INTRODUCTION TO SHIMURA CURVES, II

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Introduction to Shimura curves

I: Fundamental domains and CM-points (M. Alsina, P. Bayer)

II: Uniformization of Shimura curves (A. Travesa, P. Bayer)

III: Abelian varieties with QM (J. Guàrdia, V. Rotger, P. Bayer)

Uniformizing functions for certain Shimura curves, the case $D = 6$

$$\mathbb{H}_6 = \left(\frac{3, -1}{\mathbb{Q}}\right), \qquad \mathrm{I}^2 = 3,\ \mathrm{J}^2 = -1,\ \mathrm{IJ} = -\mathrm{JI} = \mathrm{K}$$

$$\mathcal{O}_6 := \mathbb{Z}\left[1, \mathrm{I}, \mathrm{J}, \frac{1 + \mathrm{I} + \mathrm{J} + \mathrm{K}}{2}\right]$$

$$\begin{array}{rcl} \Phi : \mathbb{H}_6 & \longrightarrow & \mathbf{M}(2, \mathbb{R}) \\ x + y\mathrm{I} + z\mathrm{J} + t\mathrm{K} & \mapsto & \begin{bmatrix} x + y\sqrt{3} & z + t\sqrt{3} \\ -(z - t\sqrt{3}) & x - y\sqrt{3} \end{bmatrix} \end{array}$$

$$\Gamma_6 = \left\{\gamma = \frac{1}{2}\begin{bmatrix} \alpha & \beta \\ -\beta' & \alpha' \end{bmatrix} :\ \alpha, \beta \in \mathbb{Z}[\sqrt{3}],\ \det\gamma = 1,\ \alpha \equiv \beta \pmod{2}\right\}$$

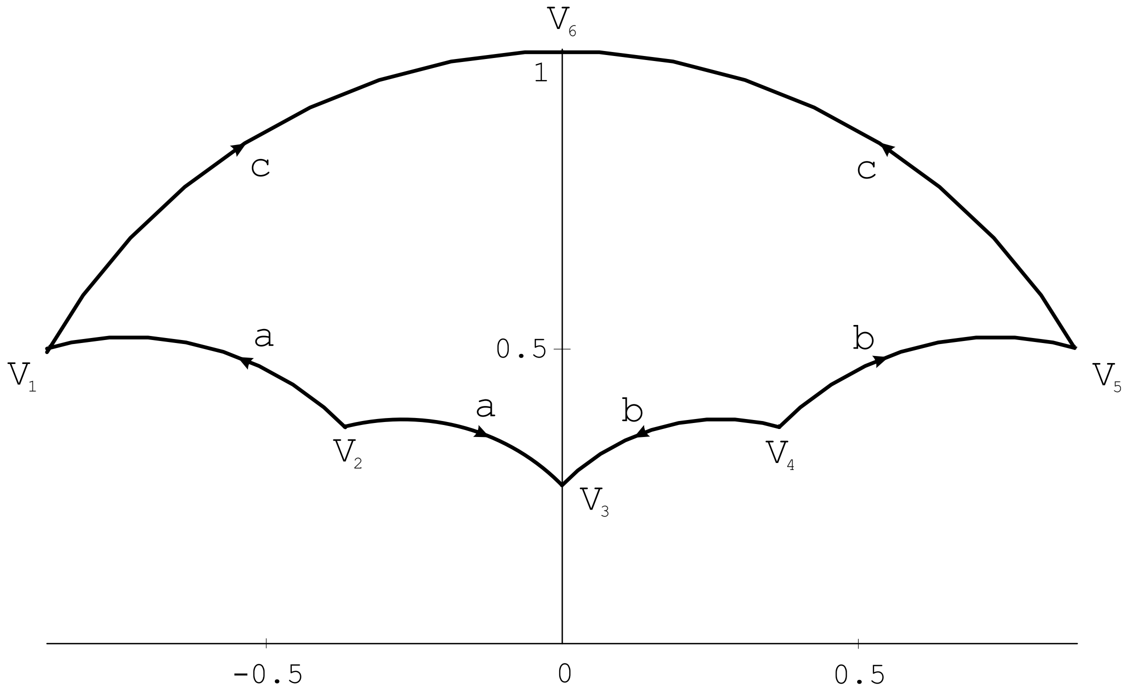
$j_6 : \Gamma_6 \backslash \mathcal{H} \sim X(6,1)(\mathbb{C}), \quad X_6 := X(6,1), \quad g(X_6) = 0$

$\mathbb{C}(X_6) = \mathbb{C}(t), \quad D_a(t) \in \mathbb{C}(t), \quad t$ *Hauptmodul*

$D_a(t) + R(t) = 0, \quad R(t) \in \mathbb{C}(t)$

- Objective 1: to compute $R(t)$ -automorphic derivative of t-
- Objective 2: to obtain a possible t
- Objective 3: to choose t with good arithmetic properties: j_6

Since $\mathbb{C}(t) = \mathbb{C}\left(\frac{at+b}{ct+d}\right)$, for any $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbf{GL}(2, \mathbb{C})$, we can prescribe the values of t at three different points.



Fundamental domain for $X_6 := X(6,1)$

Elliptic cycles of order 2: $\{P_1, P_3, P_5\}, \{P_6\}$; of order 3: $\{P_2\}, \{P_4\}$

$t(P_6) = \infty, \quad t(P_3) = 0, \quad t(P_4) = 1, \quad t(P_2) = a$

η_1	$\begin{bmatrix} \sqrt{3} & 2 \\ -2 & -\sqrt{3} \end{bmatrix}$	η_2	$\frac{1}{2}\begin{bmatrix} 1+\sqrt{3} & 3-\sqrt{3} \\ -3-\sqrt{3} & 1-\sqrt{3} \end{bmatrix}$
η_3	$\begin{bmatrix} 0 & -2+\sqrt{3} \\ 2+\sqrt{3} & 0 \end{bmatrix}$	η_4	$\frac{1}{2}\begin{bmatrix} 1+\sqrt{3} & -3+\sqrt{3} \\ 3+\sqrt{3} & 1-\sqrt{3} \end{bmatrix}$
η_5	$\begin{bmatrix} \sqrt{3} & -2 \\ 2 & -\sqrt{3} \end{bmatrix}$	η_6	$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

Matrices representing generators for the isotropy groups at the vertices of the hexagon

Theorem. *Let $\overline{\Gamma}$ be a Fuchsian group of the first kind such that the associated curve $X(\overline{\Gamma})$ is of genus 0. Assume that we are aware of a fundamental half domain for the action of $\overline{\Gamma}$ in $\mathcal{H}$. Suppose that t is a generator of the field of $\overline{\Gamma}$-automorphic functions such that its values at the vertices of the fundamental half domain belong to $\mathbf{P}^1(\mathbb{R})$. Then, there exists a rational function $R(t)$ such that $Da(t,z) + R(t) = 0$. If $\alpha_i\pi$ are the internal angles at the vertices of the fundamental half domain, then*

$$R(t) = \sum \frac{1-\alpha_i^2}{(t-a_i)^2} + \sum \frac{B_i}{t-a_i},$$

where B_i are constants and the summation extends over all the vertices of the fundamental half domain where the function t takes finite values a_i. Moreover, if the values of t at all the vertices are finite, then *(a)* $\sum B_i = 0$,

(b) $\sum a_i B_i + \sum(1 - \alpha_i^2) = 0$,

(c) $\sum a_i^2 B_i + \sum a_i(1 - \alpha_i^2) = 0$.

But if ∞ *is the value of* t *at a vertex with internal angle* $\alpha\pi$, *then*

(a) $\sum B_i = 0$,

(b) $\sum a_i B_i + \sum(1 - \alpha_i^2) - (1 - \alpha^2) = 0$. □

First approach to the differential equation

$$R(t) = \frac{8}{9(t-1)^2} + \frac{B_4}{t-1} + \frac{3}{4t^2} + \frac{B_3}{t} + \frac{8}{9(t-a)^2} + \frac{B_2}{t-a}$$

$$a \rightsquigarrow -1$$

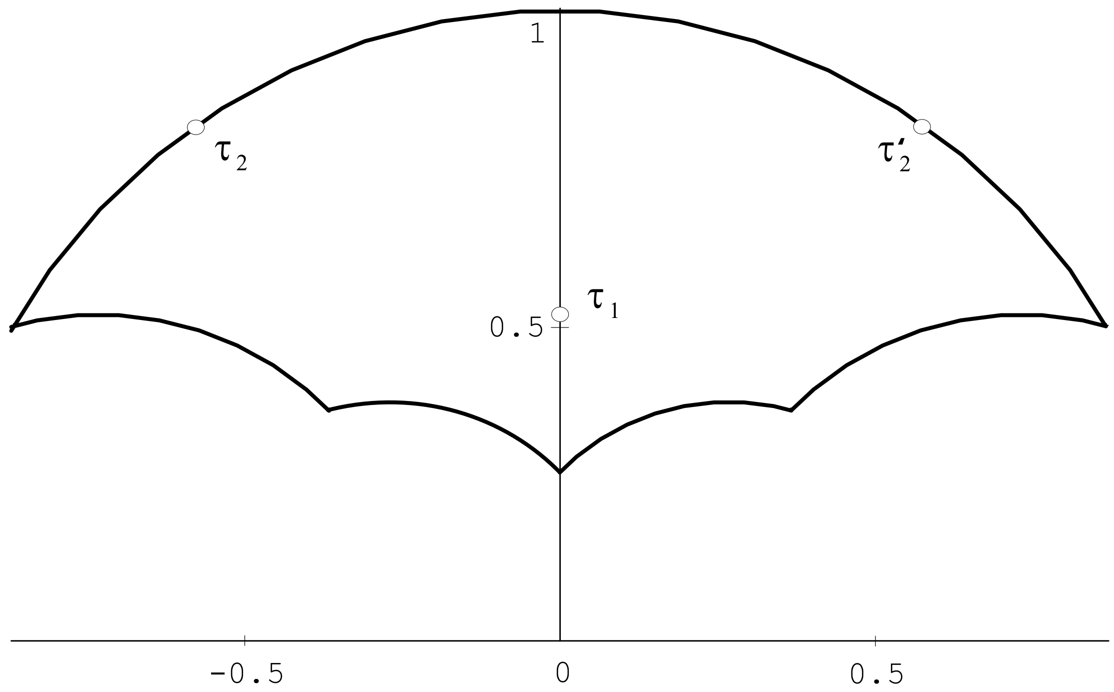
$$B_2 = \frac{16}{9} + B_4, \quad B_3 = -\frac{16}{9} - 2B_4$$

$$\frac{27 - 8(8+9B_4)t + 74t^2 + 8(8+9B_4)t^3 + 27t^4}{36t^2(-1+t^2)^2}$$

$$Da(t,z) = R(t)$$

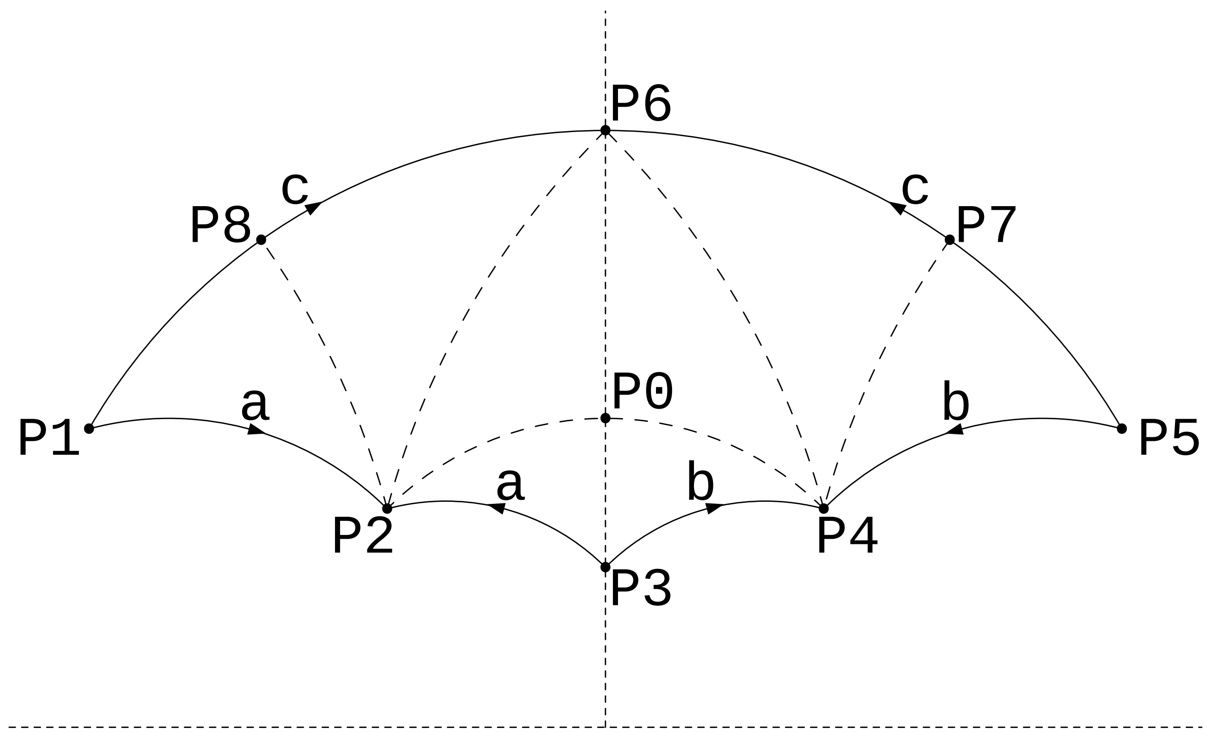
P_1	$\dfrac{-\sqrt{3}+i}{2}$	P_3	$(2-\sqrt{3})i$	P_5	$\dfrac{\sqrt{3}+i}{2}$
P_2	$\dfrac{-1+i}{1+\sqrt{3}}$	P_4	$\dfrac{1+i}{1+\sqrt{3}}$	P_6	i
P_0	$\dfrac{(\sqrt{6}-\sqrt{2})i}{2}$	P_7	$\dfrac{1+\sqrt{2}i}{\sqrt{3}}$	P_8	$\dfrac{-1+\sqrt{2}i}{\sqrt{3}}$

Vertices of a fundamental domain for X_6 and SCM-points



Fundamental domain for X_6 and SCM-points: $\{P_0, P_7 = P_8\}$

Two classes of maximal embeddings: $R(\sqrt{-6},1) \subseteq \mathcal{O}_6$,
$R(\sqrt{-6},1)$ ring of integers of $\subseteq \mathbb{Q}(\sqrt{-6})$



Fundamental domain for X_6, SCM-points: $\{P_0, P_7 = P_8\}$ and some hyperbolic lines

Involutions of X_6 (cf. Shimura, Ogg, Michon)

$$\mathcal{O}_6 \subseteq N(\mathcal{O}_6) \subseteq \mathbb{H}_6, \quad \Gamma_6^+ \subseteq \mathbf{GL}(2,\mathbb{R})^+, \quad \Gamma_6^+/\Gamma_6 = \langle \Phi(w_d) : d|6 \rangle$$

$$X_6^{(d)} := X_6/\langle \omega_d \rangle, \quad X_6^+ := X_6/\langle \{\omega_d : d|6\} \rangle$$

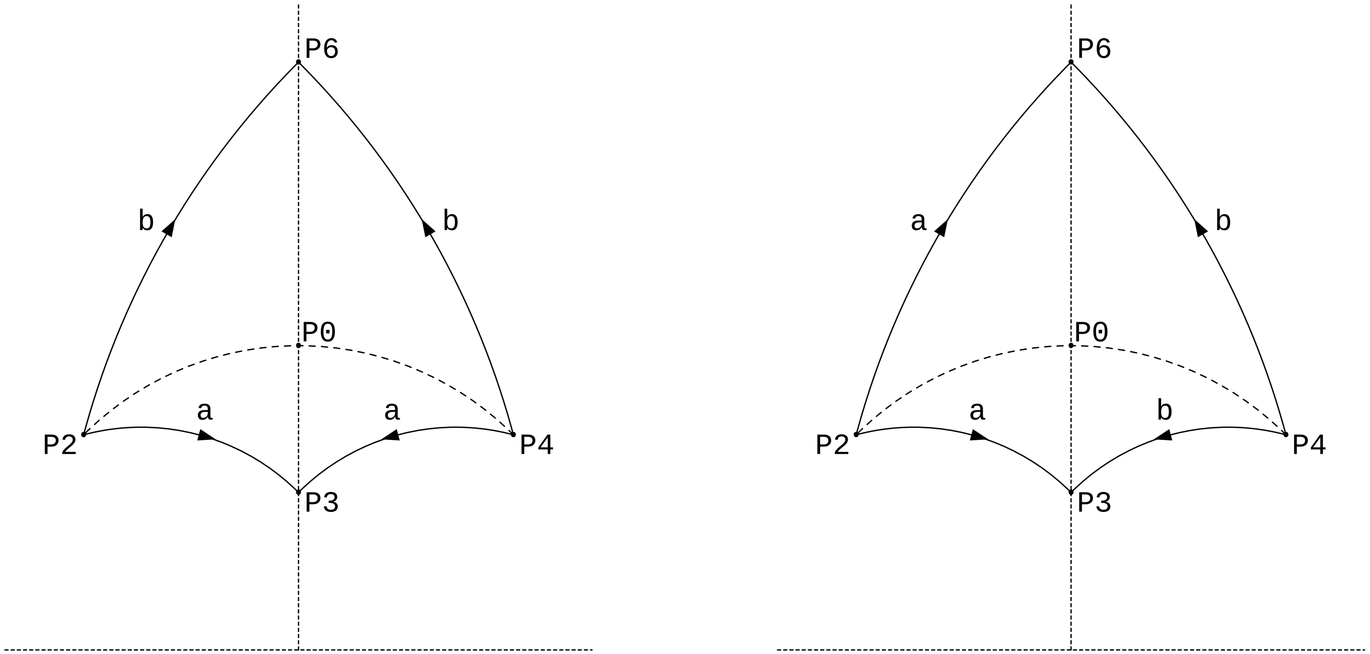
$$\begin{array}{ccc} & X_6 & \\ \swarrow & \downarrow & \searrow \\ X_6^{(2)} & X_6^{(3)} & X_6^{(6)} \\ \searrow & \downarrow & \swarrow \\ & X_6^+ & \end{array}$$

degree 2 coverings

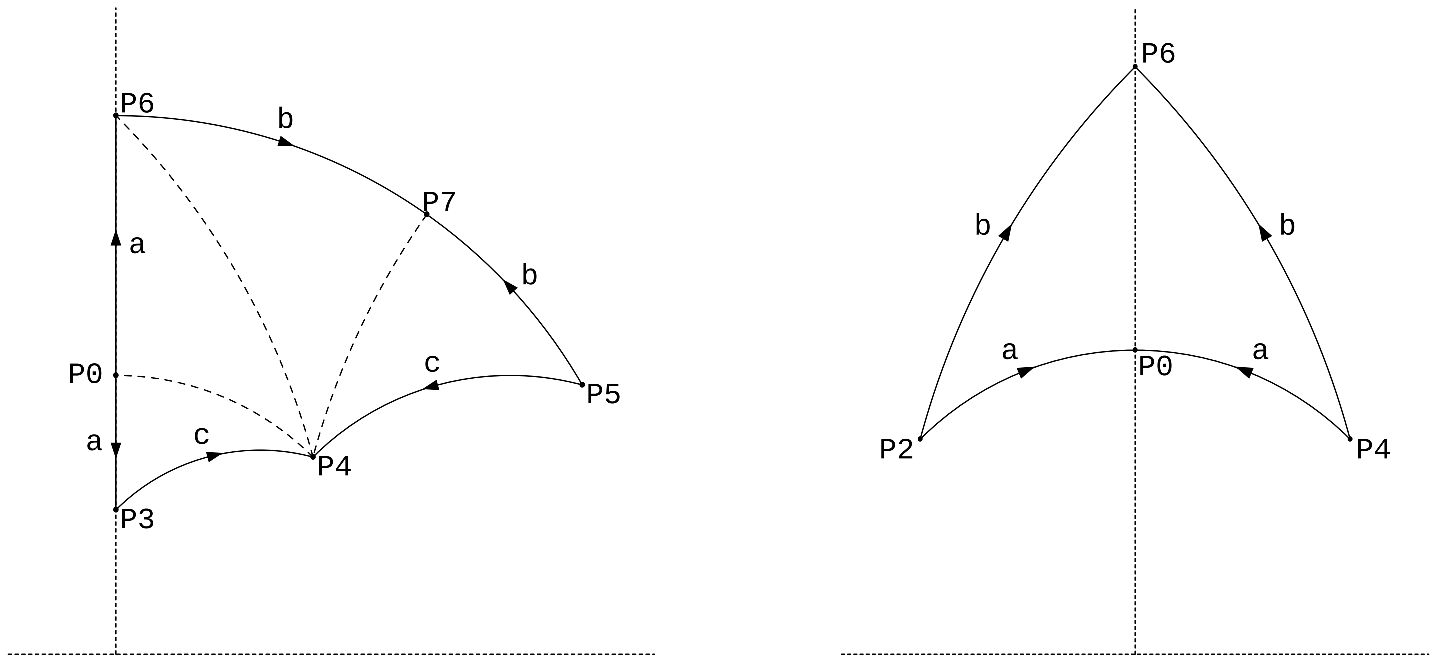
quaternions in $N(\mathcal{O}_6)$	matrix
$w_2 := 1 + \mathsf{J}$	$\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$
$w_3 := \frac{1}{2}(-3 - \mathrm{I} - 3\mathsf{J} + \mathsf{K})$	$\frac{1}{2}\begin{bmatrix} -3-\sqrt{3} & -3+\sqrt{3} \\ 3+\sqrt{3} & -3+\sqrt{3} \end{bmatrix}$
$w_3' := \frac{1}{2}(3 + \mathrm{I} - 3\mathsf{J} + \mathsf{K})$	$\frac{1}{2}\begin{bmatrix} 3+\sqrt{3} & -3+\sqrt{3} \\ 3+\sqrt{3} & 3-\sqrt{3} \end{bmatrix}$
$w_6 := w_2 w_3 = -3\mathsf{J} + \mathsf{K}$	$\begin{bmatrix} 0 & -3+\sqrt{3} \\ 3+\sqrt{3} & 0 \end{bmatrix}$

	P_0	P_1	P_2	P_3	P_4	P_6
w_2	P_7	P_3	P_4	P_5	$*$	P_6
w_3	P_8	$*$	P_2	P_6	$*$	P_1
w_3'	$*$	$*$	$*$	P_6	P_4	P_5
w_6	P_0	$*$	P_4	P_6	P_2	P_3
$w_6\eta_2^{-1}$	$*$	P_6	P_4	$*$	$*$	P_5

	$[P_1, P_2, P_6]$	$[P_2, P_3, P_6]$	$[P_3, P_4, P_6]$
ω_2	$[P_3, P_4, P_6]$	$[P_4, P_5, P_6]$	$*$
ω_3	$*$	$[P_2, P_6, P_1]$	$*$
ω_3'	$*$	$*$	$[P_6, P_4, P_5]$
ω_6	$*$	$[P_4, P_6, P_3]$	$*$
$\omega_6\eta_2^{-1}$	$[P_6, P_4, P_5]$	$*$	$*$



Fundamental domains for $X_6^{(2)}$ and $X_6^{(3)}$



Fundamental domains for $X_6^{(6)}$ and X_6^+

angles (fhd)	P_0	P_2	P_3	P_4	P_6	P_7
X_6	$*$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$*$
$X_6^{(2)}$	$*$	$*$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{4}$	$*$
$X_6^{(3)}$	π	$\frac{\pi}{6}$	$\frac{\pi}{2}$	$\frac{\pi}{6}$	$*$	$*$
$X_6^{(6)}$	$\frac{\pi}{2}$	$*$	$*$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{\pi}{2}$
X_6^{+}	$\frac{\pi}{2}$	$*$	$*$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$*$

X_6	P_0	$P_1 = P_3 = P_5$	P_2	P_4	P_6	$P_7 = P_8$
$X_6^{(2)}$	$P_0 = P_7$	P_3	$P_2 = P_4$	$P_2 = P_4$	P_6	$P_0 = P_7$
$X_6^{(3)}$	$P_0 = P_7$	$P_3 = P_6$	P_2	P_4	$P_3 = P_6$	$P_0 = P_7$
$X_6^{(6)}$	P_0	$P_3 = P_6$	$P_2 = P_4$	$P_2 = P_4$	$P_3 = P_6$	P_7
X_6^{+}	$P_0 = P_7$	$P_3 = P_6$	$P_2 = P_4$	$P_2 = P_4$	$P_3 = P_6$	$P_0 = P_7$

Identification of points

ramification	P_0	P_2	P_3	P_4	P_6	P_7
$X_6 \longrightarrow X_6^{(2)}$	P_0P_7	$*$	P_3^2	P_2P_4	P_6^2	$*$
$X_6 \longrightarrow X_6^{(3)}$	P_0P_7	P_2^2	$*$	P_4^2	P_3P_6	$*$
$X_6 \longrightarrow X_6^{(6)}$	P_0^2	$*$	$*$	P_2P_4	P_3P_6	P_7^2
$X_6^{(2)} \longrightarrow X_6^+$	P_0^2	$*$	$*$	P_4^2	P_3P_6	$*$
$X_6^{(3)} \longrightarrow X_6^+$	P_0^2	$*$	$*$	P_2P_4	P_6^2	$*$
$X_6^{(6)} \longrightarrow X_6^+$	P_0P_7	$*$	$*$	P_4^2	P_6^2	$*$

Uniformizing functions for the quotients of X_6

$\mathbb{C}(X_6^+) = \mathbb{C}(t_6^+)$

$\mathbb{C}(X_6^{(2)}) = \mathbb{C}(t_6^{(2)})$, $\mathbb{C}(X_6^{(3)}) = \mathbb{C}(t_6^{(3)})$, $\mathbb{C}(X_6^{(6)}) = \mathbb{C}(t_6^{(6)})$

$\mathbb{C}(X_6) = \mathbb{C}(t_6)$

Each function is determined by its values at three vertices.

$t_6^{(2)}, t_6^{(3)}, t_6^+$ are triangle functions; $t_6^{(6)}, t_6$ are quadrilateral functions.

initial values	P_0	P_2	P_3	P_4	P_6	P_7	
t_6	$*$	a	0	1	∞	$*$	$a \neq 0, 1, \infty \, (\Rightarrow\ a = -1)$
$t_6^{(2)}$	$*$	$*$	0	1	∞	$*$	$*$
$t_6^{(3)}$	$*$	0	$*$	1	∞	$*$	$*$
$t_6^{(6)}$	0	$*$	$*$	1	∞	b	$b \neq 0, 1, \infty \, (\Rightarrow\ b = 2)$
t_6^{+}	0	$*$	$*$	1	∞	$*$	$*$

Theorem. *The following algebraic relations are fulfilled:*

(a) $4t_6^+ t_6^{(2)} = (t_6^{(2)} + 1)^2.$

(b) $t_6^+ = (2t_6^{(3)} - 1)^2.$

(c) $4t_6^{(2)}(2t_6^{(3)} - 1)^2 = (t_6^{(2)} + 1)^2.$

(d) $t_6^2 = t_6^{(2)}.$

(e) $4t_6 t_6^{(3)} = (t_6 + 1)^2.$

(f) $t_6^+ + t_6^{(6)}(t_6^{(6)} - 2) = 0.$

(g) $2t_6 t_6^{(6)} = i(t_6 - i)^2.$

(h) $4t_6^2 t_6^+ = (t_6^2 + 1)^2.$

(i) $(t_6^{(2)} + 1)^2 + 4t_6^{(2)} t_6^{(6)}(t_6^{(6)} - 2) = 0.$

(j) $(2t_6^{(3)} - 1)^2 + t_6^{(6)}(t_6^{(6)} - 2) = 0.$

Moreover, we have the following values for the functions:

(k) $t_6^{(2)}(P_0) = -1$; (l) $t_6^{(3)}(P_0) = \dfrac{1}{2}$; (m) $b = t_6^{(6)}(P_7) = 2$;

(n) $a = t_6(P_2) = -1$; (o) $t_6(P_0) = i$; (p) $t_6(P_7) = -i$. $\square$

Proof. $c := t_6^{(2)}(P_0)$

$$\mathrm{div}(t_6^{(2)} - c)\left(1 - \frac{c}{t_6^{(2)}}\right) = \mathrm{div}(t_6^{+}); \mathrm{div}(t_6^{(2)} - 1)\left(1 - \frac{1}{t_6^{(2)}}\right) = \mathrm{div}(t_6^{+} - 1).$$

$$A(t_6^{(2)} - c)\left(1 - \frac{c}{t_6^{(2)}}\right) = t_6^{+}; B(t_6^{(2)} - 1)\left(1 - \frac{1}{t_6^{(2)}}\right) = t_6^{+} - 1.$$ [...]

Theorem. *Let $\overline{\Gamma}$ be a Fuchsian group of the first kind such that the associated curve $X(\overline{\Gamma})$ is of genus 0. Assume that we are aware of a fundamental half domain for the action of $\overline{\Gamma}$ in $\mathcal{H}$. Suppose that t is a generator of the field of $\overline{\Gamma}$-automorphic functions such that its values at the vertices of the fundamental half domain belong to $\mathbf{P}^1(\mathbb{R})$. Then, there exists a rational function $R(t)$ such that $Da(t,z)+R(t) = 0$. If $\alpha_i\pi$ are the internal angles at the vertices of the fundamental half domain, then*

$$R(t) = \sum \frac{1-\alpha_i^2}{(t-a_i)^2} + \sum \frac{B_i}{t-a_i},$$

where B_i are constants and the summation extends over all the vertices of the fundamental half domain where the function t takes finite values a_i. Moreover, if the values of t at all the vertices are finite, then *(a)* $\sum B_i = 0$,

(b) $\sum a_i B_i + \sum (1 - \alpha_i^2) = 0$,

(c) $\sum a_i^2 B_i + \sum a_i (1 - \alpha_i^2) = 0$.

But if ∞ *is the value of* t *at a vertex with internal angle* $\alpha\pi$, *then*

(a) $\sum B_i = 0$,

(b) $\sum a_i B_i + \sum (1 - \alpha_i^2) - (1 - \alpha^2) = 0$. $\square$

In our case, the relations will determine the constants B_i for three of the five functions $R(t)$: for those associated to $t^{(2)}, t^{(3)}$,and t^{+}.

We compare t_6 and t_6^2: $\rightarrow B_4 = -\frac{8}{9}$

We compare $t_6^{(6)}$ and t_6^{+}: $\rightarrow B_{(6)7} = -\frac{59}{72}$.

We deduce in this way the values of the 5 automorphic derivatives:

Curve	function	angles	$-Da(t,z)$
X_6	t_6	$[P_2, P_3, P_4, P_6]$ $[\pi/3, \pi/2, \pi/3, \pi/2]$	$\dfrac{27t^4 + 74t^2 + 27}{36t^2(t^2 - 1)^2}$
$X_6^{(2)}$	$t_6^{(2)}$	$[P_3, P_4, P_6]$ $[\pi/4, \pi/3, \pi/4]$	$\dfrac{135t^2 - 142t + 135}{144t^2(t - 1)^2}$
$X_6^{(3)}$	$t_6^{(3)}$	$[P_2, P_4, P_6]$ $[\pi/6, \pi/6, \pi/2]$	$\dfrac{27t^2 - 27t + 35}{36t^2(t - 1)^2}$
$X_6^{(6)}$	$t_6^{(6)}$	$[P_0, P_4, P_7, P_6]$ $[\pi/2, \pi/3, \pi/2, \pi/2]$	$\dfrac{27t^4 - 108t^3 + 211t^2 - 206t + 108}{36t^2(t^2 - 3t + 2)^2}$
X_6^{+}	t_6^{+}	$[P_0, P_4, P_6]$ $[\pi/2, \pi/6, \pi/4]$	$\dfrac{135t^2 - 103t + 108}{144t^2(t - 1)^2}$

Objective 2: to obtain explicit expansions of the uniformizing functions around the elliptic points and around the SCM-points.

Definition. *A local parameter at a point $P \in \mathcal{H}$ for the $\overline{\Gamma}_P$-action is any function*

$$q(z) := \left(k\frac{z-P}{z-\overline{P}}\right)^e,$$

where $e = \#\overline{\Gamma}_P$ is the order of the isotropy group at P and $k \in \mathbb{C}$ is any constant. The local parameter q is said to be adapted to a function $t = \sum_{n=m}^{\infty} a_{ne} q^n$ when, moreover, $a_e = 1$ if $m \geq 0$; and $a_{-e} = 1$ otherwise.

Suppose that $P \in \mathcal{H}$ is any elliptic point of order e for the $\overline{\Gamma}$-action. By definition, the isotropy group at P, $\overline{\Gamma}_P$, will be generated by a transformation $g \in \mathbf{PSL}(2, \mathbb{R})$ of order $e > 1$. Let $G \in \Gamma \subseteq \mathbf{SL}(2, \mathbb{R})$ be a matrix defining g. Since in all our cases $-1 \in \Gamma$, we may take the matrix G of order $2e$ and, since g is an elliptic transformation, the matrix G can be diagonalized. Let $H \in \mathbf{GL}(2, \mathbb{C})$ be such that $D := HGH^{-1} = \begin{bmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{bmatrix}$, where ζ is a $2e$-th primitive root of unity. We denote by h and d the homographic transformations of $\mathbf{P}^1(\mathbb{C})$ defined by H and D, respectively. Then

$$(*) \qquad h(g(z)) = d(h(z)) = \zeta^2 h(z).$$

By evaluating $(*)$ at the points $z = P$ and $z = \overline{P}$, we obtain

$$h(P) = h(g(P)) = \zeta^2 h(P), \quad h(\overline{P}) = h(g(\overline{P})) = \zeta^2 h(\overline{P}).$$

Since $e > 1$, is $\zeta^2 \neq 1$ and, since h is a bijective mapping of $\mathbf{P}^1(\mathbb{C})$, we must have $h(P) = 0$ and $h(\overline{P}) = \infty$ (or $h(P) = \infty$ and $h(\overline{P}) = 0$). Hence, we have

$$h(z) = k\frac{z - P}{z - \overline{P}}, \quad \left(\text{ or } h(z) = k\frac{z - \overline{P}}{z - P} \right),$$

for some constant $k \in \mathbb{C}$ to be determined. We can expand any $\overline{\Gamma}_P$-automorphic function t around the point P as a power series T in the variable $h(z) = k(z - P)/(z - \overline{P})$:

$$t(z) = T(h(z)) = \sum_{n=n_0}^{\infty} a_n h(z)^n.$$

We shall have $T(h(z)) = t(z) = t(g(z)) = T(h(g(z))) = T(\zeta^2 h(z))$. Thus $a_n = 0$ unless $n \equiv 0 \pmod{e}$.

$$q_P(z) := \left(k_P \frac{z-P}{z-\overline{P}}\right)^e$$

How to choose k_P?

cf. Carathéodory, Wolfart

Hypergeometric function

$$F(a,b,c;w) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \frac{w^n}{n!}, \qquad (a)_n := a(a+1)\dots(a+n-1), \qquad |w| < 1$$

Assume that $c \neq 1$.

• The functions $F(a,b,c;w)$ and $w^{1-c}F(a-c+1, b-c+1, 2-c; w)$ are two linearly independent solutions of the hypergeometric differential equation

$$w(1-w)D^2(f,w) + (c-(1+a+b)w)D(f,w) - ab\, f = 0.$$

The Schwarzian function

$$z = s(a,b,c;w) := \frac{w^{1-c}F(a-c+1,b-c+1,2-c;w)}{F(a,b,c;w)}$$

maps the upper half w-plane $\mathcal{H}$ onto a triangle in the z-plane. The vertices of this triangle are

$$\begin{aligned} s(a,b,c;0) &= 0, \\ s(a,b,c;1) &= \frac{\Gamma(c-a)\Gamma(c-b)\Gamma(2-c)}{\Gamma(c)\Gamma(1-b)\Gamma(1-a)}, \\ s(a,b,c;\infty) &= e^{\pi i(1-c)}\frac{\Gamma(c-a)\Gamma(b)\Gamma(2-c)}{\Gamma(c)\Gamma(b-c+1)\Gamma(1-a)}. \end{aligned}$$

The internal angles at these vertices are $\alpha\pi$, $\beta\pi$, $\gamma\pi$, where

$$\alpha = 1-c \neq 0, \quad \beta = c-a-b, \quad \gamma = b-a. \ \square$$

The computation of the constants k_P

- Local parameters adapted to the triangle functions

We compare the triangle $[s(0), s(1), s(\infty)]$ with the triangles defining our functions t_6^+, $t_6^{(2)}$, and $t_6^{(3)}$. In each case, this will allow us to obtain the local constant k_P of the adapted local parameter in closed form.

t	$[A,B,C]$	e_A	$t(A)$	ν_A	k_A
t_6^+	$[P_0,P_4,P_6]$	2	0	$2^3\cdot 3^2$	$i\dfrac{\sqrt{2}+\sqrt{3}}{2}\dfrac{\Gamma(7/24)\Gamma(11/24)}{\Gamma(19/24)\Gamma(23/24)}$
t_6^+	$[P_4,P_6,P_0]$	6	1	$\dfrac{1}{2\cdot 3^2}$	$\dfrac{2+\sqrt{3}-i}{12}\dfrac{\Gamma(1/6)\Gamma(7/24)\Gamma(19/24)}{\Gamma(5/6)\Gamma(11/24)\Gamma(23/24)}$
t_6^+	$[P_6,P_0,P_4]$	4	∞	$2^5\cdot 3$	$\dfrac{\sqrt{2}+\sqrt{3}}{4}\dfrac{\Gamma(1/4)\Gamma(13/24)\Gamma(17/24)}{\Gamma(3/4)\Gamma(19/24)\Gamma(23/24)}$

t	$[A,B,C]$	e_A	$t(A)$	ν_A	k_A
$t_6^{(2)}$	$[P_3,P_4,P_6]$	4	0	$\frac{3^2}{2^4}$	$\frac{(1+\sqrt{3})(1+i)}{8}\frac{\Gamma(1/4)\Gamma(5/12)}{\Gamma(3/4)\Gamma(11/12)}$
$t_6^{(2)}$	$[P_4,P_6,P_3]$	3	1	$\frac{2}{3}$	$\frac{2+\sqrt{3}-i}{6}\frac{\Gamma(1/3)^2\Gamma(7/12)}{\Gamma(2/3)^2\Gamma(11/12)}$
$t_6^{(2)}$	$[P_6,P_3,P_4]$	4	∞	$2^8\cdot 3$	$\frac{\sqrt{3}}{4}\frac{\Gamma(1/3)\Gamma(2/3)\Gamma(1/4)}{\Gamma(3/4)\Gamma(7/12)\Gamma(11/12)}$

t	$[A,B,C]$	e_A	$t(A)$	ν_A	k_A
$t_6^{(3)}$	$[P_2,P_4,P_6]$	6	0	$\frac{1}{2^3\cdot 3^2}$	$\frac{(1+\sqrt{3})(1+i)}{12}\frac{\Gamma(1/6)\Gamma(7/12)}{\Gamma(5/6)\Gamma(11/12)}$
$t_6^{(3)}$	$[P_4,P_6,P_2]$	6	1	$\frac{1}{2^3\cdot 3^2}$	$\frac{2+\sqrt{3}-i}{12}\frac{\Gamma(1/6)\Gamma(7/12)}{\Gamma(5/6)\Gamma(11/12)}$
$t_6^{(3)}$	$[P_6,P_2,P_4]$	2	∞	2	$\frac{(1+\sqrt{3})(1+i)}{4}\frac{\Gamma(1/4)\Gamma(5/12)}{\Gamma(3/4)\Gamma(11/12)}$

Local constants for the triangle functions

Proof. First we explain the results for the case $t(A) = 0$. By formal integration of the differential equation of the third order and taking into account that $t(A) = 0$, it follows that there exists a normalized power series in two variables

$$r(X, Y) = \sum_{n=1}^{\infty} a_{ne} X^{en} Y^{en}, \quad a_e = 1,$$

and a constant $\lambda \in \mathbb{C}$, such that

$$t(z) = r(\lambda; h_1(z)) = \sum_{n=1}^{\infty} a_{ne} \lambda^{en} h_1^{en}(z),$$

for any z in a neighbourhood of A. Here we take $h_1(z) := \dfrac{z - A}{z - \overline{A}}$.

Consider the Schwarzian function $s(a, b, c; w)$ determined by the angles $\alpha\pi$, $\beta\pi$, $\gamma\pi$. Since r satisfies the conditions

$$r(\lambda; h_1(A)) = 0, \quad r(\lambda; h_1(B)) = 1, \quad r(\lambda; h_1(C)) = \infty,$$

we can relate the inverse of the series defining $s(a, b, c; w)$ to the series defining $t(z)$. A direct computation of the first terms in both series suffices to establish the following lemma.

Lemma. *Let $u(a, b, c; z)$ denote the inverse series of $s(a, b, c; w)$. Then*

$$r(\zeta_e; h_1(z)) = u(a, b, c; h_1(z))$$

for any $z \in \mathbb{C}$ in the convergence domain and any e-th root of unity ζ_e. $\square$

To continue the calculation of λ, we may use either the condition $t(B) = 1$ or, alternatively, the condition $t(C) = \infty$. In the first case, we obtain that

$$1 = t(B) = r(\lambda; h_1(B)) = r(1; \lambda\ h_1(B)) = r(\zeta_e; \zeta_e^{-1}\lambda\ h_1(B)),$$

and

$$\zeta_e^{-1}\lambda\ h_1(B) = s(a,b,c;1) = \frac{\Gamma(c-a)\Gamma(c-b)\Gamma(2-c)}{\Gamma(c)\Gamma(1-b)\Gamma(1-a)}.$$

We can conclude that

$$\lambda = \zeta_e \frac{B-\overline{A}}{B-A}\ \frac{\Gamma(c-a)\Gamma(c-b)\Gamma(2-c)}{\Gamma(c)\Gamma(1-b)\Gamma(1-a)}.$$

• Local parameters adapted to the quadrilateral functions

Next result relates the local constants for two points in $\mathcal{H}$ in the same $\overline{\Gamma}$-orbit.

Lemma. *Let* $P \in \mathcal{H}$ *be a point of order* $e \geq 1$ *for the* $\overline{\Gamma}$*-action. For any* $w = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma \subseteq \mathbf{SL}(2, \mathbb{R})$*, the local constants* k_P *and* $k_{w(P)}$ *adapted to a* $\overline{\Gamma}$*-automorphic function* t*, at* P *and* $w(P)$*, are related by*

$$k^e_{w(P)} = k^e_P \left(\frac{cP + d}{c\overline{P} + d} \right)^e .$$

t	P	e_P	$t(P)$	ν_P	k_P
$t_6^{(6)}$	P_0	2	0	$2^2 \cdot 3^2$	$i\frac{\sqrt{2}+\sqrt{3}}{2\sqrt{2}}\frac{\Gamma(7/24)\Gamma(11/24)}{\Gamma(19/24)\Gamma(23/24)}$
$t_6^{(6)}$	P_4	3	1	3^{-1}	$\frac{(1+\sqrt{3})(1+i)}{12}\frac{\Gamma(1/6)\Gamma(7/24)\Gamma(19/24)}{\Gamma(5/6)\Gamma(11/24)\Gamma(23/24)}$
$t_6^{(6)}$	P_7	2	2	$2^2 \cdot 3^2$	$\frac{(2\sqrt{3}+3\sqrt{2})(\sqrt{2}+i)}{12}\frac{\Gamma(7/24)\Gamma(11/24)}{\Gamma(19/24)\Gamma(23/24)}$
$t_6^{(6)}$	P_6	2	∞	2^2	$i\frac{\sqrt{2}+\sqrt{3}}{4}\frac{\Gamma(1/4)\Gamma(13/24)\Gamma(17/24)}{\Gamma(3/4)\Gamma(19/24)\Gamma(23/24)}$

t	P	e_P	$t(P)$	ν_P	k_P
t_6	P_0	1	i	$2^2\cdot 3$	$i\frac{\sqrt{2}+\sqrt{3}}{2}\frac{\Gamma(7/24)\Gamma(11/24)}{\Gamma(19/24)\Gamma(23/24)}$
t_6	P_2	3	-1	3^{-1}	$\frac{1+(2+\sqrt{3})i}{6\sqrt[3]{2}}\frac{\Gamma(1/3)^2\Gamma(7/12)}{\Gamma(2/3)^2\Gamma(11/12)}$
t_6	P_3	2	0	$3\cdot 2^{-2}$	$\frac{(1+\sqrt{3})(1+i)}{8}\frac{\Gamma(1/4)\Gamma(5/12)}{\Gamma(3/4)\Gamma(11/12)}$
t_6	P_4	3	1	3^{-1}	$\frac{2+\sqrt{3}-i}{6\sqrt[3]{2}}\frac{\Gamma(1/3)^2\Gamma(7/12)}{\Gamma(2/3)^2\Gamma(11/12)}$
t_6	P_6	2	∞	2^4	$\frac{\sqrt{3}(1-i)}{4\sqrt{2}}\frac{\Gamma(1/3)\Gamma(2/3)\Gamma(1/4)}{\Gamma(3/4)\Gamma(7/12)\Gamma(11/12)}$

Example: Computation of k_{P_3} for X_6

$$X_6 \longrightarrow X_6^{(2)}, \quad t_6^2 = t_6^{(2)},$$

$$h_1(z) = \frac{z - P_3}{z - \overline{P_3}}, \quad e_6^{(2)} = 4, \quad e_6 = 2 \quad \text{isotropy at } P_3$$

$$t_6^{(2)}(z) = r_6^{(2)}(\lambda_6^{(2)}; h_1(z)) = \sum_{n=1}^{\infty} a_{4n}^{(2)} {\lambda_6^{(2)}}^{4n} h_1^{4n}(z), \quad a_4^{(2)} = 1,$$

$$t_6(z) = r_6(\lambda_6; h_1(z)) = \sum_{n=1}^{\infty} a_{2n} \lambda_6^{2n} h_1^{2n}(z), \quad a_2 = 1,$$

$$\lambda_6^4 = {\lambda_6^{(2)}}^4, \quad t_6(P_0) = i \quad \Rightarrow \quad \lambda_6 = \pm \lambda_6^{(2)}$$

At this point, it would be natural to consider the adapted local parameter

$$q_A(z) = \left(k_A \frac{z - A}{z - \overline{A}}\right)^{e_A}$$

as a uniformizing variable in the neighbourhood of the point A. By doing this, we would obtain series developments,

$$t(z) = \sum_{n=1}^{\infty} b_n q^n, \qquad b_n := a_{ne}, \;\; b_1 = 1, \quad \text{if } t(A) = 0,$$

$$t(z) = t(A) + \sum_{n=1}^{\infty} b_n q^n, \;\; b_n := a_{ne}, \;\; b_1 = 1, \quad \text{if } t(A) \neq 0, \infty,$$

$$t(z) = \sum_{n=-1}^{\infty} b_n q^n, \qquad b_n := a_{ne}, \;\; b_{-1} = 1, \;\; \text{if } t(A) = \infty.$$

Objective 3: to obtain explicit expansions of the uniformizing functions around the elliptic points and around the SCM-points with integer coefficients.

- In the classical case of $X_0(1)$:

$$j(q) = 1728\, v(q), \quad q(z) = \exp(2\pi i z),$$

$$j(q) = \frac{1}{q} + 744 + 196884q + 21493760q^2 + 864299970q^3 + 20245856256q^4 + 333202640600q^5 + 4252023300096q^6 + O(q^7)$$

Case $t(P) = 0$.

$$t(z) = \sum_{n=1}^{\infty} b'_n \frac{q(z)^n}{(en)!}, \qquad b'_1 = e!.$$

Replace q by $\nu^{-1}q$:

$$t(z) = \sum_{n=1}^{\infty} b''_n \frac{q(z)^n}{(en)!}, \qquad b''_1 = \nu e!.$$

$\mathfrak{n}_0 := \nu e!, \qquad j(P, q_P; z) := \mathfrak{n}_0^{-1} t(z)$

$$j(P, q_P; z) = \sum_{n=1}^{\infty} c_n \frac{q_P(z)^n}{(en)!}, \quad c_1 = 1, \quad q_P(z) = \frac{1}{\nu_P}\left(k_P \frac{z-P}{z-\overline{P}}\right)^{e_P}$$

Coefficients c_n $(1 \le n \le 10)$ of $j_6^+(P_0, q_{P_0}; z)$:

$$
\begin{aligned}
1 &= 1 \\
-452 &= -2^{2} \cdot 113 \\
368782 &= 2 \cdot 23 \cdot 8017 \\
-465743904 &= -2^{5} \cdot 3 \cdot 1721 \cdot 2819 \\
840330350424 &= 2^{3} \cdot 3^{3} \cdot 61 \cdot 1117 \cdot 57097 \\
-2050858105802208 &= -2^{5} \cdot 3^{3} \cdot 2373678363197 \\
6503028742464357168 &= 2^{4} \cdot 3^{4} \cdot 13 \cdot 743 \cdot 519491571737 \\
-25981826380934619350016 &= -2^{10} \cdot 3^{6} \cdot 547 \cdot 24499 \cdot 2597206657 \\
127675097928802324852258176 &= 2^{7} \cdot 3^{7} \cdot 17 \cdot 31 \cdot 41 \cdot 62501 \cdot 337727176363 \\
-756716925891887407770855224832 &= -2^{9} \cdot 3^{8} \cdot 19 \cdot 103 \cdot 947 \cdot 1307 \cdot 92998738431167
\end{aligned}
$$

●

Coefficients c_n ($1 \leq n \leq 10$) of $j_6^{(2)}(P_3, q_{P_3}; z)$:

$$\begin{aligned}
1 &= 1 \\
-448 &= -2^6 \cdot 7 \\
959904 &= 2^5 \cdot 3^3 \cdot 11 \cdot 101 \\
-6103968192 &= -2^6 \cdot 3^4 \cdot 7 \cdot 59 \cdot 2851 \\
90923623432416 &= 2^5 \cdot 3^4 \cdot 19 \cdot 1846239917 \\
-272112208073671996 8 &= -2^5 \cdot 3^5 \cdot 7 \cdot 11 \cdot 23 \cdot 197593754483 \\
1472791299519578482 91664 &= 2^4 \cdot 3^6 \cdot 7 \cdot 29 \cdot 195791 \cdot 317691018137 \\
-13341613069114979697787419072 &= -2^6 \cdot 3^6 \cdot 7 \cdot 31 \cdot 827 \cdot 15934395613479806 93 \\
1906232128795344429236197931669856 &= 2^5 \cdot 3^{10} \cdot 7^2 \cdot 11 \cdot 621799 \cdot 144613813 \cdot 20814438419 \\
-40985740764555680890794196837836887 0688 &= -2^5 \cdot 3^8 \cdot 7 \cdot 19 \cdot 322592534160773 \cdot 45499520472986
\end{aligned}$$

•

Coefficients c_n $(1 \leq n \leq 10)$ of $j_6^{(3)}(P_2, q_{P_2}; z)$:

$$
\begin{aligned}
&1\\
&-5676\\
&532178676\\
&-328466329655904\\
&845716702991756382900\\
&-6887589447952825780433918400\\
&14724982271825666368720234114339440 0\\
&-7225649830857326774763945127976142727449600\\
&73537664521563211269948154230995800941555952524 0000\\
&-14339280956756358339385770598097459207804417691601752816 0000
\end{aligned}
$$

and their factorizations:

$$
\begin{aligned}
&1\\
&-2^2 \cdot 3 \cdot 11 \cdot 43\\
&2^2 \cdot 3^2 \cdot 17 \cdot 19 \cdot 45767\\
&-2^5 \cdot 3^2 \cdot 11 \cdot 23 \cdot 4507937111\\
&2^2 \cdot 3^4 \cdot 5^2 \cdot 29 \cdot 16126171 \cdot 223259851\\
&-2^6 \cdot 3^5 \cdot 5^2 \cdot 11 \cdot 17 \cdot 19 \cdot 101 \cdot 109 \cdot 10243 \cdot 44215113463\\
&2^5 \cdot 3^5 \cdot 5^2 \cdot 41 \cdot 163 \cdot 113341004907003045999648347\\
&-2^{10} \cdot 3^6 \cdot 5^2 \cdot 11 \cdot 23 \cdot 47 \cdot 181 \cdot 4124746079 \cdot 436129280532628393331\\
&2^6 \cdot 3^8 \cdot 5^4 \cdot 17 \cdot 19 \cdot 43 \cdot 53 \cdot 727 \cdot 166723 \cdot 3140530599070642075967702 0203\\
&-2^8 \cdot 3^8 \cdot 5^4 \cdot 11 \cdot 29 \cdot 59 \cdot 7717 \cdot 6269826414944463 02399 \cdot 1499997867399813686137
\end{aligned}
$$

Coefficients c_n $(1 \leq n \leq 10)$ of $j_6^{(6)}(P_0, q_{P_0}; z)$:

$$\begin{aligned}
1 &= 1 \\
-236 &= -2^2 \cdot 59 \\
113902 &= 2 \cdot 56951 \\
-95763552 &= -2^5 \cdot 3 \cdot 571 \cdot 1747 \\
123617657304 &= 2^3 \cdot 3^3 \cdot 15919 \cdot 35951 \\
-226399919228064 &= -2^5 \cdot 3^3 \cdot 262036943551 \\
558634091378761008 &= 2^4 \cdot 3^4 \cdot 13 \cdot 181 \cdot 467 \cdot 4649 \cdot 84377 \\
-1786399367397350427648 &= -2^{10} \cdot 3^6 \cdot 971 \cdot 1279 \cdot 1926909407 \\
7185275996670852724319616 &= 2^7 \cdot 3^6 \cdot 17 \cdot 137 \cdot 149 \cdot 683 \cdot 146369 \cdot 2219629 \\
-35501246674777850728791937536 &= -2^9 \cdot 3^9 \cdot 19 \cdot 29 \cdot 6393383440755160741
\end{aligned}$$

●

Coefficients c_n $(1 \leq n \leq 10)$ of $j_6(P_3, q_{P_3}; z)$:

$$
\begin{aligned}
1 &= 1 \\
0 &= 0 \\
-48 &= -2^4 \cdot 3 \\
0 &= 0 \\
27504 &= 2^4 \cdot 3^2 \cdot 191 \\
0 &= 0 \\
-64498392 &= -2^3 \cdot 3^2 \cdot 7 \cdot 127973 \\
0 &= 0 \\
436272183216 &= 2^4 \cdot 3^4 \cdot 23 \cdot 229 \cdot 63913 \\
0 &= 0
\end{aligned}
$$

●

Case $t(P) \neq 0, \infty$.

$$t(z) = \sum_{n=0}^{\infty} b'_n \frac{q(z)^n}{(en)!}, \qquad b'_1 = e!.$$

Replace q by $\nu^{-1}q$,

$$t(z) = \sum_{n=0}^{\infty} b''_n \frac{q(z)^n}{(en)!}, \qquad b''_1 = \nu e!.$$

$\mathfrak{n}_v = \nu e!, \qquad v = t(P),$

$j(P, q_P; z) := \mathfrak{n}_v^{-1} t(z)$

$$j(P, q_P; z) = \sum_{n=0}^{\infty} c_n \frac{q_P(z)^n}{(en)!}, \quad c_1 = 1, \quad q_P(z) = \frac{1}{\nu_P}\left(k_P \frac{z-P}{z-\overline{P}}\right)^{e_P}$$

Coefficients c_n $(0 \leq n \leq 10)$ of $j_6^{+}(P_4, q_{P_4}; z)$:

$$
\begin{array}{r}
1/40 \\
1 \\
14916 \\
2639563956 \\
2632246040676384 \\
997172716867357008690 0 \\
112207190111310380168705198400 \\
3168053921557488802801671338888930400 \\
19842868236679746002102001177591900778805760 0 \\
250997845360868088796974230287647909186888429620400 00 \\
59544344605718897186542231146660681114740725221767928417600 00
\end{array}
$$

and their factorizations:

$2^{-3} \cdot 5^{-1}$
1
$2^2 \cdot 3 \cdot 11 \cdot 113$
$2^2 \cdot 3^3 \cdot 17 \cdot 151 \cdot 9521$
$2^5 \cdot 3^2 \cdot 11 \cdot 23 \cdot 31 \cdot 13183 \cdot 88397$
$2^2 \cdot 3^4 \cdot 5^2 \cdot 29 \cdot 263 \cdot 1181 \cdot 136672684427$
$2^6 \cdot 3^6 \cdot 5^2 \cdot 11 \cdot 17 \cdot 463 \cdot 787 \cdot 167777 \cdot 8414792699$
$2^5 \cdot 3^5 \cdot 5^2 \cdot 41 \cdot 59 \cdot 113 \cdot 521 \cdot 98865563057 \cdot 1157441966699$
$2^{10} \cdot 3^6 \cdot 5^2 \cdot 11 \cdot 23 \cdot 47 \cdot 563 \cdot 15882182456303747077823649554553$
$2^6 \cdot 3^{10} \cdot 5^4 \cdot 17 \cdot 31 \cdot 53 \cdot 179 \cdot 115577367943 \cdot 183901443880916341956425557$
$2^8 \cdot 3^8 \cdot 5^4 \cdot 11 \cdot 29 \cdot 59 \cdot 2647 \cdot 3491 \cdot 80387 \cdot 5611687 \cdot 123817039 \cdot 724312409 \cdot 806154743287$

●

Coefficients c_n $(0 \leq n \leq 10)$ of $j_6^{(2)}(P_4, q_{P_4}; z)$:

$$
\begin{aligned}
1/4 &= 2^{-2} \\
1 &= 1 \\
40 &= 2^3 \cdot 5 \\
4716 &= 2^2 \cdot 3^2 \cdot 131 \\
1193280 &= 2^6 \cdot 3 \cdot 5 \cdot 11 \cdot 113 \\
552688980 &= 2^2 \cdot 3 \cdot 5 \cdot 37 \cdot 47 \cdot 5297 \\
422330232960 &= 2^7 \cdot 3^3 \cdot 5 \cdot 17 \cdot 151 \cdot 9521 \\
494861914346400 &= 2^5 \cdot 3^2 \cdot 5^2 \cdot 68730821437 \\
842318733016442880 &= 2^{11} \cdot 3^2 \cdot 5 \cdot 11 \cdot 23 \cdot 31 \cdot 13183 \cdot 88397 \\
1997360876347002360000 &= 2^6 \cdot 3^6 \cdot 5^4 \cdot 29 \cdot 7561 \cdot 312386159 \\
6381905387951084855616000 &= 2^9 \cdot 3^4 \cdot 5^3 \cdot 29 \cdot 263 \cdot 1181 \cdot 136672684427
\end{aligned}
$$

●

Coefficients c_n ($0 \leq n \leq 10$) of $j_6^{(3)}(P_4, q_{P_4}; z)$:

$$
\begin{array}{r}
1/10 \\
1 \\
5676 \\
532178676 \\
328466329655904 \\
845716702991756382900 \\
688758944795282578043391840 0 \\
1472498227182566636872023411433944 00 \\
7225649830857326774763945127976142727449600 \\
735376645215632112699481542309958009415559525240000 \\
1433928095675635833938577059809745920780441769160175281600 00
\end{array}
$$

and their factorizations:

$$
\begin{array}{l}
2^{-1} \cdot 5^{-1} \\
1 \\
2^{2} \cdot 3 \cdot 11 \cdot 43 \\
2^{2} \cdot 3^{2} \cdot 17 \cdot 19 \cdot 45767 \\
2^{5} \cdot 3^{2} \cdot 11 \cdot 23 \cdot 4507937111 \\
2^{2} \cdot 3^{4} \cdot 5^{2} \cdot 29 \cdot 16126171 \cdot 223259851 \\
2^{6} \cdot 3^{5} \cdot 5^{2} \cdot 11 \cdot 17 \cdot 19 \cdot 101 \cdot 109 \cdot 10243 \cdot 44215113463 \\
2^{5} \cdot 3^{5} \cdot 5^{2} \cdot 41 \cdot 163 \cdot 11334100490700304599648347 \\
2^{10} \cdot 3^{6} \cdot 5^{2} \cdot 11 \cdot 23 \cdot 47 \cdot 181 \cdot 4124746079 \cdot 43612928053262839331 \\
2^{6} \cdot 3^{8} \cdot 5^{4} \cdot 17 \cdot 19 \cdot 43 \cdot 53 \cdot 727 \cdot 166723 \cdot 314053059907064207596770202 03 \\
2^{8} \cdot 3^{8} \cdot 5^{4} \cdot 11 \cdot 29 \cdot 59 \cdot 7717 \cdot 6269826414944463023999 \cdot 14999978673998136861 37
\end{array}
$$

●

Coefficients c_n $(0 \leq n \leq 10)$ of $j_6^{(6)}(P_4, q_{P_4}; z)$:

$$
\begin{aligned}
1/2 &= 2^{-1} \\
1 &= 1 \\
0 &= 0 \\
-1356 &= -2^2 \cdot 3 \cdot 113 \\
0 &= 0 \\
74611380 &= 2^2 \cdot 3 \cdot 5 \cdot 1243523 \\
0 &= 0 \\
-38683567274400 &= -2^5 \cdot 3^2 \cdot 5^2 \cdot 5372717677 \\
0 &= 0 \\
10178260405689996000\,0 &= 2^6 \cdot 3^4 \cdot 5^4 \cdot 139 \cdot 226002762361 \\
0 &= 0
\end{aligned}
$$

●

Coefficients c_n $(0 \leq n \leq 10)$ of $j_6^{(6)}(P_7, q_{P_7}; z)$:

$$
\begin{aligned}
1/36 &= 2^{-2} \cdot 3^{-2} \\
1 &= 1 \\
236 &= 2^2 \cdot 59 \\
113902 &= 2 \cdot 56951 \\
95763552 &= 2^5 \cdot 3 \cdot 571 \cdot 1747 \\
123617657304 &= 2^3 \cdot 3^3 \cdot 15919 \cdot 35951 \\
226399919228064 &= 2^5 \cdot 3^3 \cdot 262036943551 \\
558634091378761008 &= 2^4 \cdot 3^4 \cdot 13 \cdot 181 \cdot 467 \cdot 4649 \cdot 84377 \\
1786399367397350427648 &= 2^{10} \cdot 3^6 \cdot 971 \cdot 1279 \cdot 1926909407 \\
7185275996670852724319616 &= 2^7 \cdot 3^6 \cdot 17 \cdot 137 \cdot 149 \cdot 683 \cdot 146369 \cdot 2219629 \\
35501246674777850728791937536 &= 2^9 \cdot 3^9 \cdot 19 \cdot 29 \cdot 6393383440755160741
\end{aligned}
$$

●

Coefficients c_n $(0 \leq n \leq 10)$ of $j_6(P_4, q_{P_4}; z)$:

$$
\begin{aligned}
1/2 &= 2^{-1} \\
1 &= 1 \\
20 &= 2^2 \cdot 5 \\
1356 &= 2^2 \cdot 3 \cdot 113 \\
227040 &= 2^5 \cdot 3 \cdot 5 \cdot 11 \cdot 43 \\
74611380 &= 2^2 \cdot 3 \cdot 5 \cdot 1243523 \\
42574294080 &= 2^6 \cdot 3^2 \cdot 5 \cdot 17 \cdot 19 \cdot 45767 \\
38683567274400 &= 2^5 \cdot 3^2 \cdot 5^2 \cdot 5372717677 \\
52554612744944640 &= 2^{10} \cdot 3^2 \cdot 5 \cdot 11 \cdot 23 \cdot 4507937111 \\
101782604056899960000 &= 2^6 \cdot 3^4 \cdot 5^4 \cdot 139 \cdot 226002762361 \\
270629344957362042528000 &= 2^8 \cdot 3^4 \cdot 5^3 \cdot 29 \cdot 16126171 \cdot 223259851
\end{aligned}
$$

●

Coefficients c_n $(0 \leq n \leq 10)$ of $j_6(P_2, q_{P_2}; z)$:

$$\begin{aligned}
-1/2 &= -2^{-1} \\
1 &= 1 \\
-20 &= -2^2 \cdot 5 \\
1356 &= 2^2 \cdot 3 \cdot 113 \\
-227040 &= -2^5 \cdot 3 \cdot 5 \cdot 11 \cdot 43 \\
74611380 &= 2^2 \cdot 3 \cdot 5 \cdot 1243523 \\
-42574294080 &= -2^6 \cdot 3^2 \cdot 5 \cdot 17 \cdot 19 \cdot 45767 \\
38683567274400 &= 2^5 \cdot 3^2 \cdot 5^2 \cdot 5372717677 \\
-52554612744944640 &= -2^{10} \cdot 3^2 \cdot 5 \cdot 11 \cdot 23 \cdot 4507937111 \\
101782604056899960000 &= 2^6 \cdot 3^4 \cdot 5^4 \cdot 139 \cdot 226002762361 \\
-270629344957362042528000 &= -2^8 \cdot 3^4 \cdot 5^3 \cdot 29 \cdot 16126171 \cdot 223259851
\end{aligned}$$

•

Coefficients c_n $(0 \le n \le 10)$ of $j_6(P_0, q_{P_0}; z)$:

$$
\begin{aligned}
i/12 &= -i \cdot (1+i)^{-4} \cdot 3^{-1} \\
1 &= 1 \\
-12i &= i \cdot (1+i)^{4} \cdot 3 \\
-226 &= -2 \cdot 113 \\
5664i &= (1+i)^{10} \cdot 3 \cdot 59 \\
160728 &= 2^{3} \cdot 3 \cdot 37 \cdot 181 \\
-5467296i &= -(1+i)^{10} \cdot 3 \cdot 56951 \\
-211472208 &= -2^{4} \cdot 3^{5} \cdot 109 \cdot 499 \\
9193300992i &= -i \cdot (1+i)^{20} \cdot 3^{2} \cdot 571 \cdot 1747 \\
445513958784 &= 2^{7} \cdot 3^{3} \cdot 128910289 \\
-23734590202368i &= -(1+i)^{18} \cdot 3^{4} \cdot 15919 \cdot 35951
\end{aligned}
$$

●

Case $t(P) = \infty$.

$$t(z) = \sum_{n=-1}^{\infty} b'_n \frac{q(z)^n}{(2e(n+2))!}, \qquad b'_{-1} = (2e)!.$$

Replace q by $\nu^{-1}q$:

$$t(z) = \sum_{n=-1}^{\infty} b''_n \frac{q(z)^n}{(2e(n+2))!}, \qquad b''_{-1} = \nu(2e)!.$$

$\mathfrak{n}_\infty = \nu(2e)!$

$j(P, q_P; z) := \mathfrak{n}_\infty^{-1} t(z)$

$$j(P, q_P; z) = \sum_{n=-1}^{\infty} c_n \frac{q_P(z)^n}{(2e(n+2))!}, \qquad c_{-1} = 1, \qquad q_P(z) = \frac{1}{\nu_P}\left(k_P \frac{z-P}{z-\overline{\overline{P}}}\right)^{e_P}$$

Lemma. *Let* $f(q) := \sum_{n=1}^{\infty} \frac{a_n}{(en)!} q^n$*, be a power series such that* $a_1 = e!$ *and* $a_n \in \mathbb{Z}$*. Define* $\frac{1}{f(q)} = \sum_{n=-1}^{\infty} \frac{b_n}{(2e(n+2))!} q^n$*. Then,* $b_n \in (2e)!\mathbb{Z}$*, for any* $n \geq -1$*.* $\square$

Coefficients c_n $(-1 \leq n \leq 10)$ of $j_6^{+}(P_6, q_{P_6}; z)$:

$$
\begin{array}{r}
1 \\
3343340 \\
111948373987450 \\
-5443586937434533888 0000 \\
1409741272541205464056548352 00000 \\
-1451995069656286502661556678971134279680000 \\
492831002106249915715101794294890905721392281 60000000 \\
-4777228974509029772293671213092611285145988194147197337600000000 \\
117571961942560998519490027906605883901285736655384964649200717803520000 0000 \\
-6677591577235950140211708698754652808384366012611262796805315535238896 43274\backslash \\
\backslash 240000000000 \\
8097322334981560627048285624325266997838412729219764838718576955737893235659 \\
\backslash 524088990863360000000000 \\
-19647291423656032972174019595435700543280910750368686774092167978596223787 92\backslash \\
\backslash 5003671442667164930289664000000000000
\end{array}
$$

and their factorizations:

1
$2^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 167$
$2 \cdot 5^2 \cdot 7 \cdot 11^2 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 101 \cdot 271$
$-2^{16} \cdot 5^4 \cdot 7^3 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 4057$
$2^{15} \cdot 5^5 \cdot 7^2 \cdot 11^3 \cdot 13^3 \cdot 17 \cdot 19^2 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 2046397$
$-2^{18} \cdot 5^4 \cdot 7^4 \cdot 11^3 \cdot 13^2 \cdot 17^2 \cdot 19 \cdot 23^2 \cdot 29 \cdot 31 \cdot 37 \cdot 41 \cdot 43 \cdot 47 \cdot 2049602209$
$2^{16} \cdot 5^7 \cdot 7^5 \cdot 11^3 \cdot 13^2 \cdot 17^2 \cdot 19^2 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 41 \cdot 43 \cdot 47 \cdot 53 \cdot 17683 \cdot 28181 \cdot 14576069$
$-2^{20} \cdot 5^8 \cdot 7^7 \cdot 11^4 \cdot 13^3 \cdot 17^2 \cdot 19^2 \cdot 23^2 \cdot 29 \cdot 31^2 \cdot 37 \cdot 41 \cdot 43 \cdot 47^2 \cdot 53 \cdot 59 \cdot 61 \cdot 277 \cdot 1559 \cdot 24116461$
$2^{19} \cdot 5^8 \cdot 7^6 \cdot 11^4 \cdot 13^3 \cdot 17^2 \cdot 19^2 \cdot 23^2 \cdot 29^2 \cdot 31 \cdot 37 \cdot 41 \cdot 43 \cdot 47 \cdot 53 \cdot 59 \cdot 61 \cdot 67 \cdot 71 \cdot 311 \cdot 6031504093 \cdot$
202040107657
$-2^{23} \cdot 5^{10} \cdot 7^8 \cdot 11^4 \cdot 13^3 \cdot 17^3 \cdot 19^3 \cdot 23^2 \cdot 29 \cdot 31^2 \cdot 37 \cdot 41 \cdot 43 \cdot 47 \cdot 53 \cdot 59 \cdot 61 \cdot 67 \cdot 71 \cdot 73 \cdot 79 \cdot 57487 \cdot$
$959395789497167O1709$
$2^{20} \cdot 3 \cdot 5^9 \cdot 7^8 \cdot 11^6 \cdot 13^4 \cdot 17^3 \cdot 19^2 \cdot 23^2 \cdot 29^2 \cdot 31 \cdot 37^2 \cdot 41 \cdot 43^2 \cdot 47 \cdot 53 \cdot 59 \cdot 61 \cdot 67 \cdot 71 \cdot 73 \cdot 79 \cdot 83 \cdot$
$557 \cdot 63127 \cdot 247969188922729907134591$
$-2^{23} \cdot 3 \cdot 5^{11} \cdot 7^9 \cdot 11^5 \cdot 13^4 \cdot 17^3 \cdot 19^4 \cdot 23^3 \cdot 29^2 \cdot 31^2 \cdot 37 \cdot 41^2 \cdot 43 \cdot 47^2 \cdot 53 \cdot 59 \cdot 61 \cdot 67 \cdot 71 \cdot 73 \cdot 79 \cdot$
$83 \cdot 89 \cdot 1996739944641703 \cdot 3000365731636993457$

●

Coefficients c_n $(-1 \leq n \leq 10)$ of $j_6^{(2)}(P_6, q_{P_6}; z)$:

$$
\begin{array}{r}
1 \\
320320 \\
1897776680800 \\
25407270303655240000 \\
-308356733036737724702363000000 \\
-4142488246645479262995483421374974000 0 \\
11329110107237732941366440149044677487785875 0000 \\
3670539218120769620921601917276433901003637288446925000000 \\
18340419325124171728546151222131453593741680361009377806886262500000 \\
-10584642016145720779619088055290118389392418264002902567171569142793771562500000 \\
-13518097449053674932987377211453854768447917742745189587186761357278963468536578\backslash \\
\backslash 60218750000 \\
426347447623198386969559652711687014265693163941626902981474912318165271235796451\backslash \\
\backslash 77640598002031250000
\end{array}
$$

and their factorizations:

1

$2^6 \cdot 5 \cdot 7 \cdot 11 \cdot 13$

$2^5 \cdot 5^2 \cdot 7 \cdot 11^2 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 29$

$2^6 \cdot 5^4 \cdot 7^4 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 277$

$-2^5 \cdot 5^5 \cdot 7^2 \cdot 11^3 \cdot 13^3 \cdot 17 \cdot 19^2 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 458357$

$-2^5 \cdot 5^4 \cdot 7^5 \cdot 11^3 \cdot 13^2 \cdot 17^2 \cdot 19 \cdot 23^2 \cdot 29 \cdot 31 \cdot 37 \cdot 41^2 \cdot 43 \cdot 47 \cdot 157 \cdot 10631$

$2^4 \cdot 5^7 \cdot 7^5 \cdot 11^3 \cdot 13^2 \cdot 17^2 \cdot 19^2 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 41 \cdot 43 \cdot 47 \cdot 53 \cdot 1253333 \cdot 54568751$

$2^6 \cdot 5^8 \cdot 7^8 \cdot 11^4 \cdot 13^3 \cdot 17^2 \cdot 19^2 \cdot 23^2 \cdot 29 \cdot 31^2 \cdot 37 \cdot 41 \cdot 43 \cdot 47 \cdot 53 \cdot 59 \cdot 61 \cdot 880264179511$

$2^5 \cdot 5^8 \cdot 7^6 \cdot 11^4 \cdot 13^3 \cdot 17^2 \cdot 19^2 \cdot 23^2 \cdot 29^2 \cdot 31 \cdot 37 \cdot 41^2 \cdot 43 \cdot 47 \cdot 53 \cdot 59 \cdot 61 \cdot 67 \cdot 71 \cdot 277 \cdot 1721 \cdot$
4955694727637

$-2^5 \cdot 5^{10} \cdot 7^9 \cdot 11^4 \cdot 13^3 \cdot 17^3 \cdot 19^3 \cdot 23^2 \cdot 29^2 \cdot 31^2 \cdot 37 \cdot 41 \cdot 43 \cdot 47 \cdot 53 \cdot 59 \cdot 61^2 \cdot 67 \cdot 71 \cdot 73 \cdot 79 \cdot$
$532830271 \cdot 3473353087$

$-2^4 \cdot 3 \cdot 5^9 \cdot 7^8 \cdot 11^6 \cdot 13^4 \cdot 17^3 \cdot 19^2 \cdot 23^2 \cdot 29^2 \cdot 31 \cdot 37^2 \cdot 41 \cdot 43^2 \cdot 47 \cdot 53 \cdot 59 \cdot 61 \cdot 67 \cdot 71 \cdot 73 \cdot 79 \cdot$
$83 \cdot 47933 \cdot 210853927 \cdot 94385621098829$

$2^5 \cdot 3 \cdot 5^{11} \cdot 7^{10} \cdot 11^5 \cdot 13^4 \cdot 17^3 \cdot 19^4 \cdot 23^3 \cdot 29^2 \cdot 31^2 \cdot 37 \cdot 41^2 \cdot 43 \cdot 47^2 \cdot 53 \cdot 59 \cdot 61 \cdot 67 \cdot 71 \cdot 73 \cdot 79 \cdot$
$83 \cdot 89 \cdot 4003 \cdot 6833 \cdot 18823841 \cdot 94556917997669$

•

Coefficients c_n $(-1 \leq n \leq 10)$ of $j_6^{(3)}(P_6, q_{P_6}; z)$:

$$
\begin{array}{r}
1 \\
420 \\
385770 \\
0 \\
-5661515059200 \\
0 \\
226935548952137625600 0 \\
0 \\
-7720765145640785980553011200000 \\
0 \\
14055087151878082674316333443484950528000 0 \\
0
\end{array}
$$

and their factorizations:

$$1$$
$$2^2 \cdot 3 \cdot 5 \cdot 7$$
$$2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 167$$
$$0$$
$$-2^{12} \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19^2$$
$$0$$
$$2^{13} \cdot 3^3 \cdot 5^3 \cdot 7^3 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 225257$$
$$0$$
$$-2^{16} \cdot 3^4 \cdot 5^5 \cdot 7^3 \cdot 11^2 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 373 \cdot 346277$$
$$0$$
$$2^{17} \cdot 3^5 \cdot 5^4 \cdot 7^4 \cdot 11^3 \cdot 13^2 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 41 \cdot 43 \cdot 61 \cdot 317 \cdot 467 \cdot 3323$$
$$0$$

●

Coefficients c_n $(-1 \leq n \leq 10)$ of $j_6^{(6)}(P_6, q_{P_6}; z)$:

1
420
237930
0
6071229964800
0
2374058686017642624000
0
7640852733559978782846796800000
0
140299254334201815956492248499478558720000
0

and their factorizations:

$$1$$
$$2^2 \cdot 3 \cdot 5 \cdot 7$$
$$2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 103$$
$$0$$
$$2^9 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 163$$
$$0$$
$$2^{10} \cdot 3^3 \cdot 5^3 \cdot 7^3 \cdot 11 \cdot 13 \cdot 17 \cdot 19^2 \cdot 23 \cdot 313 \cdot 317$$
$$0$$
$$2^{13} \cdot 3^4 \cdot 5^5 \cdot 7^3 \cdot 11^2 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 197 \cdot 239 \cdot 587$$
$$0$$
$$2^{14} \cdot 3^5 \cdot 5^4 \cdot 7^4 \cdot 11^3 \cdot 13^2 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 41 \cdot 43 \cdot 239633828339$$
$$0$$

●

Coefficients c_n $(-1 \leq n \leq 10)$ of $j_6(P_6, q_{P_6}; z)$:

$$
\begin{array}{r}
1 \\
0 \\
18480 \\
0 \\
12803590800 \\
0 \\
-817993722627081000 \\
0 \\
-15607892984532655801995 0000 \\
0 \\
12285995340772011067924117938 0345000 \\
0
\end{array}
$$

$$
j_6(q) = \frac{1}{q} + 18480q + 12803590800q^3 - 817993722627081000q^5 + \ldots
$$

and their factorizations:

$$
\begin{array}{l}
1 \\
0 \\
2^4 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \\
0 \\
2^4 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 11^2 \cdot 13 \cdot 17 \cdot 19 \\
0 \\
-2^3 \cdot 3^3 \cdot 5^3 \cdot 7^3 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 47 \cdot 61 \\
0 \\
-2^4 \cdot 3^4 \cdot 5^5 \cdot 7^3 \cdot 11^2 \cdot 13 \cdot 17 \cdot 19^2 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 13729 \\
0 \\
2^3 \cdot 3^5 \cdot 5^4 \cdot 7^4 \cdot 11^4 \cdot 13^2 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 41 \cdot 43 \cdot 661 \cdot 59107 \\
0
\end{array}
$$

•

We note that each generating function $j(P, q_P; z)$ is a representative of the homothety class of the corresponding function t.

$\mathfrak{n}_\infty$ $t(P)=\infty$	$3870720 = 2^{12}\cdot 3^3\cdot 5\cdot 7$ $j_6^+(P_6, q_{P_6})$	$30965760 = 2^{15}\cdot 3^3\cdot 5\cdot 7$ $j_6^{(2)}(P_6, q_{P_6})$	$48 = 2^4\cdot 3$ $j_6^{(3)}(P_6, q_{P_6})$	$96 = 2^5\cdot 3$ $j_6^{(6)}(P_6, q_{P_6})$	$384 = 2^7\cdot 3$ $j_6(P_6, q_{P_6})$
$\mathfrak{n}_0$ $t(P)=0$	$144 = 2^4\cdot 3^2$ $j_6^+(P_0, q_{P_0})$	$\frac{27}{2} = 2^{-1}\cdot 3^3$ $j_6^{(2)}(P_3, q_{P_3})$	$10 = 2\cdot 5$ $j_6^{(3)}(P_2, q_{P_2})$	$72 = 2^3\cdot 3^2$ $j_6^{(6)}(P_0, q_{P_0})$	$\frac{3}{2} = 2^{-1}\cdot 3$ $j_6(P_3, q_{P_3})$
$\mathfrak{n}_1$ $t(P)=1$	$40 = 2^3\cdot 5$ $j_6^+(P_4, q_{P_4})$	$4 = 2^2$ $j_6^{(2)}(P_4, q_{P_4})$	$10 = 2\cdot 5$ $j_6^{(3)}(P_4, q_{P_4})$	2 $j_6^{(6)}(P_4, q_{P_4})$	2 $j_6(P_4, q_{P_4})$
$\mathfrak{n}_i$ $t(P)=i$	$*$ $*$	$*$ $*$	$*$ $*$	$*$ $*$	$12 = 2^2\cdot 3$ $j_6(P_0, q_{P_0})$

Local uniformizing functions $j(P, q_P; z) = \mathfrak{n}^{-1} t(z)$