

INTRODUCTION TO SHIMURA CURVES, III



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Introduction to Shimura curves

I: Fundamental domains and CM-points (M. Alsina, P. Bayer)

II: Uniformization of Shimura curves (A. Travesa, P. Bayer)

III: Abelian varieties with QM (J. Guàrdia, V. Rotger, P. Bayer)

$$H = \left(\frac{a,b}{\mathbb{Q}} \right), \quad \Phi : H \hookrightarrow \mathbf{M}(2, \mathbb{R}), \quad D = p_1 \cdots p_{2r}, \quad \mathcal{O} \subset H$$

$$\Gamma \subseteq \Phi(\mathcal{O}^1), \quad X(\overline{\Gamma})(\mathbb{C})$$

- Objective: To understand Shimura curves as moduli spaces

Objects that they classify: Abelian surfaces with QM =
fake elliptic curves

CM-points: fake elliptic curves with complex multiplication

- Examples in X_D , for $D = 6, 10, 15$

PEL-type (Shimura): $\Omega = (H, \Phi, *; \mathbb{T}, \mathcal{O}; \mathbb{V})$

- H indefinite quaternion algebra, over \mathbb{Q} ; $\Phi : H \hookrightarrow M(2, \mathbb{R})$;
* a positive anti-involution of H .
- $\mathcal{O} \subseteq H$ maximal order, stable under *;
 $\mathbb{T} : H \times H \rightarrow \mathbb{Q}$ non-deg. alternating form, integer valued over \mathcal{O} .
- $\mathbb{V} = (v_1, \dots, v_s)$ a level structure, provided by finitely many ordered quaternion classes $v_i \in \mathcal{O} \otimes \mathbb{Q} / \mathcal{O} = H / \mathcal{O}$.

P = polarization, **E** = endomorphism, **L** = level

- The alternating form $T : H \times H \rightarrow \mathbb{Q}$:

Consider the positive anti-involution in H : $(\alpha\beta)^* = \beta^*\alpha^*$, $\text{Tr}(\alpha\alpha^*) > 0$.

From the **Skolem-Noether** theorem: There exists $\mu \in H$ such that $\alpha^* = \mu^{-1}\bar{\alpha}\mu$, for any $\alpha \in H$; moreover

$$\mu = -\bar{\mu}, \quad \mu^2 < 0.$$

Since $\mathbb{Q}(\sqrt{-D}) \hookrightarrow H$, μ may be chosen so that $\mu \in \mathcal{O}$ and $\mu^2 = -D$.

$$T(\alpha, \beta) := \text{Tr}(\mu\alpha\bar{\beta})$$

$$\delta := (\text{Pf}(T))^{1/2}$$

Fake elliptic curves: $(A, \iota, \mathcal{L}, W)/\mathbb{C}$ of PEL-type $\Omega = (H, \Phi, *; T, \mathcal{O}; V)$

- (i) An abelian surface A/\mathbb{C} .
- (ii) An injective ring homomorphism $\iota : \mathcal{O} \hookrightarrow \text{End}(A)$ such that $H_1(A, \mathbb{Z})$ regarded as a left \mathcal{O} -module is isomorphic to \mathcal{O} .
- (iii) A principal polarization \mathcal{L} on A such that the associated Rosati anti-involution, $\phi_{\mathcal{L}} : \text{End}^0(A) \rightarrow \text{End}^0(A)$, restricts to the anti-involution $*$ on $\iota(\mathcal{O})$. $\text{End}^0(A) := \text{End}(A) \otimes \mathbb{Q}$.
- (iv) A level structure $W \subseteq H_1(A, \mathbb{Q})$ given by the image of V under the isomorphism $H \simeq H_1(A, \mathbb{Q})$ obtained from (ii).

Fake elliptic curves with CM

$R(d, m) \subseteq \mathbb{Q}(\sqrt{d})$, $d < 0$, suppose that $R(d, m) \subseteq \mathcal{O} \subseteq H$

Definition. A fake elliptic curve $(A, \iota, \mathcal{L}, N)$ is said to admit CM by $R(d, m)$ if

$$\text{End}(A, \iota, \mathcal{L}, N) \simeq R(d, m).$$

In this case,

$$\text{End}^0(A, \iota, \mathcal{L}, N) \simeq \mathbb{Q}(\sqrt{d}), \quad \mathbb{Q}(\sqrt{d}) \otimes_{\mathbb{Q}} H \simeq \mathbf{M}(2, \mathbb{Q}(\sqrt{d}))$$

$$A \sim E \times E, \text{ isogeny, } E \text{ elliptic curve with CM by } \mathbb{Q}(\sqrt{d})$$

Fake elliptic curves with CM are jacobians of genus 2 curves defined over $\overline{\mathbb{Q}}$.

Shimura's canonical model

$$\Omega : \quad \Phi : H \hookrightarrow \mathbf{M}(2, \mathbb{R}), \quad \mu^2 = -D, \quad \alpha \rightarrow \alpha^*; \quad \mathcal{O}(D, 1); \quad N$$

$X(D, N)/\mathbb{Q}$ canonical model

$$\pi : \mathcal{H} \rightarrow \Gamma(D; N) \backslash \mathcal{H}, \quad j_{D, N} : \Gamma(D, N) \backslash \mathcal{H} \longrightarrow X(D, N)(\mathbb{C})$$

$j_{D, N}(\pi(z)) \leftrightarrow [A_z, \iota_z, \mathcal{L}_z, N_z]$, isomorphism class of fake elliptic curves

- CM-points generate class fields: $j_{D, N}(\pi(z)) \in X(D, N)(\mathbb{Q}(\sqrt{d})^{ab})$
- Shimura's reciprocity law, 1959

SCM-points

$$F = \mathbb{Q}(\sqrt{d}) \quad n_{F,2}(X, Y) = X^2 - dY^2$$

$$R = R(d, m) \quad n_{R,2}(X, Y)$$

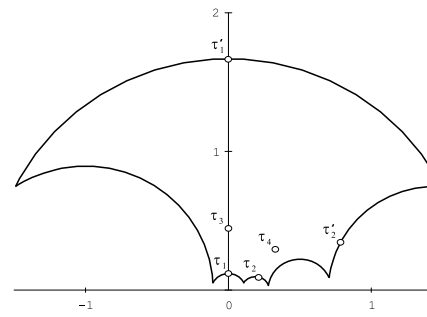
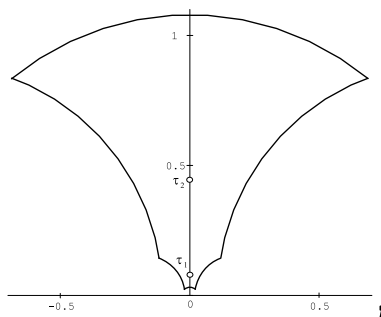
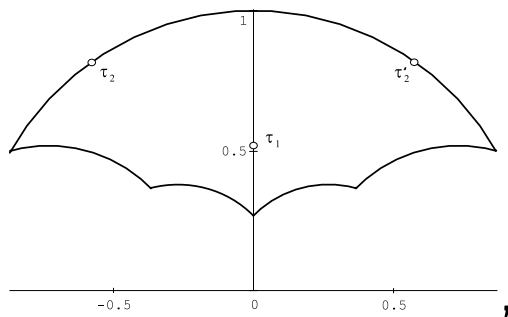
$$n_{R,2}(X, Y) = \begin{cases} X^2 - \frac{D_F}{4}m^2Y^2 & \text{if } D_F \equiv 0 \pmod{4} \\ X^2 + mXY + m^2\frac{1 - D_F}{4}Y^2 & \text{if } D_F \equiv 1 \pmod{4} \end{cases}$$

SCM points in $X(D, N)$:

$$R(d, m) \hookrightarrow \mathcal{O}(D, N), \quad n_{R(d,m),2}(X, Y) \rightarrow DN$$

The moduli interpretation depends on the chosen PEL-type Ω

$$X \rightarrow X_\Omega \hookrightarrow \mathcal{A}_2$$



$D = 6, 10, 15$

General facts on genus 2 curves

$$f(X) = a_n X^n + \cdots + a_1 X + a_0 \in \mathbb{C}[X], \quad \Delta(f) \neq 0, \quad n = 5, 6$$

$$Y^2 = f(X) \rightsquigarrow C_f \quad \text{smooth projective curve} \quad g = \lfloor (n-1)/2 \rfloor = 2$$

$$H^0(C_f, \omega_{C_f}) = \left\langle \frac{dx}{y}, \frac{x dx}{y} \right\rangle = \langle \omega_1, \omega_2 \rangle$$

$$H_1(C_f, \mathbb{Z}) = \langle \gamma_1, \gamma_2, \gamma_3, \gamma_4 \rangle \quad \text{symplectic bases for the intersection pairing}$$

$\mathcal{H}_2 = \{Z \in M(2, \mathbb{C}) : Z = {}^t Z, \text{Im}Z > 0\}$ Siegel upper half space

$$\Omega_f = \begin{bmatrix} \int_{\gamma_1} \omega_1 & \int_{\gamma_2} \omega_1 & \int_{\gamma_3} \omega_1 & \int_{\gamma_4} \omega_1 \\ \int_{\gamma_1} \omega_2 & \int_{\gamma_2} \omega_2 & \int_{\gamma_3} \omega_2 & \int_{\gamma_4} \omega_2 \end{bmatrix} = [\Omega_{1,f} | \Omega_{2,f}]$$

$$Z_f := \Omega_{1,f}^{-1} \Omega_{2,f} \in \mathcal{H}_2$$

$$\begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} := \Omega_{1,f}^{-1} \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix},$$

$$H^0(C_f, \Omega_{C_f}^1) = \langle \eta_1, \eta_2 \rangle, \quad \text{normalized period matrix: } [1_2 | Z_f]$$

$$J(C_f) := \mathbb{C}^2 / \langle 1_2, Z_f \rangle$$

Theta functions

- Riemann theta function:

$$\vartheta(z; Z) = \sum_{n \in \mathbb{Z}^2} \exp(\pi i {}^t n \cdot Z \cdot n + 2\pi i {}^t n \cdot z), \quad z \in \mathbb{C}^2, \quad Z \in \mathcal{H}_2$$

- Theta functions with characteristics: $c = \begin{bmatrix} t_r \\ t_s \end{bmatrix}$, $r, s \in \{0, 1\}^2$

$$\vartheta[c](z; Z) := \exp\left(\frac{1}{4}\pi i {}^t r \cdot Z \cdot r + \frac{1}{2}\pi i {}^t r \cdot (z + s)\right) \vartheta\left(z + \frac{1}{2}Z \cdot r + \frac{1}{2}s; Z\right)$$

$$\vartheta[c](z; Z) \text{ is } \begin{Bmatrix} \text{even} \\ \text{odd} \end{Bmatrix} \Leftrightarrow c \text{ is } \begin{Bmatrix} \text{even} \\ \text{odd} \end{Bmatrix} \Leftrightarrow r \cdot s \equiv \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \pmod{2}$$

We fix a Weierstrass point $W \in C \equiv C_f$ as the base point for the Abel-Jacobi mapping defined by

$$\begin{aligned} \Pi : C &\rightarrow J(C) \\ P &\rightarrow \int_W^P (\eta_1, \eta_2) + \kappa, \end{aligned}$$

where κ is the Riemann vector. It guarantees that

$$\Pi(C) = \Theta - \kappa,$$

being Θ the divisor of the Riemann theta function $\vartheta(z; Z_f)$.

Let α_k denote the roots of f . By means of Π , the six Weierstrass points $W_k = (\alpha_k, 0)$ of C give rise to six odd 2-torsion points in its Jacobian:

$$w_k = \Pi(W_k) \in J(C)[2]^{\text{odd}}.$$

Weierstrass points and torsion points

$$C_f : Y^2 = a_6(X - \alpha_1) \dots (X - \alpha_6), \quad \alpha_k \in \mathbb{C}, \quad J(C_f) \simeq \mathbb{C}^2 / (\text{Id} | Z_f)$$

6 Weierstrass points in $C_f \mapsto 6$ torsion points in $J(C_f)$

- $W_k = (\alpha_k, 0)$, $D_i = \sum_{j \neq i} W_j$, $w_i := \Pi(D_i) \in J(C_f)[2]^{\text{odd}}$

$$m_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad m_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad m_3 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad m_4 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \text{Abel-Jacobi}$$

$$z_{j,k} := \frac{1}{2}(m_j + Z \cdot m_k), \quad (j, k) \in \{(2, 2), (4, 2), (3, 3), (4, 3), (2, 4), (3, 4)\}$$

$$f(X) = a_6X^6 + a_5X^5 + \cdots + a_0, \quad Y^2 = f(X), \quad \Delta(f) \neq 0$$

Theorem. The roots of $f(X)$ are $\alpha_k = -x_{2,k}/x_{1,k}$, where $(x_{1,k}, x_{2,k})$ is a solution of the linear equation

$$\left[\frac{\partial \vartheta}{\partial z_1}(w_k, Z_f) \quad \frac{\partial \vartheta}{\partial z_2}(w_k, Z_f) \right] \Omega_{1,f}^{-1} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = 0,$$

$$w_k \in J(C_f)[2]^{\text{odd}}, \quad 1 \leq k \leq 6.$$

- $\Omega_f = (\Omega_{1,f} | \Omega_{2,f})$ period matrix of C_f with respect to the basis $\left\{ \frac{dx}{y}, x \frac{dx}{y} \right\}$ of $H^0(C, \Omega_C^1)$, and to any symplectic basis of $H_1(C, \mathbb{Z})$

- $Z_f = \Omega_{1,f}^{-1} \Omega_{2,f}$.

cf. Guàrdia, 2002

Theorem. $(A, \mathcal{L})/k$ polarized abelian surface, $k \subseteq \mathbb{C}$

$A(\mathbb{C}) \simeq \mathbb{C}^2 / \langle 1_2, Z \rangle$; $w_1, w_2 \in A[2]^{\text{odd}}, w \in A[2]^{\text{even}}$

$$\Omega_1(Z) := \frac{1}{\vartheta(w, Z)} \begin{bmatrix} \frac{\partial \vartheta}{\partial z_1}(w_1, Z) & \frac{\partial \vartheta}{\partial z_2}(w_1, Z) \\ \frac{\partial \vartheta}{\partial z_1}(w_2, Z) & \frac{\partial \vartheta}{\partial z_2}(w_2, Z) \end{bmatrix}$$

If $\det \Omega_1(Z) \neq 0$, the differential forms

$$(\omega_1, \omega_2) = (dz_1, dz_2) \Omega_1(Z)$$

yield a basis of $H^0(A, \Omega_{/k}^1)$ with periods $[(\Omega_1(Z) \mid \Omega_1(Z)Z)]$.

cf. Weil, Shimura

Equations for fake elliptic curves with CM: $\tau \rightsquigarrow A_\tau \rightsquigarrow C_\tau$

$\tau \in \Gamma \setminus \mathcal{H}$ zero of a binary quadratic form attached to $R \hookrightarrow \mathcal{O}$

$$(\mathbf{E}_\tau) \quad \Lambda_\tau := \Phi(\mathcal{O}) \begin{bmatrix} \tau \\ 1 \end{bmatrix}, \quad A_\tau(\mathbb{C}) = \mathbb{C}^2 / \Lambda_\tau, \quad \iota_\tau : \mathcal{O} \hookrightarrow \text{End}(A_\tau)$$

$$(\mathbf{P}_\tau) \quad E_\tau : \Lambda_\tau \times \Lambda_\tau \longrightarrow \mathbb{Z}, \quad E_\tau(\Phi(\alpha)_\tau, \Phi(\beta)_\tau) = \delta^{-1} T(\alpha, \beta)$$

$$\Lambda_\tau = \langle 1, Z_\tau \rangle, \quad E_\tau \leftrightarrow \mathcal{L}_\tau \quad \text{Appell-Humbert}$$

$$(\mathbf{L}_\tau) \quad \mathbf{V} = (v_1, v_2), \quad v_i \in H/\mathcal{O} \text{ such that } w_i := \Phi(v_i)_\tau \in \mathbb{C}^2 / \Lambda_\tau [2]^{\text{odd}};$$

$$V_\tau := (w_1, w_2)$$

$$[A_\tau, \iota_\tau, \mathcal{L}_\tau, V_\tau]$$

$W := (w_1, w_2)$, $w_i \in A[2]^{\text{odd}}$, $w_i \leftrightarrow c_i$ level structure

$$J(c_1, c_2) = \begin{bmatrix} \frac{\partial \vartheta[c_1]}{\partial z_1}(0; Z) & \frac{\partial \vartheta[c_1]}{\partial z_2}(0; Z) \\ \frac{\partial \vartheta[c_2]}{\partial z_1}(0; Z) & \frac{\partial \vartheta[c_2]}{\partial z_2}(0; Z) \end{bmatrix}, \quad [w_1, w_2] := \det J(c_1, c_2).$$

The algorithm for computing equations for our CM fake elliptic curves is based in the

Theorem. Let C/k be a genus 2 curve and let $Z \in \mathcal{H}_2$ be its normalized period matrix. Then

- (i) For any $w \in J(\mathbb{C})[2]^{\text{odd}}$, the point in $\mathbb{P}^1(\mathbb{C})$ cut by the hyperplane
- $$\begin{bmatrix} \frac{\partial \vartheta}{\partial z_1}(w; Z) & \frac{\partial \vartheta}{\partial z_2}(w; Z) \end{bmatrix} J(w_1, w_2)^{-1} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = 0$$
- is $([w_1, w] : [w_2, w])$.

(ii) Let $J(\mathbb{C})[2]^{\text{odd}} = \{w_i\}_{1 \leq i \leq 6}$. The ratios

$$\ell_{12j} := \frac{[w_2, w_j]}{[w_1, w_j]}, \quad 1 \leq j \leq 6,$$

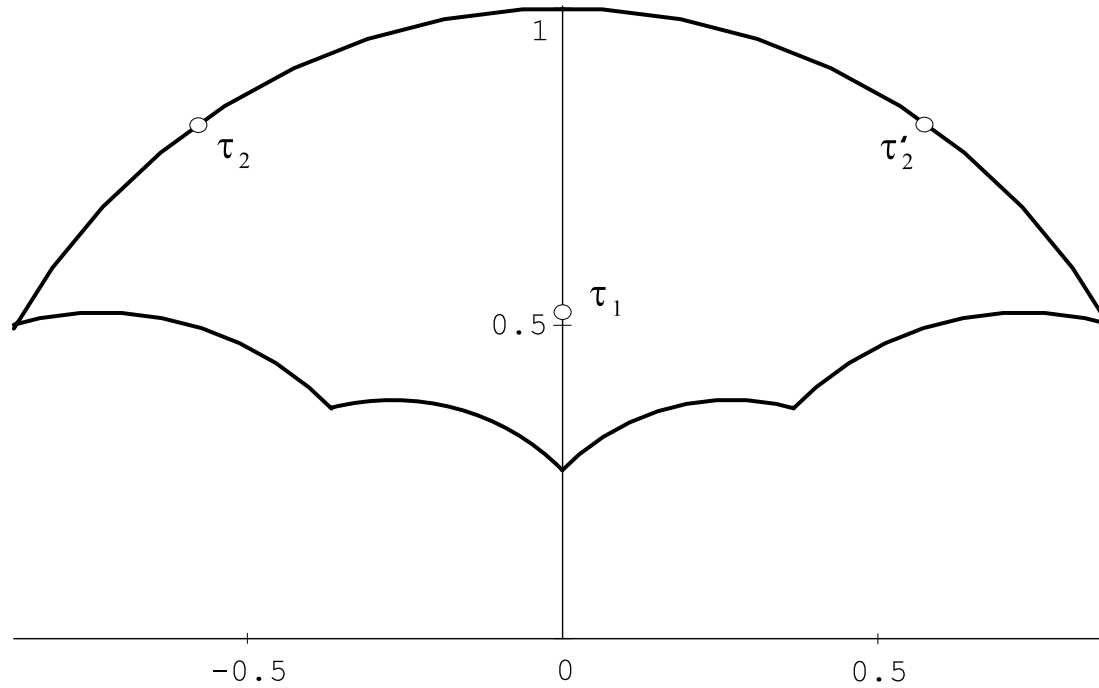
are either ∞ (for $j = 1$) or algebraic over k .

(iii) The curve C admits the hyperelliptic model

$$Y^2 = X(X - \ell_{123})(X - \ell_{124})(X - \ell_{125})(X - \ell_{126}),$$

defined over \bar{k} .

Remark: for each τ , CM-point, the algorithm provides 30 hyperelliptic models for C_τ , defined over $\bar{\mathbb{Q}}$.



Fundamental domain for X_6 and SCM-points:

$$\tau_1 = \frac{i(\sqrt{6} - \sqrt{2})}{2}, \quad \tau'_2 = \frac{\sqrt{3} + i\sqrt{6}}{3} \sim \tau_2 = \frac{-\sqrt{3} + i\sqrt{6}}{3}$$

Example 1. $(H, \Phi : H \hookrightarrow \mathbf{M}(2, \mathbb{R}), *; T, \mathcal{O}; \mathbb{V})$ for $D = 6$

We consider the PEL-type given by the following data:

$$H_6 = \left(\frac{3, -1}{\mathbb{Q}} \right), \quad \mathcal{O}_6 = \mathbb{Z} \left[1, I, J, \frac{1 + I + J + IJ}{2} \right]$$

$$\Phi(I) = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & -\sqrt{3} \end{bmatrix}, \quad \Phi(J) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

$$\mu = I + 3J, \quad \text{Nr}(\mu) = 6, \quad \mu^2 = -6$$

$$T : \mathcal{O}_6 \times \mathcal{O}_6 \rightarrow \mathbb{Z}, \quad T(x, y) := \delta^{-1} \text{Tr}(\mu x \bar{y}), \quad \begin{bmatrix} 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 2 \\ -1 & 0 & 0 & -1 \\ 0 & -2 & 1 & 0 \end{bmatrix}$$

$$\Phi(\mathcal{O}_6) = \mathbb{Z} \left[\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} \sqrt{3} & 0 \\ 0 & -\sqrt{3} \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} + \frac{\sqrt{3}}{2} & \frac{1}{2} + \frac{\sqrt{3}}{2} \\ -\frac{1}{2} + \frac{\sqrt{3}}{2} & \frac{1}{2} - \frac{\sqrt{3}}{2} \end{bmatrix} \right]$$

$$\Phi(\mathcal{O}_6) = \mathbb{Z} \left[1_2, \begin{bmatrix} \frac{-1+\sqrt{3}}{2} & \frac{-1+\sqrt{3}}{2} \\ \frac{1+\sqrt{3}}{2} & \frac{-1-\sqrt{3}}{2} \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 1 - \sqrt{3} & 0 \\ 0 & 1 + \sqrt{3} \end{bmatrix} \right]$$

$$V = (v_1, v_2) = \left(\left(\frac{1}{4}, \frac{1}{4}, -\frac{3}{4}, \frac{1}{4} \right), \left(\frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, \frac{1}{4} \right) \right), \quad v_i \in H_6/\mathcal{O}_6$$

The two CM-points by the maximal order of $\mathbb{Q}(\sqrt{-6})$

$$\tau_1 = \frac{i(\sqrt{6} - \sqrt{2})}{2}, \quad \tau_2 = \frac{\sqrt{3} + i\sqrt{6}}{3}$$

define two fake elliptic curves

$$J_i = \mathbb{C}^2 / \Lambda_i, \quad \text{where } \Lambda_i = \Phi(\mathcal{O}(6, 1)) \cdot \begin{bmatrix} \tau_i \\ 1 \end{bmatrix}$$

The polarization yields $\Lambda_i \sim \langle 1_2, Z_i \rangle$, $Z_i \in \mathcal{H}_2$

$$Z_1 = \begin{bmatrix} \frac{-1+3i\sqrt{6}}{5} & \frac{-2+i\sqrt{6}}{5} \\ \frac{-2+i\sqrt{6}}{5} & \frac{6+2i\sqrt{6}}{5} \end{bmatrix}, \quad Z_2 = \begin{bmatrix} \frac{-1+2i\sqrt{6}}{5} & \frac{-2-i\sqrt{6}}{5} \\ \frac{-2-i\sqrt{6}}{5} & \frac{6+3i\sqrt{6}}{5} \end{bmatrix}.$$

The two points in the Siegel upper half-space

$$Z_1 = \begin{bmatrix} \frac{-1+3i\sqrt{6}}{5} & \frac{-2+i\sqrt{6}}{5} \\ \frac{-2+i\sqrt{6}}{5} & \frac{6+2i\sqrt{6}}{5} \end{bmatrix}, \quad Z_2 = \begin{bmatrix} \frac{-1+2i\sqrt{6}}{5} & \frac{-2-i\sqrt{6}}{5} \\ \frac{-2-i\sqrt{6}}{5} & \frac{6+3i\sqrt{6}}{5} \end{bmatrix}$$

yield the genus 2 curves

$$C_1 : Y^2 = X^5 - 2i\sqrt{2}X^4 - \frac{11}{3}X^3 + 2i\sqrt{2}X^2 + X,$$

$$C_2 : Y^2 = X^5 + 2\sqrt[4]{-5 + 2\sqrt{6}}X^4 - \frac{10}{3}i\sqrt{2}X^3 + 2i\sqrt[4]{-5 - 2\sqrt{6}}X^2 + X$$

The Igusa invariants of C_1, C_2 are

$$i_1 = -\frac{322102}{3}, \quad i_2 = 23958, \quad i_3 = 5082$$

Observation:

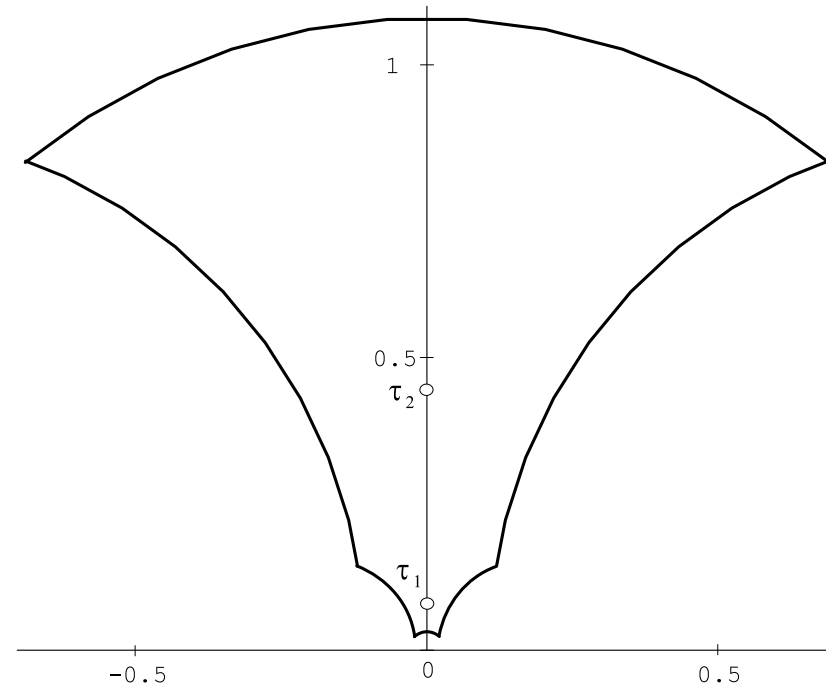
$$i_k([J_1, \mathcal{L}_1]) = i_k([J_2, \mathcal{L}_2]), \text{ for } 1 \leq k \leq 3$$

but

$$j_6(\tau_1) = j_6([J_1, \mathcal{O}_6, \mathcal{L}_1]) = j_6(P_0) = \frac{i}{12},$$

$$j_6(\tau_2) = j_6([J_2, \mathcal{O}_6, \mathcal{L}_2]) = j_6(P_7) = -\frac{i}{12}$$

Forgetful mapping: $X_6 \rightarrow X_6/\langle w_2, w_3 \rangle \hookrightarrow \mathcal{A}_2$, cf. Rotger



Fundamental domain for X_6 and SCM-points:

$$\tau_1 = \frac{i(3\sqrt{5} - 2\sqrt{10})}{5}, \quad \tau_2 = \frac{i\sqrt{5}}{5}.$$

Example 2. $H = \left(\frac{2, 5}{\mathbb{Q}} \right) = \langle 1, I, J, IJ \rangle$, $D = 10$.

We consider the PEL-type given by the following data:

$$\Phi(I) = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & -\sqrt{2} \end{bmatrix}, \quad \Phi(J) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

$$\mu = -IJ, \quad \delta = 10,$$

$$\mathcal{O}_{10} = \mathbb{Z} \left[1, I, \frac{1+J}{2}, \frac{I+IJ}{2} \right],$$

$$\mathcal{V} = \left(\left(\frac{1}{4}, \frac{1}{2}, -\frac{1}{4}, 0 \right), \left(\frac{1}{2}, \frac{1}{2}, 0, 0 \right) \right),$$

$$\Phi(\mathcal{O}_{10}) = \mathbb{Z} \left[\mathbf{1}_2, \begin{bmatrix} \sqrt{2} & 0 \\ 0 & -\sqrt{2} \end{bmatrix}, \begin{bmatrix} -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{5\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} \\ -\frac{5}{2} & -\frac{1}{2} \end{bmatrix} \right].$$

We have two CM-points by the maximal order $R(-10, 1)$:

$$\tau_1 = \frac{i(3\sqrt{5} - 2\sqrt{10})}{5}, \quad \tau_2 = \frac{i\sqrt{5}}{5}.$$

They define the points in the Siegel upper half-space

$$Z_1 = \begin{bmatrix} \frac{3i\sqrt{10}}{2} & -\frac{1}{2} + i\sqrt{10} \\ -\frac{1}{2} + i\sqrt{10} & \frac{3i\sqrt{10}}{4} \end{bmatrix}, \quad Z_2 = \begin{bmatrix} \frac{i\sqrt{10}}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{i\sqrt{10}}{4} \end{bmatrix}$$

which yield the genus 2 curves

$$Y^2 = X^5 + 2\sqrt{5}X^4 + \frac{125}{18}X^3 + 2\sqrt{5}X^2 + X,$$

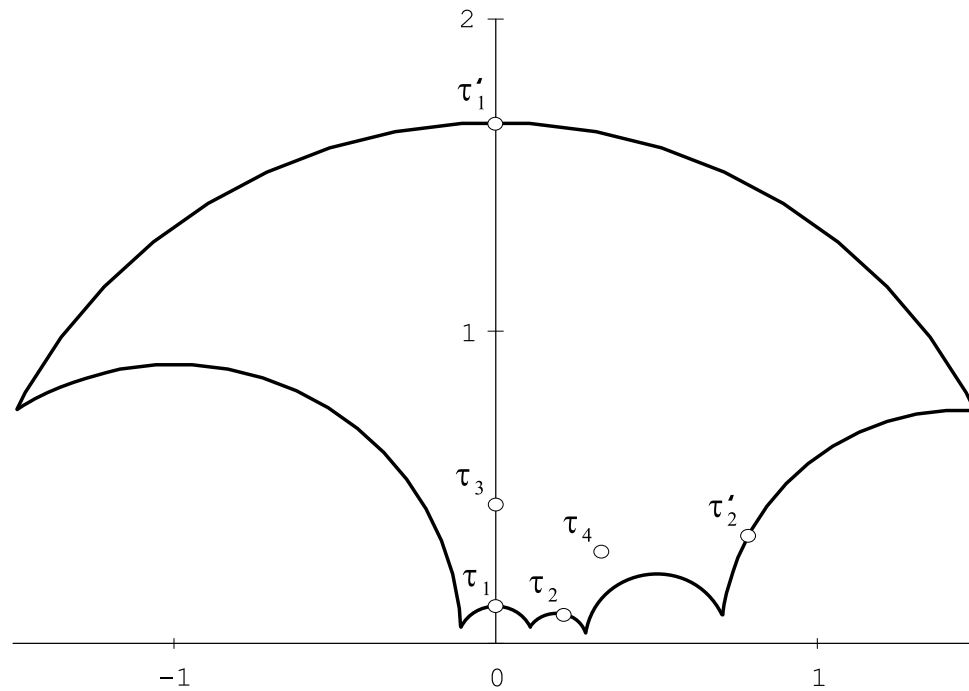
$$Y^2 = X^5 - 2\sqrt{5}X^4 + \frac{125}{18}X^3 - 2\sqrt{5}X^2 + X.$$

Their common Igusa's invariants are

$$i_1 = \frac{32844064065625}{69984} = \frac{5^5 \cdot 101^5}{2^5 \cdot 3^7},$$

$$i_2 = \frac{3219690625}{216} = \frac{5^5 \cdot 101^3}{2^3 \cdot 3^3},$$

$$i_3 = \frac{2806550125}{648} = \frac{5^3 \cdot 31 \cdot 71 \cdot 101^2}{2^3 \cdot 3^4}.$$



SCM-points in X_{15}

Example 3. $H = \left(\frac{3, 5}{\mathbb{Q}} \right) = \langle 1, I, J, K \rangle$, $D = 15$.

We consider the PEL-type given by the following data:

$$\Phi(I) = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & -\sqrt{3} \end{bmatrix}, \quad \Phi(J) = \begin{bmatrix} 0 & 1 \\ 5 & 0 \end{bmatrix},$$

$$\mu = -IJ, \quad \delta = 15,$$

$$\mathcal{O}_{15} = \mathbb{Z} \left[1, I, \frac{1+J}{2}, \frac{I+IJ}{2} \right],$$

$$\mathcal{V} = \left(\left(\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4} \right), \left(\frac{1}{2}, \frac{1}{2}, 0, 0 \right) \right),$$

$$\Phi(\mathcal{O}_{15}) = \mathbb{Z} \left[\mathbf{1}_2, \begin{bmatrix} \sqrt{3} & 0 \\ 0 & -\sqrt{3} \end{bmatrix}, \begin{bmatrix} -\frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \\ \frac{5\sqrt{3}}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} \\ -\frac{5}{2} & -\frac{1}{2} \end{bmatrix} \right].$$

We have four SCM-points.

Two of them have CM by $R(-15, 1)$:

$$\tau_1 = \frac{i(2\sqrt{5} - \sqrt{15})}{5}, \quad \tau_2 = \frac{15 - 5\sqrt{3} + i(3\sqrt{5} - \sqrt{15})}{30},$$

and the other two have CM by $R(-15, 2)$:

$$\tau_3 = \frac{i\sqrt{5}}{5}, \quad \tau_4 = \frac{5 + 2i\sqrt{5}}{15}.$$

They define the points in the Siegel upper half-space

$$Z_1 = \begin{bmatrix} i\sqrt{15} & \frac{-1+i\sqrt{15}}{2} \\ \frac{-1+i\sqrt{15}}{2} & \frac{i\sqrt{15}}{3} \end{bmatrix}, \quad Z_2 = \begin{bmatrix} \frac{-5+i\sqrt{15}}{4} & \frac{-21+i\sqrt{15}}{12} \\ \frac{-21+i\sqrt{15}}{12} & \frac{-5+i\sqrt{15}}{12} \end{bmatrix},$$

$$Z_3 = \begin{bmatrix} \frac{i\sqrt{15}}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{i\sqrt{15}}{6} \end{bmatrix}, \quad Z_4 = \begin{bmatrix} \frac{i\sqrt{15}}{3} & -\frac{4}{3} \\ -\frac{4}{3} & \frac{i\sqrt{15}}{9} \end{bmatrix}.$$

The points τ_1, τ_2 yield the genus 2 curves

$$Y^2 = X^5 + \frac{7\sqrt{5}}{4} X^4 + \frac{35}{6} X^3 + \frac{7\sqrt{5}}{4} X^2 + X,$$

$$Y^2 = X^5 - \frac{i\sqrt{5}}{4} \sqrt[4]{2111 - 168i\sqrt{15}} X^4 - \frac{3\sqrt{15}}{2} X^3 \\ + \frac{i\sqrt{5}}{4} \sqrt[4]{2111 + 168i\sqrt{15}} X^2 + X.$$

Their common Igusa's invariants are

$$i_1 = -\frac{25000}{3} = -\frac{2^3 \cdot 5^5}{3},$$

$$i_2 = -\frac{9375}{2} = -\frac{3 \cdot 5^5}{2},$$

$$i_3 = -\frac{9875}{8} = -\frac{5^3 \cdot 79}{2^3}.$$

The points τ_3, τ_4 yield the genus 2 curves

$$\begin{aligned}
 Y^2 = X^5 & - \frac{1}{28} \sqrt{5(3439 + 240\sqrt{15})} X^4 \\
 & + \frac{1}{294} (2140 + 97\sqrt{15}) X^3 \\
 & - \frac{1}{28} \sqrt{5(3439 + 240\sqrt{15})} X^2 + X,
 \end{aligned}$$

$$\begin{aligned}
 Y^2 = X^5 & - \frac{\sqrt{5}}{28} \sqrt[4]{91861516 + 21983071\sqrt{15}} X^4 \\
 & + \frac{11}{294} \sqrt{5(40772 + 9139\sqrt{15})} X^3 \\
 & - \frac{\sqrt{5}}{4} \sqrt[4]{527116 + 135377\sqrt{15}} X^2 + X.
 \end{aligned}$$

Their common Igusa's invariants are

$$i_1 = \frac{230234596815794975000}{41523861603} = \frac{2^3 \cdot 5^5 \cdot 1559^5}{3 \cdot 7^{12}},$$

$$i_2 = \frac{2877362908115625}{11529602} = \frac{3^5 \cdot 5^5 \cdot 1559^3}{2 \cdot 7^8},$$

$$i_3 = \frac{3063998328865125}{46118408} = \frac{3 \cdot 5^3 \cdot 1559^2 \cdot 3361747}{2^3 \cdot 7^8}.$$

The quaternion algebras of discriminant $D = 6, 10, 15$ are twisting quaternion algebras in the sense of Rotger. In these cases, forgetful mappings explain the coincidence of Igusa's invariants detected in the examples.