# THE PERIODIC $\mu$-$b$-EQUATION AND EULER EQUATIONS ON THE CIRCLE

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ABSTRACT. In this paper, we study the $\mu$-variant of the periodic $b$-equation and show that this equation can be realized as a metric Euler equation on the Lie group $\mathrm{Diff}^\infty(\mathbb{S})$ if and only if $b = 2$ (for which it becomes the $\mu$-Camassa-Holm equation). In this case, the inertia operator generating the metric on $\mathrm{Diff}^\infty(\mathbb{S})$ is given by $L = \mu - \partial_x^2$. In contrast, the $\mu$-Degasperis-Procesi equation (obtained for $b = 3$) is not a metric Euler equation on $\mathrm{Diff}^\infty(\mathbb{S})$ for any regular inertia operator $A \in \mathcal{L}_{\mathrm{is}}^{\mathrm{sym}}(C^\infty(\mathbb{S}))$. The paper generalizes some recent results of [13, 16, 24].

For the mathematical modelling of fluids, the so-called family of $b$-equations

$$m_t = -(m_x u + bmu_x), \quad m = u - u_{xx}, \tag{1}$$

attracted a considerable amount of attention in recent years. Here, $b$ stands for a real parameter, [17]. Each of these equations models the unidirectional irrotational free surface flow of a shallow layer of an inviscid fluid moving under the influence of gravity over a flat bed. In this model $u(t, x)$ represents the wave's height at time $t \geq 0$ and position $x$ above the flat bottom. If the wave profile is assumed to be periodic, $x \in \mathbb{S} \simeq \mathbb{R}/\mathbb{Z}$; otherwise $x \in \mathbb{R}$. For further details concerning the hydrodynamical relevance we refer to [10, 21, 22]. As shown in [11, 18, 20, 28], the $b$-equation is asymptotically integrable which is a necessary condition for complete integrability, but only for $b = 2$ and $b = 3$ for which it becomes the Camassa-Holm (CH) equation

$$u_t - u_{txx} + 3uu_x - 2u_x u_{xx} - uu_{xxx} = 0$$

and the Degasperis-Procesi (DP) equation

$$u_t - u_{txx} + 4uu_x - 3u_x u_{xx} - uu_{xxx} = 0$$

respectively. The Cauchy problems for CH and DP have been studied in detail: For the CH, there are global strong as well as global weak solutions. In addition, CH allows for finite time blow-up solutions which can be interpreted as breaking waves and there are no shock waves (see, e.g., [4, 5, 6]). Some recent global well-posedness results for strong and weak solutions, precise blow-up scenarios and wave breaking for the DP are discussed in [14, 15, 30, 31, 32].

Besides the various common properties of the CH and the DP there are also significant differences to report on, e.g., when studying geometric aspects of the

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family (1). The periodic equation (1) reexpresses a geodesic flow on the group $\mathrm{Diff}^{\infty}(\mathbb{S})$ of smooth and orientation preserving diffeomorphisms of the circle, cf. [13]. If $b = 2$, the geodesic flow corresponds to the right-invariant metric induced by the inertia operator $1-\partial_x^2$ whereas for $b \neq 2$, equation (1) can only be realized as a non-metric Euler equation, i.e., as geodesic flow with respect to a linear connection which is not Riemannian in the sense that it is compatible with a right-invariant metric, cf. [8, 9, 16, 24].

The idea of studying Euler's equations of motion for perfect (i.e., incompressible, homogeneous and inviscid) fluids as a geodesic flow on a certain diffeomorphism group goes back to [1, 12] and in a recent work [13], Escher and Kolev show that the theory is also valid for the general $b$-equation.

In this paper, we are interested in the following variant of the periodic family (1). Let $\mu(u) = \int_{\mathbb{S}} u(t,x)\,\mathrm{d}x$ and $m = \mu(u) - u_{xx}$ in (1) to obtain the family of $\mu$-$b$-equations, cf. [27]. The study of the $\mu$-variant of (1) is motivated by the following key observation: Letting $m = -\partial_x^2 u$, equation (1) for $b = 2$ becomes the Hunter-Saxton (HS) equation, cf. [19], which possesses various interesting geometric properties, cf. [25, 26], whereas the choice $m = (1 - \partial_x^2)u$ leads to the CH as explained above. In the search for integrable equations that are obtained by a perturbation of $-\partial_x^2$, the $\mu$-$b$-equation has been introduced and it could be shown that it behaves quite similarly to the $b$-equation; cf. [27] where the authors discuss local and global well-posedness as well as finite time blow-up and peakons. Peakons are peculiar wave forms: they are travelling wave solutions which are smooth except at their crests; the lateral tangents exist, are symmetric but different. Such wave forms are known to characterize the steady water waves of greatest height, [3, 7, 29], and were first shown to arise for the CH in [2].

The goal of this paper is to extend the work done in [16] to the family of $\mu$-$b$-equations. Our main result is that the periodic $\mu$-$b$-equation can be realized as a metric Euler equation on $\mathrm{Diff}^{\infty}(\mathbb{S})$ if and only if $b = 2$, for which it becomes the $\mu$CH equation. The corresponding regular inertia operator is $\mu - \partial_x^2$. Before we give a proof, we begin with some introductory remarks about Euler equations on $\mathrm{Diff}^{\infty}(\mathbb{S})$. In a first step, we comment on the operator $\mu - \partial_x^2$.

**Lemma 1.** *The bilinear map*

$$\langle\cdot,\cdot\rangle_\mu : C^{\infty}(\mathbb{S}) \times C^{\infty}(\mathbb{S}) \to \mathbb{R}, \quad \langle u, v\rangle_\mu = \mu(u)\mu(v) + \int_{\mathbb{S}} u_x(x)v_x(x)\,\mathrm{d}x$$

*defines an inner product on $C^{\infty}(\mathbb{S})$.*

*Proof.* Clearly, $\langle\cdot,\cdot\rangle_\mu$ is a symmetric bilinear form and $\langle u, u\rangle_\mu \geq 0$. If $u \in C^{\infty}(\mathbb{S})$ satisfies $\langle u, u\rangle_\mu = 0$, then $u_x = 0$ on $\mathbb{S}$ and hence $u$ is constant. The fact that $\mu(u) = 0$ implies $u = 0$. □

We obtain a right-invariant metric on the Lie group $G = \mathrm{Diff}^{\infty}(\mathbb{S})$ by defining the inner product $\langle\cdot,\cdot\rangle_\mu$ on the Lie algebra $\mathfrak{g} \simeq \mathrm{Vect}^{\infty}(\mathbb{S}) \simeq C^{\infty}(\mathbb{S})$ of smooth vector fields on $\mathbb{S}$ and transporting $\langle\cdot,\cdot\rangle_\mu$ to any tangent space of $G$ by using right translations, i.e., if $R_\varphi : G \to G$ denotes the map sending $\psi$ to $\psi \circ \varphi$, then

$$\langle u, v\rangle_{\mu;\varphi} = \left\langle D_\varphi R_{\varphi^{-1}} u, D_\varphi R_{\varphi^{-1}} v\right\rangle_\mu ,$$

for all $u, v \in T_\varphi G$. Observe that $\langle\cdot,\cdot\rangle_\mu$ can be expressed in terms of the symmetric linear operator $L\colon \mathfrak{g} \to \mathfrak{g}'$ defined by $L = \mu - \partial_x^2$, i.e.,

$$\langle u, v\rangle_\mu = \langle Lu, v\rangle = \langle Lv, u\rangle , \quad u, v \in C^\infty(\mathbb{S}),$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing on $\mathfrak{g}' \times \mathfrak{g}$.

**Definition 2.** Each symmetric isomorphism $A\colon \mathfrak{g} \to \mathfrak{g}'$ is called an *inertia operator* on $G$. The corresponding right-invariant metric on $G$ induced by $A$ is denoted by $\rho_A$.

Let $A$ be an inertia operator on $G$. We denote the Lie bracket on $\mathfrak{g}$ by $[\cdot,\cdot]$ and write $(\mathrm{ad}_u)^*$ for the adjoint with respect to $\rho_A$ of the natural action of $\mathfrak{g}$ on itself given by $\mathrm{ad}_u : \mathfrak{g} \to \mathfrak{g}$, $v \mapsto [u, v]$. Let

$$B(u, v) = \frac{1}{2}\left[(\mathrm{ad}_u)^* v + (\mathrm{ad}_v)^* u\right].$$

We define a right-invariant linear connection on $G$ via

$$\nabla_u v = \frac{1}{2}[u, v] + B(u, v), \quad u, v \in C^\infty(\mathbb{S}). \tag{2}$$

As explained in [13, 16], we have the following theorem.

**Theorem 3.** *A smooth curve $g(t)$ on the Lie group $G = \mathrm{Diff}^\infty(\mathbb{S})$ is a geodesic for the right-invariant linear connection defined by* (2) *if and only if its Eulerian velocity $u(t) = D_{g(t)} R_{g^{-1}(t)} g'(t)$ satisfies the Euler equation*

$$u_t = -B(u, u). \tag{3}$$

Observe that the topological dual space of $\mathrm{Vect}^\infty(\mathbb{S}) \simeq C^\infty(\mathbb{S})$ is given by the distributions $\mathrm{Vect}'(\mathbb{S})$ on $\mathbb{S}$. In order to get a convenient representation of the *Christoffel operator* $B$ we restrict ourselves to $\mathrm{Vect}^*(\mathbb{S})$, the set of all regular distributions which can be represented by smooth densities, i.e., $T \in \mathrm{Vect}^*(\mathbb{S})$ if and only if there is a $\rho \in C^\infty(\mathbb{S})$ such that

$$T(\varphi) = \int_{\mathbb{S}} \rho(x)\varphi(x)\,\mathrm{d}x, \quad \forall \varphi \in C^\infty(\mathbb{S}).$$

By means of the Riesz representation theorem we may identify $\mathrm{Vect}^*(\mathbb{S}) \simeq C^\infty(\mathbb{S})$. This motivates the following definition.

**Definition 4.** Let $\mathcal{L}_{\mathrm{is}}^{\mathrm{sym}}(C^\infty(\mathbb{S}))$ denote the set of all continuous isomorphisms on $C^\infty(\mathbb{S})$ which are symmetric with respect to the $L_2$ inner product. Each $A \in \mathcal{L}_{\mathrm{is}}^{\mathrm{sym}}(C^\infty(\mathbb{S}))$ is called a *regular inertia operator* on $\mathrm{Diff}^\infty(\mathbb{S})$.

The following lemma establishes that the operator $L$ belongs to the above defined class of regular inertia operators.

**Lemma 5.** *The operator $L$ is a regular inertia operator on* $\mathrm{Diff}^\infty(\mathbb{S})$.

*Proof.* One checks that applying $L$ to

$$\left(\frac{1}{2}x^2-\frac{1}{2}x+\frac{13}{12}\right)\int_0^1 u(a)\,\mathrm{d}a+\left(x-\frac{1}{2}\right)\int_0^1\int_0^a u(b)\,\mathrm{d}b\,\mathrm{d}a$$
$$-\int_0^x\int_0^a u(b)\,\mathrm{d}b\,\mathrm{d}a+\int_0^1\int_0^a\int_0^b u(c)\,\mathrm{d}c\,\mathrm{d}b\,\mathrm{d}a$$

gives back the function $u$. It is easy to see that if $u\in C^\infty(\mathbb{S})$, then its pre-image also belongs to $C^\infty(\mathbb{S})$. Assume that $Lu=0$ for $u\in C^\infty(\mathbb{S})$. We thus can find constants $c,d\in\mathbb{R}$ such that $u=\frac{1}{2}\mu(u)x^2+cx+d$. Since $u$ is periodic, $c=0$ and $\mu(u)=0$ and thus also $d=0$. Clearly, $L:C^\infty(\mathbb{S})\to C^\infty(\mathbb{S})$ is bicontinuous. □

A proof of the following theorem can be found in [16].

**Theorem 6.** *Given $A\in\mathcal{L}_{\mathrm{is}}^{\mathrm{sym}}(C^\infty(\mathbb{S}))$, the Christoffel operator $B=\frac{1}{2}[(\mathrm{ad}_u^*)v+(\mathrm{ad}_v^*)u]$ has the form*

$$B(u,v)=\frac{1}{2}A^{-1}\left[2(Au)v_x+2(Av)u_x+u(Av)_x+v(Au)_x\right],$$

*for all $u,v\in C^\infty(\mathbb{S})$.*

It may be instructive to discuss the following paradigmatic examples.

**Example 7.** Let $\lambda\in[0,1]$ and let $A$ be the inertia operator for the equation $m_t=-(m_xu+2u_xm)$.

(1) The choice $A=-\partial_x^2$ yields $B(u,u)=-A^{-1}(2u_xu_{xx}+uu_{xxx})$ and $u_t=-B(u,u)$ is the Hunter-Saxton equation

$$u_{txx}+2u_xu_{xx}+uu_{xxx}=0.$$

(2) We choose $A=1-\lambda\partial_x^2$. If $\lambda=0$, the equation $m_t=-(m_xu+2u_xm)$ becomes the periodic inviscid Burgers equation $u_t+B(u,u)=u_t+3uu_x=0$. For $\lambda\neq 0$, we obtain

$$u_t+B(u,u)=u_t+3uu_x-\lambda(2u_xu_{xx}+uu_{xxx}+u_{txx})=0,$$

a 1-parameter family of Camassa-Holm equations.

(3) Choosing $A=\mu-\partial_x^2$, we arrive at the $\mu$CH equation

$$\mu(u_t)-u_{txx}+2\mu(u)u_x=2u_xu_{xx}+uu_{xxx},$$

which is also called $\mu$HS in the literature, cf. [23].

Each regular inertia operator induces a metric Euler equation on $\mathrm{Diff}^\infty(\mathbb{S})$. We now consider the question for which $b\in\mathbb{R}$ there is a regular inertia operator such that the $\mu$-$b$-equation is the corresponding Euler equation on $\mathrm{Diff}^\infty(\mathbb{S})$. Example 7 shows that, for $b=2$, the operator $L\in\mathcal{L}_{\mathrm{is}}^{\mathrm{sym}}(C^\infty(\mathbb{S}))$ induces the $\mu$CH. Our goal is to show that this works only for $b=2$.

**Theorem 8.** *Let $b\in\mathbb{R}$ be given and suppose that there is a regular inertia operator $A\in\mathcal{L}_{\mathrm{is}}^{\mathrm{sym}}(C^\infty(\mathbb{S}))$ such that the $\mu$-$b$-equation*

$$m_t=-(m_xu+bmu_x),\quad m=\mu(u)-u_{xx},$$

*is the Euler equation on $\mathrm{Diff}^\infty(\mathbb{S})$ with respect to $\rho_A$. Then $b=2$ and $A=L$.*

*Proof.* We assume that, for given $b \in \mathbb{R}$ and $A \in \mathcal{L}_{\mathrm{is}}^{\mathrm{sym}}(C^\infty(\mathbb{S}))$, the $\mu$-$b$-equation is the Euler equation on the circle diffeomorphisms with respect to $\rho_A$. Then

$$u_t = -A^{-1}((Au)_x u + 2(Au)u_x)$$

and the $\mu$-$b$-equation can be written as

$$(Lu)_t = -((Lu)_x u + b(Lu)u_x).$$

Using that $(Lu)_t = Lu_t$ and resolving both equations with respect to $u_t$ we get that

$$A^{-1}\left(2(Au)u_x + u(Au)_x\right) = L^{-1}\left(b(Lu)u_x + u(Lu)_x\right), \tag{4}$$

for $u \in C^\infty(\mathbb{S})$. Denote by $\mathbf{1}$ the constant function with value 1. If we set $u = \mathbf{1}$ in (4), then $A^{-1}(\mathbf{1}(A\mathbf{1})_x) = 0$ and hence $(A\mathbf{1})_x = 0$, i.e., $A\mathbf{1} = c\mathbf{1}$. Scaling (4) shows that we may assume $c = 1$. Replacing $u$ by $u + \lambda$ in (4) and scaling with $\lambda^{-1}$, we get on the left-hand side

$$\begin{aligned}
&\frac{1}{\lambda}A^{-1}\big(2(A(u+\lambda))(u+\lambda)_x + (u+\lambda)(A(u+\lambda))_x\big)\\
&\quad= \frac{1}{\lambda}A^{-1}\big(2((Au)+\lambda)u_x + (u+\lambda)(Au)_x\big)\\
&\quad= A^{-1}\left(\frac{2(Au)u_x + u(Au)_x}{\lambda} + 2u_x + (Au)_x\right)\\
&\quad\to A^{-1}(2u_x + (Au)_x),\ \lambda \to \infty,
\end{aligned}$$

and a similar computation for the right-hand side gives

$$\begin{aligned}
&\frac{1}{\lambda}L^{-1}\big(b(L(u+\lambda))(u+\lambda)_x + (u+\lambda)(L(u+\lambda))_x\big)\\
&\quad\to L^{-1}(bu_x + (Lu)_x),\ \lambda \to \infty.
\end{aligned}$$

We obtain

$$A^{-1}\left(2u_x + (Au)_x\right) = L^{-1}(bu_x + (Lu)_x). \tag{5}$$

We now consider the Fourier basis functions $u_n = e^{inx}$ for $n \in 2\pi\mathbb{Z}\backslash\{0\}$ and have $Lu_n = n^2 u_n$ and

$$L^{-1}(b(u_n)_x + (Lu_n)_x) = \mathrm{i}\alpha_n u_n, \quad \alpha_n = \frac{b}{n} + n.$$

We now apply $A$ to (5) with $u = u_n$ and see that

$$2\mathrm{i}nu_n + (Au_n)_x = \mathrm{i}\alpha_n(Au_n).$$

Therefore $v_n := Au_n$ solves the ordinary differential equation

$$v' - \mathrm{i}\alpha_n v = -2\mathrm{i}nu_n. \tag{6}$$

If $b = 0$, then $\alpha_n = n$ and hence the general solution of (6) is

$$v(x) = (c - 2\mathrm{i}nx)u_n, \quad c \in \mathbb{R},$$

which is not periodic for any $c \in \mathbb{R}$. Hence $b \neq 0$ and there are numbers $\gamma_n$ so that

$$v_n = Au_n = \gamma_n e^{\mathrm{i}\alpha_n x} + \beta_n u_n, \quad \beta_n = \frac{2}{b}n^2.$$

We first discuss the case $\gamma_n = 0$ for all $n$ and show that $\gamma_p \neq 0$ for some $p \in 2\pi\mathbb{Z}\backslash\{0\}$ is not possible. If all $\gamma_n$ vanish, then $Au_n = \beta_n u_n$ and $A$ is a Fourier multiplication operator; in particular $A$ commutes with $L$. Therefore (4) with $u = u_n$ is equivalent to

$$L(2(Au_n)(u_n)_x + u_n(Au_n)_x) = A(b(Lu_n)(u_n)_x + u_n(Lu_n)_x)$$

and by direct computation

$$12\mathrm{i}n^3\beta_n u_{2n} = \mathrm{i}(b+1)n^3\beta_{2n}u_{2n}.$$

Inserting $\beta_n = 2n^2/b$ we see that $b = 2$ and $\beta_n = n^2$. Therefore $A = L$. Assume that there is $p \in 2\pi\mathbb{Z}\backslash\{0\}$ with $\gamma_p \neq 0$. Since $v_p = Au_p$ is periodic, $\alpha_p \in 2\pi\mathbb{Z}$ and hence $b = kp$ for some $k \in 2\pi\mathbb{Z}\backslash\{0\}$. Let $\alpha_p = m$. If $m = p$, then $b = 0$ which is impossible. We thus have $\langle u_m, u_p\rangle = 0$ and

$$\langle Au_p, u_m\rangle = \left\langle \gamma_p e^{\mathrm{i}mx}, u_m\right\rangle = \gamma_p.$$

The symmetry of $A$ yields

$$\gamma_p = \langle Au_p, u_m\rangle = \langle u_p, Au_m\rangle = \overline{\gamma_m}\left\langle u_p, e^{\mathrm{i}\alpha_m x}\right\rangle.$$

Since $\gamma_p \neq 0$, $\gamma_m$ is non-zero and periodicity implies $\alpha_m \in 2\pi\mathbb{Z}$. More precisely, $\alpha_m = p$ since otherwise $\left\langle u_p, e^{\mathrm{i}\alpha_m x}\right\rangle = 0 = \gamma_p$. Using $b = kp$ and the definition of $\alpha_p$, we see that $m = \alpha_p = k + p$. Furthermore,

$$p(k+p) = \alpha_m(k+p) = \alpha_{k+p}(k+p) = kp + (k+p)^2$$

and hence $0 = k^2 + 2pk$. Since $k \neq 0$, it follows that $k = -2p$ and hence $b = -2p^2$. We get $\alpha_p = -p$ and observe that $\gamma_n = 0$ for all $n \notin \{p, -p\}$, since otherwise repeating the above calculations would yield $b = -2n^2$ contradicting $b = -2p^2$. Inserting $u = u_p$ in (4) shows that

$$\mathrm{i}p\gamma_p\mathbf{1} - \frac{3\mathrm{i}p}{\beta_{2p}}u_{2p} = \mathrm{i}p^3(b+1)\frac{u_{2p}}{4p^2};$$

here we have used that $Au_p = \gamma_p/u_p + \beta_p u_p$, $\beta_p = -1$ and $A^{-1}u_{2p} = u_{2p}/\beta_{2p}$, since $2p$ does not coincide with $\pm p$ and hence $\gamma_{2p} = 0$. It follows that $p\gamma_p = 0$ in contradiction to $p, \gamma_p \neq 0$. $\square$

**Corollary 9.** *The $\mu DP$ equation on the circle*

$$m_t = -(m_x u + 3mu_x), \quad m = \mu(u) - u_{xx},$$

*cannot be realized as a metric Euler equation for any $A \in \mathcal{L}_{\mathrm{is}}^{\mathrm{sym}}(C^\infty(\mathbb{S}))$.*

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