ with the usual (flat) product metric, where $\mathbb{R}/\mathbb{Z} = \frac{1}{2\pi}\mathbb{S}^1$. Let us write $\mathbb{S}' := \mathbb{R}/\mathbb{Z}$ for simplicity of notation. We consider the Sobolev space $\mathsf{H}^1(S)$ with its canonical norm; note that $\mathsf{C}_c^\infty(S)$ is dense in $\mathsf{H}^1(S)$. Equivalently, we could work with the Sobolev space $\mathsf{H}^1_{\mathrm{per}}(\mathbb{R} \times (0,1))$ consisting of functions in $\mathsf{H}^1(\mathbb{R} \times (0,1))$ that are periodic in the y-variable.

In the Hilbert space $\mathsf{L}_2(S)$ we define our basic Laplacian, L, to be the unique (self-adjoint and non-negative) operator associated with the (closed and non-negative)

quadratic form

$$\mathsf{H}^1(S) \ni u \mapsto \int_S |\nabla u|^2 \, \mathrm{d}x \, \mathrm{d}y,$$

by the first representation theorem ([22, Thm. VI-2.1]). As on the real line, the Laplacian L is essentially self-adjoint on $\mathsf{C}_c^\infty(S)$. L is unitarily equivalent to the Laplacian $-\Delta$ acting in $\mathsf{L}_2(\mathbb{R} \times (0,1))$ with periodic boundary conditions in the y-variable.

For $M \subset \mathbb{R}$ open we denote by L_M the Friedrichs extension of $-\Delta$, defined on $\mathsf{C}_c^\infty(M \times \mathbb{S}')$, in $\mathsf{L}_2(M \times \mathbb{S}')$; in other words, the form domain of L_M is given as the closure of $\mathsf{C}_c^\infty(M \times \mathbb{S}')$ in $\mathsf{H}^1(S)$. Frequently, M will be an open interval on the real line, or a finite union of such intervals, as in $L_{(\alpha,\beta)}$ for $-\infty \le \alpha < \beta \le \infty$, or in $L_{\mathbb{R}\setminus\{\gamma\}} = L_{(-\infty,\gamma)} \oplus L_{(\gamma,\infty)}$ for $\gamma \in \mathbb{R}$. If $M = (\alpha,\beta)$ for some $-\infty < \alpha < \beta < \infty$, we say that $L_{(\alpha,\beta)}$ satisfies Dirichlet boundary conditions on the lines $\{\alpha\} \times \mathbb{S}'$ and $\{\beta\} \times \mathbb{S}'$.

Given two bounded, measurable functions $V^{(1)}, V^{(2)}\colon S \to \mathbb{R}$ we define the Schrödinger operators $H^{(k)} = L + V^{(k)}$, for $k = 1, 2$. Throughout Sections 2 and 3, we assume $V^{(k)} \ge 0$ for simplicity (and without loss of generality). For $t \in \mathbb{R}$ the dislocation potentials V_t are defined as in (1.1), and we let $H_t = L + V_t$, $t \in \mathbb{R}$, denote the family of dislocation operators.

From a technical point of view, the following three tools are fundamental for our approach:

- decoupling by Dirichlet boundary conditions on circles $\{c\} \times \mathbb{S}'$,
- exponential decay of eigenfunctions,
- a coordinate transformation with respect to the x-variable.

We provide some preliminary facts concerning these tools here. We begin with Dirichlet decoupling.

2.2. **Dirichlet Decoupling.** In this subsection we show how to control the effect of an additional Dirichlet boundary condition on the line $\{0\}\times\mathbb{S}' \subset S$; topologically, $\{0\} \times \mathbb{S}' \subset S$ is a circle. On the strip $\mathbb{R} \times (0,1)$ with periodic boundary conditions the additional Dirichlet boundary condition would be placed on the straight line segment $\{0\} \times (0,1)$. Note that it is essential for our applications later on to have estimates with constants that are uniform for certain classes of potentials.

Lemma 2.1. Let $0 \le W \in \mathsf{L}_\infty(S)$, let $H = L + W$ in the Hilbert space $\mathcal{H} = \mathsf{L}_2(S)$, and let $H_D := L_{\mathbb{R}\setminus\{0\}} + W$.

Then $(H + r)^{-1} - (H_D + r)^{-1}$ is Hilbert-Schmidt for all $r \ge 1$ and there is a constant $C \ge 0$, which is independent of W and r, such that

$$(2.1) \qquad \left\| (H + r)^{-1} - (H_D + r)^{-1} \right\|_{\mathcal{B}_2(\mathcal{H})} \le C, \qquad r \ge 1.$$

Estimates of type (2.1) are well-known and have been of great use in spectral and in scattering theory. For the convenience of the reader, we give a sketch of the proof in the appendix. Similar estimates hold for finite tubes $(-n, n) \times \mathbb{S}'$ where we compare $L_{(-n,n)}$ and $L_{(-n,n)\setminus\{0\}} = L_{(-n,0)} \oplus L_{(0,n)}$.

Lemma 2.2. *Let* $0 \leq W \in \mathsf{L}_\infty(S)$ *and let* $L_{(-n,n)}$ *and* $L_{(-n,n)\backslash\{0\}}$ *be as above. Then* $(L_{(-n,n)} + W + r)^{-1} - (L_{(-n,n)\backslash\{0\}} + W + r)^{-1}$ *is Hilbert-Schmidt for* $r \geq 1$ *and we have an estimate*

$$\left\| (L_{(-n,n)} + W + r)^{-1} - (L_{(-n,n)\backslash\{0\}} + W + r)^{-1} \right\|_{\mathcal{B}_2(\mathcal{H})} \leq C,$$

with a constant C *independent of* r *and* W.

Indications on the proof are given in the appendix.

It is easy to generalize the above results to situations where we add in Dirichlet boundary conditions on several lines of the type $\{x_0\} \times \mathbb{S}'$. This immediately gives a simple proof for the invariance of the essential spectrum.

Proposition 2.3. *For* $k = 1, 2$, *let* $V^{(k)} \in \mathsf{L}_\infty(S)$ *and define* $H^{(k)}$ *and* H_t *as above. In addition, let* $H_+^{(1)} := L_{(0,\infty)} + V^{(1)}$ *and* $H_-^{(2)} := L_{(-\infty,0)} + V^{(2)}$. *We then have*

$$(2.2) \qquad \sigma_{\mathrm{ess}}(H_t) = \sigma_{\mathrm{ess}}(H_+^{(1)}) \cup \sigma_{\mathrm{ess}}(H_-^{(2)}) \subset \sigma_{\mathrm{ess}}(H^{(1)}) \cup \sigma_{\mathrm{ess}}(H^{(2)}), \qquad t \in \mathbb{R}.$$

Proof. (cf. also [18]) For $t \geq 0$, let $H_{t,\mathrm{dec}}$ denote the operator obtained from H_t by the insertion of Dirichlet boundary conditions on the lines $\{0\} \times \mathbb{S}'$ and $\{-t\} \times \mathbb{S}'$. By Lemma 2.1, $(H_t + 1)^{-1} - (H_{t,\mathrm{dec}} + 1)^{-1}$ is compact and so H_t and $H_{t,\mathrm{dec}}$ have the same essential spectrum. The part of $H_{t,\mathrm{dec}}$ to the left of $-t$ is unitarily equivalent to $H_-^{(2)}$, and the part of $H_{t,\mathrm{dec}}$ associated with the interval $(-t, 0)$ has compact resolvent. Thus $\sigma_{\mathrm{ess}}(H_{t,\mathrm{dec}}) = \sigma_{\mathrm{ess}}(H_{0,\mathrm{dec}}) = \sigma_{\mathrm{ess}}(H_+^{(1)}) \cup \sigma_{\mathrm{ess}}(H_-^{(2)})$. This proves the equality in (2.2). The inclusion stated in (2.2) is immediate from Lemma 2.1. $\qquad \square$

The result of Prop. 2.3 can also be obtained by considering singular sequences (so-called *Weyl-sequences*). Yet another method of proof, based on a transformation of coordinates, is given in Section 2.4 below.

2.3. **Exponential Decay of Eigenfunctions.** The following contains our basic exponential decay estimate. The proof uses classic arguments as discussed in [38] or [20]; cf. also [5] for a recent version. It is of importance for the applications we are having in mind that the bound of Lemma 2.4 below is independent of W within the class of bounded, non-negative potentials with a given spectral gap (a_0, b_0). We let χ_k denote the characteristic function of the set $[-k, k] \times \mathbb{S}' \subset S$, for brevity. Again, the proofs of the statements in this subsection are deferred to the appendix.

Lemma 2.4. *For* $0 \leq a_0 < a < b < b_0$ *given there exist constants* $C \geq 0$ *and* $\gamma > 0$ *such that for all* $0 \leq W \in \mathsf{L}_\infty(S)$ *with* $\sigma(L + W) \cap (a_0, b_0) = \emptyset$ *we have*

$$\|(1 - \chi_k)u\| \leq C e^{-\gamma k} \|u\|, \qquad k \in \mathbb{N},$$

for all eigenfunctions u *of* $L_{\mathbb{R}\backslash\{0\}} + W$ *that are associated with an eigenvalue* $\lambda \in [a, b]$.

The following lemma gives an upper bound for the number of eigenvalues that are moved into a compact subset $[a, b]$ of a spectral gap (a_0, b_0) upon enforcing a Dirichlet boundary condition on the line $\{0\} \times \mathbb{S}'$. Again, it is important that the

bound is *independent* of the potential W, provided $W \geq 0$. A proof is given in the Appendix.

Lemma 2.5. *For numbers $a_0 < a < b < b_0 \in \mathbb{R}$ given there exists a constant $c \geq 0$ with the following property: If $0 \leq W \in \mathsf{L}_\infty(S)$ satisfies $\sigma(L + W) \cap (a_0, b_0) = \emptyset$, then*

$$\mathrm{tr}\, \mathbb{E}_{[a,b]}(L_{\mathbb{R} \setminus \{0\}} + W) \leq c.$$

2.4. **Transformation of Coordinates.** Some additional insight can be gained by using a transformation of coordinates which, in a sense, "undoes" the effect of the dislocation outside a finite section of the tube S. In this way, the dislocation problem can be viewed as a perturbation which acts in a compact subset of S only. To this end, we provide (smooth) diffeomorphisms $\varphi_t \colon \mathbb{R} \to \mathbb{R}$ of class C^∞ with the additional properties that

$$\varphi_t(x) = x, \quad x \geq 0, \qquad \varphi_t(x) = x - t, \quad x \leq -2;$$

we also require that there is a constant $C \geq 0$ s.th.

$$\max_{x \in \mathbb{R}} |\varphi_t(x) - x|, \quad \max_{x \in \mathbb{R}} |\varphi_t'(x) - 1|, \quad \max_{x \in \mathbb{R}} |\varphi_t''(x)| \leq Ct, \qquad t \in [0, 2].$$

In Section 5 it will be shown that, for $0 \leq t \leq 1$, the dislocation operators H_t are unitarily equivalent to (s.a.) operators $\hat{H}_t$ acting in $\mathsf{L}_2(S)$ with domain $D(\hat{H}_t) = D(L)$ where the quadratic form of $\hat{H}_t$ is given by

$$\hat{H}_t[u, u] := \int_S \left(\frac{1}{(\varphi_t')^2} |\partial_1 u|^2 + |\partial_2 u|^2 - \frac{\varphi_t''}{(\varphi_t')^3} \mathrm{Re}\, (\bar{u} \partial_1 u) + \frac{(\varphi_t'')^2}{4(\varphi_t')^4} |u|^2 \right) \mathrm{d}x\, \mathrm{d}y$$
$$+ \int_S V_t(\varphi_t(x), y) |u|^2 \, \mathrm{d}x\, \mathrm{d}y;$$

here $\hat{H}_0 = H_0 = H$. Note that $\hat{H}_t - H_0$ has support in the compact set $\{-2 \leq x \leq 0\}$. In other words, the family $(\hat{H}_t)_{0 \leq t \leq 1}$ gives an equivalent description of the dislocation problem where the perturbation is now restricted to the bounded set $\{(x, y) \in S \mid -2 \leq x \leq 0\}$.

The family $(\hat{H}_t)_{0 \leq t \leq 1}$ enjoys the following properties:
(1) The mapping $[0, 1] \ni t \mapsto (\hat{H}_t + 1)^{-1}$ is norm-continuous.
(2) For $t, t' \in [0, 1]$, the resolvent difference $(\hat{H}_t + 1)^{-1} - (\hat{H}_{t'} + 1)^{-1}$ is compact.

Both properties will be proved in Section 5. It follows from (2) that the essential spectrum of $\hat{H}_t$ is stable, and then the same property holds for the family $(H_t)_{0 \leq t \leq 1}$; cf. Prop. 2.3. Property (1) implies that the spectrum of $\hat{H}_t$ depends continuously on t in the usual Hausdorff-metric on the real line, and then the same holds for the family $(H_t)_{0 \leq t \leq 1}$.

3. The main result

In this section we give a proof of Theorem 1.1. We consider some $E \in \mathbb{R}$ satisfying the assumptions (1.2) and (1.3) of Thm. 1.1. It follows from condition (1.2) that there is an $\alpha > 0$ such that

$$\mathrm{dist}\, (E, \sigma(L + V^{(k)})) \geq 2\alpha, \qquad k = 1, 2;$$

E and α will be kept fixed throughout this section. If it happens that E is an eigenvalue of $H_0 = L + V_0$ we set $\tau_1 := 0$ and consider H_1 instead of H_0. We may therefore assume in the sequel that $E \notin \sigma(H_0)$. We now fix some $0 < \beta \le 2\alpha/3$ such that

(3.1) $$\operatorname{dist}\,(E, \sigma(H_0)) \ge 3\beta.$$

We find solutions of suitable approximating problems, and then pass to the limit. The basic idea is to restrict the problem to finite sections of the tube S of the form $(-n-t, n) \times \mathbb{S}'$, as in [18, 19] where S is a strip and the potential V is periodic. In [18, 19] periodic boundary conditions at the ends of the finite strip work nicely, but for non-periodic potentials there is no natural choice of boundary conditions on the lines $\{\pm n\} \times \mathbb{S}'$ that would keep the interval $(E - \beta, E + \beta)$ free of spectrum of the operators $H_{n,0} = L_{(-n,n)} + V_0$ and we have to resort to a more complicated construction, inspired by some work of Deift and Hempel [8] (cf. also [1]).

3.1. **The Approximating Problems.** We first introduce "correction terms" in the form of projections, sandwiched between suitable cut-offs. While we have two interacting Dirichlet boundaries, we prefer a construction where the correction term at the left end does not depend on the correction term at the right end. Let

$$H_n^+ = L_{(-\infty,n)} + V^{(1)} \text{ in } \mathsf{L}_2((-\infty, n) \times \mathbb{S}'),$$
$$H_n^- = L_{(-n,\infty)} + V^{(2)} \text{ in } \mathsf{L}_2((-n, \infty) \times \mathbb{S}'),$$

for $n \in \mathbb{N}$, where we have chosen the upper indices $\pm$ of $H_n^\pm$ in reference to the Dirichlet boundary condition on the lines $\{\pm n\} \times \mathbb{S}'$. As in Prop. 2.3, we have $\sigma_{\mathrm{ess}}(H_n^+) \subset \sigma_{\mathrm{ess}}(H^{(1)})$ and $\sigma_{\mathrm{ess}}(H_n^-) \subset \sigma_{\mathrm{ess}}(H^{(2)})$ so that $(E - 3\beta, E + 3\beta)$ is a gap in the essential spectrum of $H_n^\pm$. We are now going to construct a family of operators $\tilde{H}_{n,t}$ on $(-n-t, n) \times \mathbb{S}'$ that will serve as approximations to H_t and which enjoy the property that the interval $(E - \beta, E + \beta)$ is free of spectrum of $\tilde{H}_{n,0}$.

Let $\Phi_{n,k}^\pm$, $k = 1, \ldots, J_n^\pm$, denote a (maximal) orthonormal set of eigenfunctions of $H_n^\pm$ corresponding to its eigenvalues in $[E - 2\beta, E + 2\beta]$. By Lemma 2.5, there is a constant c_0 such that $J_n^\pm \le c_0$ for all n; here we apply Lemma 2.5 twice, with the choice $a_0 := E - 3\beta$, $a := E - 2\beta$, $b := E + 2\beta$, $b_0 := E + 3\beta$, and $W := V^{(1)}(.+n)$ or $W := V^{(2)}(.-n)$, respectively.

Next, we introduce the projections $P_n^\pm$ onto the span of the $\Phi_{n,k}^\pm$, given by

$$P_n^\pm = \mathbb{E}_{[E-2\beta, E+2\beta]}(H_n^\pm).$$

As a consequence,

(3.2) $$\sigma(H_n^\pm + 4\beta P_n^\pm) \cap [E - 2\beta, E + 2\beta] = \emptyset.$$

Here the eigenfunctions $\Phi_{n,k}^\pm$ are localized near $\{\pm n\} \times \mathbb{S}'$ and decay (exponentially) as x increases or decreases from $\pm n$, cf. Lemma 2.4. We are now going to make this more precise.

Let us first introduce some cut-off functions. Let $\chi_1^+ \in \mathsf{C}^\infty(-\infty, 1)$ with $0 \le \chi_1^+ \le 1$, $\chi_1^+(x) = 1$ for $x > 3/4$, and $\chi_1^+(x) = 0$ for $x < 1/2$, be given. Now set $\chi_n^+(x) = \chi_1(x/n)$, so that $\chi_n^+ \in \mathsf{C}^\infty(-\infty, n)$, $\chi_n^+(x) = 1$ for $x > 3n/4$, and

$\chi_n^+(x) = 0$ for $x < n/2$. We define $\chi_n^- \in C^\infty(-n, \infty)$ analogously by setting $\chi_n^-(x) = \chi_n^+(-x)$. Furthermore, choose $\varphi_1 \in C_c^\infty(-1/2, 1/2)$ with $0 \le \varphi_1 \le 1$ and $\varphi_1(x) = 1$ for $|x| < 1/4$, and set $\varphi_n(x) = \varphi_1(x/n)$ and $\psi_n = 1 - \varphi_n$. Finally, we decompose $\psi_n = \psi_n^- + \psi_n^+$ and note that $\psi_n^\pm \chi_n^\pm = \chi_n^\pm$. By Lemma 2.4 there are constants $c \ge 0$ and $n_0 \in \mathbb{N}$ such that

$$\left\| (1 - \chi_n^\pm)\Phi_{n,k}^\pm \right\| \le c/n, \qquad n \ge n_0,$$

and we infer that there is a constant $C \ge 0$ such that

$$(3.3) \qquad \left\| \chi_n^\pm P_n^\pm \chi_n^\pm - P_n^\pm \right\| \le \frac{C}{n};$$

in fact, a stronger estimate of the form $\left\| \chi_n^\pm P_n^\pm \chi_n^\pm - P_n^\pm \right\| \le C' e^{-\gamma n}$, for some $\gamma > 0$, holds true. We now define

$$\tilde{H}_n^\pm = H_n^\pm + 4\beta \chi_n^\pm P_n^\pm \chi_n^\pm$$

and observe that, by (3.2) and (3.3),

$$\sigma(\tilde{H}_n^\pm) \cap [E - \beta, E + \beta] = \emptyset,$$

for n large. In particular, for any $u \in D(\tilde{H}_n^\pm) = D(H_n^\pm)$ we have

$$(3.4) \qquad \|u\| \le \frac{1}{\beta} \left\| (\tilde{H}_n^\pm - E)u \right\|.$$

Now the dislocation enters the game: let $T_t(x, y) = (x + t, y)$, for $(x, y) \in S$, and define

$$P_{n,t}^- = \sum_{k \in J_n^-} \left\langle \,\cdot\,, \Phi_{n,k}^- \circ T_t \right\rangle \Phi_{n,k}^- \circ T_t,$$

as well as $\chi_{n,t}^- := \chi_n^- \circ T_t$. Finally, let

$$\mathcal{P}_{n,t} = 4\beta \left(\chi_n^+ P_n^+ \chi_n^+ + \chi_{n,t}^- P_{n,t}^- \chi_{n,t}^- \right)$$

and

$$\tilde{H}_{n,t} = L_{(-n-t,n)} + V_t + \mathcal{P}_{n,t}$$

in $L_2((-n - t, n) \times \mathbb{S}')$. The operators $\tilde{H}_{n,t}$ are the principal players in our approximating problems. We first establish that the operators $\tilde{H}_{n,0}$ have no spectrum in the interval $[E - \beta, E + \beta]$, for n large.

Lemma 3.1. *Let $E \in \mathbb{R} \setminus \sigma(H_0)$ satisfy condition (1.2) of Thm. 1.1 and let β be as in (3.1). Then there is an $n_0 \in \mathbb{N}$ such that*

$$\sigma(\tilde{H}_{n,0}) \cap (E - \beta, E + \beta) = \emptyset, \qquad n \ge n_0.$$

Proof. Else there exists a sequence $n_j \to \infty$ and there exist $E_j \in [E - \beta, E + \beta]$ such that E_j is an eigenvalue of $\tilde{H}_{n_j,0}$, for $j \in \mathbb{N}$. Let u_{n_j} denote an associated normalized eigenfunction. With the cut-off functions φ_k and $\psi_k^\pm$ defined above, we see that $\varphi_{n_j/4} u_{n_j} \in D(H_0)$ and $\psi_{n_j/4}^\pm u_{n_j} \in D(\tilde{H}_{n_j}^\pm)$ with estimates

$$(3.5) \qquad \left\| (H_0 - E_j)(\varphi_{n_j/4} u_{n_j}) \right\| \le c/n_j, \qquad \left\| (\tilde{H}_{n_j}^\pm - E_j)(\psi_{n_j/4}^\pm u_{n_j}) \right\| \le c/n_j,$$

for j large; here $c \ge 0$ is a suitable constant. Since $\sigma(H_0) \cap (E - 2\beta, E + 2\beta) = \emptyset$ and $E_j \in [E - \beta, E + \beta]$, we have $\|(H_0 - E_j)u\| \ge \beta \|u\|$ for all $u \in D(H_0)$, so that

$\varphi_{n_j/4} u_{n_j} \to 0$ as $j \to \infty$. Similarly, the second estimate in (3.5) and (3.4) imply that $\psi^{\pm}_{n_j/4} u_{n_j} \to 0$, as $j \to \infty$. Therefore $u_{n_j} \to 0$ as $j \to \infty$, in contradiction to $\|u_{n_j}\| = 1$. $\qquad\square$

3.2. **Solution of the Approximating Problems.** We are now going to show that, for large $n \in \mathbb{N}$, there exist parameters $t_n \geq 0$ such that E is an eigenvalue of $\tilde{H}_{n,t_n}$. Since all the operators involved have purely discrete spectrum we can use a simple eigenvalue counting argument.

Proposition 3.2. *Let $E \in \mathbb{R} \setminus \sigma(H_0)$ satisfy conditions (1.2) and (1.3) of Theorem 1.1. Then there are $n_0 \in \mathbb{N}$ and $\gamma_0 > 0$ such that for any $n \in \mathbb{N}$ with $n \geq n_0$ there exists $0 < t_n \leq \gamma_0$ such that E is an eigenvalue of $\tilde{H}_{n,t_n}$.*

In preparation for the proof, we introduce variants of our operators with Dirichlet boundary conditions on suitable lines. Let $\tilde{H}_{n,t;\mathrm{dec}}$ denote the operator $\tilde{H}_{n,t}$ with additional DBCs on the lines $\{0\} \times \mathbb{S}'$ and $\{-t\} \times \mathbb{S}'$; note that—by virtue of the cut-offs χ_n^+ and $\chi_{n,t}^-$—the non-local operators $\mathcal{P}_{n,t}$ are not affected by these boundary conditions. For $n \geq n_0(t)$ the operators $\tilde{H}_{n,t;\mathrm{dec}}$ can be written as direct sums

$$\tilde{H}_{n,t;\mathrm{dec}} = \tilde{h}_{n,t;1} \oplus h_{t;2} \oplus \tilde{h}_{n;3},$$

with

$$\tilde{h}_{n,t;1} := L_{(-n-t,-t)} + V_t + 4\beta \chi_{n,t}^- P_{n,t}^- \chi_{n,t}^-,$$

acting in $\mathsf{L}_2((-n-t,-t) \times \mathbb{S}')$ with DBCs on $\{-n-t\} \times \mathbb{S}'$ and on $\{-t\} \times \mathbb{S}'$,

$$h_{t;2} := L_{(-t,0)} + V_t$$

acting in $\mathsf{L}_2((-t,0) \times \mathbb{S}')$ with DBCs on $\{-t\} \times \mathbb{S}'$ and on $\{0\} \times \mathbb{S}'$, and, finally,

$$\tilde{h}_{n;3} := L_{(0,n)} + V^{(1)} + 4\beta \chi_n^+ P_n^+ \chi_n^+,$$

acting in $\mathsf{L}_2((0,n) \times \mathbb{S}')$ with DBCs on $\{0\} \times \mathbb{S}'$ and on $\{n\} \times \mathbb{S}'$.

We now collect some properties of the operators $\tilde{H}_{n,t;\mathrm{dec}}$ that we need in the proof of Proposition 3.2.

(1) For $t = 0$ we have

$$\tilde{H}_{n,0;\mathrm{dec}} = \tilde{h}_{n,0;1} \oplus \tilde{h}_{n;3}$$
$$= (L_{(-n,0)} + V^{(2)} + 4\beta \chi_n^- P_n^- \chi_n^-) \oplus (L_{(0,n)} + V^{(1)} + 4\beta \chi_n^+ P_n^+ \chi_n^+).$$

The following lemma compares the number of eigenvalues below E for the operators $\tilde{H}_{n,0}$ and $\tilde{H}_{n,0;\mathrm{dec}}$.

Lemma 3.3. *Let E and β satisfy (3.1), let $\tilde{H}_{n,0}$ and $\tilde{H}_{n,t;\mathrm{dec}}$ be as above, and let n_0 as in Lemma 3.1. Then there is a constant $c_0 \geq 0$ such that*

$$\operatorname{tr} \mathbb{E}_{(-\infty,E]}(\tilde{H}_{n,0}) \geq \operatorname{tr} \mathbb{E}_{(-\infty,E]}(\tilde{H}_{n,0;\mathrm{dec}}) \geq \operatorname{tr} \mathbb{E}_{(-\infty,E]}(\tilde{H}_{n,0}) - c_0, \qquad n \geq n_0.$$

In the proof of Lemma 3.3 we use a proposition, based on the Birman-Schwinger principle (cf. [35], [39]) to control the spectral shift across E, produced by the Dirichlet boundary condition on $\{0\} \times \mathbb{S}'$. Recall that $\mathcal{B}_p$ denotes the p-th Schatten-von Neumann class, for $1 \leq p < \infty$.

Proposition 3.4. *Let $1 \leq T \leq S$ be self-adjoint operators with compact resolvent in the Hilbert-space $\mathcal{H}$, and suppose that $T^{-1} - S^{-1} \in \mathcal{B}_p(\mathcal{H})$ for some $p \in [1, \infty)$. Then for any $E \in \mathbb{R} \setminus \sigma(T)$ we have*

$$\operatorname{tr} \mathbb{E}_{(-\infty, E)}(S) \geq \operatorname{tr} \mathbb{E}_{(-\infty, E)}(T) - \operatorname{dist}(E, \sigma(T))^{-p} \left\| T^{-1} - S^{-1} \right\|_{\mathcal{B}_p}^p .$$

Proof. The proof is immediate from Proposition 1.1 in [17] with $A := (T+1)^{-1}$, $B := (S+1)^{-1}$, and $\eta := (E+1)^{-1}$. $\qquad \square$

Proof (of Lemma 3.3). The first inequality follows immediately from $\tilde{H}_{n,0;\mathrm{dec}} \geq \tilde{H}_{n,0}$. To prove the second inequality, we apply Prop. 3.4 with $T := \tilde{H}_{n,0} + 1$, $S := \tilde{H}_{n,0,\mathrm{dec}} + 1$, and $p = 2$. Here $(H_{n,0}+1)^{-1} - (H_{n,0,\mathrm{dec}}+1)^{-1}$ is Hilbert-Schmidt by Lemma 2.2 with a bound c_1 on the HS-norm which is independent of n. Simple perturbational arguments ([17, Lemma 1.4]) yield that there exists a constant $c_2 \geq 0$ such that

$$\left\| (\tilde{H}_{n,0}+1)^{-1} - (\tilde{H}_{n,0,\mathrm{dec}}+1)^{-1} \right\|_{\mathcal{B}_2} \leq c_2, \qquad n \geq n_0.$$

Now Prop. 3.4 implies

$$\operatorname{tr} \mathbb{E}_{(-\infty, E)}(\tilde{H}_{n,0,\mathrm{dec}}) \geq \operatorname{tr} \mathbb{E}_{(-\infty, E)}(\tilde{H}_{n,0}) - \beta^{-2} c_2^2;$$

here the left hand side is enlarged if we replace $\mathbb{E}_{(-\infty, E)}$ with $\mathbb{E}_{(-\infty, E]}$ while the right hand side remains unchanged under this replacement since $E \notin \sigma(\tilde{H}_{n,0})$. $\qquad \square$

(2) The operator $\tilde{h}_{n,t;1}$ is unitarily equivalent to $\tilde{h}_{n,0;1}$ via a right-translation through t so that

$$(3.6) \qquad \operatorname{tr} \mathbb{E}_{(-\infty, E]}(\tilde{h}_{n,t;1}) + \operatorname{tr} \mathbb{E}_{(-\infty, E]}(\tilde{h}_{n;3}) = \operatorname{tr} \mathbb{E}_{(-\infty, E]}(\tilde{H}_{n,0;\mathrm{dec}}).$$

(3) The operators $h_{t;2}$ are unitarily equivalent to $L_{(0,t)} + V^{(2)} \upharpoonright (0,t)$ for all $t > 0$ by a right translation and we have the following lemma.

Lemma 3.5. *Let $h_{t;2}$ as above and let E and $V^{(2)}$ satisfy condition (1.3). Then*

$$\operatorname{tr} \mathbb{E}_{(-\infty, E)}(h_{t;2}) \to \infty, \qquad t \to \infty.$$

For the proof we prepare a lemma.

Lemma 3.6. *Let A and A_n, for $n \in \mathbb{N}$, be bounded, symmetric operators in some Hilbert space and suppose that $A_n \to A$ strongly. Then, for any $\lambda_0 \in \sigma_{\mathrm{ess}}(A)$ and any $\varepsilon > 0$ we have $\operatorname{tr} \mathbb{E}_{(\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)}(A_n) \to \infty$ as $n \to \infty$.*

In Lemma 3.6 we allow for $\operatorname{tr} \mathbb{E}_{(\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)}(A_n) = \infty$; a precise statement would read as follows: For any $k \in \mathbb{N}$ there exists $n_0 \in \mathbb{N}$ such that $\operatorname{tr} \mathbb{E}_{(\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)}(A_n) \in [k, \infty]$ for all $n \geq n_0$.

Proof. Assume for a contradiction that there exist $\lambda_0 \in \sigma_{\mathrm{ess}}(A)$, $k_0 \in \mathbb{N}$, and a sequence $(n_j) \subset \mathbb{N}$ with $n_j \to \infty$, as $j \to \infty$, such that $\operatorname{tr} \mathbb{E}_{(\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)}(A_{n_j}) \leq k_0$ for all $j \in \mathbb{N}$.

Let $0 < \varepsilon' < \varepsilon$ and choose a continuous function $f \colon \mathbb{R} \to [0,1]$ such that $f(x) = 1$ for $|x - \lambda_0| \leq \varepsilon'$ and $f(x) = 0$ for $|x - \lambda_0| \geq \varepsilon$. By routine arguments, it follows from the assumptions that $p(A_n) \to p(A)$ strongly for all real-valued polynomials

and then that $f(A_n) \to f(A)$ strongly; here we also use that the norms $\|A_n\|$ form a bounded sequence.

There exists an ONS $\{u_1, \dots, u_{k_0+1}\} \subset \operatorname{Ran} \mathbb{E}_{(\lambda_0-\varepsilon', \lambda_0+\varepsilon')}(A)$. As $\chi_{(\lambda_0-\varepsilon, \lambda_0+\varepsilon)} \geq f \geq 0$, monotonicity of the trace yields

$$\operatorname{tr} \mathbb{E}_{(\lambda_0-\varepsilon, \lambda_0+\varepsilon)}(A_{n_j}) \geq \operatorname{tr} f(A_{n_j}) \geq \sum_{m=1}^{k_0+1} \langle f(A_{n_j}) u_m, u_m \rangle$$

with $\sum_{m=1}^{k_0+1} \langle f(A_{n_j}) u_m, u_m \rangle \to \sum_{m=1}^{k_0+1} \langle f(A) u_m, u_m \rangle = k_0 + 1$, as $j \to \infty$. □

Proof (of Lemma 3.5). Since $h_{t;2}$ and $L_{(0,t)} + V^{(2)}$ are unitarily equivalent, we only have to show that $\operatorname{tr} \mathbb{E}_{(-\infty, E)}(L_{(0,t)} + V^{(2)}) \to \infty$ as $t \to \infty$. Here we may use Lemma 3.6, applied to the operators

$$A := (L_{(0,\infty)} + V^{(2)} + 1)^{-1}, \qquad A_t := (L_{(0,t)} + V^{(2)} + 1)^{-1} \oplus 0,$$

with the operator 0 acting in $\mathsf{L}_2((t, \infty) \times \mathbb{S}')$. Indeed, it follows from a result in [37] that $A_t \to A$ strongly, as $t \to \infty$. □

Let us note as an aside that there is a kind of converse to the statement of Lemma 3.5: If $\eta < \inf \sigma_{\mathrm{ess}}(L_{(0,\infty)} + V^{(2)})$, then min-max and $L_{(0,t)} + V^{(2)} \geq L_{(0,\infty)} + V^{(2)}$ imply that

$$\operatorname{tr} \mathbb{E}_{(-\infty, \eta]}(L_{(0,t)} + V^{(2)}) \leq \operatorname{tr} \mathbb{E}_{(-\infty, \eta]}(L_{(0,\infty)} + V^{(2)}) < \infty, \qquad t > 0,$$

and thus $\operatorname{tr} \mathbb{E}_{(-\infty, \eta]}(L_{(0,t)} + V^{(2)})$ is a bounded function of $t > 0$.

We are now ready for the proof of Proposition 3.2.

Proof of Prop. 3.2. Let $E \in (a, b) \setminus \sigma(H_0)$. By Lemma 3.1 there exist $\beta > 0$ and $n_0 \in \mathbb{N}$ such that

$$(E - \beta, E + \beta) \cap \sigma(\tilde{H}_{n,0}) = \emptyset, \qquad n \geq n_0.$$

Adding in Dirichlet boundary conditions raises eigenvalues and we thus have

$$\begin{aligned}
\operatorname{tr} \mathbb{E}_{(-\infty, E]}(\tilde{H}_{n,t}) &\geq \operatorname{tr} \mathbb{E}_{(-\infty, E]}(\tilde{H}_{n,t;\mathrm{dec}}) \\
&= \operatorname{tr} \mathbb{E}_{(-\infty, E]}(\tilde{h}_{n,t;1}) + \operatorname{tr} \mathbb{E}_{(-\infty, E]}(h_{t;2}) + \operatorname{tr} \mathbb{E}_{(-\infty, E]}(\tilde{h}_{n;3}) \\
&= \operatorname{tr} \mathbb{E}_{(-\infty, E]}(\tilde{H}_{n,0;\mathrm{dec}}) + \operatorname{tr} \mathbb{E}_{(-\infty, E]}(h_{t;2}),
\end{aligned}$$

where we have used (3.6) in the last step. It now follows from Lemma 3.3 that

$$\operatorname{tr} \mathbb{E}_{(-\infty, E]}(\tilde{H}_{n,t}) \geq \operatorname{tr} \mathbb{E}_{(-\infty, E]}(\tilde{H}_{n,0}) - c_0 + \operatorname{tr} \mathbb{E}_{(-\infty, E]}(h_{t;2}),$$

with the constant c_0 from Lemma 3.3. Since $V^{(2)}$ satisfies condition (1.3), Lemma 3.5 implies that there exists $\gamma_0 > 0$ such that $\operatorname{tr} \mathbb{E}_{(-\infty, E]}(h_{\gamma_0;2}) > c_0$ and we conclude that

$$(3.7) \qquad \operatorname{tr} \mathbb{E}_{(-\infty, E]}(\tilde{H}_{n,\gamma_0}) > \operatorname{tr} \mathbb{E}_{(-\infty, E]}(\tilde{H}_{n,0}), \qquad n \geq n_0.$$

The operators $\tilde{H}_{n,t}$ have purely discrete spectrum and their eigenvalues depend continuously on $t \geq 0$, as can be easily seen by arguments similar to the ones used at the end of Section 2. Therefore, (3.7) implies that at least one eigenvalue of $\tilde{H}_{n,t}$ has crossed E at some $0 < t_n \leq \gamma_0$, and we are done. □

3.3. **Convergence of the Approximative Solutions.** The above Prop. 3.2 shows that there exists a bounded sequence of parameters t_n such that E is an eigenvalue of $\tilde{H}_{n,t_n}$. Then there is a convergent subsequence $t_{n_j} \to \bar{t}$, as $j \to \infty$, and we expect that E is an eigenvalue of $H_{\bar{t}}$.

Lemma 3.7. *Suppose we are given sequences $(t_n) \subset [0, \infty)$ and $(E_n) \subset [E - \beta, E + \beta]$ with $t_n \to \bar{t}$ and $E_n \to E$, as $n \to \infty$, with the property that E_n is an eigenvalue of $\tilde{H}_{n,t_n}$ for $n \geq n_0$. Then E is an eigenvalue of $H_{\bar{t}}$.*

Proof. Let $f_n \in D(H_{n,t_n})$ be normalized eigenfunctions of $\tilde{H}_{n,t_n}$ associated with the eigenvalue E_n. There is a subsequence $(f_{n_j}) \subset (f_n)$ and a function $f \in \mathsf{H}^1(S)$ such that $f_{n_j} \to f$ weakly in $\mathsf{H}^1(S)$ and weakly in $\mathsf{L}_2(S)$; furthermore, we may assume that $f_{n_j} \to f$ in $\mathsf{L}_{2,\mathrm{loc}}(S)$ and that $f_{n_j} \to f$ pointwise a.e. All these properties are standard. As a consequence, for all $\varphi \in \mathsf{C}_c^\infty(S)$ we have $\langle \nabla f_{n_j}, \nabla \varphi \rangle \to \langle \nabla f, \nabla \varphi \rangle$ and $\langle f_{n_j}, \varphi \rangle \to \langle f, \varphi \rangle$. In addition, since translation is continuous in L_1 and since $V^{(2)}$ is bounded, we have $V_{t_n} \to V_{\bar{t}}$ in $\mathsf{L}_{2,\mathrm{loc}}(S)$ whence

$$\left\langle V_{t_{n_j}} \varphi, f_{n_j} \right\rangle \to \langle V_{\bar{t}} \varphi, f \rangle, \qquad j \to \infty.$$

Now let $\mathbf{h}_t$, $\mathbf{h}_{n,t}$ and $\tilde{\mathbf{h}}_{n,t}$ denote the quadratic forms associated with the operators H_t, $H_{n,t}$ and $\tilde{H}_{n,t}$, respectively, for $t \geq 0$. We note that, for any $g \in \mathsf{H}^1(S)$, $\varphi \in \mathsf{C}_c^\infty(S)$, and $t \geq 0$ we have $\mathbf{h}_{n,t}[g, \varphi] = \tilde{\mathbf{h}}_{n,t}[g, \varphi]$ for all sufficiently large n. Using this and the above convergence properties we may conclude that

$$\mathbf{h}_{\bar{t}}[f, \varphi] - E \langle f, \varphi \rangle = \lim_{j \to \infty} (\tilde{\mathbf{h}}_{n_j, t_{n_j}}[f_{n_j}, \varphi] - E \langle f_{n_j}, \varphi \rangle) = 0.$$

Therefore, f belongs to the domain of $H_{\bar{t}}$ and satisfies $H_{\bar{t}} f = Ef$.

It remains to show that $f \neq 0$. This follows from the fact that the eigenfunctions f_n are localized near the interface, i.e., near $\{0\} \times \mathbb{S}'$. To prove this, we consider $m \in \mathbb{N}$, $m \leq n$, where we observe that $\psi_m^+ f_n \in D(H_n^+)$ and that

$$\left\| (\tilde{H}_n^+ - E_n)(\psi_m^+ f_n) \right\| \leq 2 \left\| \nabla \psi_m^+ \nabla f_n \right\| + \left\| (\Delta \psi_m^+) f_n \right\| \leq \frac{c}{m},$$

where we have also used $[\mathcal{P}_{n,t}, \psi_m^+] = 0$. As a consequence of (3.4) it follows that

$$\left\| \psi_m^+ f_n \right\| \leq c/(\beta m), \qquad n \geq m,$$

for some constant $c \geq 0$. A similar estimate works near $-n - t$. We may thus pick $m = m_0$ so large that $\left\| \psi_{m_0}^\pm f_n \right\| \leq 1/4$ for all $n \geq m_0$, whence $\left\| \varphi_{m_0} f \right\| \geq 1/2$. □

We are now ready for the proof of Thm. 1.1.

Proof of Theorem 1.1. If $E \in \sigma(H_0)$ let $\tau_1 := 0$. Else Prop. 3.2 and Lemma 3.7 directly yield a $\tau_1 \geq 0$ such that $E \in \sigma(H_{\tau_1})$; in this case we would in fact know that $\tau_1 > 0$.

If E happens to be an eigenvalue of $H_{\tau_1 + 1}$, we let $\tau_2 := \tau_1 + 1$. Else we replace $V^{(2)}$ with $V^{(2)} \circ T_{\tau_1 + 1}$, to obtain some $\tau_2 \geq \tau_1 + 1$ with $E \in \sigma(H_{\tau_2})$, and so on. □

4. Examples and Applications

For simplicity of notation we restrict our attention in this section mostly to the case where $V^{(1)} = V^{(2)}$; we may then simply write V. Also note that, in this section, we do not necessarily require $V \geq 0$.

We first give two examples of potentials $V = V(x,y)$, depending on the two variables $(x,y) \in \mathbb{R} \times \mathbb{S}'$, with a (non-trivial) spectral gap in the spectrum of the associated Hamiltonian.

Example 4.1. In a particularly simple example, V is given as the sum of an almost periodic (or periodic) potential $V_1 = V_1(x)$ and a potential $V_2 = V_2(y)$,

$$V(x,y) := V_1(x) + V_2(y), \qquad (x,y) \in S;$$

both V_1 and V_2 are bounded, measurable, and real-valued functions. Without restriction of generality, we may assume that the spectrum of the one-dimensional Schrödinger operators $h_1 := -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_1(x)$, acting in $\mathsf{L}_2(\mathbb{R})$, and $h_2 := -\frac{\mathrm{d}^2}{\mathrm{d}y^2} + V_2(y)$, acting in $\mathsf{L}_2(\mathbb{S}')$, begins at the point 0.

In addition, let us assume that h_1 has a gap (a,b) in its spectrum, where $0 \leq a < b$. In view of condition (1.3) let us also assume that the operator $\ell_{(0,\infty)} + V_1 \upharpoonright (0,\infty)$ has some essential spectrum in $(-\infty, a]$; here $\ell_{(0,\infty)}$ denotes the self-adjoint realization of $-\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ in $\mathsf{L}_2(0,\infty)$ with Dirichlet boundary condition at 0. One may think of potentials V_1 of the type $V_1(x) := \cos x + \varepsilon \cos \pi x$, with $\varepsilon > 0$ small. More generally, there are many results that establish the existence of gaps for certain classes of almost periodic potentials in one dimension (cf. e.g. Johnson and Moser [21], Avila et al. [2], Goldstein and Schlag [16], Puig [31]), many of them dealing with the lattice case.

By compactness, h_2 has purely discrete spectrum consisting of a sequence of real eigenvalues $(\lambda_k)_{k \in \mathbb{N}_0}$ with $\lambda_0 = 0$, $\lambda_k < \lambda_{k+1}$ for all $k \in \mathbb{N}_0$, and $\lambda_k \to \infty$ as $k \to \infty$. The condition $\lambda_0 = 0$ is equivalent to $\int_0^1 V_2(y)\,\mathrm{d}y = 0$.

The spectrum of the associated Hamiltonian $H_0 = L + V = h_1 \otimes I + I \otimes h_2$ on S is then given by the algebraic sum $\sigma(h_1) + \sigma(h_2)$. In particular, H_0 has a gap in the essential spectrum if $a + \lambda_1 < b$ or if $\lambda_1 > a$. It is clear that condition (1.3) is satisfied as well. We may now apply Thm. 1.1 and find that for any given $E > a$ in a gap of H_0 there exists a sequence $\tau_k \to \infty$ such that E is an eigenvalue of H_{τ_k}.

Example 4.2. Another natural class of examples comes with potentials of the form

$$V(x,y) = V_1(x,y) + V_2(x,y)$$

where V_1 has a gap in the essential spectrum and V_2 decreases at both ends of S. Then (multiplication by) V_2 is a relatively compact perturbation of $L + V_1$ which preserves the essential spectrum. If, in addition, condition (1.3) is met by V_1, we can again apply Thm. 1.1.

Example 4.3. We now give an application of Thm. 1.1 to a dislocation problem on the plane $\mathbb{R}^2$. Suppose $V \colon \mathbb{R}^2 \to \mathbb{R}$ is a bounded, measurable function which is periodic with period 1 in the y-variable. Let L denote the (unique) self-adjoint

extension of $-\Delta$ defined on $\mathsf{C}_{\mathrm{c}}^\infty(\mathbb{R}^2)$ and let $H = L + V$. It is well known ([6]) that H has a direct integral decomposition of the form

$$H = \int_{[0,2\pi)}^\oplus H(\vartheta)\frac{\mathrm{d}\vartheta}{2\pi},$$

where $H(\vartheta)$ denotes the self-adjoint realization of $-\Delta + V$ in $\mathsf{L}_2(\mathbb{R} \times (0,1))$ with ϑ-periodic boundary conditions in the y-variable, for $\vartheta \in \mathbb{R}$. The cases $\vartheta \in 2\pi\mathbb{Z}$ correspond to periodic boundary conditions. The spectrum of $\sigma(H(\vartheta))$ depends continuously on ϑ in $[0, 2\pi]$ in the usual way and it follows that the spectrum of H has a band-gap structure. In particular, H has no discrete spectrum. Furthermore, the singular continuous spectrum of H is empty provided the fiber operators $H(\vartheta)$ have empty singular continuous spectrum for all $\vartheta \in [0, 2\pi]$ ([6]).

Let us assume now, in addition, that H has a (non-trivial) spectral gap (a,b), so that, in particular, $\sigma(H(0)) \cap (a,b) = \emptyset$, and that V meets condition (1.3). We obtain V_t from the potential V by (1.1) (with $V^{(1)} = V^{(2)} = V$ and $(x,y) \in \mathbb{R}^2$), and we let $H_t(0)$ denote $H(0) + V_t$. By Thm. 1.1, for any $E \in (a,b)$ there is a sequence of parameters $\tau_k > 0$ such that E is a (discrete) eigenvalue of $H(0) + V_{\tau_k}$. It is our aim to show that $H_{\tau_k} = L + V_{\tau_k}$ has a non-zero contribution to the *surface i.d.s.*, the *surface integrated density of states*, in any interval $(\alpha, \beta) \subset \mathbb{R}$ containing E; recall from [13, 27] that the existence of a surface i.d.s. for H_t inside a spectral gap (a,b) of H_0 is equivalent to the existence of the limits

$$(4.1) \qquad \mu_{\mathrm{surf}}((\alpha,\beta)) = \lim_{n\to\infty} \frac{1}{2n} \mathrm{tr}\, \mathbb{E}_{(\alpha,\beta)}(H_t^{(n)})$$

for all $a < \alpha < \beta < b$, where $H_t^{(n)}$ denotes the operator $-\Delta + V_t$ acting in $\mathsf{L}_2(Q_n)$ with periodic boundary conditions, with $Q_n := (-n,n)^2$. For the sake of comparison, let us also recall that the corresponding formula for the *bulk i.d.s.* comes with a normalization factor $1/(4n^2)$.

Under the assumptions made above the limits in (4.1) may or may not exist, but there are upper and lower bounds for $\frac{1}{2n}\mathrm{tr}\, \mathbb{E}_{(\alpha,\beta)}(H_t^{(n)})$: First, by Thm. 3.2 of [19] we have an upper bound of the form

$$\limsup_{n\to\infty} \frac{1}{n\log n} \mathrm{tr}\, \mathbb{E}_{(\alpha,\beta)}(H_t^{(n)}) < \infty,$$

provided $[\alpha, \beta]$ is contained in a spectral gap of H_0; note that the relevant result in [19] does not require any periodicity of V. As for a lower bound, Thm. 1.1 implies the following result:

Theorem 4.4. *Let $V = V(x,y)\colon \mathbb{R}^2 \to \mathbb{R}$ be periodic in y with period 1 and suppose that $H = -\Delta + V$ has a (non-trivial) spectral gap (a,b). Let us furthermore assume that*

$$\min \sigma_{\mathrm{ess}}(L_{(0,\infty)} + V \upharpoonright S^+) \le a,$$

where $S^+ := (0,\infty) \times (0,1)$.

Then for any interval $\emptyset \ne (\alpha, \beta) \subset [a,b]$ there is a sequence $\tau_k \to \infty$ such that

$$\liminf_{n\to\infty} \frac{1}{n}\mathrm{tr}\, \mathbb{E}_{(\alpha,\beta)}(H_{\tau_k}^{(n)}) > 0, \qquad k \in \mathbb{N}.$$

Thm. 4.4 generalizes [18, Cor. 4.5] in the sense that periodicity in x is no longer required. Dropping also the assumption of periodicity in y appears to be much harder.

Proof of Thm. 4.4. The proof closely follows the line of argument in the proof of Thm. 4.3 and Cor. 4.5 of [18]: For any $E \in (\alpha, \beta)$ given, Thm. 1.1 implies that there is a sequence $(\tau_k) \subset \mathbb{R}$ such that $\tau_k \to \infty$, as $k \to \infty$, and such that E is an eigenvalue of $H(0) + V_{\tau_k} \upharpoonright S$. For $k \in \mathbb{N}$, let $u_{E,\tau_k} \in D(H(0))$ denote an associated eigenfunction. Extending u_{E,τ_k} by periodicity to all of $\mathbb{R}^2$ we obtain functions v_{E,τ_k} which we finally multiply with cut-off functions $\varphi_m(x)\varphi_m(y)$ as in the proof of Thm. 4.3 of [18] to obtain

$$\Psi_{E,\tau_k,m}(x,y) := \varphi_m(x)\varphi_m(y)v_{E,\tau_k}(x,y), \qquad k,m \in \mathbb{N}.$$

Then the functions $\Psi_{E,\tau_k,m}$ belong to the domain of the self-adjoint Laplacian H on $\mathbb{R}^2$ and satisfy

$$\left\| (H + V_{t_k} - E)\Psi_{E,\tau_k,m} \right\| \leq \frac{c}{m} \left\| \Psi_{E,\tau_k,m} \right\|.$$

We may now continue as in the proof of Cor. 4.5 in [18]. $\qquad\square$

Example 4.5 (Half-space problems)**.** Suppose $V^{(2)}$ satisfies the conditions of Theorem 1.1 and let $V^{(1)} = 0$. Then H_t is a model for a semi-infinite nano-tube while the corresponding y-periodic extension to all of $\mathbb{R}^2$ yields a model for a half-crystal, i.e., for a facet of a crystal. Consider some E inside a gap (a_0, b_0). Then Thm. 1.1 states that, for suitable $t = \tau_k$, we have a bound state of H_{τ_k} at E for the half-infinite tube. Furthermore, in the case of a half-crystal, we will have a non-trivial i.d.s. for $t = \tau_k$ near E by Thm. 4.4.

5. Lipschitz continuity of discrete eigenvalues

In this section, we establish continuity of the discrete eigenvalues of dislocation problems and, under an additional assumption on the potential, also (local) Lipschitz continuity, as stated in Thm. 1.2.

As a preparation for the proof of Thm. 1.2 let us first give a detailed account of the coordinate transformation used in Section 2. It will be convenient to reverse the direction of dislocation in order to avoid a host of irritating minus-signs. Furthermore, we restrict our attention to the case where $V^{(1)} = V^{(2)}$ which allows us to write V; the proof in the general case is virtually the same. Hence the dislocation potential V_t for $t \in \mathbb{R}$ is now given by $V_t(x,y) = V(x,y)$, for $x < 0$, and by $V_t(x,y) = V(x-t,y)$, for $x > 0$. For simplicity, we assume again $V \geq 0$. It is our aim to compare the quadratic forms for the dislocation problem in the coordinates (x,y) with the corresponding forms in suitably transformed variables.

We will work in the following well-known setting: Let $\mathcal{H}_1$, $\mathcal{H}_2$ be Hilbert spaces and let $\mathbf{a}_1$ and $\mathbf{a}_2$ be (densely defined) sesquilinear forms with domains $D(\mathbf{a}_1) \subset \mathcal{H}_1$ and $D(\mathbf{a}_2) \subset \mathcal{H}_2$. We say that the forms $\mathbf{a}_1$ and $\mathbf{a}_2$ are *unitarily equivalent* if there exists a unitary map $U \colon \mathcal{H}_1 \to \mathcal{H}_2$ such that $D(\mathbf{a}_2) = UD(\mathbf{a}_1)$ and $\mathbf{a}_2[u,v] = \mathbf{a}_1[U^{-1}u, U^{-1}v]$, for all $u,v \in D(\mathbf{a}_2)$. If, in addition, the forms $\mathbf{a}_1$, $\mathbf{a}_2$

are semi-bounded and closed, then the associated self-adjoint operators H_1 and H_2 are unitarily equivalent: $H_2 = U H_1 U^{-1}$ and $D(H_2) = U D(H_1)$. This is immediate from the first representation theorem for quadratic forms.

We will need a family of simple diffeomorphisms of the real line. Here a particularly efficient construction is to start from piecewise linear (continuous) functions and then to apply a Friedrichs mollifier; this gives sufficient control of the function and its derivatives. For $t \in [-1/2, 1/2]$, let $\tilde{\varphi}_t \colon \mathbb{R} \to \mathbb{R}$ be defined by $\tilde{\varphi}_t(x) = x$ for $x \le 0$, $\tilde{\varphi}_t(x) = (1+t)x$ for $0 \le x \le 1$, and $\tilde{\varphi}_t(x) = x + t$ for $x \ge 1$. We now let

$$\varphi_t := j_1 * \tilde{\varphi}_t, \qquad |t| \le 1/2,$$

where j_η, for $\eta > 0$, is the usual kernel of the Friedrichs mollifier. Also note that $\varphi_0 = \mathrm{id}$. There exists a constant $c \ge 0$ such that

$$(5.1) \qquad |\varphi_t(x) - x| \le c|t|, \quad |\varphi_t'(x) - 1| \le |t|, \quad |\varphi_t''(x)| \le c|t|,$$

for $x \in \mathbb{R}$ and $|t| \le 1/2$. The first and second estimate follow directly from standard properties of the Friedrichs mollifier and the definition of $\tilde{\varphi}_t$. As for the last inequality, we note that the second order distributional derivative of $\tilde{\varphi}_t$ is given by $t\delta_0 - t\delta_1$ with δ_0 and δ_1 denoting the Dirac distributions at the points 0 and 1, respectively, and then $\varphi_t'' = tj_1 - tj_1(\cdot - 1)$. For the remainder of this section we will denote by c various non-negative constants with the understanding that the value of c may change from one line to the next.

We define diffeomorphisms $\Phi_t \colon S \to S$ by setting

$$(\xi, \eta)^T = \Phi_t(x, y) = (\varphi_t(x), y)^T;$$

notice that $\eta = y$. The determinant of the Jacobi matrix of $\Phi_t(x, y)$ is $\varphi_t'(x)$, with $1/2 \le \varphi_t'(x) \le 3/2$, for $|t| \le 1/2$ and $x \in \mathbb{R}$. Then the mapping

$$U_t \colon \mathsf{L}_2(S; \mathrm{d}\xi\mathrm{d}\eta) \to \mathsf{L}_2(S; \varphi_t'(x)\,\mathrm{d}x\mathrm{d}y), \qquad u \mapsto v = U_t u = u \circ \Phi_t,$$

is unitary; furthermore, it is clear that $U_t \mathsf{H}^1(S) = \mathsf{H}^1(S)$. Routine calculations show that, for $|t| \le 1/2$, the form

$$\mathbf{a}_t[u] := \int_S |\nabla u(\xi, \eta)|^2 \mathrm{d}\xi\mathrm{d}\eta + \int_S V_t(\xi, \eta)|u(\xi, \eta)|^2 \mathrm{d}\xi\mathrm{d}\eta,$$

defined on $D(\mathbf{a}_t) := \mathsf{H}^1(S)$ in the Hilbert space $\mathsf{L}_2(S; \mathrm{d}\xi\mathrm{d}\eta)$, is unitarily equivalent to the form

$$\mathbf{b}_t[v] := \int_S \left(\frac{1}{\varphi_t'}|\partial_1 v|^2 + \varphi_t'|\partial_2 v|^2 \right) \mathrm{d}x\mathrm{d}y + \int_S V_t(\Phi_t(x, y))|v(x, y)|^2 \varphi_t' \, \mathrm{d}x\mathrm{d}y$$

on $D(\mathbf{b}_t) = \mathsf{H}^1(S)$ in the Hilbert space $\mathsf{L}_2(S; \varphi_t' \, \mathrm{d}x\mathrm{d}y)$. The mapping

$$\tilde{U}_t \colon \mathsf{L}_2(S; \varphi_t' \, \mathrm{d}x\mathrm{d}y) \to \mathsf{L}_2(S; \mathrm{d}x\mathrm{d}y), \qquad v \mapsto w = \sqrt{\varphi_t'}v,$$

is unitary and $\tilde{U}_t\mathsf{H}^1(S) = \mathsf{H}^1(S)$. From $v = \frac{1}{\sqrt{\varphi_t'}}w$ we get by straight-forward calculations that the form $\mathbf{b}_t$ is unitarily equivalent to the form

$$\mathbf{c}_t[w] := \int_S \left(\frac{1}{(\varphi_t')^2}|\partial_1 w|^2 + |\partial_2 w|^2 - \frac{\varphi_t''}{(\varphi_t')^3}\,\mathrm{Re}\,(\bar{w}\partial_1 w) + \frac{(\varphi_t'')^2}{4(\varphi_t')^4}|w|^2 \right)\mathrm{d}x\mathrm{d}y$$
$$+ \int_S V_t(\Phi_t(x,y))|w|^2\mathrm{d}x\mathrm{d}y,$$

in $\mathsf{L}_2(S)$, again with domain $D(\mathbf{c}_t) = \mathsf{H}^1(S)$.

We wish to apply the perturbation theorem for quadratic forms [22, Thm. VI-3.9] to the family $(\mathbf{c}_t)_{|t|\leq 1/2}$ with $\mathbf{c}_0$ playing the rôle of the unperturbed form. This means that we take

$$\mathbf{d}_t[w] := \mathbf{c}_t[w] - \mathbf{c}_0[w]$$
$$= \int_S \left(\left(\frac{1}{(\varphi_t')^2} - 1 \right)|\partial_1 w|^2 - \frac{\varphi_t''}{(\varphi_t')^3}\,\mathrm{Re}\,(\bar{w}\partial_1 w) + \frac{(\varphi_t'')^2}{4(\varphi_t')^4}|w|^2 \right)\mathrm{d}x\mathrm{d}y$$
$$\tag{5.2} + \int_S \left(V_t(\Phi_t(x,y)) - V(x,y) \right)|w|^2\mathrm{d}x\mathrm{d}y$$

with domain $\mathsf{H}^1(S)$ as a perturbation and check that the assumptions of [22, Thm. VI-3.9] are satisfied. Define C_t as the self-adjoint operator associated with the quadratic form $\mathbf{c}_t$, for $|t| \leq 1/2$. We have the following lemma.

Lemma 5.1. *Let $0 \leq V \in \mathsf{L}_\infty(S)$ and let C_t, for $|t| \leq 1/2$, be as above. We then have:*

(a) $(C_t + 1)^{-1} - (C_0 + 1)^{-1}$ is compact for all $|t| \leq 1/2$.

(b) Let $\zeta \in \varrho(C_0)$. Then there exists $\tau_0 = \tau_0(\zeta) > 0$ such that $\zeta \in \varrho(C_t)$ for $|t| \leq \tau_0$ and $\left\| (C_t - \zeta)^{-1} - (C_0 - \zeta)^{-1} \right\| \to 0$ as $t \to 0$.

(c) Assume, in addition, that $\partial_1 V$ is a signed Borel measure, and let $\zeta \in \varrho(C_0)$. Then there exists $\tau_1 = \tau_1(\zeta) > 0$ such that $\zeta \in \varrho(C_t)$ for $|t| \leq \tau_1$ and

$$\tag{5.3} \left\| (C_t - \zeta)^{-1} - (C_0 - \zeta)^{-1} \right\| \leq c|t|, \qquad 0 \leq |t| \leq \tau_1.$$

The first statement of the lemma implies $\sigma_{\mathrm{ess}}(C_t) = \sigma_{\mathrm{ess}}(C_0)$ which in turn implies $\sigma_{\mathrm{ess}}(H_t) = \sigma_{\mathrm{ess}}(H_0)$ by the unitary equivalence of C_t and H_t.

Proof. (a) By the second resolvent equation and (5.2), we have to study several terms like

$$I_t := (C_t + 1)^{-1}\partial_1(1 - 1/(\varphi_t')^2)\partial_1(C_0 - 1)^{-1},$$

and also the term

$$J_t := (C_t + 1)^{-1}(V_t \circ \Phi_t - V)(C_0 + 1)^{-1}.$$

Notice that the functions $1 - 1/\varphi_t'(x)^2$ and $V_t \circ \Phi_t - V$ have support in $K := [-2, 3] \times \mathbb{S}'$ and are (uniformly) bounded. For all these terms compactness is immediate by the Rellich Compactness Theorem.

(b) With the aid of (5.1) it is easy to see that the first three terms on the right hand side of (5.2) are bounded by $c|t|(\mathbf{c}_0[w] + \|w\|^2)$, for $w \in \mathsf{H}^1(S)$, where $c \geq 0$ is

a suitable constant. We next provide an estimate for $\int_S |V_t \circ \Phi_t - V||w|^2 \mathrm{d}x\mathrm{d}y$; let us write $Y_t := |V_t \circ \Phi_t - V|$ for short.

From a simple Sobolev estimate (which we will prove below in part (d)) one obtains that there is a constant $c \geq 0$ such that $\left\| \chi_K(C_0 + 1)^{-1/2}u \right\|_{\mathsf{L}_3(S)} \leq c\,\|u\|$ for $u \in \mathsf{L}_2(S)$, whence

$$\left\langle Y_t(C_0 + 1)^{-1/2}u, (C_0 + 1)^{-1/2}u \right\rangle = \int_S Y_t|\chi_K(C_0 + 1)^{-1/2}u|^2 \, \mathrm{d}x \, \mathrm{d}y$$

$$\leq \|Y_t\|_{\mathsf{L}_3(S)} \cdot \left\| \chi_K(C_0 + 1)^{-1/2}u \right\|_{\mathsf{L}_3(S)}^2 \leq c\,\|Y_t\|_3 \cdot \|u\|^2 .$$

We have thus shown that

$$\int_S |V_t \circ \Phi_t - V||w|^2 \, \mathrm{d}x \, \mathrm{d}y \leq c\,\|V_t \circ \Phi_t - V\|_{\mathsf{L}_3(S)} \left(\mathbf{c}_0[w] + \|w\|^2\right), \qquad w \in D(\mathbf{c}_0).$$

Combining what we have obtained so far and writing $\gamma(t) := \|V_t \circ \Phi_t - V\|_3$, we now see that

$$(5.4) \qquad |\mathbf{d}_t[w]| \leq c(|t| + \gamma(t))(\mathbf{c}_0[w] + \|w\|^2), \qquad w \in D(\mathbf{c}_0).$$

Here dominated convergence yields $\gamma(t) \to 0$, as $t \to 0$, and the statement (b) now follows from [22, Thm. VI-3.9]. Note that both of the constants a and b in [22, loc. cit.] correspond to the same factor $c(|t| + \gamma(t))$ in (5.4) and that $\left\| C_0(C_0 - \zeta)^{-1} \right\| \leq 1 + \gamma_0|\zeta|$ with $\gamma_0 := \left\| (C_0 - \zeta)^{-1} \right\|$, whence

$$\left\| (a + bC_0)(C_0 - \zeta)^{-1} \right\| \leq c(|t| + \gamma(t))(1 + |\zeta|)\gamma_0.$$

(c) Suppose in addition that $\partial_1 V$ is a signed measure. Let $\zeta \in \mathsf{C}_c^\infty(\mathbb{R})$ satisfy $\zeta(x) = 1$ for $-2 \leq x \leq 3$ and write $\tilde{V} := \zeta V$. We then have $V_t \circ \Phi_t - V = \tilde{V}_t \circ \Phi_t - \tilde{V}$ for $|t| \leq 1/2$. Applying Lemma 5.2 with $W := \tilde{V}$ we find that

$$(5.5) \qquad \|(V_t \circ \Phi_t - V)\|_1 \leq c|t|, \qquad |t| \leq t_0.$$

Since V is bounded and $\mathrm{spt}(V_t \circ \Phi_t - V) \subset K$, this implies $\|V_t \circ \Phi_t - V\|_3 \leq c|t|$, for $|t| \leq 1/2$. Inserting this in (5.4), we obtain

$$|\mathbf{d}_t[u]| \leq ct(\mathbf{c}_0[u] + \|u\|^2), \qquad u \in D(\mathbf{c}_0),$$

and the desired result follows as in part (b) from [22, Thm. VI-3.9].

(d) We finally give the details of the Sobolev estimate used in part (b). Let B denote a disc of radius $1/2$ in S and let $\psi \in \mathsf{C}_c^\infty(B)$. For any $v \in \mathsf{H}^1(S)$, we have $\psi v \in \mathsf{W}_p^1(B)$, for any $1 \leq p \leq 2$; furthermore, there are constants $c_p \geq 0$ such that

$$\|\psi v\|_{\mathsf{W}_p^1(B)} \leq c_p \|v\|_{\mathsf{H}^1(B)}, \qquad v \in \mathsf{H}^1(S).$$

The specific choice $p := 6/5$ and the Gagliardo-Nirenberg-Sobolev inequality ([14]) yields an estimate $\|\psi v\|_{\mathsf{L}_3(B)} \leq c\,\|v\|_{\mathsf{H}^1(B)}$, for all $v \in \mathsf{H}^1(S)$. Using a covering of K by a finite number of discs B_i of radius $1/2$ and a suitable partition of unity $\{\psi_i\}$ subordinate to this covering, we easily obtain an estimate of the form

$$\|v\|_{\mathsf{L}_3(K)} \leq C\,\|v\|_{\mathsf{H}^1(S)}, \qquad v \in \mathsf{H}^1(S),$$

with a constant $C \geq 0$. $\qquad \qquad \qquad \qquad \square$

Before passing to the proof of Thm. 1.2 let us give some details on what the notion of continuity of eigenvalues is supposed to mean (cf. also the appendix in [18]). Suppose $a_0 < b_0 \in \mathbb{R}$ are such that $a_0, b_0 \in \sigma_{\mathrm{ess}}(H_0)$ and $(a_0, b_0) \cap \sigma_{\mathrm{ess}}(H_0) = \emptyset$. Then the discrete eigenvalues of H_t in (a_0, b_0) can be described by a countable family (f_i) of continuous functions $f_i \colon (\alpha_i, \beta_i) \to (a_0, b_0)$ (where $\alpha_i < \beta_i$ and $\alpha_i \in \mathbb{R} \cup \{-\infty\}$, $\beta_i \in \mathbb{R} \cup \{+\infty\}$) such that any compact set $K_{T,a,b} := [-T, T] \times [a, b] \subset \mathbb{R} \times (a_0, b_0)$ meets only finitely many of the graphs $\Gamma(f_i)$. Notice that we may have to deal with multiple eigenvalues. We may agree that each of the functions f_i carries precisely spectral multiplicity 1, so some of the functions might be identical, or their graphs may overlap or intersect. We do not discuss continuity of the associated eigenprojections here. Also note that, in general, the description of the eigenvalues by the family of functions (f_i) will not be unique.

More precisely, if $E \in (a_0, b_0)$ and $t \in \mathbb{R}$ are such that $E \in \sigma(H_t)$, then there exists some $i \in \mathbb{N}$ with $f_i(t) = E$. Conversely, if $\alpha_j < t < \beta_j$ for some j, then $f_j(t)$ is an eigenvalue of H_t. In addition, the graphs $\Gamma(f_i)$ *leave any of the sets* $K_{T,a,b}$ *as* $t \downarrow \alpha_i$ *and* $t \uparrow \beta_i$ in the following sense: if, e.g., $-T \leq \alpha_i < T$ for some i, then there exists some $\alpha_i' \in (\alpha_i, T)$ such that $f_i(t) \notin [a, b]$ for $\alpha_i < t < \alpha_i'$ and $f_i(\alpha_i') = a$ or $f_i(\alpha_i') = b$. Similar statements hold for the case where $\alpha_i < -T$ and for $\beta_i \in (-T, T]$ or $\beta_i > T$. It follows, in particular, that $f_i(t) \to a_0$ or $f_i(t) \to b_0$ as $t \downarrow \alpha_i$, if α_i is finite, and similarly for $t \uparrow \beta_i$.

Proof of Thm. 1.2. Since H_t and C_t are unitarily equivalent for $|t| \leq 1/2$, it is enough to prove the corresponding statements for the operators C_t. It is obvious that this implies the statements for all $t \in \mathbb{R}$ since, in view of Remark 5.5, we may replace V with any V_t; indeed, if $\partial_1 V$ is (locally) a signed measure, the same is true for V_t.

(a) By the preceding Lemma 5.1, the resolvents $(C_t + 1)^{-1}$ depend continuously in norm on t. Suppose E_0 is a discrete eigenvalue of C_{t_0} for some $t_0 \in -[1/4, 1/4]$. Then there are $\varepsilon_0 > 0$ and $\tau_0 > 0$ such that E_0 is the only eigenvalue of C_{t_0} in $(E_0 - 3\varepsilon_0, E_0 + 3\varepsilon_0)$ and such that C_t has no spectrum in $(E_0 - 3\varepsilon_0, E_0 - \varepsilon_0) \cup (E_0 + \varepsilon_0, E_0 + 3\varepsilon_0)$ for $|t - t_0| \leq \tau_0$. It follows by well-known results [33, Thm. VIII-20] that $\mathrm{tr}\, \mathbb{E}_{(E_0 - 2\varepsilon_0, E_0 + 2\varepsilon_0)}$ is constant for $|t - t_0| \leq \tau_0$. Consider $\zeta' := E_0 - 2\varepsilon_0$. Then min-max ([35, Exercise XIII.2]) implies that the maximum value of the spectrum of $(C_t - \zeta')^{-1}$ is a continuous eigenvalue branch, for $|t - t_0| \leq \tau_0$. This branch corresponds to a continuous eigenvalue branch of C_t which passes through the point (t_0, E_0) and has the largest value among all branches passing through (t_0, E_0). If the eigenvalue E_0 has multiplicity greater than 1, the above method will yield a corresponding number of continuous eigenvalue branches below (or equal) to the first branch.

This local description of the eigenvalues of H_t in terms of continuous functions can be patched together by means of simple compactness arguments to yield a "global" picture; cf. also the appendix in [18]. Indeed, if $a_n \downarrow a_0$, $b_n \uparrow b_0$, and $T_n \to \infty$ monotonically (with $a_0 < a_n < b_n < b_0$), we let $K_n := K_{T_n, a_n, b_n}$ so that $K_n \subset K_{n+1}$ and $\cup_n K_n = \mathbb{R} \times (a_0, b_0)$. Then for each pair $(t_0, E_0) \in K_n$ with the

property that $E_0 \in \sigma(C_{t_0})$ we have an open neighborhood

$$U_{t_0,E_0;\tau_0,\varepsilon_0} := (t_0 - \tau_0, t_0 + \tau_0) \times (E_0 - 3\varepsilon_0, E_0 + 3\varepsilon_0)$$

with the properties obtained above; here τ_0 and ε_0 are positive numbers.

If (t_0, E_0) is such that $E_0 \notin \sigma(C_{t_0})$ there is an open neighborhood $\tilde{U}_{t_0,E_0;\tau_0,\varepsilon_0}$ with the property that C_t has no spectrum in $(E_0 - \varepsilon_0, E_0 + \varepsilon_0)$ for $|t| \leq \tau_0$. By compactness, a finite selection of the sets $U_{t_0,E_0;\tau_0,\varepsilon_0}$ and $\tilde{U}_{t_0,E_0;\tau_0,\varepsilon_0}$ covers K_n, etc.

(b) We will show that the functions $f_i \colon (\alpha_i, \beta_i) \to \mathbb{R}$ introduced above are locally (uniformly) Lipschitz, i.e., for any compact interval $[\tilde{\alpha}, \tilde{\beta}] \subset (\alpha_i, \beta_i)$ there exists a constant $c \geq 0$ such that $|f_i(t) - f_i(t')| \leq c|t - t'|$, for all $t, t' \in [\tilde{\alpha}, \tilde{\beta}]$. We continue in the same setting as in part (a) of this proof. Let E_0 be a discrete eigenvalue of C_{t_0} for some $|t_0| < 1/4$, and let $\tau_0 > 0$, $\varepsilon_0 > 0$, and ζ' as above. Replacing V with V_t in (5.3) (and making also use of Remark 5.5) we obtain an estimate analogous to (5.3) for the pair (t, t') in place of the pair $(0, t)$, viz.,

$$|\mathbf{c}_t[u] - \mathbf{c}_{t'}[u]| \leq c|t - t'|(\mathbf{c}_t[u] + \|u\|^2), \qquad u \in D(\mathbf{c}_0),$$

with a constant $c \geq 0$ which can be chosen independently of $t, t' \in (t_0 - \tau_0, t_0 + \tau_0)$. Evoking again [22, Thm. VI-3.9] we obtain a resolvent estimate

$$\left\| (C_t - \zeta')^{-1} - (C_{t'} - \zeta')^{-1} \right\| \leq \frac{c}{\varepsilon_0^2} |t - t'|, \qquad t, t' \in (t_0 - \tau_0, t_0 + \tau_0),$$

where $c \geq 0$ is a constant, and min-max as in [35, Exercise XIII.2] gives the desired result. $\qquad\square$

We finally discuss how to establish the crucial estimate (5.5).

Lemma 5.2. *Let $\alpha \in \mathsf{C}^1(\mathbb{R}, \mathbb{R})$ with $\|\alpha\|_\infty < \infty$ and $\|\alpha'\|_\infty \leq 1/2$ be given, and let $\varphi \colon \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $\varphi(x_1, x_2) = (x_1 + \alpha(x_1), x_2)$. Let $W \in \mathsf{L}_1(\mathbb{R}^2)$, and assume that the distributional derivative $\partial_1 W$ is a (signed) measure μ of finite total variation $\|\mu\|_1$.*

Then $\|W \circ \varphi - W\|_1 \leq 2\|\mu\|_1 \|\alpha\|_\infty$.

Proof. (1) In a first step, we show the assertion under the additional hypothesis that $W \in \mathsf{C}^1(\mathbb{R}^2)$ and $\partial_1 W \in \mathsf{L}_1(\mathbb{R}^2)$. From

$$\frac{\mathrm{d}}{\mathrm{d}s} W(x_1 + s\alpha(x_1), x_2) = \partial_1 W(x_1 + s\alpha(x_1), x_2)\alpha(x_1)$$

we then obtain

$$\|W \circ \varphi - W\|_1 = \int \left| \int_0^1 \partial_1 W(x_1 + s\alpha(x_1), x_2)\, \mathrm{d}s\, \alpha(x_1) \right| \mathrm{d}x$$

$$(5.6) \qquad\qquad \leq \|\alpha\|_\infty \int_0^1 \|\partial_1 W \circ \varphi_s\|_1\, \mathrm{d}s$$

with $\varphi_s \colon \mathbb{R}^2 \to \mathbb{R}^2$, $\varphi_s(x_1, x_2) = (x_1 + s\alpha(x_1), x_2)$. From $1 + s\alpha'(x_1) \geq 1/2$ for all $s \in [0, 1]$, $x_1 \in \mathbb{R}$, one obtains that φ_s has an inverse ψ_s with the property that $|\det \psi_s'| \leq 2$. Using the substitution $x = \psi_s(y)$, $\mathrm{d}x = |\det \psi_s'(y)|\, \mathrm{d}y$ one obtains

$$\int |\partial_1 W(\varphi_s(x))|\, \mathrm{d}x = \int |\partial_1 W(y)||\det \psi_s'(y)|\, \mathrm{d}y \leq 2\|\partial_1 W\|_1.$$

Inserting this inequality into (5.6) one obtains the assertion for the present special case.

(2) In order to prove the general case, let $(\rho_k) \subset \mathsf{C}_c^\infty(\mathbb{R}^2)$ be a δ-sequence. It is then standard to show that $\rho_k * W \to W$ in $\mathsf{L}_1(\mathbb{R}^2)$, $\rho_k * W \in \mathsf{C}^1(\mathbb{R}^2)$ and $\partial_1(\rho_k * W) \in \mathsf{L}_1(\mathbb{R}^2)$, where $\partial_1(\rho_k * W)(x) = \int \rho_k(x - y) \, d\mu(y)$ and $\|\partial_1(\rho_k * W)\|_1 \leq \|\mu\|_1$. Using the case treated in (1) we conclude that

$$\|(\rho_k * W) \circ \varphi - \rho_k * W\|_1 \leq 2 \|\partial_1(\rho_k * W)\|_1 \|\alpha\|_\infty$$
$$\leq 2 \|\mu\|_1 \|\alpha\|_\infty$$

and for $k \to \infty$ we obtain the assertion. $\qquad\square$

Remark 5.3. If $W \in \mathsf{L}_1(S)$ and the distributional derivative $\partial_1 W$ is a measure μ (on S) of finite total variation, the same inequality as in Lemma 5.2 holds.

In fact, our assumptions on the potential V are almost optimal, which follows from the next lemma.

Lemma 5.4. Let $f \in \mathsf{L}_1(\mathbb{R}^n)$ and $C \geq 0$.

Then the following properties are equivalent:

(1) The mapping $\mathbb{R} \to \mathsf{L}_1(\mathbb{R}^n)$, $t \mapsto f(\cdot - te_1)$ is Lipschitz continuous with Lipschitz constant C.
(2) The distributional derivative $\partial_1 f$ is a (signed) Borel-measure of finite total variation $\leq C$.

Proof. (1) $\Longrightarrow$ (2) : Let $\varphi \in \mathsf{C}_c^\infty(\mathbb{R}^n)$. Then

$$\partial_1 f(\varphi) = -\int f(x)\partial_1\varphi(x)\,dx$$
$$= -\lim_{t \to 0} \int f(x)\frac{1}{t}(\varphi(x + te_1) - \varphi(x))\,dx$$
$$= -\lim_{t \to 0} \frac{1}{t} \int (f(x - te_1) - f(x))\varphi(x)\,dx.$$

Using $\|f(\cdot - te_1) - f\|_1 \leq Ct$ we obtain

$$|\partial_1 f(\varphi)| \leq C \|\varphi\|_\infty.$$

Now the Riesz-Markov Theorem implies the assertion.

(2) $\Longrightarrow$ (1) : For $t > 0$, $\varphi \in \mathsf{C}_c^\infty(\mathbb{R}^n)$ we compute, using Fubini's theorem,

$$\int (f(x - te_1) - f(x))\varphi(x)\,dx = \int f(x)(\varphi(x + te_1) - \varphi(x))\,dx$$
$$= \int_0^t \int f(x)\partial_1\varphi(x + se_1)\,dx\,ds$$
$$= -\int_0^t \partial_1 f(\varphi(\cdot + se_1))\,ds.$$

This implies

$$\left| \int (f(x - te_1) - f(x))\varphi(x)\,dx \right| \leq t \|\partial_1 f\|_{\text{var}} \|\varphi\|_\infty$$

and therefore

$$\|f(\cdot - te_1) - f\|_1 = \sup\left\{\left|\int (f(x - te_1) - f(x))\varphi(x)\,dx\right|\,;\,\varphi \in C_c^\infty(\mathbb{R}^n),\,\|\varphi\|_\infty \le 1\right\}$$
$$\le t\,\|\partial_1 f\|_{var}$$
$$\le Ct.$$

$\square$

Remark 5.5. Let $W \in L_1(S) \cap L_\infty(S)$ satisfy the assumptions on f in Lemma 5.4 and let $v \in L_\infty(S)$ satisfy the condition

$$\|v(. - te_1) - v\|_1 \le c|t|, \qquad t \in \mathbb{R}.$$

Then the product vW enjoys the same properties as W. Indeed,

$$\|(vW)(. - te_1) - vW\|_1 \le \|W\|_\infty\,\|v(. - te_1) - v\|_1 + \|v\|_\infty\,\|W(. - te_1) - W\|_1$$
$$\le C|t|,$$

and the assertion follows from Lemma 5.4.

On $\mathbb{S}'$, characteristic functions χ of the form $\chi = \chi_{(a,\infty)\times\mathbb{S}'}$ satisfy the conditions on v in this remark. We therefore see, in particular, that $\partial_1(V_t)$ is (locally) a signed measure (for any $t \in \mathbb{R}$) if $\partial_1 V$ has this property.

6. Appendix: Hilbert-Schmidt properties

In the following, we prove Lemma 2.1 and Lemma 2.2.

(1) We start from the free resolvent of H_0, the Laplacian in $\mathbb{R}^2$; note that we now write $x = (x_1, x_2) \in \mathbb{R}^2$ etc. It is well-known that $(H_0 + 1)^{-1}$ is an integral operator with kernel given by

$$G_0(x, y) := (H_0 + 1)^{-1}(x, y) = \mathbf{K}_0(|x - y|), \qquad x, y \in \mathbb{R}^2,$$

where $\mathbf{K}_0 \in C^\infty(0, \infty)$ is the modified Hankel function of order 0 ([28, p. 127], [36, Éqn. VII-10.15, p. 286], [29]; also [34, p. 128, Exercise 49]). $\mathbf{K}_0$ is a smooth, monotonically decreasing function. The asymptotic behavior of $\mathbf{K}_0$ for $r \downarrow 0$ and for $r \to \infty$ as in [28, Par. 28, Lemma 8 and Thm. 3] gives an estimate

$$(6.1) \qquad \mathbf{K}_0(r) \le \begin{cases} c\,(1 + |\log r|), & 0 < r \le 1, \\ c\,e^{-r}, & r \ge 1, \end{cases}$$

for some constant $c \ge 0$.

(2) Let $S' := \mathbb{R} \times (0, 1)$ and let $H_{S'}$ denote the Laplacian of S' with periodic boundary conditions in the x_2-variable. $H_{S'}$ is non-negative and self-adjoint. The operator $(H_{S'}+1)^{-1}$ has an integral kernel $G_{S'} = G_{S'}(x, y)$ which can be computed in terms of $\mathbf{K}_0$ by the classical method of image charges

$$G_{S'}(x, y) = \sum_{k\in\mathbb{Z}} G_0(x + ke_2, y) = \sum_{k\in\mathbb{Z}} \mathbf{K}_0(|x + ke_2 - y|), \qquad x, y \in S'.$$

Convergence of the sum in (2) follows from the exponential decay of $\mathbf{K}_0$. Also note that $G_{S'} \ge 0$.

(3) We now introduce an additional Dirichlet boundary condition on the line segment $\ell := \{(x_1, x_2) \in \mathbb{R}^2; x_1 = 0, 0 < x_2 < 1\}$ which decouples the strip S' into a left part S'_- and a right part S'_+. Let $H_{S'_\pm}$ denote the Laplacian of $S'_\pm$ with Dirichlet boundary condition on ℓ and periodic boundary conditions with respect to the x_2-variable. Let $H_{S',\mathrm{dec}} = H_{S'_-} \oplus H_{S'_+}$, and denote the associated resolvent kernels by $G_{S'_\pm}$ and $G_{S',\mathrm{dec}}$, respectively. Path integral methods imply that

$$0 \leq G_{S',\mathrm{dec}}(x,y) \leq G_{S'}(x,y), \qquad x,y \in S'_+ \text{ or } x,y \in S'_-;$$

cf. [9] or [7, Thm. 2.1.6]. In order to apply the method of image charges, we write $x^* := (-x_1, x_2)$ for $x = (x_1, x_2) \in S'$. We then have

$$G_{S'_+}(x,y) = G_{S'}(x,y) - G_{S'}(x^*, y), \qquad x,y \in S'_+,$$

and similarly for S'_-, and we see that

$$0 \leq G_{S'}(x,y) - G_{S'_\pm}(x,y) = G_{S'}(x^*, y) = \sum_{k \in \mathbb{Z}} \mathbf{K}_0(|x^* + k\mathbf{e}_2 - y|),$$

for $x, y \in S'_+$ and for $x, y \in S'_-$.

If $x \in S'_\pm$ and $y \in S'_\mp$ we extend $G_{S',\mathrm{dec}}$ by zero and have

$$G_{S'}(x,y) - G_{S',\mathrm{dec}}(x,y) = G_{S'}(x,y) = \sum_{k \in \mathbb{Z}} \mathbf{K}_0(|x + k\mathbf{e}_2 - y|),$$

for $(x,y) \in S'_+ \times S'_- \cup S'_- \times S'_+$.

(4) Writing $K(x,y) := G_{S'}(x,y) - G_{S'_-}(x,y) - G_{S'_+}(x,y)$ we now show that $K \in \mathsf{L}_2(S' \times S')$, using the estimate of (6.1). Let us first consider the case where $x, y \in S'_+$. Then

$$|x^* + k\mathbf{e}_2 - y| = \sqrt{(x_1 + y_1)^2 + (k + x_2 - y_2)^2}, \qquad k \in \mathbb{Z},$$

so that by the monotonicity of $\mathbf{K}_0$, (6.1), and $-1 < x_2 - y_2 < 1$,

$$K(x,y) \leq 3\mathbf{K}_0(\sqrt{(x_1+y_1)^2}) + 2\sum_{k \in \mathbb{N}} \mathbf{K}_0(\sqrt{(x_1+y_1)^2 + k^2}).$$

Here we can estimate

$$\mathbf{K}_0(x_1 + y_1) \leq \begin{cases} c\,(1 + |\log(x_1 + y_1)|), & 0 < x_1 + y_1 \leq 1, \\ c\,\mathrm{e}^{-x_1-y_1}, & x_1 + y_1 \geq 1. \end{cases}$$

For $k \in \mathbb{N}$ we have $\sqrt{(x_1 + y_1)^2 + k^2} \geq 1$ for all $x_1, y_1 \geq 0$ and thus

$$\mathbf{K}_0(\sqrt{(x_1+y_1)^2 + k^2}) \leq c\,\mathrm{e}^{-\sqrt{(x_1+y_1)^2 + k^2}}.$$

It is now easy to see that $K \upharpoonright (S'_+ \times S'_+)$ is square integrable: indeed, it is easy to estimate the contribution coming from the region $0 < x_1 + y_1 \leq 1$ (note that only the case $k = 0$ comes with a logarithmic contribution). As for the region $x_1 + y_1 \geq 1$, we may use the elementary estimate

$$\int_0^\infty \int_0^\infty \mathrm{e}^{-2\sqrt{(x_1+y_1)^2 + k^2}}\, \mathrm{d}x_1\, \mathrm{d}y_1 \leq \int_0^k t\mathrm{e}^{-2k}dt + \int_k^\infty t\mathrm{e}^{-2t}\, \mathrm{d}t$$

$$\leq \frac{5}{4}k^2\mathrm{e}^{-2k}, \qquad k \in \mathbb{N}.$$

The other cases can be treated in a similar fashion.

(5) In this step, we add in a (bounded) potential $W \geq 0$. Here the Trotter product formula can be applied as in [41, Prop. 1.3(b)] to yield an inequality for the semigroups. Pointwise information for the integral kernels is also obtained directly as in the proof of Thm. 7 in [9] via the Feynman-Kac formula which implies that

$$0 \leq \mathrm{e}^{-t(H_{S'}+W)}(x,y) - \mathrm{e}^{-t(H_{S',\mathrm{dec}}+W)}(x,y) \leq \mathrm{e}^{-tH_{S'}}(x,y) - \mathrm{e}^{-tH_{S',\mathrm{dec}}}(x,y),$$

for all $x, y \in S'$, and then the corresponding inequality holds for the resolvent kernels, i.e.,

$$0 \leq (H_{S'} + W + 1)^{-1}(x,y) - (H_{S',\mathrm{dec}} + W + 1)^{-1}(x,y)$$
$$\leq (H_{S'} + 1)^{-1}(x,y) - (H_{S',\mathrm{dec}} + 1)^{-1}(x,y),$$

which is in $\mathsf{L}_2(S' \times S')$ by step (4). We thus see that $(H_{S'} + W + 1)^{-1} - (H_{S',\mathrm{dec}} + W + 1)^{-1}$ is Hilbert-Schmidt; furthermore, there is a bound on the Hilbert-Schmidt norm of $(H_{S'} + W + 1)^{-1} - (H_{S',\mathrm{dec}} + W + 1)^{-1}$ which is independent of $W \geq 0$. This proves Lemma 2.1 for $r = 1$ and $W \geq 0$.

(6) If we consider $(-n, n) \times \mathbb{S}'$ instead of $\mathbb{R} \times \mathbb{S}'$, we may again use path integral methods to show that

$$0 \leq \mathrm{e}^{-t(L_{(-n,n)}+W)}(x,y) - \mathrm{e}^{-t(L_{(-n,0)}\oplus L_{(0,n)}+W)}(x,y) \leq \mathrm{e}^{-tH_S}(x,y) - \mathrm{e}^{-tH_{S,\mathrm{dec}}}(x,y),$$

so that $(L_{(-n,n)} + W + 1)^{-1} - (L_{(-n,0)} \oplus L_{(0,n)} + W + 1)^{-1}$ is Hilbert-Schmidt with a bound on the Hilbert-Schmidt norm that is independent of n and $W \geq 0$. This proves Lemma 2.2 for $r = 1$.

6.1. **Proof of Lemma 2.4.** (1) As in Lemma 3.1 of [5], let $f \colon \mathbb{R} \to \mathbb{R}$ be a bounded, monotonically increasing function of class C^2 with $|f'(x)| \leq m$ and $|f''(x)| \leq m$ for some constant m. We write $H := L + W$ with arbitrary W as in Lemma 2.4. Then

$$\mathrm{e}^{-f} H \mathrm{e}^{f} = H - 2f'\partial_1 - f'' - |f'|^2.$$

We first observe that for any $\eta > 0$

$$\|\nabla u\|^2 \leq \langle Hu, u \rangle \leq \frac{\eta}{2} \|Hu\|^2 + \frac{1}{2\eta} \|u\|^2, \qquad u \in D(L).$$

Thus the term $f'\partial_1$ is relatively bounded with respect to H with relative bound zero; the constants in the estimate can be chosen to be independent of the potentials W under consideration. Write $H_f := \mathrm{e}^{-f} H \mathrm{e}^{f}$ with $D(H_f) = D(H)$. As in [5, Lemma 3.1] there exists an $\varepsilon_0 > 0$ (which is independent of W) such that $H_{\varepsilon f} - \lambda$ is invertible for all $|\varepsilon| \leq \varepsilon_0$ and all $\lambda \in [a, b]$ with a bound

$$\left\|(H_{\varepsilon f} - \lambda)^{-1}\right\| \leq c_0, \qquad \lambda \in [a, b], \quad |\varepsilon| \leq \varepsilon_0.$$

Furthermore, $(H_{\varepsilon f} - \lambda)^{-1} = \mathrm{e}^{-\varepsilon f}(H - \lambda)^{-1}\mathrm{e}^{\varepsilon f}$.

(2) With ξ_{2k}^+ and ξ_{2k}^- denoting the characteristic functions of the sets $(2k, \infty) \times \mathbb{S}'$ and $(-\infty, -2k) \times \mathbb{S}'$, respectively, we have $1 - \chi_{2k} = \xi_{2k}^+ + \xi_{2k}^-$. We then have

$$\left\|\xi_{2k}^+(H - \lambda)^{-1}\chi_k\right\| = \left\|\xi_{2k}^+\mathrm{e}^{\varepsilon f}\mathrm{e}^{-\varepsilon f}(H - \lambda)^{-1}\mathrm{e}^{\varepsilon f}\mathrm{e}^{-\varepsilon f}\chi_k\right\|$$
$$\leq \left\|\xi_{2k}^+\mathrm{e}^{\varepsilon f}\right\| \left\|\mathrm{e}^{-\varepsilon f}(H - \lambda)^{-1}\mathrm{e}^{\varepsilon f}\right\| \left\|\mathrm{e}^{-\varepsilon f}\chi_k\right\|,$$

with $\left\| \xi_{2k}^+ \mathrm{e}^{\varepsilon f} \right\|$ denoting the norm of the operator of multiplication by the function $\xi_{2k}^+ \mathrm{e}^{\varepsilon f}$. We now specifically pick f such that $f(x) = -x$ for $|x| \leq 2k$ and we let $\varepsilon = \varepsilon_0$. Then $\left\| \mathrm{e}^{-\varepsilon f}(H-\lambda)^{-1}\mathrm{e}^{\varepsilon f} \right\| \leq c_0$ by part (1), and $\left\| \mathrm{e}^{-\varepsilon f}\chi_k \right\| \leq \mathrm{e}^{\varepsilon_0 k}$, while $\left\| \xi_{2k}^+ \mathrm{e}^{\varepsilon f} \right\| \leq \mathrm{e}^{-2k\varepsilon}$. Dealing with $\xi_{2k}^-(H-\lambda)^{-1}\chi_k$ in a similar fashion, we obtain that

$$(6.2) \qquad \left\| (1-\chi_{2k})(H-\lambda)^{-1}\chi_k \right\| \leq c_1 \mathrm{e}^{-\varepsilon_0 k}.$$

(3) We now turn to an eigenfunction u of $L_{\mathbb{R}\setminus\{0\}} + W$ associated with an eigenvalue $\lambda \in [a,b]$. Let φ_k and $\psi_k = 1 - \varphi_k$ be as in Section 2. Then $\psi_k u \in D(H)$ with

$$(H-\lambda)(\psi_k u) = (L_{\mathbb{R}\setminus\{0\}} + W - \lambda)(\psi_k u) = -2\psi_k' \partial_1 u - \psi_k'' u =: \Phi_k,$$

whence, using $\mathrm{spt}(\Phi_k) \subset \{k/4 \leq |x| \leq k/2\}$ and $\|\Phi_k\| \leq (c/k)\|u\|$, we get

$$(1-\chi_k)u = (1-\chi_k)(H-\lambda)^{-1}\chi_{k/2}\Phi_k$$

and the desired estimate follows from (6.2).

6.2. Proof of Lemma 2.5.

If the statement were not true we could find a sequence of bounded, non-negative potentials W_n such that $L + W_n$ has no spectrum in the interval (a_0, b_0) while $L_{\mathbb{R}\setminus\{0\}} + W_n$ has at least n eigenvalues in $[a,b]$ (counting multiplicities). Then, for each n, there exists an ONS of eigenfunctions $u_{n,j}$, $j = 1,\ldots,n$, associated with eigenvalues of $L_{\mathbb{R}\setminus\{0\}} + W_n$ in $[a,b]$. With φ_k and $\psi_k = 1 - \varphi_k$ as in Section 3 we let $\mathcal{M}_{n,k}$ denote the subspace spanned by the functions $\varphi_{4k} u_{n,j}$, $j = 1,\ldots,n$. Let C and γ denote the constants in Lemma 2.4. We then have the following:

Claim. For $k = k_n := \lceil (\log n)/\gamma \rceil$ and n large, $\mathcal{M}_{n,k}$ has dimension n and there exists a constant $M \geq 0$ such that

$$\|\nabla u\|^2 \leq M \|u\|^2, \qquad u \in \mathcal{M}_{n,k}.$$

By min-max, it follows from this claim that $\mathrm{tr}\, \mathbb{E}_{(-\infty,M]}(L_{(-2k_n, 2k_n)}) \geq n$ for n large, in contradiction to Weyl's Law by which $\mathrm{tr}\, \mathbb{E}_{(-\infty,M]}(L_{(-2k_n, 2k_n)})$ is bounded by $c \log n$.

Let us now prove the claim: any $u \in \mathcal{M}_{n,k}$ can be written as $u = \varphi_{4k}\sum_{j=1}^n \alpha_j u_{n,j}$ with suitable $\alpha_j \in \mathbb{C}$. Here we first note that $v := \sum_j \alpha_j u_{n,j}$ has norm $\|v\| = (\sum_j |\alpha_j|^2)^{1/2}$ and satisfies $\|\nabla v\|^2 \leq b\|v\|^2$ since the functions $u_{n,j}$, $j = 1,\ldots,n$, form an ONS of eigenfunctions of $L + W_n$ with eigenvalues $\leq b$. We next observe that, by Lemma 2.4,

$$\|\psi_{4k} u_{n,j}\| \leq \|(1-\chi_k)u_{n,j}\| \leq C\mathrm{e}^{-\gamma k} \leq C/n,$$

for $j = 1,\ldots,n$, with χ_k denoting the characteristic function of the set $(-k,k) \times \mathbb{S}'$. Using the Schwarz inequality we get

$$\|(1-\chi_k)v\| \leq \left(\sum_j |\alpha_j|^2\right)^{1/2} \cdot \left(\sum_j \|(1-\chi_k)u_{n,j}\|^2\right)^{1/2} \leq \|v\|\,(nC^2/n^2)^{1/2} = \frac{C}{\sqrt{n}}\|v\|,$$

so that also $\|\psi_{4k}v\| \leq \frac{C}{\sqrt{n}}\|v\|$. We then find for n large,

$$\|u\| = \|\varphi_{4k}v\| = \|v - \psi_{4k}v\| \geq \|v\| - \|\psi_{4k}v\| \geq (1 - \frac{C}{\sqrt{n}})\|v\| \geq \frac{1}{2}\|v\|.$$

In particular, the functions $\psi_{4k}u_{n,j}$, $j = 1,\ldots,n$ are linearly independent for n large. Furthermore,

$$\|\nabla u\| = \|\nabla(\varphi_{4k}v)\| \leq \|\nabla\varphi_{4k}\|_\infty \|(1-\chi_k)v\| + \|\nabla v\| \leq \frac{c}{kn} + \sqrt{b}\,\|v\|$$

so that $\|\nabla u\|^2 \leq b_0 \|v\|^2$ for n large. In view of the inequality $\|u\| \geq \frac{1}{2}\|v\|$, obtained above, we have therefore shown that, for n large and $k = k_n$, $\|\nabla u\|^2 \leq 2b_0\|u\|^2$ for all $u \in \mathcal{M}_{n,k}$.

References

[1] S Alama, PA Deift, and R Hempel, *Eigenvalue branches of the Schrödinger operator $H - \lambda W$ in a gap of $\sigma(H)$*, Commun. Math. Phys. **121** (1989), 291–321

[2] A Avila, J Bochi, and D Damanik, *Opening gaps in the spectrum of strictly ergodic Schrödinger operators*, J. Eur. Math. Soc. (JEMS) **14** (2012), 61–106

[3] M Baake, M Birkner, and RV Moody, *Diffraction of stochastic point sets: Explicitly computable examples*, Commun. Math. Phys. **293** (2010), 611–660

[4] M Baake and U Grimm, *Surprises in aperiodic diffraction*, J. Phys.: Conf. Ser. **226** (2010), 012023 (9 pp.)

[5] M van den Berg, R Hempel, and J Voigt, *L_1-estimates for eigenfunctions of the Dirichlet Laplacian*, J. Spectral Theory, to appear

[6] EB Davies and B Simon, *Scattering theory for systems with different spatial asymptotics on the left and right*, Commun. Math. Phys. **63** (1978), 277–301

[7] EB Davies, *Heat Kernels and Spectral Theory*. Cambridge Univ. Press, Cambridge 1989

[8] PA Deift and R Hempel, *On the existence of eigenvalues of the Schrödinger operator $H - \lambda W$ in a gap of $\sigma(H)$*, Commun. Math. Phys. **103** (1986), 461–490

[9] P Deift and B Simon, *On the Decoupling of Finite Singularities from the Question of Asymptotic Completeness in Two Body Quantum Systems*, J. Funct. Anal. **23** (1976), 218–238.

[10] T Dohnal, M Plum, and W Reichel, *Localized modes of the linear periodic Schrödinger operator with a nonlocal perturbation*, SIAM J. Math. Anal. **41** (2009), 1967–1993

[11] ———, *Surface gap soliton ground states for the nonlinear Schrödinger equation*, Commun. Math. Phys. **308** (2011), 511–542

[12] MSP Eastham, *The Spectral Theory of Periodic Differential Equations*. Scottish Academic Press, Edinburgh-London, 1973

[13] H Englisch, W Kirsch, M Schröder, and B Simon, *Random Hamiltonians ergodic in all but one direction*, Commun. Math. Phys. **128** (1990), 613–625

[14] LC Evans, *Partial Differential Equations*. Amer. Math. Soc. 1998.

[15] LC Evans and R Gariépy, *Measure Theory and Fine Properties of Functions*. Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1992

[16] M Goldstein and W Schlag, *On resonances and the formation of gaps in the spectrum of quasi-periodic Schrödinger equations*, Ann. Math. **173** (2011), 337–475

[17] R Hempel, *Eigenvalues in gaps and decoupling by Neumann boundary conditions*, J. Math. Anal. Appl. **169** (1992), 229–259

[18] R Hempel and M Kohlmann, *A variational approach to dislocation problems for periodic Schrödinger operators*, J. Math. Anal. Appl. **381** (2011), 166–178

[19] _____, *Spectral properties of grain boundaries at small angles of rotation*, J. Spectral Theory **1** (2011), 197–219

[20] PD Hislop and IM Sigal, *Introduction to Spectral Theory. With Applications to Schrödinger Operators.* Applied Mathematical Sciences, **113**. Springer-Verlag, New York, 1996

[21] R Johnson and J Moser, *The rotation number for almost periodic potentials*, Commun. Math. Phys. **84** (1982), 403–438

[22] T Kato, *Perturbation Theory for Linear Operators.* Springer, New York, 1966

[23] BA Khoruzhenko and LA Pastur, *The localization of surface states: an exactly solvable model.* Physics Reports **288** (1997), 109–126.

[24] E Korotyaev, *Lattice dislocations in a 1-dimensional model*, Commun. Math. Phys. **213** (2000), 471–489

[25] _____, *Schrödinger operators with a junction of two 1-dimensional periodic potentials*, Asymptotic Anal. **45** (2005), 73–97

[26] _____ and KM Schmidt, *On the resonances and eigenvalues for a 1D half-crystal with localised impurity*, J. Reine Angew. Math. **670** (2012), 217–248

[27] V Kostrykin and R Schrader, *Regularity of the surface density of states*, J. Funct. Anal. **187** (2001), 227–246

[28] C Müller, *Analysis of Spherical Symmetries in Euclidean Spaces.* Springer, New York, 1988

[29] N Ortner, *Regularisierte Faltung von Distributionen. II. Eine Tabelle von Fundamentallösungen* (German), Z. Angew. Math. Phys. **31** (1980), 155–173

[30] R Penrose, *Pentaplexity – a class of non-periodic tilings of the plane*, The Mathematical Intelligencer **2** (1979), 32–37

[31] J Puig, *Cantor spectrum for the almost Mathieu operator*, Comm. Math. Phys. **244** (2004), 297–309

[32] C Radin, *Aperiodic tilings, ergodic theory, and rotations,* in: *The Mathematics of Long-Range Aperiodic Order*, ed. RV Moody. NATO-ASI series C 489, Kluwer Acad. Publ., Dordrecht 1997, pp. 499–519

[33] M Reed and B Simon, *Methods of Modern Mathematical Physics.* Vol I. *Functional Analysis. Revised and Enlarged Edition.* Academic Press, New York, 1980

[34] M Reed and B Simon, *Methods of Modern Mathematical Physics.* Vol II. *Fourier Analysis, Self-Adjointness.* Academic Press, New York, 1975

[35] M Reed and B Simon, *Methods of Modern Mathematical Physics.* Vol IV. *Analysis of Operators.* Academic Press, New York, 1978

[36] L Schwartz, *Théorie des Distributions.* Nouvelle Édition. Hermann, Paris, 1973

[37] B Simon, *A canonical decomposition for quadratic forms with applications to monotone convergence theorems,* J. Funct. Anal. **28** (1978), 267–273

[38] B Simon, *Schrödinger semigroups*, Bull. Amer. Math. Soc. **7** (1982), 447–526

[39] B Simon, *Trace Ideals and their Applications. Second Edition.* Amer. Math. Soc., 2005

[40] M Stautz, *Eigenwerte in Spektrallücken bei Dislokationsproblemen mit allgemeinen Potentialen.* Master thesis, TU Braunschweig, 2010

[41] J Voigt, *Absorption semigroups.* J. Operator Theory **20** (1988), 117–131.

INSTITUTE FOR COMPUTATIONAL MATHEMATICS, TECHNISCHE UNIVERSITÄT BRAUNSCHWEIG, POCKELSSTRASSE 14, 38106 BRAUNSCHWEIG, GERMANY
   E-mail address: `r.hempel@tu-bs.de`
   E-mail address: `martin.kohlmann@plri.de`
   E-mail address: `m.stautz@tu-bs.de`

TU DRESDEN, FACHRICHTUNG MATHEMATIK, INSTITUT FÜR ANALYSIS, HELMHOLTZSTRASSE 10, 01069 DRESDEN
   E-mail address: `juergen.voigt@tu-dresden.de`