Jacquet's Functors in the Representation Theory of Reductive p-adic Groups

Diplomarbeit

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Introduction

Let G be a reductive p-adic group and let k be a field in which the number $p = \sum_{i=1}^{p} 1$ is not zero and has a square root. We have $k = \mathbb{C}, \overline{\mathbb{Q}}_\ell$ or $\overline{\mathbb{F}}_\ell$ in mind, where we have to exclude $\ell = p$ in the last case. This diploma thesis is concerned with smooth k-valued representations of such a group G.
We will use the well-known characterization that G-representations are nothing but G-modules, that is, modules over the group algebra kG. Although this characterization seems useless at first glance (arbitrary G-modules have no reason to be smooth), we can define Jacquet's functors in this language as they restrict to the smooth subcategories:

$$\begin{aligned} \boldsymbol{i}_P^G &\colon \text{smooth } M\text{-representations} \longrightarrow \text{smooth } G\text{-representations},\\ \boldsymbol{r}_P^G &\colon \text{smooth } G\text{-representations} \longrightarrow \text{smooth } M\text{-representations}, \end{aligned}$$

where $P = MU$ is a parabolic subgroup of G. We realize these functors as tensoring with appropriate bimodules.

Using this realization, we give a proof of **Frobenius Reciprocity** (this is the fact that $\boldsymbol{i}_P^G$ is right adjoint to $\boldsymbol{r}_P^G$) by directly constructing a unit and a counit. Giving these natural transformations then means giving bimodule homomorphisms. This could give a hint on a very direct proof of the Second Adjointness Theorem.

Let $P = MU$ and $Q = NV$ be two standard parabolic subgroups of G. Then Bernstein's **Geometric Lemma** gives a filtration of the functor

$$\Gamma = \boldsymbol{r}_P^G \circ \boldsymbol{i}_Q^G \colon \text{smooth } N\text{-representations} \longrightarrow \text{smooth } M\text{-representations}.$$

The name of this lemma may be explained by the fact that this filtration is indexed by the finite set ${}_P\mathcal{W}_Q = P\backslash G/Q$. This set is not representation theoretic in nature, indeed it can be thought of as some geometrical object attached to G. The proof using bimodule techniques seems to make it quite clear how and why Bruhat decomposition dominates the structure of Γ. Unfortunately, some arguments are fairly technical, especially as one has to keep track about the normalizations in terms of delta factors.

Now $\boldsymbol{i}_P^G$ is right adjoint to $\boldsymbol{r}_P^G$, and one may ask whether there is an adjointness relation in the other direction. Bernstein's famous **Second Adjointness Theorem** answers this (in the case $k = \mathbb{C}$): $\boldsymbol{i}_P^G$ is left adjoint to $\boldsymbol{r}_{\overline{P}}^G$, where $\overline{P} = M\overline{U}$ is the parabolic opposite to P. This fact is much harder to prove than Frobenius Reciprocity and relies on a deep theorem, called the Stabilization Theorem. We give a proof that Second Adjointness follows from a slightly weaker assumption than the original Stabilization Theorem.

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Notation

We denote an equivalence of categories as $\mathscr{C} \sim \mathscr{D}$.
When we are talking about an object and a subobject (or quotient) and there is no place for ambiguity, we will sometimes use the symbol $\imath$ for the injection (resp. π for the projection) without introducing it each time.
$A \subset B$ means $A \subseteq B$.
The symbol $\sqcup$ denotes a disjoint union.
When talking about a vector space, module or something similar X and a quotient space (or quotient group or something similar) X/Y, we will denote by $[x]$ the image of $x \in X$ under the projection $X \xrightarrow{\pi} X/Y$.
If A is a matrix, we denote its transpose by $A^\top$. The notion support always means the set-theoretic support:

$$\mathrm{supp}(f) = \{x \in \mathrm{source}(f) \,|\, f(x) = 0\}$$

If $A \subset B$ are sets, we denote the complement by $B - A$. The reason is, that our sets usually have additional structure and we are going to mod out subobjects from the left and the right all the time. Hence, even if $B \backslash A$ would not make sense as a quotient, one could unnecessarily ponder upon what this symbol shall mean (as the author did a few times while reading his own notes, until he changed the notation).

Chapter 1

Preparations

1.1 Explicit Description of Projective and Inductive Limits

In this section we describe a useful characterisation of some special limits. As this description is well-known, we do not go too deep into details.
Let $\mathcal{C}$ be one of the following categories: groups, rings or R-modules for some ring R.

Definition 1.1.1 (**Projective Subcategory**)**.** A small subcategory $\mathcal{P}$ of $\mathcal{C}$ is projective if

(i) $\mathrm{Hom}_{\mathcal{P}}(x,x) = \{1_x\}$ for all $x \in \mathrm{ob}(\mathcal{P})$,

(ii) $|\mathrm{Hom}_{\mathcal{P}}(x,y)| + |\mathrm{Hom}_{\mathcal{P}}(y,x)| \leq 1$ for all $x \neq y \in \mathrm{ob}(\mathcal{P})$,

(iii) For two objects x, y in $\mathcal{P}$ we find a third one t such that $|\mathrm{Hom}_{\mathcal{P}}(x,t)| = |\mathrm{Hom}_{\mathcal{P}}(y,t)| = 1$.

Observe that a projective subcategory is nothing but a projective system, where conditions (i) and (ii) correspond to orderedness and condition (iii) to directedness. If there is no danger of confusion, we will swap these notions without further explanation.
The inclusion functor $\imath : \mathcal{P} \hookrightarrow \mathcal{C}$ admits a limit $\varprojlim(\imath)$ and we have a very tangible characterization at hand:

Observation 1.1.2.

$$\varprojlim(\imath) \cong \left\{ \gamma = (\gamma_x)_{x\in\mathrm{ob}(\mathcal{P})} \in \prod_{x\in\mathrm{ob}(\mathcal{P})} x \;\middle|\; f(\gamma_x) = \gamma_y \textit{ for all } x, y \in \mathrm{ob}(\mathcal{P}), f \in \mathrm{Hom}_{\mathcal{P}}(x,y) \right\} \subset \prod_{x\in\mathrm{ob}(\mathcal{P})} x$$

where $\prod$ denotes the cartesian product of sets and the operation(s) on $\varprojlim(\imath)$ are defined component-wise.

This gives rise to projection maps

$$\pi_x : \varprojlim(\imath) \longrightarrow x$$

at the x-component. In general, they do not have to be onto.
Moreover, there is a down-to-earth description of the colimit:

Observation 1.1.3.

$$\varinjlim(\imath) \cong \Big(\bigsqcup_{x\in\mathrm{ob}(\mathcal{P})} x \Big)/\sim$$

where $\sqcup$ is the coproduct of sets (disjoint union) and the equivalence relation is given like this: $x \ni \gamma_x \sim \gamma_y \in y$ if and only if there exists an object t and arrows $f_x^t : x \to t$ and $f_y^t : y \to t$ such that $f_x^t(\gamma_x) = f_y^t(\gamma_y) \in t$.
The operation(s) are given as follows: Take $\gamma_x \in x$ and $\gamma_y \in y$ in the union. Let t be a common target of x and y, delivered by condition (iii), then we can understand γ_x and γ_y as elements of t and take $\gamma_x + \gamma_y$ (or $\gamma_x \cdot \gamma_y$) there.

There are natural injections

$$\imath_x : x \longrightarrow \varinjlim(\imath)$$

but the name is somewhat misleading since they do not necessarily have to be one-to-one. We say "$\gamma \in \varinjlim(\imath)$ occurs in the x-th component of the colimit" if $\imath_x^{-1}(\gamma) \neq \emptyset$.
Moreover, for any object x_0 we have a natural map $\imath_{x_0} \circ \pi_{x_0}$:

$$\begin{array}{rcl} \varprojlim(\imath) & \longrightarrow & \varinjlim(\imath) \\ (\gamma_x) & \longmapsto & \imath_{x_0}(\gamma_{x_0}). \end{array}$$

It clearly does not depend on the choice of x_0. $\gamma \in \varinjlim(\imath)$ is in the image precisely if it occurs at all components.

One more thing about notation: Suppose we are talking about the colimit

$$\varinjlim \Big(x_0 \to x_1 \to x_2 \to \cdots \Big)$$

with all occurring components being subobjects of a big object x, and $\xi \in x$ happens to be contained in x_0, x_1 and x_2 for example.
Then there are multiple possibilities of understanding ξ as an element of the colimit: We say "ξ in the i-th component" and mean $\imath_{x_i}(\xi) \in \varinjlim$ for $i = 0, 1, 2$.
If all the objects in the system are the same and we consequently do not put subscripts on them, we start counting from left with 0.

1.2 Stable Mappings

For this section, let V be a vector space over some field k and $T\colon V \to V$ a linear map.

Definition 1.2.1 (Stable Map). $T\colon V \to V$ is stable if $\ker(T) = \ker(T^2)$ and $\operatorname{im}(T) = \operatorname{im}(T^2)$.

We have the following equivalent characterizations:

Proposition 1.2.2. *For (T, V) as above, the following properties are equivalent:*

(i) *T is stable;*

(ii) *V decomposes as $\operatorname{im}(T) \oplus \ker(T)$;*

(iii) *$T|\operatorname{im}(T)$ is an isomorphism.*

Proof. (i)⇒(ii): Let $v \in V$. Then $Tv \in \operatorname{im}(T) = \operatorname{im}(T^2)$. Hence there is a $\tilde{v}$ such that $T^2\tilde{v} = Tv$. But then $v - T\tilde{v} \in \ker(T)$ and we have a presentation

$$v = (T\tilde{v}) + (v - T\tilde{v})$$

with the first summand clearly being in the image and the latter in the kernel of T.
Now take $v \in \operatorname{im}(T) \cap \ker(T)$. Then $Tv = 0$ and we have a $\tilde{v}$ with $T\tilde{v} = v$. But then $\tilde{v} \in \ker(T^2) = \ker(T)$, hence $v = T\tilde{v} = 0$.

(ii)⇒(iii): First, $\operatorname{im}(T) \cap \ker(T) = \{0\}$ immediately implies that $T|\operatorname{im}(T)$ is injective.
Take $v \in \operatorname{im}(T)$. Then there is a $\tilde{v}$ such that $T\tilde{v} = v$. We can write $\tilde{v} = v_1 + v_0$ with $v_1 \in \operatorname{im}(T)$ and $v_0 \in \ker(T)$. This provides us with a pre-image v_1 for v, yielding surjectivity.

(iii)⇒(i): The inclusion $\operatorname{im}(T^2) \subset \operatorname{im}(T)$ is clear. Take $v \in \operatorname{im}(T)$. Then, as $T|\operatorname{im}(T)$ is invertible, we find a pre-image $\tilde{v} \in \operatorname{im}(T)$. $\tilde{v}$ itself has a pre-image $\hat{v} \in V$. Therefore, $T^2\hat{v} = v$ and $v \in \operatorname{im}(T^2)$.
$\ker(T) \subset \ker(T^2)$ is clear. Take $v \in \ker(T^2)$. This implies $Tv \in \operatorname{im}(T) \cap \ker(T)$. But $T|\operatorname{im}(T)$ is invertible, and $T(Tv) = 0$, forcing Tv to equal 0. Therefore, $v \in \ker(T)$. □

There is an obvious weakening of this notion:

Definition 1.2.3. (T,V) is called **eventually stable** if it fulfills one of the following, equivalent properties:

(i) There is an $n \in \mathbb{N}$ such that (T^n, V) is stable;

(ii) There are $k, l \in \mathbb{N}$ such that
$$\operatorname{im}(T^k) = \operatorname{im}(T^{k+i}) \text{ and } \ker(T^l) = \ker(T^{l+i}) \text{ for all } i \in \mathbb{N}.$$

Now denote the **common kernel subspace** and the **common image subspace** by
$$\ker^\infty(T) = \bigcup_{n\in\mathbb{N}} \ker(T^n) \quad \text{and} \quad \operatorname{im}^\infty(T) = \bigcap_{n\in\mathbb{N}} \operatorname{im}(T^n).$$

Observe that $\ker^\infty(T)$ is indeed a subspace since $\ker(T^n) \subset \ker(T^{n+1})$.

Definition 1.2.4. (V,T) is called **weakly stable** if V decomposes as
$$V = \operatorname{im}^\infty(T) \oplus \ker^\infty(T)$$
and $T|\operatorname{im}^\infty(T)$ is surjective.

Observe, that (T,V) gives rise to a projective system
$$\mathcal{T} = \left(\cdots \xrightarrow{T} V \xrightarrow{T} V \xrightarrow{T} V \xrightarrow{T} \cdots\right)$$
infinite both to the right and to the left. The natural map
$$\eta : \varprojlim(\mathcal{T}) \longrightarrow \varinjlim(\mathcal{T})$$
is easily seen to be a linear map (as for any projective system of vector spaces).

Lemma 1.2.5. *The following statements are equivalent:*

(i) (T,V) *is weakly stable;*

(ii) η *provides an isomorphism* $\varprojlim(\mathcal{T}) \cong \varinjlim(\mathcal{T})$.

Proof. (i)⇒(ii): Let $(v_i)_{i\in\mathbb{Z}} \in \varprojlim(\mathcal{T})$ be a pre-image of $[0] \in \varinjlim(\mathcal{T})$ under η. Then $v_i \in \operatorname{im}^\infty(T)$ (because it appears in the limit) and $v_i \in \ker^\infty(T)$ (because $[v_i] = [0]$ in $\varinjlim$) for all $i \in \mathbb{Z}$. This implies $v_i = 0$ and shows injectivity of η.
Now let $[v] \in \varinjlim$ with $v = v_1 + v_0$, $v_1 \in \operatorname{im}^\infty(T)$ and $v_0 \in \ker^\infty(T)$. We want to construct a pre-image $(v^{(i)})_{i\in\mathbb{Z}}$ of $[v]$ under η. Set $v^{(0)} = v_1$ and $v^{(i)} = T^i(v)$ for $i \geq 1$. As $v^{(0)}$ lies in $\operatorname{im}^\infty(T)$, we can pick a pre-image $v^{(-1)} \in \operatorname{im}^\infty(T)$ under T. Repeating this, we can define $v^{(i)}$ inductively for all $i < 0$. It is clear that $(v^{(i)})_{i\in\mathbb{Z}}$ lies in $\varprojlim$ and is mapped to $[v]$.

(ii)⇒(i): Take $v_1 \in \ker^\infty(T) \cap \operatorname{im}^\infty(T)$. Define $v^{(i)}$ as above, then $(v^{(i)})_{i\in\mathbb{Z}} \in \varprojlim$ is mapped to $[v_1] = [0]$ in $\varinjlim$. Because η is injective, all the $v^{(i)}$ vanish, in particular $v^{(0)} = v_1$.
Now, take $v \in V$. As η is surjective, there is an element $(v_i)_{i\in\mathbb{Z}}$ in $\varprojlim$ that is mapped to $[v] \in \varinjlim$. Therefore $[v_0] = [v]$. This implies that we find an $n \in \mathbb{N}$ such that $T^n v_0 = T^n v$. But this means
$$v = (v_0) + (v - v_0)$$
with the first summand in $\operatorname{im}^\infty(T)$ and the latter in $\ker^\infty(T)$.
We have to prove that $T|\operatorname{im}^\infty(T)$ is surjective. Let $v \in \operatorname{im}^\infty(T)$. As η is surjective, we find a pre-image $(v_i) \in \varprojlim$ of $[v] \in \varinjlim$. This implies that we find two numbers $i \in \mathbb{Z}, n \in \mathbb{N}$ such that $T^n(v_i) = T^n(v)$. Hence $v_i - v \in \ker^\infty(T)$. But, as $v_i \in \operatorname{im}^\infty(T)$ (because it appears in the limit) and $v \in \operatorname{im}^\infty(T)$, we conclude
$$v_i - v \in \operatorname{im}^\infty(T) \cap \ker^\infty(T) = \{0\}.$$
Therefore, $v = v_i$ and has the pre-image $v_{i-1} \in \operatorname{im}^\infty(T)$. □

Lemma 1.2.6. *stable $\Rightarrow$ eventually stable $\Rightarrow$ weakly stable.*

Proof. The first implication is clear. For the second, let (T, V) be eventually stable, with kernel stabilizing from $l \in \mathbb{N}$ on and image stabilizing from $k \in \mathbb{N}$ on.
Take a $v \in \ker^{\infty}(T) \cap \operatorname{im}^{\infty}(T) = \ker(T^l) \cap \operatorname{im}^{\infty}(T)$.
Hence, $T^l v = 0$ and there is a $\tilde{v}$ such that $T^l \tilde{v} = v$. But then $\tilde{v} \in \ker(T^{2l}) = \ker(T^l)$, and this implies $v = T^l \tilde{v} = 0$.
Now, take $v \in V$. Then $T^k v \in \operatorname{im}(T^k) = \operatorname{im}^{\infty}(T)$. Therefore we find a $\tilde{v} \in \operatorname{im}^{\infty}(T)$ such that $T^k \tilde{v} = T^k v$. This says

$$v = (\tilde{v}) + (v - \tilde{v})$$

with the first summand in $\operatorname{im}^{\infty}(T)$ and the latter in $\ker^{\infty}(T)$.
We have to show that $T|\operatorname{im}^{\infty}(T)$ is surjective. Take $n \in \mathbb{N}$ sufficiently large such that $T^n|\operatorname{im}(T^n) = T^n|\operatorname{im}^{\infty}(T)$ is an isomorphism. Then for any $v \in \operatorname{im}^{\infty}(T)$ we find a $v' \in \operatorname{im}^{\infty}(T)$ such that $T^n(v') = v$. Thus $T^{n-1}(v') \in \operatorname{im}^{\infty}(T)$ is a pre-image of v. $\square$

It is easy to see that the opposite implications do not hold:

Example 1.2.7 (eventually stable $\not\Rightarrow$ stable). Take

$$V_2 := \mathbb{C}^2 \text{ and } T_2 := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Then $\operatorname{im}(T_2) = \ker(T_2) = \binom{*}{0}$ whence T_2 is not stable. But, as $T_2^2 = 0$, T_2 is eventually stable. Defining $V_k = \mathbb{C}^k$ and T_k as the matrix with ones in the upper-right secondary diagonal and zeros everywhere else, we get an example of a map that stabilizes in level k and not earlier.
By this we mean

$$V = \operatorname{im}(T_k^i) \oplus \ker(T_k^i) \text{ if and only if } i \geq k.$$

Example 1.2.8 (weakly stable $\not\Rightarrow$ eventually stable). Define the vector space

$$\mathcal{X} = \prod_{k \geq 2} V_k = \big\{ (v_2, v_3, \dots) \,\big|\, v_i \in V_i \big\}$$

where addition is defined component-wise and scalar-multiplication in the obvious way and with V_i as in Example 1.2.7. We remark that we do not require the sequence $(v_i)_{i \in \mathbb{N}}$ to vanish on a cofinite set. As linear map consider

$$T \colon \mathcal{X} \longrightarrow \mathcal{X} \qquad (v_i)_{i \geq 2} \mapsto \big(T_i(v_i)\big)_{i \geq 2}.$$

with T_i as in Example 1.2.7. Now, for any $l \in \mathbb{N}$, we can define a T-stable subspace

$$\mathcal{X}^{\leq l} = \big\{ (v_i) \in \mathcal{X} \,\big|\, T_i^l(v_i) = 0 \ \forall i \geq 2 \big\}.$$

This provides us with a nested sequence

$$\mathcal{X}^{\leq 1} \subsetneq \mathcal{X}^{\leq 2} \subsetneq \dots$$

and we define the subspace of "bounded" elements

$$\mathcal{X}^b = \bigcup_{l \in \mathbb{N}} \mathcal{X}^{\leq l}.$$

Now $\ker\big((T|\mathcal{X}^b)^l\big) = \mathcal{X}^{\leq l}$, hence the kernel-sequence never stabilizes and $T|\mathcal{X}^b$ cannot be eventually stable.
On the other hand, $\ker^{\infty}(T|\mathcal{X}^b)$ equals $\mathcal{X}^b$ by definition. Now, let $(v_i) \in \mathcal{X}^b$ be non-zero. Then there is an $l \in \mathbb{N}$ such that $v_l \neq 0$. But this clearly means that (v_i) cannot be in the image of T^l. Therefore we have

$$\mathcal{X}^b = \{0\} \oplus \mathcal{X}^b = \operatorname{im}^{\infty}(T|\mathcal{X}^b) \oplus \ker^{\infty}(T|\mathcal{X}^b)$$

and therefore $(\mathcal{X}^b, T|\mathcal{X}^b)$ is weakly stable.

1.3 Local Fields

In this section we list the definitions and facts about local fields that we will need in the sequel. Our purpose is mainly to fix notation, therefore we will not prove anything (especially since there is a lot of good literature, see [Jan96] or the comprehensive [Ser79]).

1.3.1 p-adic Numbers

We sketch the algebraic approach: Fix a prime number p, then we can form a projective system of rings

$$\mathscr{Z}_p = \left(\mathbb{Z}/p\,\mathbb{Z} \longleftarrow \mathbb{Z}/p^2\mathbb{Z} \longleftarrow \mathbb{Z}/p^3\mathbb{Z} \longleftarrow \cdots \right)$$

where each map $\mathbb{Z}/p^n\mathbb{Z} \to \mathbb{Z}/p^{n-1}\mathbb{Z}$ is reducing mod p^{n-1}.
Then we can form $\mathbb{Z}_p := \varprojlim \mathscr{Z}_p$ – the **ring of p-adic integers**. If we understand the $\mathbb{Z}/p^n\mathbb{Z}$ discretely topologized, we can take this limit in the category of topological rings and hence equip $\mathbb{Z}_p$ with a (non-trivial) topology, the so-called **Krull topology**.
Let $\pi_n : \mathbb{Z}_p \to \mathbb{Z}/p^n\mathbb{Z}$ be the projection at the n-th component of the limit, then

$$\left\{ \pi_n^{-1}(x) \,|\, n \in \mathbb{N}, x \in \mathbb{Z}/p^n\mathbb{Z} \right\}$$

forms a basis for this topology.

Proposition 1.3.1. *$\mathbb{Z}_p$ is a compact Hausdorff space.*

Proposition 1.3.2. $\mathbb{Z}_p^\times = \{x \in \mathbb{Z}_p \mid \pi_n(x) \in (\mathbb{Z}/p^n\mathbb{Z})^\times$ *for all* $n \in \mathbb{N}\} = \{x \in \mathbb{Z}_p \mid \pi_1(x) \neq 0\}$

We introduce some subsets:

Proposition 1.3.3. $\mathfrak{p} := p \cdot \mathbb{Z}_p = \{x \in \mathbb{Z}_p \,|\, \pi_1(x) = 0\}$ *is the unique maximal ideal in* $\mathbb{Z}_p$.
Moreover, $\mathbb{Z}_p/\mathfrak{p} \cong \mathbb{F}_p$ *– the field with* p *elements.*

We call p a **uniformizing element** because it generates the maximal ideal.

Lemma 1.3.4. *The* $\mathfrak{p}^n = p^n \cdot \mathbb{Z}_p, n \in \mathbb{N}$ *form a filtration of open subsets which constitutes a fundamental system of neighborhoods for* $0 \in \mathbb{Z}_p$*:*

$$\mathbb{Z}_p \supset \mathfrak{p} \supset \mathfrak{p}^2 \supset \mathfrak{p}^3 \supset \dots$$

Lemma 1.3.5. *The* n***-th Unit Subgroups*** $U^{(n)} := 1 + \mathfrak{p}^n = \{x \in \mathbb{Z}_p \,|\, \pi_n(x) = 1\}$ *form a filtration of open subsets which constitutes a fundamental system of neighborhoods for* $1 \in \mathbb{Z}_p$*:*

$$\mathbb{Z}_p \supset U := U^{(1)} \supset U^{(2)} \supset U^{(3)} \supset \dots$$

To concretize these conditions "$\pi_n(x) = \dots$" we are led to introduce an exponential valuation

$$\tilde{\nu}_p \colon \mathbb{Z}_p \to \mathbb{Z} \cup \{\infty\} \qquad x \mapsto \begin{cases} \infty & \text{if } x = 0, \\ \min\{n \in \mathbb{N} \,|\, \pi_n(x) \neq 0\} & \text{if } x \neq 0. \end{cases}$$

Observe that we have an obvious ring embedding $\mathbb{Z} \hookrightarrow \mathbb{Z}_p$.

Proposition 1.3.6. *Let* $x \in \mathbb{Z} \subset \mathbb{Z}_p$ *and write* $x = p^n \cdot a$ *with* $(p, a) = 1$*, then* $\tilde{\nu}_p(x) = n$.
On the other hand, any $x \in \mathbb{Z}_p$ *with* $\tilde{\nu}_p(x) = n$ *can be written as* $x = p^n \cdot a$ *with* $a \in \mathbb{Z}_p^\times$.

It is a key observation that $\mathbb{Z}_p$ contains no zero divisors, so we can take the field of fractions $\mathbb{Q}_p$ – the p**-adic numbers**. The integral ring of $\mathbb{Q}_p$ is $\mathbb{Z}_p$. There is an equivalent definition which gives rise to a better understanding of $\mathbb{Q}_p$:

First, the map $\mathbb{Z}_p \to \mathfrak{p}$ defined by $x \mapsto p \cdot x$ is an isomorphism of additive (topological) groups. Surjectivity is clear by definition. Injectivity is seen as follows: If $px = py = z$ for $x, y \in \mathbb{Z}_p$ and

$z \in \mathfrak{p}$, we can extract x and y from z. For $i > 1$, $\pi_i(x) = \pi_i(y) = \frac{\pi_i(z)}{p}$. But this determines $\pi_1(x) = \pi_1(y)$, hence $x = y$ and the map is injective.
The inclusion $\mathfrak{p} \subset \mathbb{Z}_p$ provides us with an injection

$$\mathbb{Z}_p \cong \mathfrak{p} \hookrightarrow \mathbb{Z}_p$$

that commutes with addition. This leads to a projective system

$$\mathscr{Q}_p = \left(\mathbb{Z}_p \hookrightarrow \mathbb{Z}_p \hookrightarrow \mathbb{Z}_p \hookrightarrow \cdots \right)$$

of groups and we can take its colimit $\varinjlim(\mathscr{Q}_p)$.

Proposition 1.3.7. $\varinjlim(\mathscr{Q}_p) \cong (\mathbb{Q}_p, +)$.

The injections are not multiplicative. Hence, in order to understand $\mathbb{Q}_p$ as a field, we have to redefine multiplication. This is done like this:
Let $x \in \mathbb{Z}_p$ be in the ith component of the colimit, y in the jth component, then their product is defined to be xy (taken in $\mathbb{Z}_p$) in the $(i + j)$th component.
This defines a multiplication with neutral element 1 in the 0th component (the first $\mathbb{Z}_p$ appearing on the left). We can nicely see how every element has an inverse: Let $x = p^i a$ (with a invertible) in the jth component, then we find an inverse in the following way:

- If $i \geq j$: The inverse is given by a^{-1} in the $(i - j)$th component.
- If $i < j$: The inverse is given by $p^{(j-i)}a^{-1}$ in the 0th component.

Moreover, there is an obvious way to extend the exponential valuation:

Definition 1.3.8 (Exponential Valuation).

$$\nu \colon \mathbb{Q}_p \longrightarrow \mathbb{Z} \qquad \imath_i(x) \longmapsto \tilde{\nu}(x) - i$$

That is, ν maps x in the ith component to the valuation of x in $\mathbb{Z}_p$ minus i.
Now ν gives rise to

Definition 1.3.9 (p-adic Absolute Valuation).

$$|\dots| \colon \mathbb{Q}_p \longrightarrow \mathbb{Q} \qquad x \longmapsto p^{-\nu(x)}$$

Moreover, we mention

Lemma 1.3.10. *Multiplication gives a group-homeomorphism*

$$p^{\mathbb{Z}} \times \mathbb{Z}_p^\times \simeq \mathbb{Q}_p^\times.$$

1.3.2 The General Case

Definition 1.3.11 (Valuation Field). Let K be a field. A valuation of K is a map $|\dots| : K \to \mathbb{R}$ such that

- $|x| > 0$ for $x \in K^\times$ and $|0| = 0$
- $|xy| = |x||y|$
- $|x + y| \leq |x| + |y|$.

Two valuations are called **equivalent** if they induce the same topology.
A valuation is called **archimedean** if the set $\{ |n| \mid n \in \mathbb{N} \}$ is not bounded above.
The pair $(K, |\dots|)$ is called a valuation field, where we usually suppress mentioning the map $|\dots|$.

In the non-archimedean case, there are various objects of interest:

- $\mathcal{O} = \{x \mid |x| \leq 1\}$ – the **valuation ring**
- $\mathcal{O}^\times = \{x \mid |x| = 1\}$ – the **unit group**
- $\mathfrak{p} = \{x \mid |x| < 1\}$ – the **valuation ideal**

$\mathcal{O}$ is a local ring with unit group $\mathcal{O}^\times$ as above and unique maximal ideal $\mathfrak{p}$. The field $\mathbf{k} = \mathcal{O}/\mathfrak{p}$ is called the **residue field** of K.

Definition 1.3.12 (Local Field). A (0-dimensional) local field is a valuation field such that the induced topology is complete. If the valuation is non-archimedean, we include the property that the residue field is finite.

Proposition 1.3.13. *The induced topology on K is locally compact.*

Theorem 1.3.14. *Up to field-homeomorphism, there are only these local fields:*

- $\mathbb{R}$ *and* $\mathbb{C}$*: archimedean local fields*
- $\mathbb{Q}_p$ *and its finite extensions: non-archimedean local fields with characteristic* 0 (*the "number theory case"*)
- $\mathbb{F}_q((T))$ (*fields of formal Laurent series over* $\mathbb{F}_q$)*: non-archimedean local fields with positive characteristic* (*the "geometric case"*)

We call an element $\varpi \in \mathcal{O}$ a uniformizing element if it generates $\mathfrak{p}$.
Lemma 1.3.4 and Lemma 1.3.5 remain true if one replaces $\mathbb{Z}_p$ by $\mathcal{O}$ and p by ϖ.

1.4 Haar Measure on Locally Compact Groups

Recall that a **topological group** is a group object in the category of topological spaces. It is called **locally compact** if we find for every point some compact neighborhood. Moreover, we will include the Hausdorff property in the definition. We begin with

Definition 1.4.1. Let X be a topological space, then a measure μ_X on X is called **Borel measure** if $\mu_X(C) < \infty$ for $C \subset X$ compact.

Definition 1.4.2. A measure μ_X on X is called **regular** if

$$\mu_X(A) = \inf\{\mu_X(B) \,|\, B \supset A \text{ open}\} = \sup\{\mu_X(B) \,|\, B \subset A \text{ compact}\}$$

for any measurable set A.

Definition 1.4.3. A regular positive Borel measure on a topological group G is called a left **Haar measure**, if it is invariant from the left: $\mu_G(gA) = \mu_G(A)$, and assigns positive values to nonempty open sets.

A right Haar measure is defined in the analogous way. The main theorem here is:

Theorem 1.4.4 (Existence of the Haar measure)**.** *Let G be locally compact, then there exists a left Haar measure. Moreover, if μ_G, λ_G are two of them, there is some real number $z > 0$ with $\mu_G = z\lambda_G$.*

Proof. The proof is quite lengthy and treats things we do not want to go into here. For a proof (and everything else about the theory) see [Loo53]. □

We have an analogue for the finite group statement $|G| = [G : H] \cdot |H|$:

Proposition 1.4.5. *Let $K, C \subset G$ be open, compact subgroups with $K \subset C$, then*

$$\mu_G(C) = [C : K] \cdot \mu_G(K)$$

Proof. This follows easily from additivity, invariance and the decomposition

$$C = \bigsqcup_{c \in C/K} cK.$$ □

Lemma 1.4.6. *Let K, C be open compact subgroups of G, then KC is an open and compact subset of G and we have*

$$\mu_G(KC) = \frac{\mu_G(K) \cdot \mu_G(C)}{\mu_G(K \cap C)}.$$

Proof. The first claim is clear, since multiplication $G \times G \to G$ is open and continuous. Write

$$KC = \bigcup_{k \in K} k \cdot C = \bigsqcup_{k \in K/(K \cap C)} k \cdot C.$$

Then additivity yields $\mu_G(KC) = [K : K \cap C] \cdot \mu_G(C)$ and the result follows from the above proposition. □

A left Haar measure μ_G gives rise to an integral on G, and the invariance amounts to

$$\int_G f(g)\, \mathrm{d}\mu_G(g) = \int_G f(g'g)\, \mathrm{d}\mu_G(g) \qquad \text{for all } g' \in G. \tag{1.1}$$

Of course, the existence theorem is also true for the right Haar measure. If there is a left Haar measure on G that is a right one at the same time, we call G **unimodular**. The difference between a left and a right Haar measure is recorded by

Definition 1.4.7. Let μ_G be a left Haar measure, then we can define

$$\lambda_g \colon A \longmapsto \mu_G(Ag).$$

It is clear that this defines again a left Haar measure, therefore we have some $\delta_G(g) \in \mathbb{C}$ with $\lambda_g = \delta_G(g)\mu_G$. Call the G-character

$$\delta_G \colon G \longrightarrow \mathbb{R}_+ \subset \mathbb{C}^\times \qquad g \mapsto \delta_G(g)$$

the **modular character** of G.

The link between the measures is the following: The measure

$$A \longmapsto \int_A \delta_G(g) \, \mathrm{d}\mu_G(g) \quad \text{ for } A \subset G \text{ measurable}$$

is a right Haar measure.

Remark 1.4.8. It is not hard to see that δ_G is smooth. Very smooth, to be accurate: For every k in any open compact subgroup $K \subset G$ we have

$$0 < \mu_G(K) = \mu_G(Kk) < \infty,$$

hence $\delta_G(K) = 1$.

The remark tells us that compactness implies unimodularity.
We shortly discuss the standard examples:

Example 1.4.9 ($G = \mathrm{GL}_n(F)$ for a local field F)**.** Recall from [JS06], App. B, the identities

$$\mathrm{Center}(G) = \{z \cdot \mathbf{1} \mid z \in F^\times\} \quad \text{and} \quad [G,G] = \mathrm{SL}_n(G).$$

Since $\mathbb{R}_+$ is abelian, $\delta_G([G,G]) = 1$. Furthermore, it is clear from the definition that δ_G vanishes on the center of G. Therefore, δ_G vanishes on the subgroup

$$\Omega \coloneqq \mathrm{Center}(G) \cdot [G,G] = \{M \in \mathrm{GL}_n(G) \mid \det(M) \in (F^\times)^n)\}.$$

This means that δ_G factorizes through

$$G \twoheadrightarrow G/\Omega \rightarrow \mathbb{R}_+$$

and

$$G/\Omega \cong F^\times/(F^\times)^n$$

is finite ([Neu90], Korollar 5.8). So the image of δ_G must be a finite subgroup of $\mathbb{R}_+$, but there is just the trivial one, hence δ_G vanishes on all of G: $G = \mathrm{GL}_n(F)$ is unimodular.

Example 1.4.10. In general, any reductive algebraic group is unimodular.

Example 1.4.11. An example for a non-unimodular group is $G =$ upper triangular matrices in $\mathrm{GL}_n(F)$. The measures and the modular character can be found in [Bum97], p. 426.

Remark 1.4.12 (Integration of vector-valued functions)**.** Denote the space of integrable functions temporarily by I. Moreover, let V be a $\mathbb{C}$-vector space. There is an obvious integral on $I \otimes V$: that one defined by linear extension of the rule

$$f \otimes v \longmapsto \Big(\int_G f(g) \, \mathrm{d}\mu_G(g) \Big) \cdot v.$$

Since $I \otimes V$ injects into $\mathrm{Hom}_{\mathrm{Sets}}(G, V)$ (as a vector space) via linear extension of

$$f \otimes v \longmapsto \big(g \mapsto f(g) \cdot v \big),$$

this gives rise to an integral on a certain space of integrable vector-valued functions on G with the same invariance property (1.1) as in the $\mathbb{C}$-valued case.

Virtually all functions we will be dealing with are of the following type:

Example 1.4.13 (Locally constant compactly supported functions)**.** Denote by $\mathscr{C}_c^\infty(G)$ the space of locally constant and compactly supported[1] functions $G \to \mathbb{C}$. Such a function $f \in \mathscr{C}_c^\infty(G)$ is clearly integrable since integration is reduced to a finite sum over finite values:

$$\int_G f(g)\, \mathrm{d}\mu_G(g) = \sum_{\lambda \in \mathbb{C}} \lambda \cdot \mu_G\big(f^{-1}(\lambda)\big) < \infty.$$

It is straightforward to verify that $\mathscr{C}_c^\infty(G) \otimes V$ is isomorphic to the space of locally constant and compactly supported functions $G \to V$. In the view of Remark 1.4.12 we hence have a left-invariant integral on these functions.

We should moreover mention

Theorem 1.4.14 (**Fubini's Theorem**)**.** *Let A, B be locally compact unimodular groups. Fix Haar measures μ_A, μ_B on them. Then, if $f\colon A \times B \longrightarrow \mathbb{C}$ is locally constant and compactly supported, we have the identity*

$$\int_A \int_B f(a,b)\, \mathrm{d}\mu_B(b)\, \mathrm{d}\mu_A(a) = \int_B \int_A f(a,b)\, \mathrm{d}\mu_A(a)\, \mathrm{d}\mu_B(b).$$

Proof. This is easy to prove by hand. For a general proof (that implies that Fubini's Theorem holds in a broader context as just locally constant and compactly supported functions) see 16C of [Loo53]. ☐

Later, we will have to use the second part of Proposition 2.1.5 in [Bum97]:

Proposition 1.4.15. *Let G be unimodular, P, K be closed subgroups such that $P \cap K$ is compact and $G = P \cdot K$. Let μ_P denote a left Haar measure on P, μ_K a right one on K. Then we find a Haar measure μ_G on G such that we have for any integrable function f an identity*

$$\int_G f(g)\, \mathrm{d}\mu_G(g) = \int_P \int_K f(pk)\, \mathrm{d}\mu_K(k)\, \mathrm{d}\mu_P(p).$$

A slight modification (Theorem 5.3.1 in [Far08]) looks as follows:

Proposition 1.4.16. *Allow G to be not unimodular, then take a left Haar measure μ_G and its modular character δ_G. Assume that there are two closed, unimodular subgroups P and Q such that $G = PQ$ and the multiplication $P \times Q \longrightarrow G$ is in fact a homeomorphism. Then there are Haar measures μ_P and μ_Q such that*

$$\int_G f(g)\, \mathrm{d}\mu_G(g) = \int_Q \int_P f(pq)\delta_G(q)\, \mathrm{d}\mu_P(p)\, \mathrm{d}\mu_Q(q)$$

for any integrable function f.

1.4.1 Invariant Measures on Homogeneous Spaces

We will be interested in the following situation: Let G be a locally compact group and U a closed subgroup. Then, of course, we can develop a Haar measure theory on U in just the same way we did for G.
The coset space G/U (or $U\backslash G$) is locally compact and Hausdorff.[2] Therefore, it is natural to ask whether there is an invariant measure on G/U, where invariance means

$$\mu_{G/U}(X) = \mu_{G/U}(gX) \quad \forall g \in G.$$

[1] This means that $\overline{\mathrm{supp}}(f)$ is compact.
[2] A proof can be found on page 38 in [HR63].

The answer is

Theorem 1.4.17. *The following statements are equivalent:*

- $\delta_G|U = \delta_U$,
- *On G/U exists an invariant positive measure $\mu_{G/U}$.*

Our main application is

Corollary 1.4.18. *Let G be unimodular, U a closed, unimodular subgroup, then there exists an invariant positive measure $\mu_{G/U}$ on G/U.*

Now, let $f\colon G \to \mathbb{C}$ be smooth and compactly supported. Then

$$U \longrightarrow \mathbb{C} \qquad u \mapsto f(gu)$$

is smooth and compactly supported in U and

$$G/U \to \mathbb{C} \qquad [x] \mapsto \int_U f(xu)\,\mathrm{d}\mu_U(u)$$

is smooth and compactly supported in G/U. We can make use of a version of

Theorem 1.4.19 (**Weil Integration Formula**)**.** *Let G, U be as in the corollary. Fix a Haar measure μ_U and an invariant measure $\mu_{G/U}$, then there exists a Haar measure μ_G such that*

$$\int_G f(g)\,\mathrm{d}\mu_G(g) = \int_{G/U}\int_U f(xu)\,\mathrm{d}\mu_U(u)\,\mathrm{d}\mu_{G/U}(x).$$

For a more detailed treatment as well as proofs, see pages 42 – 45 of [Wei40].

Chapter 2

p-adic Groups

This thesis is concerned with the representation theory of p-adic groups. We use the following chapter to introduce these groups. Moreover, we explain some relevant structure theorems.
There exists a more general concept: Each p-adic group is an ℓ-group, and many representation theoretic considerations only rely on this structure. Therefore, ℓ-groups are the topic of our first section.

2.1 ℓ-Groups

Definition 2.1.1. An **ℓ-group**, or sometimes called a **locally profinite group**, is a Hausdorff topological group G such that every open neighborhood of 1 contains an open, compact subgroup of G.

For an ℓ-group G we find that

$$\{\, gK \mid g \in G, K \subset G \text{ an open, compact subgroup}\}$$

is a basis of the topology.

Remark 2.1.2. The term "locally profinite" makes sense: One can show that G is locally profinite if we find a neighborhood of 1 that is a profinite group. Locally profinite and compact means profinite.

Let L be a non-archimedean local field with ring of integers $\mathcal{O}$ and uniformizing element ϖ.

Example 2.1.3 (L additive)**.** The groups $\varpi^i\mathcal{O}$ make the additive group L into an ℓ-group.

Example 2.1.4 ($L^\times$)**.** The higher unit groups U^i make the group $L^\times$ into an ℓ-group.

We make a brief aside on the underlying topological space of an ℓ-group:

Definition 2.1.5 (**ℓ-space**)**.** A topological space is called an ℓ-space if it is

- locally compact,
- Hausdorff,
- totally disconnected (every point is its own connected component).

Lemma 2.1.6. *Let U be the intersection of an open and a closed subset of an ℓ-space. Then U, equipped with the subset topology, is an ℓ-space on its own.*

Proof. This is Lemma 1.2 in [BZ76]. $\square$

Remark 2.1.7. One can show (see the remark in chapter one of [BH06]) that a topological group is an ℓ-group if and only if its underlying space is an ℓ-space.
The general theory of these groups is not so old: The structure has been widely clarified by Willis in [Wil94] in terms of tidy subgroups and the scale function. The latter is an interesting continuous map

$$\mathbf{s} : G \longrightarrow \mathbb{N}$$

defined as the index of certain subgroups. For example, one can characterize the modular character as

$$\delta_G(g) = \mathbf{s}(g)\mathbf{s}(g^{-1})^{-1}.$$

This implies that δ_G takes only rational values. We will not need anything from this theory but encourage the reader to have a look at the very readable paper.

We cite one more lemma:

Lemma 2.1.8. *Let $H \subset G$ be a closed subgroup of an ℓ-group. Then G/H (equipped with the quotient topology) is an ℓ-space. Consequently, if H is normal, the quotient group is an ℓ-group.*

Proof. This is Proposition 1.4 of [BZ76]. □

We introduce an important technical property of some ℓ-spaces:

Definition 2.1.9 (Countable at Infinity). An ℓ-space is said to be countable at infinity if it can be written as the union of countably many compact subsets. An ℓ-group is called countable at infinity if its underlying ℓ-space is.

In the sequel, nearly all of our ℓ-spaces and ℓ-groups will fulfill this property. We remark that being countable at infinity is hereditary with respect to

- closed subspaces,
- quotient spaces,
- finite product spaces.

Remark 2.1.10. If an ℓ-space X is countable at infinity, this implies a stronger condition: X can in fact be written as a countable union of open and compact subsets. This is seen as follows: Write

$$X = \bigcup_{n \in \mathbb{N}} X_n \qquad \text{with the } X_n \text{ compact.}$$

For n fixed, we can assign to each $x \in X_n$ an open, compact neighborhood U_x. As X_n is compact, we find a finite subset $\Lambda_n \subset X_n$ such that $X_n \subset \bigcup_{x \in \Lambda_n} U_x$. Then write X as the countable union

$$X = \bigcup_{\substack{n \in \mathbb{N} \\ x \in \Lambda_n}} U_x.$$

Using this, we can prove

Proposition 2.1.11. *Let X be an ℓ-space that is countable at infinity. Then X can be written as the disjoint union of countably many open and compact subsets.*

Proof. Write the space as a countable union of open, compact subsets: $X = \bigcup_{n \in \mathbb{N}} X_n$. Consider

$$X^n := \bigcup_{k \leq n} X_k.$$

It is clear that the X^n are open and compact on its own.
In general, if we have two open and compact subsets A and B of X, it is not hard to see that

$A - (A \cap B)$ is open and compact as well. Therefore, define $X'_n = X_n - X^{n-1}$. Then we have a decomposition into open and compact subsets:

$$X = \bigsqcup_{n \in \mathbb{N}} X'_n \qquad \square$$

In order to state the next lemma, we have to introduce some notation:
Assume, we have a covering

$$X = \bigcup_{i \in I} U_i \qquad \text{with the } U_i \text{ open.}$$

Then, a decomposition $X = \sqcup_{\omega \in \Omega} X_\omega$ is said to be **compatible with respect to the covering** if we can assign to each $\omega \in \Omega$ an $i \in I$ such that X_ω is contained in U_i.
If $Y \subset X$ is a subset, it is clear what we mean by a compatible decomposition of Y with respect to $\{U_i\}_{i \in I}$.

Lemma 2.1.12. *Let X be an ℓ-space and consider an open covering $\{U_i\}_{i \in I}$ of X. Then:*

(i) *Let $K \subset X$ be open and compact. Then there exists a compatible (disjoint) decomposition of K into finitely many open and compact subsets.*

(ii) *Let X be countable at infinity. Then there exists a compatible (disjoint) decomposition of X into countably many open and compact subsets.*

Proof. For the first part, recall from [Fed90], Theorem 5, that K is 0-dimensional as a consequence of being totally disconnected. By "dimension" we mean the Lebesgue covering dimension: For each finite open covering of K we find a refinement by a finite, disjoint and open covering. As K is open, the constituents of this covering are open in X, too.
It is clear how the second part follows from the first part and Proposition 2.1.11. $\square$

2.1.1 ℓ-Actions

Definition 2.1.13 (ℓ-action). A (left) ℓ-action is a continuous group action $G \curvearrowright X$ where G is an ℓ-group and X is an ℓ-space. A right ℓ-action is defined in the analogue way.

Proposition 2.1.14. *Let $G \curvearrowright X$ be an ℓ-action where G is countable at infinity and assume that X decomposes into finitely many G-orbits. Then there exists an open orbit.*

Proof. This is Proposition 1.4 in [BZ76]. $\square$

Of great importance will be

Corollary 2.1.15. *Let $G \curvearrowright X$ as in the preceding proposition, moreover assume that this action admits only finitely many orbits. Then we can enumerate the orbits $\{X_i\}_{1 \le i \le n}$ such that*

$$\bigcup_{1 \le i \le (k-1)} X_i \overset{\text{open}}{\subset} X \tag{2.1}$$

for any k between 2 and n.

Proof. Start with the open orbit X_1. Then $G \curvearrowright X - X_1$ is an ℓ-action and consequently admits an open orbit X_2. Proceed in this manner and enumerate the set of orbits such that

$$X_k \subset X - \Big(\bigcup_{1 \le j < k} X_j \Big) \tag{2.2}$$

is open for any k between 1 and $n-1$. Now recall the following fact from general topology: If $A \subset B$ is an open subspace of a topological space, then

$$H \subset (B - A) \text{ is open} \Longrightarrow H \cup A \subset B \text{ is open.} \tag{2.3}$$

Now we can prove (2.1): We know from (2.2) that

$$X_k \subset \Big(X - \bigcup_{1\le j<k-1} X_j\Big) - X_{k-1}$$

is open. Hence, using (2.3), we get that

$$X_k \cup X_{k-1} \subset X - \Big(\bigcup_{1\le j<k-1} X_j\Big) = \Big(X - \bigcup_{1\le j<k-2} X_j\Big) - X_{k-2}$$

is open. Repeat this argument $k-1$ times and the claim is settled. $\square$

2.2 Matrix Groups

Since there are good reasons[1] for being interested primarily in $\mathrm{GL}_n(F)$ with F a non-archimedean local field, we will carefully go through the structure theory of matrix groups before mentioning more general p-adic groups. In fact, most results generalize straightforwardly and the main difficulty is to find the right definitions to replace matrix-theoretic conditions and properties.

Definition 2.2.1 (General Linear Group). Let V be a vector space over F. Then denote the group of invertible linear transformations $V \to V$ by $\mathrm{GL}(V)$.
For $n \in \mathbb{N}$, we set

$$\mathrm{GL}_n(F) := \mathrm{GL}(F^n)$$

and observe that this is isomorphic to the group of invertible $n \times n$-matrices with entries in F. The isomorphism depends on a choice of basis.

As a subset of $\mathrm{Mat}_{n\times n}(F) = F^{n^2}$, $\mathrm{GL}_n(F)$ inherits a topology from F (and it is not hard to see that this does not depend on the choice of the basis). Multiplication and inversion are continuous with respect to this topology, making $\mathrm{GL}_n(F)$ into a topological group. If we are talking about open, closed or compact subgroups, we always mean with respect to this topology.

There are some (closed) subgroups of $\mathrm{GL}_n(F)$ that are of particular importance:

- The (standard) **Borel** subgroup B of upper triangular matrices;
- The (standard) **torus subgroup** $T \cong F^n$ of diagonal matrices;
- The (standard) **unipotent radical subgroup** U of upper triangular matrices that are unipotent: Every diagonal entry equals 1;
- The **congruence subgroups** $K = \mathrm{GL}_n(\mathcal{O})$ and $K_i = 1 + \varpi^i \operatorname{Mat}_{n\times n}(\mathcal{O})$ for $i \ge 1$.
(It is easy to see that any $x \in K_i$ is invertible: For example, we can use the Leibniz formula in order to see that $\det(x) \in U^i$.)

Generalizing Example 2.1.4, we can state:

Proposition 2.2.2. *The K_i are open, compact subgroups, forming a neighborhood basis of* 1 *and making $G = \mathrm{GL}_n(F)$ into an ℓ-group.*

Proof. That these groups are open and compact is clear from the fact that $\mathcal{O}$ is open and compact. Now let N be some open neighborhood of $1 \in \mathrm{GL}_n(F)$, then there is an open subgroup $N' \subset \mathrm{Mat}_{n\times n}(F)$ with $\imath^{-1}(N') = N$. In fact, we can take $N' = N$ as $\mathrm{GL}_n(F)$ is open in $\mathrm{Mat}_{n\times n}(F)$. Then for any pair a_1, a_2 with $1 \le a_1, a_2 \le n$ we have the projection

$$\mathrm{pr}_{(a_1,a_2)} \colon \mathrm{Mat}_{n\times n}(F) \longrightarrow F \qquad M = (M_{ij}) \mapsto M_{a_1 a_2}.$$

[1] The groups $\mathrm{GL}_n(F)$ occur somehow naturally in the Langlands correspondence.

Because of Example 2.1.3 and Example 2.1.4 we find numbers $i_{(a_1,a_2)}$ with

$$\mathrm{pr}_{(a_1,a_2)}(N') \supset \begin{cases} U^{i_{(a_1,a_2)}} & \text{if } a_1 = a_2, \\ \varpi^{i_{(a_1,a_2)}}\mathcal{O} & \text{if } a_1 \neq a_2. \end{cases}$$

whence $N' \supset K_m$ with $m = \max\{i_{(a_1,a_2)} \,|\, 1 \leq a_1, a_2 \leq n\}$. Hence $N \supset K_m$. □

We should remark that K is the unique maximal compact subgroup of $\mathrm{GL}_n(F)$ (up to conjugacy). The proof is not hard but involves some lattice theory.

Proposition 2.2.3. *$B = T \ltimes U$, and this decomposition holds topologically: the multiplication $T \times U \longrightarrow B$ is a homeomorphism.*

Proof. It is straightforward to check that U is normal in B. In order to see that B is the semidirect product, take $b = (b_{i,j})_{1 \leq i,j \leq n} \in B$ and define $t \in T$ and $u \in U$ as

$$t_{i,j} = \begin{cases} 0 & \text{if } i \neq j \\ b_{i,j} & \text{if } i = j \end{cases} \qquad \text{and} \qquad u_{i,j} = \frac{b_{i,j}}{b_{i,i}}$$

and observe $b = tu$. Since $T \cap U = 1$ we indeed have $B = T \ltimes U$.
Now for the topological statement: It is clear that the multiplication is a continuous bijection. We have to show that it is an open mapping. In order to show this, it suffices to consider a neighborhood basis for $(1,1)$. Therefore, it suffices to consider the open subsets

$$\begin{pmatrix} U^{i_1} & & & \\ & U^{i_2} & & \\ & & \ddots & \\ & & & U^{i_n} \end{pmatrix} \subset T \qquad \text{and} \qquad \begin{pmatrix} 1 & \varpi^{i_{1,2}}\mathcal{O} & \dots & \varpi^{i_{1,n}}\mathcal{O} \\ & 1 & \ddots & \vdots \\ & & \ddots & \varpi^{i_{n-1,n}}\mathcal{O} \\ & & & 1 \end{pmatrix} \subset U$$

for which it is clear that their product is open in B. □

2.2.1 Parabolic Subgroups

We have introduced B as the (standard) Borel subgroup. This suggests that there are "non-standard" ones, too.

Definition 2.2.4 (**Flags**)**.** Let V be a finite-dimensional vector space. A flag is an increasing sequence of subspaces

$$\mathbf{F} = \big(0 = V_0 \subsetneq V_1 \subsetneq \ldots \subsetneq V_m = V\big).$$

It is called **complete** if $\dim(V_{i+1}/V_i) = 1$ for all i. The sequence $\big(\dim(V_0), \dim(V_1), \ldots, \dim(V_m)\big)$ is called the **signature** of $\mathbf{F}$. The number $l(\mathbf{F}) \coloneqq m$ is called the **length** of $\mathbf{F}$.

Definition 2.2.5 (**Parabolic Subgroups**)**.** Let $\mathbf{F}$ be a flag. The stabilizing subgroup

$$\{g \,|\, gV_i = V_i \text{ for all } i\} \subset \mathrm{GL}(V)$$

is called the **parabolic subgroup with respect to F**. More generally, a subgroup P is called parabolic if there is a flag with stabilizer P. If this flag is complete, P is called a **Borel subgroup**.

Proposition 2.2.6. *Let $\mathbf{F}, \mathbf{F}'$ be flags in V with same signature and P, P' the corresponding parabolics. Then P and P' are conjugate: there is a $\gamma \in G$ with $\gamma P \gamma^{-1} = P'$.*

Proof. Take a basis $(b_i)_{1 \leq i \leq n}$ of V such that $\langle b_1, \ldots, b_{\dim(V_t)}\rangle = V_t$ for all $1 \leq t \leq l(\mathbf{F})$. Then take a basis $(b'_i)_{1 \leq i \leq n}$ with the corresponding property for $\mathbf{F}'$.
The linear transformation $\gamma\colon V \to V$ defined by $b_i \mapsto b'_i$ clearly works. □

One more word about notation: If a Borel subgroup B is fixed, we call a parabolic subgroup a **standard parabolic subgroup** if it contains B. Usually, we will fix the standard Borel subgroup and consequently the standard parabolic subgroups look like this:

$$\begin{pmatrix} \boxed{*} & & & * & \\ & & * & & \\ & & & * & \\ & & & \boxed{*} & \\ 0 & & & & * \end{pmatrix} \cap \mathrm{GL}_n(F)$$

Definition 2.2.7 (**Unipotent Radical**)**.** Let $\mathbf{F}$ be a flag and P the associated parabolic subgroup.
Then define the unipotent radical as the subgroup that operates trivially on all V_{i+1}/V_i:

$$U \coloneqq \{u \in P \mid [uv] = [v] \text{ in } V_{i+1}/V_i \ \text{ for all } v \in V_{i+1} \text{ and } 1 \leq i < l(\mathbf{F})\}.$$

Definition 2.2.8 (**Levi Factor**)**.** Let $\mathbf{F}$ and P be as above. Then define the Levi factor as the subgroup that does not see V_i when working on V_{i+1}:

$$M \coloneqq \{m \in P \mid [v] = [w] \text{ in } V_{i+1}/V_i \Rightarrow mv = mw \ \text{ for all } v, w \in V_{i+1} \text{ and } 1 \leq i < l(\mathbf{F})\}.$$

We have

$$M \cong \prod_{i=0}^{m-1} \mathrm{GL}(V_{i+1}/V_i).$$

We may call a V-basis B compatible with a flag $\mathbf{F}$ if we can enumerate it as $B = (b_j)_{1 \leq j \leq n}$ such that $\langle b_1, \ldots, b_{\dim V_i} \rangle = V_i$ for all $1 \leq i \leq l(\mathbf{F})$. Each choice of a compatible basis (b_j) gives rise to an imbedding of the torus into the parabolic subgroup associated to the flag:

$$\begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{pmatrix} \longmapsto \Big(b_j \mapsto x_j b_j \Big) \in M \subset P$$

We may express this as follows: The bigger the parabolic subgroup is, the more tori lie inside M.

2.2.2 Structure Theory

A straightforward generalization of Proposition 2.2.3 is:

Theorem 2.2.9 (**Levi decomposition**)**.** *Let $P \subset \mathrm{GL}_n(F)$ be a parabolic subgroup, then*

$$P = M \ltimes U$$

and this decomposition holds topologically.

Proof. There are no new ideas in comparison to the Borel case, but calculations become elaborate. We refer to the general theory, coarsely outlined in the next chapter. □

Let us clarify this situation a bit: P being parabolic means that V decomposes as $W_1 \oplus W_2 \oplus \ldots \oplus W_k$ and any $p \in P$ looks like this

$$\begin{tikzcd} W_1 \arrow[d] & \oplus & W_2 \arrow[d] \arrow[dll] & \oplus & W_3 \arrow[d] \arrow[dll] \arrow[dllll] & \oplus & \cdots & \oplus & W_k \arrow[d] \arrow[dll] \\ W_1 & \oplus & W_2 & \oplus & W_3 & \oplus & \cdots & \oplus & W_k \end{tikzcd} \tag{2.4}$$

where the vertical arrows are required to be invertible. Then M is the subgroup acting strictly vertically: The arrows to the left are 0. U is the subgroup of arrows that basically go only to the left: the vertical parts are all 1. The associated flag is

$$\mathbf{F} = \Big(W_1, W_1 \oplus W_2, W_1 \oplus W_2 \oplus W_3, \ldots, \bigoplus_{i=1}^{k} W_i\Big).$$

Definition/Lemma 2.2.10 (Opposite Parabolic). We are going to define the very important term of the opposite to a parabolic subgroup P. This is a parabolic subgroup $\overline{P}$ of $\mathrm{GL}_n(F)$ that may be defined in any of the following ways:

1. Let $\mathbf{F} = (\oplus_{1\le i\le m_1}\langle v_i\rangle, \oplus_{1\le i\le m_2}\langle v_i\rangle, \ldots, \oplus_{1\le i\le m_k}\langle v_i\rangle)$ the flag (with signature $(m_1, m_2-m_1, \ldots, m_k-m_{k-1})$) that is stabilized by P, where $(v_1, \ldots, v_{m_k})$ denotes a suitable basis of V.
   Then define $\overline{P}$ as the stabilizer of the "opposite" flag
   $\mathbf{F}' = (\oplus_{m_{k-1}\le i\le m_k}\langle v_i\rangle, \oplus_{m_{k-2}\le i\le m_k}\langle v_i\rangle, \ldots, \oplus_{m_1\le i\le m_k}\langle v_i\rangle)$ with signature $(m_k - m_{k-1}, m_{k-1} - m_{k-2}, \ldots, m_1)$.

2. Let $\overline{P}$ be the unique parabolic with $P \cap \overline{P} = M$ – the Levi factor of P.

3. If we understand $\mathrm{GL}_n(F)$ as a matrix group, $\overline{P}$ equals $P^\top = \{p^\top \mid p \in P\}$.

As any of these criteria suggests, $\overline{\overline{P}} = P$.

Proof. Statement 1. ⇔ 2. is quite clear, looking at (2.4): Taking the opposite means replacing the word "left" by "right." This does not affect the Levi part, but U is completely killed. Moreover, again looking at (2.4), there is only one parabolic possible that contains M but nothing of U.
For 1. ⇔ 3. recall that taking transposes is the matrix-theoretic expression of taking the dual map between the dual spaces (identified with the basis dual to the initial choice).
If we carry the decomposition over to the dual spaces, we immediately get

$$\begin{tikzcd}
W_1^* & \oplus & W_2^* & \oplus & W_3^* & \oplus & \cdots & \oplus & W_k^* \\
W_1^* \arrow[u] \arrow[urr] \arrow[urrrr] & \oplus & W_2^* \arrow[u] \arrow[urr] & \oplus & W_3^* \arrow[u] & \oplus & \cdots \arrow[urr] & \oplus & W_k^* \arrow[u]
\end{tikzcd}$$

from what, after identifying back with our original space, the equivalence follows. □

Now define the subset $\Lambda^+ \subset \mathrm{GL}_n(F)$ consisting of diagonal matrices of the form

$$\begin{pmatrix} \varpi^{m_1} & & \\ & \ddots & \\ & & \varpi^{m_n} \end{pmatrix}$$

with integers m_i satisfying $m_i \le m_{i+1}$.

Theorem 2.2.11 (Cartan Decomposition).

$$K\backslash G/K \cong \Lambda^+ \text{ or, alternatively, } G = \bigsqcup_{a\in\Lambda^+} KaK.$$

Proof. We proceed as follows: Take $g = (g_{i,j}) \in G$ and find a $g_{i,j}$ with maximal absolute value. Then we kill all other entries in the ith row and jth column via multiplication by matrices in K. To illustrate this, let

$$(\ldots a\varpi^k \ldots b\varpi^m \ldots)$$

be the ith row with $b\varpi^m = g_{i,j}$ the maximal entry and $a\varpi^k = g_{i,l}$ an entry we want to eliminate. To conduct the elimination, we can subtract $ab^{-1}\varpi^{k-m}$-times the jth column from the lth column. Since $ab^{-1}\varpi^{k-m} \in \mathcal{O}$, this corresponds to multiplication from the right by a K-matrix.
This can be done with all remaining entries in the i, j-cross, leaving $b\varpi^m$ or, after multiplication with $b^{-1}\mathbf{1}$, ϖ^m as the i, j-th entry. Then we swap the ith and the first row, and the jth and the first column, yielding a matrix of the form

$$\begin{pmatrix} \varpi^m & \\ & \boxed{\;g'\;} \end{pmatrix}$$

with $g' \in \mathrm{GL}_{n-1}(F)$ and whose entries' absolute values are not exceeding $|\varpi^m|$. It is clear how to proceed.
Proving disjointness is not so straightforward. If $g \in G$, we have to show that it determines uniquely a $\lambda \in \Lambda^+$ such that $g \in K\lambda K$. Except for special cases,[2] one has to apply a lattice-theoretic argument: The claim follows from applying Theorem 2 of Chapter 2, §2 of [Wei74] to the standard lattice $\Gamma = \bigoplus_{1\leq i\leq n} \mathcal{O}e_i$ and $\Gamma' = g\Gamma$. ◻

Observe that Cartan's decomposition tells us that $\mathrm{GL}_n(F)$ is countable at infinity.
Another important structure information is given by

Theorem 2.2.12 (Iwasawa Decomposition). *We have*

$$G = KB.$$

Proof. We use induction on n: For $\mathrm{GL}_1(F) = F^\times$ the statement is trivial since B equals all of the group.
Now let us assume we know the Iwasawa decomposition for $\mathrm{GL}_{n-1}(F)$ and take some $g = (a_{i,j}) \in \mathrm{GL}_n(F)$. Since column-swapping is implemented in K via permutation matrices, we may assume that $a_{1,1} \neq 0$ has maximal valuation within the first row. Moreover, since we can multiply by $a_{1,1}^{-1}\mathbf{1} \in B$ from the right, we can assume that $a_{1,1}$ equals 1. With this, we find that

$$k = \begin{pmatrix} 1 & & & \\ -a_{2,1} & 1 & & \\ -a_{3,1} & & 1 & \\ \vdots & & & \ddots \end{pmatrix} \in K \text{ and } \gamma := kg = \begin{pmatrix} 1 & \boxed{\;*\;} \\ & \boxed{\;*\;} \end{pmatrix}.$$

In some analogous manner we have

$$b = \begin{pmatrix} 1 & -\gamma_{1,2} & -\gamma_{1,3} & \cdots \\ & 1 & & \\ & & 1 & \\ & & & \ddots \end{pmatrix} \in B \text{ and } kgb = \begin{pmatrix} 1 & \\ & \boxed{\;g'\;} \end{pmatrix}$$

for a $g' \in \mathrm{GL}_{n-1}(F)$. But this is helpful, since g' decomposes as $k'b'$ with $k' \in K, b' \in B$ as we assumed. So we can write

$$g = k^{-1} \begin{pmatrix} 1 & \\ & \boxed{\;k'\;} \end{pmatrix} \begin{pmatrix} 1 & \\ & \boxed{\;b'\;} \end{pmatrix} b^{-1} \in KB$$

and we are done. ◻

[2] See (7.2.2) in [BH06] for an easy argument in the $n = 2$ case.

We straightly proceed to

Theorem 2.2.13 (**Iwahori Decomposition for** K_m)**.** *If we denote by* U^- *the group of lower-left unipotent matrices, we have for* $m > 1$*:*

$$K_m = (K_m \cap U^-) \cdot (K_m \cap T) \cdot (K_m \cap U)$$

Proof. We will proceed in the same pseudo-induction style[3] as in the Iwasawa case: $\mathrm{GL}_1(F)$ is clear, so let us discuss the step $(n-1) \rightsquigarrow n$:
Let $g = (a_{i,j}) \in K_m \subset \mathrm{GL}_n(F)$, then we firstly realize that $a_{1,1}^{-1}\mathbf{1} \in (K_m \cap T)$. Moreover, $a_{1,1}\mathbf{1}$ lies in the center of $\mathrm{GL}_n(F)$.
We manipulate now $g' := a_{1,1}^{-1}\mathbf{1}g$: Define the matrices

$$l = \begin{pmatrix} 1 & & & \\ -a_{2,1} & 1 & & \\ -a_{3,1} & & 1 & \\ \vdots & & & \ddots \end{pmatrix} \in (K_m \cap U^-)$$

and, with $(b_{i,j}) := lg'$,

$$r = \begin{pmatrix} 1 & -b_{1,2} & -b_{1,3} & \cdots \\ & 1 & & \\ & & 1 & \\ & & & \ddots \end{pmatrix} \in (K_m \cap U).$$

We find that $q := lg'r$ takes the shape

$$\begin{pmatrix} 1 & \\ & \boxed{\quad \tilde{g} \quad} \end{pmatrix}$$

with $\tilde{g} \in K_m \subset \mathrm{GL}_{n-1}(F)$. As we assumed, $\tilde{g}$ decomposes as $\tilde{l}\tilde{t}\tilde{r}$ with $\tilde{l} \in (K_m \cap U^-) \subset \mathrm{GL}_{n-1}(F)$ and so on. We can understand $\tilde{l}, \tilde{t}, \tilde{r}$ as injected into $\mathrm{GL}_{n-1}(F)$ in the obvious way (1 as the additional upper-left entry). Then $\tilde{l} \in (K_m \cap U^-) \subset \mathrm{GL}_n(F)$ and so on. We subsume

$$g = a_{1,1}\mathbf{1}g' = a_{1,1}\mathbf{1}l^{-1}qr^{-1} = l^{-1}a_{1,1}\mathbf{1}qr^{-1} = l^{-1}a_{1,1}\mathbf{1}\tilde{l}\tilde{t}\tilde{r}r^{-1} = (l^{-1}\tilde{l}) \cdot (a_{1,1}\mathbf{1}\tilde{t}) \cdot (\tilde{r}r^{-1}),$$

whence the statement. □

In general, we say that a subgroup $H \subset G$ **admits an Iwahori decomposition** with respect to a given parabolic subgroup $P = MU \subset G$ if H decomposes as

$$H = H^+ H^0 H^-$$

with $H^+ = H \cap U$, $H^0 = H \cap M$ and $H^- = H \cap U^-$. Here $U^- = \overline{U}$ is the unipotent radical of the parabolic opposite of P. Another notation, used in [Ber92], is that H and P **are in good position**.
The preceding lemma told us that K_m admits an Iwahori decomposition with respect to the standard Borel B. In fact, we can replace B with any standard parabolic subgroup, see Lemma 3.11 in [BZ76]. Because any parabolic subgroup is conjugate to a standard one, we find suitable replacements for the congruence subgroups that admit Iwahori factorizations in the non-standard case.

[3]Our proofs essentially do not rely on an induction argument, but we can get along with less indices and writing efforts avoiding the straightforward way.

Now recall the definition of Λ^+. If B is a standard Borel, we see that $\lambda \in \Lambda^+$ is **dominant** with respect to B and K_m for any $m \in \mathbb{N}$. By this condition we mean

$$\lambda K^+ \lambda^{-1} \supset K^+, \quad \lambda K^0 \lambda^{-1} = K^0 \text{ and } \lambda K^- \lambda^{-1} \subset K^-. \tag{2.5}$$

Define $\Lambda^{++} \subset \Lambda^+$ to be the subset of matrices

$$\begin{pmatrix} \varpi^{m_1} & & \\ & \ddots & \\ & & \varpi^{m_n} \end{pmatrix}$$

such that $m_i < m_{i+1}$. Then a $\lambda \in \Lambda^{++}$ is **strictly dominant**, that is the inclusions in (2.5) are strict.

The existence of such strictly dominant elements will become crucial, and we remark that this fact is not a unique feature of the Borel subgroups. If P is a standard parabolic subgroup, it is clear how to define the corresponding sets Λ_P^+ and Λ_P^{++}. One can write down (strictly) dominant elements similar as in the Borel case (taking into consideration the signature), so these sets are not empty.

As any parabolic subgroup is conjugate to a standard one, we can work with conjugates of Λ_P^{++} that contain strictly dominant elements for non-standard parabolic subgroups.

Now fix a $\lambda \in \Lambda_B^{++}$ and $m \in \mathbb{N}$. It is not hard to see that any $u \in U$ occurs in some $\lambda^k K_m^+ \lambda^{-k}$. We write this as

$$U = \bigcup_{k \in \mathbb{N}} \lambda^k K_m^+ \lambda^{-k}.$$

As usual, this is not limited to the standard Borel case, and we formulate this as a lemma:

Lemma 2.2.14. *Let $P = MU$ be a parabolic subgroup of $\mathrm{GL}_n(F)$. Then U can be written as a union of compact subgroups.*

2.3 General Groups

Many of the observations and theorems of the preceding section carry over to a more general setting. We will now briefly describe this setting.
This short section is not self-contained at all. We assume familiarity with the basic notions of algebraic geometry and (linear) algebraic groups. A word about literature: As a good (and very brief) introduction into the subject, the author can recommend the 12-page survey of F. Murnaghan [Mur05]. The triad of linear algebraic groups is [Bor91], [Hum75] and [Spr81]. Classics are the articles by Borel-Tits [BT65] and by Bruhat-Tits [BT72]. Moreover, we should mention the overview article [Tit79], where J. Tits gives an introduction into reductive groups over local fields. In this section, we give no proofs at all. If the results are not standard, we give a reference.

2.3.1 Overview and Definitions

As usual, denote by F a non-archimedean local field and define p to be the characteristic of the residue field of F. Let $\mathbf{G}$ be a connected, reductive linear algebraic group defined over F. We are interested in the F-rational points $G = \mathbf{G}(F)$, and for simplicity we will call G itself a reductive p-adic group.

Proposition 2.3.1. *G is an ℓ-group.*

Proof. This is Proposition 22 (1) in [Ber92]. □

Let us moreover remark that G is countable at infinity, as we can embed it as a closed subgroup into GL_n for some n.

Definition 2.3.2 (**Borel Subgroup**)**.** A maximal connected solvable algebraic subgroup of G is called a Borel subgroup.

Definition/Lemma 2.3.3 (**Parabolic Subgroup**)**.** A Zariski-closed subgroup P of G is called parabolic if the quotient space G/P is a projective variety. This is equivalent to demanding that P contains a Borel subgroup.

We carry over the following notation: If a Borel subgroup B is fixed, any parabolic containing B will be called a standard parabolic. As in the GL_n-case, any parabolic subgroup is conjugate to a standard one.
If P is a parabolic subgroup, let $U = \mathfrak{R}_U(P)$ be the unipotent radical of P. Then, as in the GL_n-case, we find a (reductive) F-subgroup $M \subset P$ such that P admits a **Levi decomposition** $P = MU$, $M \cap U = 1$.
To be accurate, P is again a semidirect product and M is homeomorphic to P/U. This is the reason why we call M the **Levi component** of P, see Definition 11.22 of [Bor91]. We may moreover remark that the tori lying inside M are of some interest.
For a parabolic P, we define the **opposite parabolic** as the (unique) parabolic subgroup $\overline{P}$ that intersects with P in a common Levi component.

2.3.2 Structure Theory

Of great importance is the following deep result of Borel (see II.2 in [Ber92]):

Theorem 2.3.4. *Let P be a parabolic subgroup of G. Then there are arbitrarily small open compact subgroups that admit an Iwahori decomposition with respect to P. Moreover, to each such K we find a strictly dominant element in G.*

Here the terms "Iwahori decomposition" and "dominant" are generalized in the obvious way.
If a parabolic subgroup P and an open, compact subgroup K with Iwahori decomposition is fixed, we denote the set of (strictly) dominant elements by Λ^+ (resp. Λ^{++}).

Observation 2.3.5. *Let $P = MU \subset G$ be a parabolic subgroup, then take K and λ according to Borel's Theorem. As in the GL_n-case we see that U is the union of compact subgroups:*

$$U = \bigcup_{i \in \mathbb{N}} \lambda^i K^+ \lambda^{-i}.$$

This tells us that U is unimodular.
Much of the structure theory for GL_n carries over. As an example we give the following decompositions which may be found in Chapter 4 of [BT72]:

Theorem 2.3.6 (**Cartan Decomposition**). *Fix a Borel subgroup $B \subset G$, then there is a maximal compact subgroup $K \subset G$ that admits an Iwahori decomposition with respect to B. Let Λ^+ denote the corresponding set of dominant elements. Then*

$$G = K\Lambda^+K.$$

Theorem 2.3.7 (**Iwasasa Decomposition**). *Let B, K be as in the Cartan decomposition, then*

$$G = BK.$$

2.4 Weyl Group and Bruhat Decomposition

Before we treat the important Bruhat decomposition, we make an aside on Coxeter groups and the Bruhat order on them. A very readable treatment is [BB05]. We will give no proofs but indicate where the reader can find them.

2.4.1 Coxeter Groups and the Bruhat Order

Let W be a group and let $S \subset W$ be a generating subset. Moreover, assume we have numbers $m(s,s') \in \mathbb{N} \cup \infty$ for all $s, s' \in S$ such that

(i) $m(s,s') = m(s',s)$,

(ii) $m(s,s') = 1 \iff s = s'$,

(iii) $(ss')^{m(s,s')} = 1$ in W if $m(s,s') < \infty$.

Definition 2.4.1 (Coxeter System). $\big(W, S, m(\,,\,)\big)$ is called a Coxeter system if W has a presentation

$$W = \langle s \in S \mid (ss')^{m(s,s')} = 1 \rangle.$$

Usually, we suppress m in the notation. Moreover, we will simply say that W is a **Coxeter Group**. We call the elements of S the **Simple Reflections**. Moreover, we have the **Reflections**

$$T = \{wsw^{-1} \,|\, w \in W, s \in S\}.$$

The simple reflections are a minimal generating subset of W, see Cor. 1.4.8 in [BB05].

Definition 2.4.2 (Length Function). Let $w \in W$ be an element of a Coxeter group. Then we can write $w = s_1 \cdots s_n$ $(s_i \in S)$.
We call $s_1 \cdots s_n$ a reduced expression of w if w cannot be written as the product of fewer than n simple reflections. If this is the case, define the length of w as $l(w) \coloneqq n$. With $l(1) \coloneqq 0$, this defines a map $W \to \mathbb{N}_0$.

Now let (W,S) be a Coxeter system with reflections T, then we can define an order relation on W like this:

Definition 2.4.3 (Bruhat Order). Set $v \leq w$ if and only if there is $u \in W$ with $w = vu$ and $l(w) = l(v) + l(u)$.

We have a combinatorial characterization of this partial order:

Theorem 2.4.4 (Subword Property). *Let $w = s_1 \cdots s_n$ be a reduced expression. Then $v \leq w$ precisely if we find a reduced expression*

$$v = s_{i_1} \cdots s_{i_k} \qquad (1 \leq i_1 \leq \cdots \leq i_k \leq n).$$

This means that v is obtained from w by dropping some elements in the expression $s_1 \cdots s_n$.

Proof. This is Thm. 2.2.2 of [BB05]. □

Prop. 2.2.9 of [BB05] tells us that the Bruhat order makes W into a directed poset.
If $|W| < \infty$, there exists a unique **longest element** w_0 (this means $w \leq w_0$ for $w \in W$).

Proposition 2.4.5. *From* [BB05] *we collect some properties:*

(i) $w_0^2 = 1$,

(ii) $l(w_0) = |T|$,

(iii) $l(ww_0) = l(w_0w) = |T| - l(w)$ *and* $l(w_0ww_0^{-1}) = l(w)$,

(iv) $w \mapsto w_0 w w_0^{-1}$ *is an automorphism.*

Example 2.4.6 (**Symmetric Group**)**.** Let S_n denote the symmetric group in n letters. This is the most popular example of a Coxeter group. We then have

$$S = \{(i, i+1) \,|\, 1 \leq i \leq n-1\} \text{ and } T = \{(i,j) \,|\, 1 \leq i \leq j \leq n\}.$$

To describe the Bruhat order, denote $a \leq b$ by $a \to b$. Then we can illustrate the situation for S_3:

$$\begin{tikzcd}
& (12) \arrow[r] \arrow[ddr] & (132) \arrow[dr] & \\
\mathrm{id} \arrow[ur] \arrow[rrr] \arrow[dr] & & & (13) \\
& (23) \arrow[r] \arrow[uur] & (123) \arrow[ur] &
\end{tikzcd}$$

The longest element is (13). A picture for S_4 can be found on p. 31 of [BB05].

2.4.2 Bruhat Decomposition

Definition 2.4.7. Let G be a reductive algebraic group and let T be a torus in G. Then T acts on the tangent space $\mathfrak{g}$ via the adjoint representation. As T is abelian, this representation decomposes into characters $\alpha : T \to \mathbb{C}^\times$. These characters are called the **weights** of G.
The nonzero weights Φ span a euclidean space and meet some additional symmetry requirements, making it into a **root system**.
The roots Φ^+ that cannot be written as the sum of others are called **simple roots**.

Now we can define the Weyl group:

Definition 2.4.8 (**Weyl Group**)**.** Define $\mathcal{W}$ as the group generated by reflections through the hyperplanes orthogonal to the roots. $\mathcal{W}$ is independent of the choice of the torus and depends only on G.

Theorem 2.4.9. *Let S be the subset of $\mathcal{W}$ that consists of reflections through the hyperplanes orthogonal to the simple roots. Then $(\mathcal{W}, S)$ is a Coxeter System.*

Proof. See Section 29.4 in [Hum75]. □

There is another characterisation of $\mathcal{W}$ as the normalizer of T modulo T, where T denotes a (maximal) torus in G. From this characterization it is immediately clear that the Weyl group does not depend on the torus, since two tori are conjugate.

Example 2.4.10 ($\mathrm{GL}_n(F)$)**.** One can easily calculate that $N_G(T)$ is the subgroup of monomial matrices (in every row and column there is exactly one nonzero entry). Hence

$$\mathcal{W} = S_n.$$

The main theorem is

Theorem 2.4.11 (**Bruhat Decomposition**)**.** *Let B be a Borel subgroup with torus T. Then understand $\mathcal{W}$ as a set of representatives in G via the characterization $\mathcal{W} = \mathrm{N}_G(T)/T$. G decomposes as*

$$G = \bigsqcup_{\mathcal{W}} BwB.$$

Proof. For the general case see Theorem 8.3.5 of [Spr98], we will say something about the case $\mathrm{GL}_n(F)$:
It suffices to prove $G = \bigsqcup_w B^- wB$ with $B^- = w_0 B w_0$ the lower-diagonal matrices. This is clear, since $\mathcal{W} = \{w\} = \{w_0 w\}$, hence $\bigsqcup_w B w_0 w B = \bigsqcup_w BwB$. Moreover, $G = w_0 G$.

The equation $G = \bigsqcup_w B^- w B$ is just the Gauss-Jordan Algorithm:
By multiplying appropriate lower-diagonal elementary matrices from the left and upper-diagonal ones from the right to an invertible matrix, we end up with a monomial matrix:

$$\begin{pmatrix} 0 & & & & & \\ 0 & & & & & \\ 0 & & & & & \\ * & & & & & \\ & & & & & \\ & & & & & \end{pmatrix} \rightsquigarrow \begin{pmatrix} 0 & & & & & \\ 0 & & & & & \\ 0 & & & & & \\ * & 0 & 0 & 0 & 0 & 0 \\ 0 & & & & & \\ 0 & & & & & \end{pmatrix} = \begin{pmatrix} 0 & 0 & & & & \\ 0 & * & & & & \\ 0 & & & & & \\ * & 0 & 0 & 0 & 0 & 0 \\ 0 & & & & & \\ 0 & & & & & \end{pmatrix} \rightsquigarrow \begin{pmatrix} 0 & 0 & & & & \\ 0 & * & 0 & 0 & 0 & 0 \\ 0 & 0 & & & & \\ * & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & & & & \\ 0 & 0 & & & & \end{pmatrix} \text{ and so on.}$$

That $BwB = Bw'B$ implies $w = w'$ is not so straightforward. In Chapter 30 of [Bum04] the reader can find a proof based on induction on the length $l(w)$. $\square$

In the sequel, we will not distinguish between $\mathcal{W}$ as an abstract group and as a set of representatives in G if a Borel subgroup is understood.

2.4.3 Enumerating the Double Cosets

Definition 2.4.12 (**Flag Variety**)**.** Let $P \subset G$ be a parabolic subgroup, then G/P is a homogeneous space for G (acting from the left). G/P corresponds to the set of all flags with signature provided by P.

Observation 2.4.13. *The Bruhat decomposition tells us that B acts with finitely many orbits on G/B (on the left). Write this as*

$$\mathcal{W} = B\backslash G/B.$$

As G is a topological space, $B\backslash G/B$ inherits a topology τ (finite, hence Alexandroff). Because of the equivalence between Alexandroff topologies and preorders,[4] we find a preorder $\leq$ on $B\backslash G/B$ (and, hence, on $\mathcal{W}$) such that τ is characterized by

$$U \text{ open } \Leftrightarrow x \in U, x \leq y \text{ implies } y \in U.$$

One can show that $\leq$ is the Bruhat order (see for example Section 8.5.4 of [Spr81]).

Observation 2.4.14. *We can enumerate $\mathcal{W}$ and decompose G as $\bigsqcup_I Bw_iB$ such that the subsets $\bigsqcup_{1\leq i\leq k} Bw_iB \subset G$ are open for any $1 \leq k \leq |I|$.*

Proof. As we are talking about the induced topology, we can check that in $B\backslash G/B = \mathcal{W}$: First of all, the longest element is clearly open, hence $w_1 := w_0$. Now take any $w \leq w_1$ such that there is no w' with $w \leq w' \leq w_1$ as w_2. As w_3 take any w fulfilling the same condition as w_2 or fulfilling $w \leq w_2$ but there is no w' such that $w \leq w' \leq w_2$. It is clear how to proceed. $\square$

Remark 2.4.15. The $X(w) := BwB/B \subset G/B$ are called **Bruhat cells**, and $X(w_0)$ is the **big cell**.

Remark 2.4.16. An analogue of Observation 2.4.14 is true for $B\backslash G/B'$ with any Borel subgroup B': We find a $g \in G$ such that G decomposes as $\bigsqcup_I Bw_igB'$ with the $\bigsqcup_{1\leq i\leq k} Bw_igB'$ open.

Proof. Assume $B' = \gamma B \gamma^{-1}$, then consider

$$\alpha_\gamma : G/B \longrightarrow G/B' \qquad gB \mapsto gB\gamma^{-1} = g\gamma^{-1}B'.$$

α_γ is a continuous isomorphism (with inverse $\alpha_{\gamma^{-1}}$) and thus identifies $G/B \cong G/B'$. Hence the observation leads to the successive unions $\bigsqcup_{1\leq i\leq k} Bw_i\gamma^{-1}B'$ being open. $\square$

[4]See [Ale37].

Now let P, Q be parabolic subgroups of G with Borel subgroups $B \subset P, B' \subset Q$. Then there is the natural projection

$$p : B\backslash G/B' \twoheadrightarrow P\backslash G/Q$$

yielding $P\backslash G/Q = {}_P\mathcal{W}_Q$ for some quotient ${}_P\mathcal{W}_Q$ of $\mathcal{W}\gamma^{-1}$.
Using the enumeration $(w_i)_I$ obtained from the last remark, $\{p(w_i)\}_I$ provides us, after ruling out the double entries, with a suitable enumeration $(w'_j)_J$ of ${}_P\mathcal{W}_Q$ such that the subsets $\bigsqcup_{1\leq j\leq k} Pw'_jQ$ are open in G.

Remark 2.4.17. It is easy to describe ${}_P\mathcal{W}_Q$ in terms of the root system, see the first pages of [Cas95].

Remark 2.4.18. In this context, the standard way to prove the existence of this enumeration of ${}_P\mathcal{W}_Q$ (as used by Bernstein and others) is different: It suffices to realize that G decomposes into finitely many cosets PwQ. This says that P(which is countable at infinity) acts with finitely many orbits on the ℓ-space G/Q. Then we can immediately apply Corollary 2.1.15 and we are done.

2.5 p-adic Groups and Measures

A reductive p-adic group G is an ℓ-group, hence locally compact and there exists a Haar measure. In the first part, we recall some integration formulae in this setting from the relevant literature.
G itself is unimodular (as is any reductive group), but this is not true for its parabolic subgroups. The "error term" δ_P will become an important technical aggravation in what follows. In the second part of this section we will prove two formulae involving these delta factors.
The aim of the third part is to establish an invariant measure on G with values in other fields, for example in $k = \overline{\mathbb{Q}}_\ell$ or $k = \overline{\mathbb{F}}_\ell$ (with $\ell \neq p$ in the latter case). This will allow us to say something about the representation theory of G with ground field different than $\mathbb{C}$.

2.5.1 Integration Formulae

Let G be a reductive p-adic group with parabolic subgroup $P = MU$ and modular character δ_P. The Cartan decomposition provides us with a maximal compact subgroup K such that $G = KP$. Then we have the following integration formulae:

Proposition 2.5.1. *Denote by φ a locally constant and compactly supported function $G \longrightarrow \mathbb{C}$.*

(i) *Pick Haar measures μ_M, μ_U and μ_K on M, U and K, respectively. Then the assignment*

$$\varphi \longmapsto \int_M \int_U \int_K \delta_P^{-1}(m)\varphi(muk)\, \mathrm{d}\mu_M(m)\, \mathrm{d}\mu_U(u)\, \mathrm{d}\mu_K(k)$$

is a Haar integral on G. That is, there is a Haar measure μ_G on G such that

$$\int_G \varphi(g)\, \mathrm{d}\mu_G(g) = \int_M \int_U \int_K \delta_P^{-1}(m)\varphi(muk)\, \mathrm{d}\mu_M(m)\, \mathrm{d}\mu_U(u)\, \mathrm{d}\mu_K(k)$$

for any $\varphi \in \mathscr{C}_c^\infty(G)$.

(ii) *Pick Haar measures μ_U, μ_M and $\mu_{\overline{U}}$ on U, M and $\overline{U}$, respectively. Then there is a Haar measure μ_G on G such that*

$$\int_G \varphi(g)\, \mathrm{d}\mu_G(g) = \int_U \int_M \int_{\overline{U}} \varphi(um\overline{u}) \cdot \delta_P(m)^{-1}\, \mathrm{d}\mu_U(u)\, \mathrm{d}\mu_M(m)\, \mathrm{d}\mu_{\overline{U}}(\overline{u})$$

for any $\varphi \in \mathscr{C}_c^\infty(G)$.

(iii) *Pick Haar measures μ_M and μ_U on M and U, respectively. Then there is a left Haar measure μ_P on P such that*

$$\int_P \varphi(p)\, \mathrm{d}\mu_P(p) = \int_M \int_U \varphi(mu)\, \mathrm{d}\mu_U(u)\, \mathrm{d}\mu_M(m)$$

holds for any $\varphi \in \mathscr{C}_c^\infty(P)$.

Proof. Part (i) is Proposition 18 in [Wal01]. We just remark that this follows from our Proposition 1.4.15 and the integration formula on p. 45 in [Wei40]. For (ii) see p. 12 in [Art91] and for (iii) see Chapter 13.10 in [Kot05]. □

2.5.2 Delta Factor Computations

Recall that we denote the modulus of a parabolic subgroup P of a p-adic group G by δ_P. There is another characterization which Bernstein and Zelevinskii use in [BZ77]. This characterization is more manifest by any means, but became non-standard in the literature, unfortunately.

Take a closed subgroup $H \subset G$. Any element x in its normalizer $\mathrm{N}_G(H)$ induces a homeomorphism

$$\sigma_x \colon H \longrightarrow H \qquad h \mapsto x^{-1}hx.$$

The module (in the sense of Bourbaki) of σ_x is denoted by $\mathrm{mod}_H(x)$. This gives rise to a (smooth) character

$$\mathrm{mod}_H \colon \mathrm{N}_G(H) \longrightarrow \mathbb{C}^\times$$

which fulfills

$$\mathrm{mod}_H(x) \int_H \varphi(x^{-1}hx)\, \mathrm{d}\mu_H(h) = \int_H \varphi(h)\, \mathrm{d}\mu_H(h) \qquad \forall x \in \mathrm{N}_G(H)$$

for any integrable function $\varphi \colon H \longrightarrow \mathbb{C}$ and any left Haar measure μ_H on H.

Lemma 2.5.2. *Let $P = MU$ be a parabolic subgroup of G. Then*

$$\mathrm{mod}_U | P = \delta_P.$$

Proof. Fix arbitrary (non-zero) open, compact subgroups $K_M \subset M$ and $K_U \subset U$. Then write $\overline{p} \in P$ as $\overline{um}$. Now we can fix Haar measures on P, M and U according to 1.4.16 and calculate

$$\delta_P(\overline{p}) = \delta_P(\overline{m}) = \frac{\int_P \mathbf{1}_{K_M K_U}(p)\, \mathrm{d}\mu_P(p)}{\int_P \mathbf{1}_{K_M K_U}(\overline{m}^{-1} p \overline{m})\, \mathrm{d}\mu_P(p)} = \frac{\int_P \mathbf{1}_{K_M}(m) \mathbf{1}_{K_U}(u)\, \mathrm{d}\mu_P(mu)}{\int_P \mathbf{1}_{K_M}(\overline{m}^{-1} m \overline{m}) \mathbf{1}_{K_U}(\overline{m}^{-1} u \overline{m})\, \mathrm{d}\mu_P(mu)}$$
$$= \frac{\mu_M(K_M) \int_U \mathbf{1}_{K_U}(u)\, \mathrm{d}\mu_U(u)}{\mu_M(K_M) \int_U \mathbf{1}_{K_U}(\overline{m}^{-1} u \overline{m})\, \mathrm{d}\mu_U(u)} = \frac{\int_U \mathbf{1}_{K_U}(u)\, \mathrm{d}\mu_U(u)}{\int_U \mathbf{1}_{K_U}((\overline{um})^{-1} u \overline{um})\, \mathrm{d}\mu_U(u)} = \mathrm{mod}_U(\overline{um}) = \mathrm{mod}_U(\overline{p}).$$

□

There is a nice lemma:

Lemma 2.5.3 (mod **is "multiplicative").** *Let $A, B \subset G$ be closed subgroups such that AB is a closed subgroup of its own and multiplication induces a homeomorphism $A \times B \cong AB \subset G$. Moreover, assume that A, B and AB are unimodular. Then*

$$\mathrm{mod}_A(x)\, \mathrm{mod}_B(x) = \mathrm{mod}_{AB}(x)$$

holds for any $x \in \mathrm{N}_G(A) \cap \mathrm{N}_G(B) \subset \mathrm{N}_G(AB)$.

Proof. As in the preceding proof, take open, compact (non-zero) subgroups $K_A \subset A$ and $K_B \subset B$ and use Proposition 1.4.16 to calculate

$$\operatorname{mod}_{AB}(x) = \frac{\int_{AB} \mathbf{1}_{K_A K_B}(ab)\, \mathrm{d}\mu_{AB}(ab)}{\int_{AB} \mathbf{1}_{K_A K_B}(x^{-1}abx)\, \mathrm{d}\mu_{AB}(ab)} = \frac{\int_A \mathbf{1}_{K_A}(a)\, \mathrm{d}\mu_A(a)\ \int_B \mathbf{1}_{K_B}(b)\, \mathrm{d}\mu_B(b)}{\int_A \mathbf{1}_{K_A}(x^{-1}ax)\, \mathrm{d}\mu_A(a)\ \int_B \mathbf{1}_{K_B}(x^{-1}bx)\, \mathrm{d}\mu_B(b)}.$$

This evidently equals $\operatorname{mod}_A(x)\operatorname{mod}_B(x)$. □

The reason why we are interested in all of this is the following technical result that we will need to prove the Geometric Lemma:

Theorem 2.5.4. *Let $P = MU$ and $Q' = N'V'$ be standard parabolics in G. For $w \in {}_P\mathcal{W}_{Q'}$ set $Q = wQ'w^{-1}$. This is a parabolic with Levi decomposition $Q = NV = (wN'w^{-1})(wV'w^{-1})$ and we have*

$$\delta_P \cdot \delta_Q \cdot \delta_{M\cap Q} \cdot \delta_{N\cap P} = \delta_{P\cap Q}^2$$

as characters of $M \cap N$.

Proof. Let us rewrite

$$\delta_P \cdot \delta_Q \cdot \delta_{M\cap Q} \cdot \delta_{N\cap P} = \operatorname{mod}_U \cdot \operatorname{mod}_V \cdot \operatorname{mod}_{M\cap V} \cdot \operatorname{mod}_{N\cap U}$$

using the fact that $M \cap Q$ is a parabolic subgroup of M with Levi decomposition $(M\cap N)\cdot(M\cap V)$ (see [Cas95], Proposition 1.3.3 (c)) and the analogous result for $N \cap P$.
Now, citing the calculations in the second part of Part 6.4 of [BZ77], we can replace $\operatorname{mod}_U \cdot \operatorname{mod}_V$ by $\operatorname{mod}_{U\cap Q} \cdot \operatorname{mod}_{V\cap P}$.
Therefore, using the Multiplicativity Lemma for the decompositions $U \cap Q = (U \cap N) \cdot (U \cap V)$ and $V \cap P = (V \cap U) \cdot (V \cap M)$ (which are stated as Proposition 2.8.6 in [Car93]), we have

$$\delta_P \cdot \delta_Q \cdot \delta_{M\cap Q} \cdot \delta_{N\cap P} = \Big(\operatorname{mod}_{U\cap N} \cdot \operatorname{mod}_{M\cap V} \cdot \operatorname{mod}_{U\cap V}\Big)^2.$$

But this is exactly what we want: According to Theorem 2.8.7 of [Car93], we have a decomposition $P\cap Q = X \cdot L$ with uniqueness, where L is reductive and X the largest normal unipotent subgroup. Analyzing the proof, it is easy to see that all the prerequisites for the proof of Lemma 2.5.2 are met (even if $P \cap Q$ is not parabolic). We conclude

$$\delta_{P\cap Q} = \operatorname{mod}_X .$$

Carter decomposes X as

$$(U \cap V) \cdot (U \cap N) \cdot (V \cap M)$$

with uniqueness. The multiplication maps

$$(U \cap V) \times (U \cap N) \longrightarrow U \cap Q \quad \text{and} \quad (U \cap Q) \times (V \cap M) \longrightarrow (U \cap Q) \cdot (V \cap M) \subset UM = P$$

are easily seen to be homeomorphisms and all occurring groups are closed and unimodular (except for P, of course).
Applying two times the Multiplicativity Lemma, we can decompose

$$\delta_{P\cap Q} = \operatorname{mod}_{U\cap V} \cdot \operatorname{mod}_{U\cap N} \cdot \operatorname{mod}_{V\cap M}$$

and we are done. □

We need one more result:

Theorem 2.5.5. *Let $P = MU$ be a parabolic subgroup of a reductive p-adic group G. Then, as characters of M, we have the identity*

$$\delta_P = \delta_{\overline{P}}^{-1}.$$

Proof. Take any (non-zero) open, compact subgroup $K \subset G$ that admits an Iwahori decomposition with respect to P. Then, using the integration formula from p. 12 in [Art91] (it is written down here as the first part of Proposition 2.5.1), we have

$$\begin{aligned}
1 &= \frac{\int_G \mathbf{1}_K(g)\,\mathrm{d}\mu_G(g)}{\int_G \mathbf{1}_K(g)\,\mathrm{d}\mu_G(g)} = \frac{\int_G \mathbf{1}_K(g)\,\mathrm{d}\mu_G(g)}{\int_G \mathbf{1}_K(x^{-1}gx)\,\mathrm{d}\mu_G(g)} \\
&= \frac{\int_U \mathbf{1}_{K^+}(u)\,\mathrm{d}\mu_U(u) \int_M \mathbf{1}_{K^0}(m)\delta_P(m)\,\mathrm{d}\mu_M(m) \int_{\overline{U}} \mathbf{1}_{K^-}(\overline{u})\,\mathrm{d}\mu_{\overline{U}}(\overline{u})}{\int_U \mathbf{1}_{K^+}(x^{-1}ux)\,\mathrm{d}\mu_U(u) \int_M \mathbf{1}_{K^0}(x^{-1}mx)\delta_P(x^{-1}mx)\,\mathrm{d}\mu_M(m) \int_{\overline{U}} \mathbf{1}_{K^-}(x^{-1}\overline{u}x)\,\mathrm{d}\mu_{\overline{U}}(\overline{u})} \\
&= \mathrm{mod}_U(x) \cdot \mathrm{mod}_{\overline{U}}(x) = \delta_P(x) \cdot \delta_{\overline{P}}(x)
\end{aligned}$$

for $x \in M$. □

Remark 2.5.6. The proofs of these two theorems seem appropriate as they are totally self-contained (except for the citation of the Bernstein-Zelevinskii paper, of course). But the author does not want to conceal that there is a much more direct approach to these delta-factor computations (but this needs some Lie theory).
First of all, if $P = MU$ is a parabolic subgroup of G, denote the corresponding Lie algebras by $\mathfrak{p}$, $\mathfrak{m}$ and $\mathfrak{u}$. Then, one can show

$$\delta_P(m) = \left|\det\left(\mathrm{Ad}_{\mathfrak{p}}(m)\right)\right|$$

where Ad is the adjoint representation of P on $\mathfrak{p}$. As M acts trivially on $\mathfrak{m}$, we can write

$$\delta_P(m) = \left|\det\left(\mathrm{Ad}_{\mathfrak{m}+\mathfrak{u}}(m)\right)\right| = \left|\det\left(\mathrm{Ad}_{\mathfrak{u}}(m)\right)\right|.$$

If $P = P_\Theta$ for some subset Θ of the simple roots $\Delta \subset \Sigma^+$ (with respect to some Borel $P_\emptyset$), δ_P is therefore uniquely characterized by the property that its restriction to A_Θ equals

$$\prod_{\alpha \in \Sigma^+ - \Sigma^+_\Theta} \alpha^{|\alpha|}$$

where

- A_Θ is the connected component of the identity in $\cap_{\alpha\in\Theta} \ker(\alpha)$,
- $|\alpha|$ is the dimension of the α-eigenspace $\mathfrak{g}_\alpha$ in the Lie algebra of G,
- Σ^+ is the set of positive roots (with respect to the chosen Borel subgroup $P_\emptyset$) in a reduced root system Σ,
- Σ^+_Θ is the subset of Σ^+ of positive linear combinations of the roots in Θ.

Casselman calculates delta-factors using this characterization, the cited fact from [BZ77] is proved in this way and it should be possible to prove the above theorems by calculating the representation as a product over the roots.

2.5.3 Haar Measure with Values in $\mathbb{Z}[\frac{1}{p}]$

Definition 2.5.7. Let p be a prime number. A compact group K' is called a **pro-p-group** if the number $[K' : K]$ is a power of p for any open subgroup $K \subset K'$.

Let G be a p-adic group. We have the following result by Vigneras:

Theorem 2.5.8. *There exists a (left) Haar measure μ_G on G with $\mu_G(K) \in \mathbb{Z}[\frac{1}{p}]$ for any open, compact subgroup $K \subset G$ if and only if there exists an open, compact pro-p-subgroup $K' \subset G$. If this is the case, we can find such a measure with the additional normalization property $\mu_G(K') = 1$.*

Proof. See [Vig96], Theorem 2.4. □

Moreover, there is an analogue of Corollary 1.4.18:

Corollary 2.5.9. *Let G be unimodular and such that Theorem 2.5.8 does hold. If $U \subset G$ is a closed, unimodular subgroup, then there exists a left invariant, positive measure $\mu_{G/U}$ on G/U such that any open, compact subset has measure in $\mathbb{Z}[\frac{1}{p}]$.*

Proof. This is Proposition 2.8 in [Vig96]. □

Lemma 2.5.10. *In our case (G is a reductive p-adic group), there exists an open, compact pro-p-subgroup.*

Proof. See [MS09], Lemma 1.1. □

One defines the modular character δ_G just as in Definition 1.4.7. We already mentioned that δ_G takes only rational values. Surprisingly, we can encircle the possible values even better:

Remark 2.5.11. Take $x \in G$. We know that

$$\delta_G(x) = \frac{\mu_G(x^{-1}Kx)}{\mu_G(K)}$$

for any open, compact subgroup $K \subset G$. Thus we can take $K = K'$ and read off that $\delta_G(x) \in \mathbb{Z}[\frac{1}{p}]$. But, as $\delta_G(x)$ is invertible (with inverse $\delta_G(x^{-1})$), we indeed have $\delta_G(x) = \pm p^n$ for some $n \in \mathbb{Z}$.

Now, let k be a field in which the number $p = \sum_{i=1}^{p} 1$ does not equal 0 and moreover there is a $q \in k$ such that $q^2 = p$. We will abbreviate this condition as "p is a non-zero square in k". The examples we have in mind are $k = \mathbb{C}, \overline{\mathbb{Q}}_\ell$ and $\overline{\mathbb{F}}_\ell$, where we need $\ell \neq p$ in the last case.
We have a mapping

$$\imath\colon \mathbb{Z}\left[\frac{1}{p}\right] \longrightarrow k$$

which is injective if and only if $\operatorname{char}(k) = 0$. Thus, we can talk about a k-valued Haar measure on G and, consequently, we can integrate smooth, compactly supported functions $\varphi\colon G \longrightarrow k$. It is clear that all identities and formulae involving measures, integrals or modular characters carry over. For example, it is easy to see that Fubini's theorem does hold. From now on, if we are talking about k-valued representations and ask the reader to fix some Haar measure, we always mean with values in k.
In contrast to the $\mathbb{C}$-valued case, it may well happen that an open, compact subgroup has measure 0. We remark this because many of our proofs will rely on the fact that we have arbitrary small open, compact subgroups K such that

- $\mu_G(K) \neq 0$,
- $\mu_A(A \cap K) \neq 0$ for any closed subgroup $A \subset G$ with k-valued Haar measure μ_A.

As pro-p-subgroups always have measure $\pm p^n$ with $n \in \mathbb{Z}$ (this follows immediately from Theorem 2.5.8), we can luckily handle these needs using

Proposition 2.5.12. *Let G be a reductive p-adic group, $P = MU \subset G$ a parabolic subgroup.*

(i) *The set*

$$\mathscr{K}_P(G) = \left\{ K \subset G \;\middle|\; \substack{K \text{ is an open, compact pro-}p\text{-subgroup such that } K \text{ admits an Iwahori decomposition} \\ \text{with respect to } P \text{ and there is a strictly dominant element with respect to } P \text{ and } K} \right\}$$

is a neighborhood basis of $1 \in G$. For $K \in \mathscr{K}_P(G)$, all constituents in the Iwahori decomposition

$$K = K^+ K^0 K^-$$

are pro-p-groups.

(ii) *The set*

$$\mathscr{K}(G) = \{ K \subset G \mid K \text{ is an open, compact pro-}p\text{-subgroup} \}$$

is a neighborhood basis of $1 \in G$*. Consequently, the set*

$$\mathscr{K}(H) = \{ K \subset H \mid K \text{ is an open, compact pro-}p\text{-subgroup} \}$$

is a neighborhood basis of $1 \in H$ *if* $H \subset G$ *is a closed subgroup.*

(iii) $K \in \mathscr{K}(G) \Rightarrow gKg^{-1} \in \mathscr{K}(G)$ *for any* $g \in G$*.*

(iv) *Let* $K \in \mathscr{K}_P(G)$ *with strictly dominant element* λ*. Then the set*

$$\{ \lambda^{-m} K^+ \lambda^m \mid m \in \mathbb{N} \}$$

is a neighborhood basis for $1 \in U$*.*

(v)

$$U = \bigcup_{K \in \mathscr{K}(U)} K$$

(vi) *Any closed subgroup of a pro-p-group is again a pro-p-group.*

Proof. (i): Let $X \subset G$ be open. Our task is to find a $K \in \mathscr{K}_P(G)$ that is contained in X. But this means putting together Lemma 2.5.10 and Theorem 2.3.4: Take an open, compact pro-p-subgroup K'. Then $X \cap K'$ is open, consequently we find an open, compact $K \subset X \cap K'$ with the desired properties. K is closed in G, hence in K', and closed subgroups of a pro-p-group are again closed (this is stated for example on p. 139 in [PR94]).
As $U \subset G$ is closed, $K^+ = K \cap U$ is closed in K, therefore the same argument yields the second statement, analogous for K^0 and K^-.
(ii): Because $\mathscr{K}_P(G) \subset \mathscr{K}(G)$ for any parabolic subgroup P, the first claim is obvious. The second statement follows from this and some very basic topological considerations.
(iii): This follows from the basic observation that for subgroups $J \leq H \leq G$ and an element $g \in G$ we have

$$[H : J] = [gHg^{-1} : gJg^{-1}].$$

(iv): As Bernstein states in the proof of Lemma 5.2 in [Ber87], the $\lambda^{-m} K^+ \lambda^m \subset U$ get arbitrary small. As they are closed in K, the claim follows from this observation.
(v): This is stated in the proof of Proposition 2.3 in [HW08].
(vi): We already used this fact and gave a reference. $\square$

The condition that p has a square root is needed because we will have to twist some of our representations with the character $\delta_P^{\frac{1}{2}}$ or $\delta_P^{-\frac{1}{2}}$. According to Remark 2.5.11, we have to choose a square root of p in order to give sense to these symbols.

Chapter 3

Representation Theory of p-adic Groups

Here we will treat the standard facts of the representation theory of p-adic groups. There are good references for this chapter, my favourite is [BH06], from where we took various arguments. Classics are [BZ76], [BZ77] and [Cas95].

3.1 Definitions and Properties

If one wants to investigate a group G, it is often useful to consider a concrete realization of G (or one of its quotients) as a matrix group over some (usually algebraically closed) field k. Recall

Definition 3.1.1 (**G-Representation**)**.** A G-representation (π, V) consists of an k-vector space V and a group homomorphism

$$\pi\colon G \longrightarrow \mathrm{GL}(V),$$

where we abbreviate $gv = \pi(g)v$.

If $V \cong k^n$, we talk about (π, V) as an n-dimensional representation. The arrows on G-representations are the following:

Definition 3.1.2 (**G-Intertwiner**)**.** Let (π, V) and (ρ, W) be two G-representations. Then a linear map

$$\tau : V \longrightarrow W$$

is called a G-intertwiner if $\tau(gv) = g\tau(v)$ for all $v \in V$, $g \in G$.
The set of these maps is called $\mathrm{Hom}_G(V, W)$.

Later on, we will suppress the symbol π in most situations and talk about a G-representation V. Recall that a representation is called **irreducible** if there is no G-invariant subspace in V. This means that we cannot find a subspace $W \subset V$ such that $\sum_{i\in I} g_i w_i \in W$ for any finite index set I with $g_i \in G, w_i \in W$.
Group representations are of great importance in many areas, as they make other methods than group theory applicable, for instance, linear algebra. But, besides the case where G is finite, these objects are too general to be accessible. Therefore, one is led to impose certain smoothness restrictions on π. In our situation (G is a reductive p-adic group, or, more general, an ℓ-group), the suitable concept of smoothness turns out to be the following:

Definition 3.1.3. A representation (π, V) of G is called **smooth** if

$$V = \bigcup_K V^K$$

where K ranges over all open, compact subgroups of G and

$$V^K = \{v \in V \mid kv = v \text{ for all } k \in K\}$$

denotes the subspace of K-invariant vectors.[1] Denote the associated category by G-Rep_k and abbreviate G-Rep $= G$-$\mathrm{Rep}_{\mathbb{C}}$.

It is easily seen that a subquotient of a smooth representation (taken in the category of all representations) is smooth. To an arbitrary representation we can associate a smooth subrepresentation:

Definition 3.1.4 (Smoothening). Let (π, V) be a G-representation, then define the **smooth part** of V as

$$V^\infty := \bigcup_K V^K$$

where K ranges over the open, compact subgroups. This defines a smooth representation (π^∞, V^∞) of G. Any $v \in V^\infty$ is called a **smooth vector** of V.

It is not hard to see that the process of smoothening is left exact. Moreover, we have

Proposition 3.1.5. *Let V, W be G-representations, V be smooth. Then*

$$\mathrm{Hom}_G(V, W) = \mathrm{Hom}_G(V, W^\infty).$$

Proof. "$\supset$" is obvious.
Now, take a G-intertwiner τ out of the left Hom-set. If K is an open, compact subgroup of G and $v \in V^K$, we have

$$\tau(v) = \tau(kv) = k\tau(v) \qquad \text{for all } k \in K.$$

Hence $\tau(V^K) \subset W^K$. But this says

$$\mathrm{im}(\tau) = \tau\Big(\bigcup_K V^K\Big) \subset \bigcup_K W^K = W^\infty. \qquad \square$$

Definition 3.1.6. A smooth representation (π, V) is called **admissible** if each V^K is finite-dimensional.

One very desirable feature of a representation is semisimplicity, which we want to define as follows

Definition 3.1.7. If $K \subset G$ is a subgroup, a G-representation (π, V) is said to be K**-semisimple** if V is the sum of its irreducible subspaces (as a representation of K). In the case $K = G$ we simply say **semisimple**. A representation category is semisimple if every object is.

Example 3.1.8. Let K be open and compact, then any smooth G-representation over $\mathbb{C}$ is K-semisimple, see [BH06] Lemma 2.2. In particular, G-Rep is semisimple for G compact.

If V is a vector space, we can form the dual V^*. If V carries a G-representation, one can define a G-representation on V^* by the rule

$$\langle gv^*, v\rangle := \langle v^*, g^{-1}v\rangle.$$

One can show by counterexample that V^* need not to be smooth, even if V is. Therefore, one is naturally led to consider

Definition 3.1.9 (Smooth Dual). Let (π, V) be a representation of G, then we define the smooth representation

$$(\check{\pi}, \check{V}) := ((\pi^*)^\infty, (V^*)^\infty)$$

as the smooth dual[2] of V.

[1] Observe that this is just an economic way of writing down the condition $\forall v\ \exists K$ such that $kv = v\ \forall k \in K$.
[2] Some authors call this the contragredient representation.

Lemma 3.1.10. *A smooth G-representation (π, V) over $\mathbb{C}$ is admissible if and only if $V \cong \check{\check{V}}$.*

Proof. See Proposition 2.9 in [BH06]. $\square$

The next thing to mention is

Lemma 3.1.11 (Schur's Lemma)**.** *Let (π, V) be an irreducible smooth G-representation, where G is assumed to be countable at infinity and k to be uncountable. Then* $\mathrm{Hom}_G(V, V) \cong k$.

Proof. Understand $\mathrm{Hom}_G(V, V)$ as a division algebra over k. This is possible since any $\varphi \neq 0$ is invertible because V is irreducible.
It is easy to deduce from the irreducibility of V and the countability at infinity of G the fact that $\dim_k(V)$ is countable. Now fix any $v_0 \in V$. As V is irreducible, φ is determined by the value it assigns to v_0. We conclude that $\mathrm{Hom}_G(V, V)$ has countable dimension.
But then we are done: Assume that $\mathrm{Hom}_G(V, V)$ is bigger than k. Then any element $\alpha \in \mathrm{Hom}_G(V, V) - k$ is transcendent over k (as we assumed k to be algebraically closed). But this clashes with Corollary 2' of [Ami56] which assures in this situation that the cardinal number of k is not greater than $\dim_k(\mathrm{Hom}_G(V, V))$. $\square$

Observe that this proves Schur's lemma for smooth $\mathbb{C}$- and $\overline{\mathbb{Q}}_\ell$-valued representations, but not for $\overline{\mathbb{F}}_\ell$-valued ones.

3.2 Restriction and Induction

Let H be a closed subgroup of G, then there is a straightforward (and functorial) way of obtaining an H-representation from a G-representation (π, V). If V is smooth, it is clear that the obtained representation is smooth as well.

Definition 3.2.1 (**Restriction**)**.** Denote by

$$\mathrm{Res}_H^G \colon G\text{-}\mathrm{Rep}_k \longrightarrow H\text{-}\mathrm{Rep}_k$$

the functor that assigns to (π, V) the representation $(\pi|H, V)$ and to a G-intertwiner $\tau \colon V \to W$ itself.

Much more interesting is of course the other direction: Roughly speaking, a difficult group may often possess subgroups whose representation theory we do understand to some extent. We want to lift our knowledge (what means the representations) to G, and there is substantially one way:

Definition 3.2.2 (**Induction**)**.** For an H-representation (π, V) obtain the vector space

$$\mathrm{IND}_H^G(V) \coloneqq \{f : G \longrightarrow V \mid f(hg) = hf(g)\ \forall h \in H, g \in G\}$$

and observe that this defines a G-representation via

$$\big(gf\big)(g') \coloneqq f(g'g).$$

This bare induction is almost useless since it respects smoothness by no means. Therefore, we do what we usually do in such cases:

Definition 3.2.3 (**Smooth Induction**)**.** Let (π, V) be a (usually smooth) H-representation, then define

$$\mathrm{Ind}_H^G(V) \coloneqq (\mathrm{IND}_H^G(V))^\infty.$$

This defines a functor $H\text{-}\mathrm{Rep}_k \to G\text{-}\mathrm{Rep}_k$ acting on arrows simply as $\tau \mapsto \tau_*$.

There is a slight modification:

Definition 3.2.4 (Compact smooth Induction). Let (π, V) be as above, then set

$$\mathrm{ind}_H^G(V) := \{f \in \mathrm{Ind}_H^G(V) \,|\, f \text{ compactly supported modulo } H\}$$

where the condition means that $\mathrm{supp}(f)$ is contained in a compact set when projected onto $H\backslash G$.

The basic properties of induction are summarized by the following theorem:

Theorem 3.2.5 (Properties). *Let $H \subset G$ and $T \subset H$ be closed subgroups and set $\Delta = \delta_H^{-1}\delta_G$. Assume, $k = \mathbb{C}$.*

(i) *Transitivity:* $\mathrm{Ind}_H^G \circ \mathrm{Ind}_T^H \cong \mathrm{Ind}_T^G$, *the same is true for* ind.

(ii) Ind_H^G *and* ind_H^G *are additive and exact functors* H-Rep $\to$ G-Rep.

(iii) *If $H\backslash G$ is compact,* $\mathrm{Ind}_H^G = \mathrm{ind}_H^G$ *and induction respects admissibility.*

(iv) *Frobenius Reciprocity: We have functorial isomorphisms:*

$$\begin{aligned}
\mathrm{Hom}_G(V, \mathrm{Ind}_H^G(W)) &\cong \mathrm{Hom}_H(\mathrm{Res}_H^G(V), W) \\
\mathrm{Hom}_G(\mathrm{ind}_H^G(V), W) &\cong \mathrm{Hom}_H(\Delta^{-1}V, \mathrm{Res}_H^G(W))
\end{aligned}$$

(v) *Duality: Understand $\Delta\check{V}$ as the representation $h\colon \check{v} \mapsto \Delta(h) \cdot h\check{v}$ on the space V. Then*

$$\mathrm{ind}_H^G(V)\check{} \cong \mathrm{Ind}_H^G(\Delta\check{V})$$

Proof. For (i) we give the maps

$$\begin{aligned}
\mathrm{Ind}_H^G \circ \mathrm{Ind}_T^H(V) \longrightarrow \mathrm{Ind}_T^G(V) \qquad & \zeta \longmapsto \big(g \mapsto \zeta(g)(1)\big) \\
\mathrm{Ind}_T^G(V) \longrightarrow \mathrm{Ind}_H^G \circ \mathrm{Ind}_T^H(V) \qquad & \xi \longmapsto \Big(g \mapsto \big(h \mapsto \xi(hg)\big)\Big).
\end{aligned}$$

It is not hard to check that they make sense and are inverse to each other.
(ii) is easily checked by hand, see [BH06] p. 18.
The first statement of (iii) is clear by definition, for the second, according to [BH06], we argue like this: Take some open, compact subgroup $K \subset G$. We have to show that $\big(\mathrm{Ind}_H^G(V)\big)^K$ is finite-dimensional. If ξ is an element of this space, it is clearly determined by its values on the set $\Omega := \{HgK\}_{g \in G}$ which is finite since $H\backslash G$ is compact.
Moreover, we know a bit about possible values of ξ: Let $g \in G$, then

$$\xi(g) = t\xi(g) \qquad \text{for all } t \in \Gamma_g := H \cap gKg^{-1}.$$

Hence $\xi(g)$ is an element of the finite-dimensional (since V is admissible) vector space V^{Γ_g}.
We conclude

$$\big(\mathrm{Ind}_H^G(V)\big)^K \subset \bigoplus_{\omega \in \Omega} V^{\Gamma_\omega}.$$

The first equation of (iv) is verified like (i): We give the maps

$$\begin{aligned}
\mathrm{Hom}_G(V, \mathrm{Ind}_H^G(W)) \longrightarrow \mathrm{Hom}_H(\mathrm{Res}_H^G(V), W) \quad &: \quad f \longmapsto \big(v \mapsto f(v)(1)\big) \\
\mathrm{Hom}_H(\mathrm{Res}_H^G(V), W) \longrightarrow \mathrm{Hom}_G(V, \mathrm{Ind}_H^G(W)) \quad &: \quad f \longmapsto \Big(v \mapsto \big(g \mapsto f(gv)\big)\Big)
\end{aligned}$$

and observe that they are inverse to each other. For the second part we refer the reader to 2.29 in [BZ76]. The proof for (v) can be found as Theorem 3.5 in [BH06]. □

Remark 3.2.6. The assumption $k = \mathbb{C}$ is too strong. For a more general result, assume that G is a reductive p-adic group and k is a field in which p is a non-zero square, then the theorem holds: (i) is Section 5.3 in [Vig96], (ii) is 5.10, (iii) is 5.6, (iv) is 5.7 and (v) is 5.11.

We will need the construction from part (v) in the sequel, therefore we introduce the following notation:

Definition 3.2.7 (Character Twist). Let (π, V) be a G-representation, χ a character of G, then define the G-representation $(\chi \circledast \pi, V)$ as follows:

$$g \longmapsto \Big(V \to V \quad v \mapsto \chi(g)gv\Big)$$

In what follows, we will abbreviate this as $\chi \circledast V$. If both V and χ are smooth, it is clear that $\chi \circledast V$ is smooth as well.

3.3 The Hecke Algebra

It is a major feature of the representation theory of finite groups that representations of G over a (suitable) field k are the same as modules over the group algebra kG. At first, this is nothing but a matter of notation. But consequently it makes module theory applicable.
Of course, one can do the same in our context, but these modules correspond to arbitrary representations. Since we are interested in smooth ones, we are led to replace kG by a more suitable algebra.
For this section, we assume that G is unimodular and $k = \mathbb{C}$. This is not necessary, because virtually all can be generalized and we explain this in a remark at the end. These restrictions allow us to cite common references and help avoiding technical issues which the author considers pointless at the moment because we do not need these results in the sequel.
Again, our exposition is inspired by [BH06], from where we adopt some arguments.

3.3.1 Definitions

Definition 3.3.1 (**Hecke Algebra**). Let G be a unimodular ℓ-group, then define

$$\mathcal{H}(G) \coloneqq \{f\colon G \longrightarrow \mathbb{C} \mid f \text{ locally constant and } \overline{\operatorname{supp}(f)} \text{ compact}\}.$$

It is clear that the elements of $\mathcal{H}(G)$ are measurable with respect to some Haar measure μ_G on G. Define the convolution

$$*\colon \mathcal{H}(G) \times \mathcal{H}(G) \longrightarrow \mathcal{H}(G) \qquad (f, g) \longmapsto \left(x \mapsto \int_G f(\gamma) g(\gamma^{-1}x) \, \mathrm{d}\mu_G(\gamma) \right).$$

$*$ is associative, as can readily be checked. Moreover, $\mathcal{H}(G)$ is a $\mathbb{C}$-vector space via $(\lambda f + g)(\gamma) = \lambda f(\gamma) + g(\gamma)$.

Clearly the definition of the convolution (and, hence, the definition of $\mathcal{H}(G)$) depends on the choice of Haar measure. But it is not hard to see that two different choices give rise to isomorphic Hecke algebras.

Example 3.3.2. For any G equipped with the discrete topology, $\mathcal{H}(G) \cong \mathbb{C}G$ via $f \leftrightarrow \sum_G f(g) \cdot g$.

The following proposition exhibits some of the basic properties of the convolution:

Proposition 3.3.3. (i) $(f * g)(1) = (g * f)(1)$.

(ii) $(f * g)^\dagger = g^\dagger * f^\dagger$ *where* $f^\dagger$ *denotes the function that maps* x *to* $f(x^{-1})$.

(iii) $\overline{f * g} = \overline{f} * \overline{g}$ *for* $\overline{f}(x) \coloneqq \overline{f(x)}$ *– the complex conjugate.*

(iv) $(f + f') * g = f * g + f' * g$ *and* $f * (g + g') = f * g + f * g'$.

Proof. (i):

$$(f * g)(1) = \int_G f(\gamma) g(\gamma^{-1}) \, \mathrm{d}\mu_G(\gamma) = \int_G f(\gamma^{-1}) g(\gamma) \, \mathrm{d}\mu_G(\gamma) = \int_G g(\gamma) f(\gamma^{-1}) \, \mathrm{d}\mu_G(\gamma) = (g * f)(1)$$

(ii):

$$(f * g)^\dagger(x) = \int_G f(\gamma) g(\gamma^{-1} x^{-1}) \, \mathrm{d}\mu_G(\gamma) = \int_G f(x^{-1}\gamma) g(\gamma^{-1}) \, \mathrm{d}\mu_G(\gamma) = (g^\dagger * f^\dagger)(x)$$

(iii) and (iv) are obvious when writing down the claim. □

Now, for S an open, compact subset of G, denote by e_S the characteristic function of S normalized so that $\int_G e_S(\gamma) \, \mathrm{d}\mu_G(\gamma) = 1$. e_S is clearly an element of the Hecke algebra, and if $S = K$ is a subgroup of G, we have $e_K(kg) = e_K(gk) = e_K(g)$ for $g \in G, k \in K$.

Proposition 3.3.4. *Let $K \subset G$ be an open, compact subgroup. Then $e_K * f = f$ if and only if $f(kg) = f(g)$ for all $k \in K, g \in G$.*

Proof. "$\Leftarrow$" is easy:

$$e_K * f(g) = \int_G e_K(\gamma) f(\gamma^{-1}g) \, \mathrm{d}\mu_G(\gamma) = \frac{1}{\mu_G(K)} \int_K f(\gamma^{-1}g) \, \mathrm{d}\mu_G(\gamma) = \frac{f(g)}{\mu_G(K)} \int_K 1 \, \mathrm{d}\mu_G(\gamma).$$

For the other direction, write

$$f(g) = e_K * f(g) = \int_G e_K(\gamma) f(\gamma^{-1}g) \, \mathrm{d}\mu_G(\gamma)$$

and observe that this coincides with

$$f(kg) = e_K * f(kg) = \int_G e_K(\gamma) f(\gamma^{-1}kg) \, \mathrm{d}\mu_G(\gamma) = \int_G e_K(k\gamma) f(\gamma^{-1}g) \, \mathrm{d}\mu_G(\gamma).$$ □

Corollary 3.3.5. *e_K is idempotent: $e_K * e_K = e_K$.*

Definition 3.3.6 (Hecke Algebra with respect to K). Let $K \subset G$ be an open, compact subgroup, then define the subalgebra

$$\mathcal{H}(G//K) \coloneqq e_K * \mathcal{H}(G) * e_K.$$

Proposition 3.3.7. *Let K run through all open, compact subgroups of G, then we have*

$$\mathcal{H}(G) = \bigcup_K \mathcal{H}(G//K).$$

Proof. All we have to show is this: Let f be in $\mathcal{H}(G)$, then we find some open, compact subgroup K such that $f(kg) = f(g)$ for all $k \in K, g \in G$. This is enough, since then we can argue in the same manner to get some K' with $f(gk') = f(g)$, and we find that $f \in \mathcal{H}(G//K'')$ for some open, compact subgroup K'' contained in $K \cap K'$.
Since f is locally constant, we find for every g in the support some open neighborhood and hence an open, compact subgroup K_g such that $f(K_g g) = f(g)$.
The support of f is compact, hence we find a finite subset Γ of G such that

$$\operatorname{supp}(f) \subset \bigcup_\Gamma K_\gamma \gamma.$$

Take K to be an open, compact subgroup contained in the open neighborhood $\bigcap_\Gamma K_\gamma$ of 1.
Let us now calculate $f(kg)$ for $g \in G, k \in K$. We will, moreover, assume that g is in the support of f. Then we have $g = k'\gamma$ for a $k' \in K_\gamma$. Hence $kg = kk'\gamma$ with $kk' \in K_\gamma$. This yields $f(g) = f(kg) = f(\gamma)$. Indeed, it shows that kg is in the support of f.
Doing the same with kg and $k^{-1}(kg)$ tells us that g is in the support if and only if kg is, hence $f(g) = f(kg) = 0$ if g is not in the support. □

Remark 3.3.8. Proposition 3.3.4 and its obvious right version tell us that some element f of the Hecke algebra is contained in $\mathcal{H}(G//K)$ precisely if $f(KgK) = f(g)$ for all $g \in G$.
Moreover, e_K is the unit in $\mathcal{H}(G//K)$. This is remarkable since $\mathcal{H}(G)$ itself is by no means unital.
Indeed, $\mathcal{H}(G)$ is what one calls an **idempotented algebra**: For every finite subset $\digamma$ there is some idempotent $e_\digamma$ such that

$$e_\digamma * f = f * e_\digamma \qquad \text{for all } f \in \digamma.$$

For $\mathcal{H}(G)$ take K to be an open, compact subgroup with $\digamma \subset \mathcal{H}(G//K)$. Such a K exists, see Proposition 3.3.7. Then clearly $e_\digamma = e_K$ works.

3.3.2 Equivalence of Categories

Definition 3.3.9 (**Non-degenerate Module**)**.** Let $\mathcal{A}$ be an idempotented algebra, then an $\mathcal{A}$-module M is called non-degenerate if

$$\mathcal{A}M = M.$$

Denote the subcategory of non-degenerate modules by $\mathcal{A}$-mod.

Proposition 3.3.10. *An $\mathcal{H}(G)$-module M is non-degenerate if and only if for any $m \in M$ we find some open, compact subgroup K such that $e_K m = m$.*

Proof. Let m be an element of the non-degenerate module M, then we find $f \in \mathcal{H}(G)$ and $n \in M$ such that $fn = m$.
Since $\mathcal{H}(G) = \bigcup_K \mathcal{H}(G//K)$, there is some open, compact subgroup K and some $\tilde{f}$ with $e_K * \tilde{f} * e_K = f$. Then we have

$$e_K * \tilde{f} * e_K \cdot n = m$$

and therefore m is clearly invariant under multiplication by e_K.
The other way is clear: If e_K leaves m fixed, m is obviously contained in $\mathcal{H}(G)M$. ☐

Now we want to establish an equivalence of categories G-Rep $\sim \mathcal{H}(G)$-mod.
In order to do so, we need

Definition 3.3.11. Let (π, V) be a smooth G-representation and f an element of the Hecke algebra of G. Then set

$$fv := \int_G f(\gamma) \cdot \gamma v \, \mathrm{d}\mu_G(\gamma) \ \in V$$

where μ_G denotes the same Haar measure we are using for convolution in $\mathcal{H}(G)$. The function $g \mapsto f(g) \cdot gv$ is locally constant and compactly supported, hence integrable.

Now we can define the first direction

Definition/Lemma 3.3.12 (Functor G-Rep $\to \mathcal{H}(G)$-mod)**.** Let (π, V) be a smooth G-representation. Then V carries the structure of an $\mathcal{H}(G)$-module via $f \cdot v = fv$.
Moreover, if (ρ, W) is another such representation, any G-intertwiner $\tau \colon V \to W$ is at the same time an $\mathcal{H}(G)$-module homomorphism.

Proof. That $(f * f')v = f(f'v)$ is a straightforward calculation using Fubini's Theorem. Since v is smooth, there is some K leaving v fixed. This implies $e_K v = v$, and Proposition 3.3.10 yields that V is non-degenerate.
For the statement about the morphisms we have to understand fv better: Take an open subgroup K_1 with $f(gK_1) = f(g)$ and K_2 with $K_2 v = v$. Then for any open, compact subgroup K of G contained in $K_1 \cap K_2$ we find that

$$fv = \int_G f(\gamma) \cdot \gamma v \, \mathrm{d}\mu_G(\gamma) = \mu_G(K) \sum_{\gamma \in G \setminus K} f(\gamma) \cdot \gamma v. \tag{3.1}$$

This sum is finite since f is compactly supported. Hence

$$\tau(fv) = \tau\Big(\mu_G(K) \sum_{\gamma \in G\setminus K} f(\gamma) \cdot \gamma v\Big) = \mu_G(K) \sum_{\gamma \in G \setminus K} f(\gamma) \cdot \gamma \tau(v) = f\big(\tau(v)\big).$$ ☐

Definition/Lemma 3.3.13 (Functor $\mathcal{H}(G)$-mod $\to G$-Rep)**.** Let M be a non-degenerate $\mathcal{H}(G)$-module. We have to equip the vector space M with a G-action. For this, take $m \in M$, then we find some open, compact subgroup K with $e_K m = m$.
Let λ_g denote the left shift operator on $\mathcal{H}(G)$: $\lambda_g(f)(x) = f(g^{-1}x)$. Define

$$gm := \lambda_g(e_K)m.$$

This makes M into a smooth G-representation. Moreover, $\mathcal{H}(G)$-module morphisms are at the same time G-intertwiners.

Proof. First of all, we should check that this is well-defined. This means, for two K, K' with $e_K m = e_{K'} m$ we should have $\lambda_g(e_K)m = \lambda_g(e_{K'})m$. To see this, we calculate for $f, f' \in \mathcal{H}(G)$:

$$\big(\lambda_g(f)*f'\big)(t) = \int_G f(g^{-1}\gamma)f'(\gamma^{-1}t)\,\mathrm{d}\mu_G(\gamma) = \int_G f(\gamma)f'(\gamma^{-1}g^{-1}t)\,\mathrm{d}\mu_G(\gamma) = \lambda_g\big(f*f'\big)(t) \tag{3.2}$$

Knowing this, take some open, compact subgroup $\Gamma \subset K \cap K'$ and calculate using Proposition 3.3.4:

$$\begin{aligned}\lambda_g(e_K)m &= \lambda_g(e_K)(e_\Gamma m) = \big(\lambda_g(e_K) * e_\Gamma\big)m = \lambda_g\big(e_K * e_\Gamma\big)m \\ &= \lambda_g\big(e_\Gamma * e_K\big)m = \big(\lambda_g(e_\Gamma) * e_K\big)m = \lambda_g(e_\Gamma)(e_K m) = \lambda_g(e_\Gamma)m.\end{aligned}$$

The same calculation applies to $\lambda_g(e_{K'})m$, hence the G-action is well-defined.
Now we have to check that $(gh)m = g(hm)$ holds. For this take $hm = \lambda_h(e_K)m$ and fix some K' with $e_{K'}\lambda_h(e_K)m = \lambda_h(e_K)m$. We have to verify

$$\lambda_{gh}(e_K)m = \lambda_g(e_{K'})\lambda_h(e_K)m. \tag{3.3}$$

For this, take Ξ to be some open, compact subgroup contained in $K' \cap K \cap hKh^{-1} \cap h^{-1}Kh$. Proposition 3.3.4 then yields

$$\lambda_h(e_K) * e_\Xi = \lambda_h(e_K) = e_\Xi * \lambda_h(e_K).$$

Moreover, since $\Xi \subset K$, we have $m = e_\Xi m$. Applying (3.2) to the left hand side of (3.3) we get

$$\lambda_{gh}(e_K)m = \lambda_g(\lambda_h(e_K))m = \lambda_g(e_\Xi * \lambda_h(e_K))m = \big(\lambda_g(e_\Xi) * \lambda_h\big)(e_K)m$$

and to the right hand side

$$\begin{aligned}\big(\lambda_g(e_{K'}) * \lambda_h(e_K)\big)m &= \lambda_g\big(e_{K'} * \lambda_h(e_K)\big)m = \lambda_g\big(e_{K'} * \lambda_h(e_K) * e_\Xi\big)m \\ &= \lambda_g\big(e_\Xi * e_{K'} * \lambda_h(e_K)\big)m = \big(\lambda_g(e_\Xi) * e_{K'} * \lambda_h(e_K)\big)m\end{aligned}$$

and the claim follows.
Smoothness is obvious: Take some K with $e_K m = m$, then for all $k \in K$ we have

$$km = \lambda_k(e_K)m = e_K m = m.$$

The statement about morphisms is not much harder: Applying an algebra homomorphism τ to gm yields $\tau(\lambda_g(e_K)m) = \lambda_g(e_K)\tau(m)$. □

Observation 3.3.14. *The process G-Rep $\to \mathcal{H}(G)$-mod $\to G$-Rep equals $\mathrm{id}_{G\text{-Rep}}$:*
Concerning the arrows, this is clear since none of our functors touches them.
Let us check this for some object (π, V). Equip V with the module structure. Take some $v \in V$ and let K be in its (representation-theoretic) stabilizer, then clearly $e_K v = v$. Hence

$$gv = \lambda_g(e_K)v = \int_G e_K(g^{-1}\gamma)\gamma v\,\mathrm{d}\mu_G(\gamma) = \int_G e_K(\gamma)g\gamma v\,\mathrm{d}\mu_G(\gamma) = \frac{1}{\mu_G(K)}\int_K gv\, d\mu_G(\gamma)$$

what clearly equals the original action.

Observation 3.3.15. *The process $\mathcal{H}(G)$-mod $\to G$-Rep $\to \mathcal{H}(G)$-mod equals $\mathrm{id}_{\mathcal{H}(G)\text{-mod}}$:*
Again, we do not have to talk about arrows.
Let M be a non-degenerate module over $\mathcal{H}(G)$. Make it into a smooth G-representation and equip it with the associated $\mathcal{H}(G)$-module structure. We have to show that it coincides with the original structure. It clearly suffices to consider $f = \lambda_g(e_K)$ for $g \in G$ and an open, compact subgroup $K \subset G$ with the property $e_K m = m$. But then it is easy to check

$$\begin{aligned}\int_G \lambda_g(e_K)(\gamma)\gamma m\,\mathrm{d}\mu_G(\gamma) &= \int_G e_K(g^{-1}\gamma)e_K(\gamma^{-1}\text{␣})m\,\mathrm{d}\mu_G(\gamma) \\ &= \int_G e_K(\gamma)e_K(\gamma^{-1}g^{-1}\text{␣})m\,\mathrm{d}\mu_G(\gamma) = \lambda_g(e_K)m.\end{aligned}$$

These observations tell us

Corollary 3.3.16. *The categories* G-Rep *and* $\mathcal{H}(G)$-mod *are isomorphic. In particular, they are equivalent.*

3.3.3 $\mathcal{H}(G//K)$-Modules

Now, we want to examine modules over $\mathcal{H}(G//K)$ for an open, compact subgroup $K \subset G$. The biggest difference between $\mathcal{H}(G//K)$ and $\mathcal{H}(G)$ is the existence of a unit e_K:

Observation 3.3.17. *All* $\mathcal{H}(G//K)$*-modules are non-degenerate.*

Recall that for a (smooth) G-representation (π, V) we denote by V^K the space of K-invariant vectors.

Proposition 3.3.18. *Let* (π, V) *be a smooth* G*-representation, then* V^K *carries the structure of an* $\mathcal{H}(G//K)$*-module. If* V *is irreducible, then* V^K *either vanishes or is simple.*

Proof. First take V as an $\mathcal{H}(G)$-module, then $e_K V$ clearly is an $\mathcal{H}(G//K)$-module. We have to prove that $e_K V$ coincides with V^K. This is seen like this: $e_K V \subset V^K$ follows immediately from the definition:

$$e_K v = \int_K \gamma v \, \mathrm{d}\mu_G(\gamma)$$

for any $v \in V$. On the other hand, $e_K v$ equals v for $v \in V^K$, what yields the opposite inclusion. Now let V be irreducible. Suppose $V^K \neq 0$. Let W be a non-zero submodule of V^K, then $\mathcal{H}(G)W$ is a submodule of V. Since V is irreducible, we have $\mathcal{H}(G)W = V$. But this means

$$V^K = e_K V = e_K\big(\mathcal{H}(G)W\big) = (e_K * \mathcal{H}(G) * e_K)W = \mathcal{H}(G//K)W = W$$

proving the second statement. ∎

We can state

Corollary 3.3.19. V *is irreducible if and only if for every open, compact subgroup* $K \subset G$ *the module* V^K *is either simple or zero.*

Proof. One direction follows from the proposition. For the other direction we refer the reader to the brief argument in Chapter 4.3 of [BH06]. ∎

Moreover, we see that $\mathcal{H}(G//K)$ allows just the modules we are interested in:

Lemma 3.3.20. *Every simple* $\mathcal{H}(G//K)$*-module occurs as* V^K *for one* (*and, up to isomorphism, only one*) *irreducible* G*-representation* V*.*

Proof. See Proposition 4.3 in [BH06]. ∎

Corollary 3.3.21. *Two irreducible* G*-representations* V *and* W *are isomorphic if and only if there is an open, compact subgroup* K *such that* $V^K \cong W^K \neq 0$.

Another way to put this down is:

Corollary 3.3.22. *Take an open, compact subgroup* $K \subset G$*, then:*

$$\left\{\begin{matrix}\text{irreducible smooth} \\ G\text{-representations with a } K\text{-fixed vector}\end{matrix}\right\}_{/\mathrm{Iso}} \overset{1:1}{\longleftrightarrow} \Big\{\text{simple } \mathcal{H}(G//K)\text{-modules}\Big\}_{/\mathrm{Iso}}$$

Remark 3.3.23. Since $\{K \subset G \,|\, K \text{ is an open, compact subgroup}\}$ forms an inductive system, $\mathcal{H}(G//K)$-mod forms a projective system in **CAT**. A simple consideration then shows

$$\mathcal{H}(G)\text{-mod} = \varinjlim \mathcal{H}(G//K)\text{-mod}.$$

Remark 3.3.24. We give the general remark that the definition of the Hecke algebra is not limited to the case that G is unimodular. Most of the results in this chapter carry over to the general case, sometimes with an additional modular character in the formula. The reader may have a look at chapter I.3 of [Vig96] or the article [How02].
Moreover, the restriction $k = \mathbb{C}$ can be weakened. The right condition for k is that G contains an open, compact subgroup with pro-order invertible in k, see [Vig96]. A suitable definition for a Hecke algebra $\mathcal{H}_k(G)$ whose elements are k-valued is then straight-forward and the convolution can be constructed in the obvious way: Because of Theorem 2.4 in [Vig96] we have a suitable Haar measure. This gives rise to a Haar integral, allowing us to define the convolution as in the complex case. For example, Vigneras then shows in Theorem 4.4 of [Vig96] that the categories G-Rep_k and $\mathcal{H}_k(G)$-mod are equivalent.

3.4 Parabolic Induction and Jacquet Restriction

Let G be a reductive p-adic group and let k be a field in which p is a non-zero square. We define the main tools for the investigation of smooth representations:

Definition 3.4.1 (**Parabolic Induction Functor**)**.** Let $P = MU \subset G$ be a parabolic subgroup, then we define the functor

$$i_P^G : M\text{-Rep}_k \longrightarrow G\text{-Rep}_k$$

by firstly extending an M-representation (π, V) trivially across U: $p = mu$ acts on v as $pv := mv$, then inducing the obtained representation of P up to G. Observe that Iwasawa's decomposition tells us that it does not matter whether we induce compactly or not.

Definition 3.4.2 (**Jacquet Restriction Functor**)**.** Let $P = MU \subset G$ be as above, then we define the functor

$$r_P^G : G\text{-Rep}_k \longrightarrow M\text{-Rep}_k$$

as follows: Let (π, V) be a G-representation, then set $V(U) = \langle v - uv \,|\, u \in U, v \in V\rangle$ and define $r_P^G(V) = V/V(U)$ – the space of coinvariants. This is the largest quotient of V on which U works trivially. Since M normalizes U, $r_P^G(V)$ is an M-representation. Moreover, it is not hard to see how this process may treat arrows.

There is a useful criterion whether a vector is contained in the space we mod out:

Lemma 3.4.3. *Fix some Haar measure μ_U on U. Then $v \in V$ is contained in $V(U)$ precisely if there is an open, compact pro-p-subgroup $K \subset U$ such that*

$$\int_K kv \,\mathrm{d}\mu_U(k) = 0.$$

If $\mathrm{char}(k) = 0$*, we can drop the "pro-p" condition in the formulation.*

Proof. One direction is easy: Let $v = \sum_I v_i - u_i v_i \in V(U)$ and take $K \in \mathscr{K}(U)$ which contains all u_i, see Proposition 2.5.12 (v). Then

$$\int_K kv \,\mathrm{d}\mu_U(k) = \sum_I \int_K kv_i \,\mathrm{d}\mu_U(k) - \int_K ku_iv_i \,\mathrm{d}\mu_U(k) = 0.$$

On the other hand, let $v \in V$ be such that $\int_K kv \,\mathrm{d}\mu_U(k)$ vanishes for some $K \in \mathscr{K}(U)$. Define K' to be the intersection of K with the stabilizer of v, then we can write

$$\frac{-1}{[K:K']} \sum_{k \in K/K'} kv - v = \left(\frac{-1}{\mu_G(K)} \int_K kv \,\mathrm{d}\mu_U(k)\right) - \left(\frac{-1}{[K:K']} \sum_{k \in K/K'} v\right) = 0 - \frac{-[K:K']}{[K:K']}\, v = v.$$

The only reason why we took a pro-p-subgroup is that we certainly do not want $[K : K']$ or $\mu_G(K)$ to vanish. This cannot happen in the case that $\mathrm{char}(k) = 0$, hence we can drop the "pro-p" condition. $\square$

Remark 3.4.4. This usage of pro-p-groups is typical for our arguments in the sequel: We will always take Ks in $\mathscr{K}(G)$, but in the case $\mathrm{char}(k) = 0$ we could get along with the Ks just being open, compact subgroups.

We state a first theorem, summarizing the basic properties of the functors defined:

Theorem 3.4.5. (1) *Both r_P^G and i_P^G are exact and additive,*

(2) ***Frobenius Reciprocity**: r_P^G is left adjoint to i_P^G:*

$$\mathrm{Hom}_G(V, i_P^G W) \cong \mathrm{Hom}_M(r_P^G V, W) \qquad \text{for } V \in G\text{-Rep}_k, W \in M\text{-Rep}_k,$$

(3) i_P^G *respects admissibility.*

Proof. (1): Concerning i_P^G, this is clear as we know that Ind_P^G is exact and additive and the same is true about the process of inflating across U.
For r_P^G, right exactness is done by some diagram chasing. The tricky part is to show that for $V \overset{\tau}{\hookrightarrow} W$, the obtained arrow $V/V(U) \to W/W(U)$ is injective (or, in other words, that $\tau^{-1}\big(W(U)\big) \subset V(U)$). For this, we may use the above characterization of $V(U)$: Take a $v \in V$ such that $\tau(v) \in W(U)$. This means that $0 = \int_K k\tau(v) = \tau(\int_K kv)$ for some K. Injectivity of τ yields the result. Additivity is clear.
(2): Frobenius Reciprocity can be derived from ordinary Frobenius Reciprocity:

$$\mathrm{Hom}_G(V, i_P^G W) \cong \mathrm{Hom}_P(\mathrm{Res}_P^G V, W) \cong \mathrm{Hom}_M(r_P^G V, W)$$

where the second isomorphism is seen like this: A P-intertwiner $V \xrightarrow{\tau} W$ is taken to $[v] \mapsto \tau(v)$. This process is well-defined since W is trivially inflated from M. An M-intertwiner $r_P^G V \xrightarrow{\rho} W$ is taken to $v \mapsto \rho([v])$. This map intertwines with all of P since W is trivially inflated from M.
Now for (3): It goes without saying that inflating respects admissibility: To compute V^K for $K \subset P$ open compact, remark that $M \cap K$ is open in M. Hence there is some $K' \subset M \cap K$ open and compact in M. Hence $\dim(V^K) \leq \dim(V^{K'}) < \infty$.
That induction up to G respects admissibility as well means putting together Iwasawa's decomposition and part (iii) of Proposition 3.2.5.

□

It is convenient to twist the induction and restriction a bit:

Definition 3.4.6 (Normalized induction and restriction). Take the M-character $\Delta = \delta_P^{-1}\delta_G$, then we may define functors between M-Rep_k and G-Rep_k as follows

$$\boldsymbol{i}_P^G : V \mapsto i_P^G(\Delta^{\frac{1}{2}} \circledast V) \quad \text{and} \quad \boldsymbol{r}_P^G : W \mapsto \Delta^{-\frac{1}{2}} \circledast r_P^G(W).$$

This is a good idea since they fulfill the properties just established and additionally we have

Observation 3.4.7. $\boldsymbol{i}_P^G(V)\check{} = \boldsymbol{i}_P^G(\check{V})$

Proof. It is easily seen that $(\chi \circledast V)\check{} = \chi^{-1} \circledast \check{V}$. Then Theorem 3.2.5 (v) yields

$$\boldsymbol{i}_P^G(V)\check{} = \mathrm{ind}_P^G(\Delta^{\frac{1}{2}} \circledast V)\check{} = \mathrm{ind}_P^G(\Delta \circledast (\Delta^{\frac{1}{2}} \circledast V)\check{}\,) = \mathrm{ind}_P^G(\Delta \circledast \Delta^{-\frac{1}{2}} \circledast \check{V}) = \mathrm{ind}_P^G(\Delta^{\frac{1}{2}} \circledast \check{V}) = \boldsymbol{i}_P^G(\check{V}).$$ □

As the reader may have observed, we could have used δ_P^{-1} instead of Δ, because G is reductive, hence unimodular. We gave the definition of Δ because this is the right definition if we are working with a non-unimodular group G. Moreover, this Δ is standard in the literature.
For simplicity, we will use the symbol δ_P^{-1} in the sequel.

3.5 On Exact Sequences

Proposition 3.5.1. *Let $\tau : V \longrightarrow W$ be an intertwining map between two smooth G-representations. Let $K \in \mathscr{K}(G)$, then*

$$\tau(V^K) = \tau(V) \cap W^K.$$

Proof. "$\subseteq$" is easy: Let $v \in V^K$, then $\tau(v) \in \tau(V)$ and $k\tau(v) = \tau(kv) = \tau(v)$ for all $k \in K$.
Now for "$\supseteq$": Take $w \in \tau(V) \cap W^K$. Then we find a preimage $v \in V$ that is mapped to w under τ. Set

$$v_0 = \int_G e_K(\gamma)\, \gamma v \, \mathrm{d}\mu_G(\gamma).$$

v_0 lies in V^K and is mapped to w:

$$\tau(v_0) = \int_G e_K(\gamma)\, \tau(\gamma v)\, \mathrm{d}\mu_G(\gamma) = \int_G e_K(\gamma)\, \gamma\tau(v)\, \mathrm{d}\mu_G(\gamma) = \int_G e_K(\gamma)\, \gamma w\, \mathrm{d}\mu_G(\gamma) = w.$$ □

We can prove a useful criterion that allows us to decide whether a sequence of G-representations is exact or not:

Lemma 3.5.2. *Let V, W, X be smooth G-representations and $\mathscr{J} \subset \mathscr{K}(G)$ be a subset such that*

$$W = \bigcup_{K \in \mathscr{J}} W^K.$$

Then, for a G-sequence

$$V \xrightarrow{\tau} W \xrightarrow{\rho} X$$

the following is equivalent:

(i) *The sequence is exact at W;*

(ii) *The related sequence of vector spaces*

$$V^K \longrightarrow W^K \longrightarrow X^K$$

is exact at W^K for each $K \in \mathscr{J}$.

Proof. (i) $\Rightarrow$ (ii):
Call the maps in the induced sequence τ_K and ρ_K. We have

$$w \in \operatorname{im}(\tau_K) \Rightarrow w \in \operatorname{im}(\tau) \Rightarrow w \in \ker(\rho) \Rightarrow w \in \ker(\rho_K).$$

And

$$w \in \ker(\rho_K) \Rightarrow w \in \ker(\rho) \Rightarrow w \in \operatorname{im}(\tau),$$

but $w \in W^K$, hence $w \in \operatorname{im}(\tau_K)$ because of the preceding proposition.
(ii) $\Rightarrow$ (i):
Let $w \in \operatorname{im}(\tau)$. Take a $K \in \mathscr{J}$ such that $w \in W^K$. The proposition gives $w \in \operatorname{im}(\tau_K)$. Hence $w \in \ker(\rho_K)$, and this says $w \in \ker(\rho)$.
Let $w \in \ker(\rho)$ and take again a $K \in \mathscr{J}$ such that $w \in W^K$. We have

$$w \in \ker(\rho_K) \Rightarrow w \in \operatorname{im}(\tau_K) \subset \operatorname{im}(\tau).$$

□

Chapter 4

Bimodule Techniques

Let G be a reductive p-adic group and k be a field in which p is a non-zero square. Any smooth G-representation can be understood as a G-module, that is, a module over the group ring kG. As we remarked, it is usually of no avail to use this characterization, as an arbitrary G-module has no reason to be smooth. This is why one defines the Hecke algebra as a substitute for kG.
We will go back and use the naive G-module point of view. We will develop Jacquet functors as tensoring with certain bimodules. These functors restrict to the smooth categories.

We give the following general remark about the groups we are considering: As already said, we are interested in the case where G is a reductive p-adic group. Nevertheless, many proofs will hold in the more general setting where G is a closed subgroup of a reductive p-adic group. Observe that such a subgroup does not have to be reductive or unimodular.
Moreover, we would like to remark that this restriction to closed subgroups is still too strong. There are some definitions or proofs which work for general ℓ-groups, at least if we assume $\mathrm{char}(k) = 0$. If the characteristic does not vanish, we would have to introduce another assumption on G in order to have an invariant integral, see 3.3.24. But, as the methods will only be used for closed subgroups, we restrict ourselves to this case and avoid technical and notational efforts from which, at the end, we would not profit.

4.1 Definitions

We start with

Definition 4.1.1. Let X be an ℓ-space and let G and H be ℓ-groups. Assume, moreover, that there are ℓ-actions $G \curvearrowright X \curvearrowleft H$. Define

$$\mathcal{D}(X) = \mathscr{C}_c^\infty(X) = \left\{ \varphi : X \longrightarrow k \;\middle|\; \varphi \text{ locally constant, } \overline{\mathrm{supp}(\varphi)} \text{ compact} \right\}.$$

The first condition is called **smoothness** and the second **compactness of support**. We make $\mathcal{D}(X)$ into a G-H-bimodule (that is short for kG-kH-bimodule) as follows:

$$\lambda g \varphi \mu h = \Big(x \longmapsto \lambda\mu\varphi(g^{-1}xh^{-1}) \Big) \qquad \text{for } g \in G, h \in H, \lambda, \mu \in k.$$

We will primarily be concerned with the following examples:

- As G acts on itself via multiplication (from the left and from the right), we are provided with a G-G-bimodule $\mathcal{D}(G)$.
- Let $U \subset G$ be a closed subgroup, $M \subset G$ another closed subgroup which is contained in the normalizer of U in G. Then $\mathcal{D}(G/U)$ is a G-M-bimodule.

- The product space $G \times H$ is an ℓ-space on which G acts from the left and H from the right. This gives rise to a G-H-bimodule $\mathcal{D}(G \times H)$.

We should introduce some more notation: If $P \subset G$ is a subgroup, we may understand $\mathcal{D}(G)$ as a G-G-bimodule. Or, say, as a G-P-bimodule. If we want to make clear what we mean, we put subscripts like ${}_G\mathcal{D}(G)_G$ or ${}_G\mathcal{D}(G)_P$.

There is a general principle: Assume we have a continuous and proper map between ℓ-spaces

$$f \colon X \longrightarrow Y.$$

It is easy to see that f induces a map

$$f^* \colon \mathscr{C}_c^\infty(Y) \longrightarrow \mathscr{C}_c^\infty(X) \qquad \varphi \mapsto \Big(x \mapsto \varphi(f(x)) \Big) .$$

If now X and Y are acted on by G from the left and by H from the right and if f is equivariant with respect to these actions, f^* in fact defines a G-H-bimodule homomorphism $\mathcal{D}(Y) \longrightarrow \mathcal{D}(X)$.

We need one more general result about twisting with characters:

Proposition 4.1.2. (i) *Assume we have a right G-module M, a left G-module N and a G-character χ. Then we have the identity*

$$M \circledast \chi \otimes_G N \cong M \otimes_G \chi^{-1} \circledast N$$

where we abbreviate $\otimes_G$ for $\otimes_{kG}$.

(ii) *Consider the G-G-bimodule $\mathcal{D}(G)$. We have an isomorphism*

$$\chi \circledast \mathcal{D}(G) \cong \mathcal{D}(G) \circledast \chi^{-1}$$

of G-G-bimodules.

Proof. The first claim is straightforward. For the second, we may observe that the assignment

$$\varphi \longmapsto \Big(x \mapsto \chi(x^{-1})\varphi(x) \Big)$$

provides us with an isomorphism

$$\mathcal{D}(G) \cong \chi \circledast \mathcal{D}(G) \circledast \chi. \qquad \square$$

4.2 Basic Properties

Throughout this chapter, G denotes a closed subgroup of a reductive p-adic group and μ_G some fixed (left) Haar measure on G (with values in k).

We want to recall the definition of e_K from our Hecke algebra chapter and disburden it from the unimodularity assumption:

Definition 4.2.1 (Normalized Indicator Function). Let $S \subset G$ be an open and compact subset such that $\mu_G(S) \neq 0$. Then define $e_S \in \mathcal{D}(G)$ as

$$e_S \colon g \longmapsto \mu_G(S)^{-1} \cdot \mathbf{1}_S(g).$$

Recall, moreover, that for any $\varphi \in \mathcal{D}(G)$ we find a $K \in \mathscr{K}(G)$ such that

$$\varphi(kxk') = \varphi(x) \qquad \forall\, x \in G, k, k' \in K.$$

The following proposition deals with the G-module ${}_G\mathcal{D}(G)_G \otimes_G V$, where V is a G-representation.

Lemma 4.2.2 (Properties). *Let V be a smooth G-representation.*

(i) ${}_G\mathcal{D}(G)_G \otimes_G V$ *is a smooth G-representation, where G acts on the first factor from the left.*

(ii) *If $v \in V$ and $K, C \in \mathscr{K}(G)$ such that $Kv = v = Cv$, we have*

$$e_K \otimes v = e_C \otimes v.$$

(iii) *If $v \in V, g \in G$ and $K \in \mathscr{K}(G)$ is a subgroup such that $Kv = v$ and $Kgv = gv$, we have*

$$e_{gK} \otimes v = \delta_G^{-1}(g) \cdot e_K \otimes gv.$$

Proof. Part (i) follows easily from the definitions. Let us treat (ii):
Considering $K \geq K \cap C \leq C$, the problem boils down to the special case that K is a subgroup of C. As $C \in \mathscr{K}(G)$ and K is an open subgroup of C, we have $m := [C : K] \in k^\times$. Because δ_G vanishes on compact subgroups, we can write

$$e_C = \frac{1}{\mu_G(C)} \cdot \mathbf{1}_C = \frac{1}{m} \sum_{x \in K\backslash C} e_{Kx}$$

where the sum is taken over a set of representatives for $K\backslash C$. Hence

$$e_C \otimes v = \frac{1}{m} \sum_{x \in K\backslash C} e_{Kx} \otimes v = \frac{1}{m} \sum_{x \in K\backslash C} e_K x \otimes v = \frac{1}{m} \sum_{x \in K\backslash C} \Big(e_K \otimes xv\Big) = e_K \otimes v.$$

Now for (iii): First of all, we should remark that we can always find such a K: We have a lot of C's fixing v alone, and we can simply take $K = C \cap gCg^{-1}$.
Since K and gKg^{-1} both fix gv, we can apply part (ii) and write

$$e_{gK} \otimes v = \delta_G^{-1}(g) e_{gKg^{-1}} \otimes gv = \delta_G^{-1}(g) e_K \otimes gv. \qquad \square$$

Our first serious result is

Lemma 4.2.3. *Suppose that G is unimodular. Then, for a smooth G-representation V, we have an isomorphism of G-modules*

$$\mathcal{D}(G) \otimes_G V \cong V.$$

Proof. The identification is given by

$$\varphi \otimes v \longmapsto \int_G \varphi(g) gv \, \mathrm{d}\mu_G(g).$$

It is straightforward to see that this is a well-defined G-module homomorphism. Surjectivity is clear: If $v \in V$, take a $K \in \mathscr{K}(G)$ fixing v, then $e_K \otimes v$ is mapped to v.
Injectivity is seen like this: Let $\sum_{(\varphi,v)} \varphi \otimes v$ be in the kernel. Then take a $K \in \mathscr{K}(G)$ such that $\varphi K = \varphi$ and $Kv = v$ for all φ and v occurring in the sum.
Take a set of representatives Γ' for G/K. Then

$$\Gamma := \Gamma' \cap \Big(\bigcup_{\varphi} \mathrm{supp}(\varphi)\Big)$$

is finite. For each φ we have then

$$\begin{aligned} \varphi &= \sum_{x \in \Gamma} \mathbf{1}_{xK} \cdot \varphi(x) \\ \int_G \varphi(g) gv \, \mathrm{d}\mu_G(g) &= \sum_{x \in \Gamma} \mu_G(K) \varphi(x) \cdot xv. \end{aligned}$$

Hence we can write

$$\varphi \otimes v = \sum_{x\in\Gamma} \mathbf{1}_{xK} \cdot \varphi(x) \otimes v = \sum_{x\in\Gamma} \mathbf{1}_{xKx^{-1}} \otimes \varphi(x)xv = \sum_{x\in\Gamma} e_{xKx^{-1}} \otimes \mu_G(K)\varphi(x)xv.$$

But, taking $C = \bigcap_\Gamma xKx^{-1}$, we can use Lemma 4.2.2 (ii) to reduce this to

$$\varphi \otimes v = \sum_{x\in\Gamma} e_C \otimes \mu_G(K)\varphi(x)xv = e_C \otimes \sum_{x\in\Gamma} \mu_G(K)\varphi(x)xv = e_C \otimes \int_G \varphi(x)xv \, \mathrm{d}\mu_G(x).$$

But then we have

$$\sum_{(\varphi,v)} \varphi \otimes v = e_C \otimes \sum_{(\varphi,v)} \int_G \varphi(g)gv \, \mathrm{d}\mu_G(g) = 0. \qquad \square$$

Remark 4.2.4. It is easy to give a generalization to the case that G is not unimodular:

$$\left(\mathcal{D}(G) \circledast \delta_G^{-\frac{1}{2}}\right) \otimes_G \left(\delta_G^{\frac{1}{2}} \circledast V\right) \cong V$$

Carefully investigating the proof above, we see that the only obstacles are occurring delta-factors, but luckily they eventually cancel because of the relation $\mu_G(K) = \delta_G(x^{-1})\mu_G(x^{-1}Kx)$ we already used in part (iii) of Lemma 4.2.2.

In the unimodular case, we immediately get

Corollary 4.2.5. *Every element in $\mathcal{D}(G) \otimes_G V$ is of the form $e_K \otimes v$ for a $v \in V$ and a subgroup $K \in \mathscr{K}(G)$ such that $Kv = v$.*

Proof. Any element in $\mathcal{D}(G)$ can be written as $\sum_{(\varphi,v)} \varphi \otimes v$. Take $v_0 \coloneqq \int_G \sum_{(\varphi,v)} \varphi(g)gv$ and $K \in \mathscr{K}(G)$ a subgroup that fixes v_0. Then both $\sum_{(\varphi,v)} \varphi \otimes v$ and $e_K \otimes v_0$ are mapped to v_0 under the isomorphism of Lemma 4.2.3, hence

$$\sum_{(\varphi,v)} \varphi \otimes v = e_K \otimes v_0. \qquad \square$$

Remark 4.2.6. We can understand $\mathcal{D}(G)$ as the regular representation in the following sense: Any irreducible, smooth G-representation V occurs as a quotient of $\mathcal{D}(G)$. In order to see this, let $v \in V$ be an arbitrary, non-zero vector. Then the G-mapping

$$\mathcal{D}(G) \longrightarrow \mathcal{D}(G) \otimes_G V \cong V \qquad \varphi \mapsto \varphi \otimes v$$

is surjective. Indeed, let $w \in V$. Then we can write $w = \sum_G \lambda_g gv$. A pre-image is given by $\sum_G \lambda_g e_K g \in \mathcal{D}(G)$, where $K \in \mathscr{K}(G)$ is a subgroup fixing all gv for which $\lambda_g \neq 0$.

Remark 4.2.7. As Remark 4.2.4 and various other formulae in the sequel suggest, it may be a good idea to include the delta factor in the notion $\mathcal{D}(G)$: Let φ be an element, then redefine

$$\lambda g\varphi\mu h = \left(x \longmapsto \delta_G^{\frac{1}{2}}(gh^{-1})\lambda\mu\varphi(g^{-1}xh^{-1})\right) \qquad \text{for } g, h \in G, \lambda, \mu \in k.$$

Depending on the context, this may seem more natural and lead to nicer formulae and calculations. In this thesis, it would in fact make them more complicate and would provoke confusion at various places. Therefore, we do *not* include the delta factor.

4.3 Jacquet Restriction

Again, let G be a closed subgroup of a reductive p-adic group and k a field in which p is a non-zero square. Moreover, fix a (left) Haar measure μ_G. We need a first technical result:

Proposition 4.3.1 (Integration is possible). *Consider a continuous, proper ℓ-group action $G \curvearrowright X$ on an ℓ-space. Let $\varphi \in \mathcal{D}(X)$, $x \in X$. Then the integral*

$$\int_G \varphi(\gamma x)\, \mathrm{d}\mu_G(\gamma)$$

exists.

Proof. By definition, the map

$$\alpha : G \times X \longrightarrow X \times X \qquad (g, x) \mapsto (gx, x)$$

is proper (and continuous). For $x \in X$, consider the "restricted" map

$$\alpha_x : G \times \{x\} \longrightarrow X \times \{x\} \qquad (g, x) \mapsto (gx, x).$$

Now α_x is proper as well: Take $C \times \{x\}$ compact in $X \times \{x\}$ (and, hence, in $X \times X$). Then $\alpha^{-1}\big(C \times \{x\}\big)$ is compact in $G \times X$. But

$$\alpha_x^{-1}\big(C \times \{x\}\big) = \alpha^{-1}\big(C \times \{x\}\big) \cap G \times \{x\}$$

is compact in $G \times \{x\}$. Moreover, α_x is continuous:

$$\begin{tikzcd} G \times X \arrow[r, "\alpha"] & X \times X \arrow[d, "\mathrm{id} \times \mathrm{const.}"] \\ G \times \{x\} \arrow[u, "\imath"] \arrow[r, dotted, "\alpha_x"'] & X \times \{x\} \end{tikzcd}$$

Therefore, α_x gives rise to a proper and continuous map

$$\alpha_x : G \longrightarrow X \qquad g \mapsto gx.$$

According to our general principle, this gives rise to

$$\mathscr{C}_c^\infty(X) \xrightarrow{\alpha_x^*} \mathscr{C}_c^\infty(G) \qquad \varphi \mapsto \varphi(\text{\textvisiblespace} \cdot x).$$

and all functions in the latter set are integrable. ☐

We remark that obviously an analogous result does hold if G acts on X from the right or if we take a right Haar measure.

Example 4.3.2. Let H be a closed subgroup of G, then the action

$$H \curvearrowright G \qquad g \overset{h}{\longmapsto} hg$$

is proper.

Proof. We could apply part (a) of Lemma 3.1 in [Bil03] with $A = \{*\}$ – the one-point space. But it is not hard to prove this fact directly:
Recall from Chapter 6 of [Mey01] that it suffices to give for any two points g and $g' \in G$ neighborhoods U_g and $U_{g'}$ such that the set

$$\{h \in H \,|\, hU_g \cap U_{g'} \neq \emptyset\} \subset H$$

is compact.
Take any open, compact subgroup $K \subset G$ and set $U_g = gK$ and $U_{g'} = g'K$. Then we have

$$\{h \in H \,|\, hU_g \cap U_{g'} \neq \emptyset\} = \{h \in H \,|\, \exists k, \tilde{k} \in K \text{ s. t. } hgk = g'\tilde{k}\} = H \cap g'Kg^{-1}.$$

As H is closed in G, this set is compact in H. ☐

From now on, assume that G reductive. Let $P = MU \subset G$ be a parabolic subgroup and fix a Haar measure μ_U on U. Considering $\mathcal{D}(G)$, we generate the "error submodule"

$$\mathcal{D}(G)(U) = \langle \varphi - u\varphi \,|\, \varphi \in \mathcal{D}(G), u \in U \rangle.$$

Call the quotient of $\mathcal{D}(G)$ by this submodule ${}_U\backslash\mathcal{D}(G)$ and temporarily denote the projection $\mathcal{D}(G) \longrightarrow {}_U\backslash\mathcal{D}(G)$ by π. ${}_U\backslash\mathcal{D}(G)$ is known under the name (left) U-**coinvariants**.
Denote $\pi(\varphi)$ as ${}_U[\varphi]$ for a $\varphi \in \mathcal{D}(G)$. We can express our crucial result about the quotient space as follows:

Lemma 4.3.3. *A $\varphi \in \mathcal{D}(G)$ vanishes under π precisely if*

$$\int_U \varphi(ug)\, \mathrm{d}\mu_U(u) = 0 \qquad \forall g \in G.$$

Proof. That no function in $\mathcal{D}(G)(U)$ survives this integration process is clear. The interesting part is the other direction: Take a $\varphi \in \mathcal{D}(G)$ with $\int_U \varphi(u\_)\, \mathrm{d}\mu_U(u) = 0$.
We find a $K \in \mathscr{K}(G)$ that fixes φ from the right, so let us have a look at the projections

$$G \xrightarrow{\pi_1} G/K \xrightarrow{\pi_2} U\backslash G/K\ .$$

We fix a set of representatives Ω for G/K. For $x \in \Omega$ we introduce the following notation:

$$\langle x \rangle := \{ y \in \Omega \,|\, \varphi(y) \neq 0, \pi_2(x) = \pi_2(y) \}.$$

Then, take a subset $\Gamma \subset \Omega$ such that

$$\Omega \cap \operatorname{supp}(\varphi) = \bigsqcup_{x \in \Gamma} \langle x \rangle.$$

Clearly, Γ and all the $\langle x \rangle$ are finite (or can be chosen so, resp.).
Now, let us turn our attention to the functions

$$\xi_x : G \longrightarrow k \qquad g \mapsto \int_U \mathbf{1}_{xK}(ug)\, \mathrm{d}\mu_U(u).$$

for $x \in G$. Since $UxK = UyK$ implies $y \in UxK$, we clearly have

$$x, y \in \Omega,\, y \in \langle x \rangle \implies \xi_x = \xi_y.$$

Moreover, $\operatorname{supp}(\xi_x) \subset UxK$ and the ξ_x are not zero (for example, $\xi_x(x) = \mu_U(xKx^{-1} \cap U) \neq 0$), hence the set $\{\xi_x\}_{x \in \Gamma}$ is linearly independent in the k-vector space $\mathcal{D}(G)$. Then write

$$\varphi = \sum_{x \in \Omega} \mathbf{1}_{xK} \cdot \varphi(x) = \sum_{y \in \Gamma} \sum_{x \in \langle y \rangle} \mathbf{1}_{xK} \cdot \varphi(x)$$

and, hence,

$$0 = \int_U \varphi(u\_)\, \mathrm{d}\mu_U(u) = \sum_{y \in \Gamma} \sum_{x \in \langle y \rangle} \xi_x \cdot \varphi(x) = \sum_{y \in \Gamma} \xi_y \cdot \sum_{x \in \langle y \rangle} \varphi(x).$$

But this means $\sum_{x \in \langle y \rangle} \varphi(x) = 0$ for all $y \in \Gamma$. Now, if $y \in \Gamma$, then for any $x \in \langle y \rangle$ there exists a $u_x \in U$ such that $u_x xK = yK$. Therefore we can write

$$\varphi = \sum_{y \in \Gamma} \sum_{x \in \langle y \rangle} \mathbf{1}_{xK} \cdot \varphi(x) = \sum_{y \in \Gamma} \sum_{x \in \langle y \rangle} (\mathbf{1}_{xK} - \mathbf{1}_{yK}) \cdot \varphi(x) = \sum_{y \in \Gamma} \sum_{x \in \langle y \rangle} \big(\mathbf{1}_{xK} \cdot \varphi(x)\big) - \big(\mathbf{1}_{u_x xK} \cdot \varphi(x)\big).$$

□

Proposition 4.3.4 (**Cut-off function**). *Let $P = MU \subset G$ be a parabolic subgroup and denote the projection $G \to U\backslash G$ by π. Then there exists a smooth $\kappa : G \to k$ such that*

(i) *For any open and compact subset $C \subset U\backslash G$, $\operatorname{supp}(\kappa) \cap \pi^{-1}(C)$ is open and compact in G.*

(ii) *For any $g \in G$, we have $\int_U \kappa(ug)\, \mathrm{d}\mu_U(u) = 1$.*

Proof. Take a subgroup $K \in \mathscr{K}(G)$. Moreover, fix a U-K-transversal $\Gamma \subset G$. This says $\bigsqcup_{\gamma\in\Gamma} U\gamma K = G$. Defining $\Xi = \bigcup_{\gamma\in\Gamma} \gamma K$, we have $\pi(\Xi) = U\backslash G$. Define the smooth $\kappa' = \mathbf{1}_\Xi$.
Now, suppose we are given a $C \subset U\backslash G$ open and compact. Because $\pi^{-1}(\pi(\gamma K)) = U\gamma K$ is open, we find that $\pi(\gamma K)$ itself is open. Hence, there is a finite $\Omega \subset \Gamma$ with $C \subset \bigcup_{\gamma\in\Omega} \pi(\gamma K)$. Then we have $\pi^{-1}(C) \subset \bigcup_\Omega U\gamma K$. Hence $\pi^{-1}(C) \cap \bigcup_{\Gamma-\Omega} U\gamma K = \emptyset$. Hence $\pi^{-1}(C) \cap \bigcup_{\Gamma-\Omega} \gamma K = \emptyset$. Then we have

$$\operatorname{supp}(\kappa') \cap \pi^{-1}(C) = \Big(\bigcup_{\gamma\in\Gamma} \gamma K\Big) \cap \pi^{-1}(C) = \Big(\bigcup_{\gamma\in\Omega} \gamma K\Big) \cap \pi^{-1}(C).$$

The next step is to observe that $U\backslash G$ is Hausdorff as a quotient of a Hausdorff group by a closed subgroup. Hence, C is closed as a consequence of being compact. Since the projection is continuous, the same is true about $\pi^{-1}(C)$. Moreover, K is closed (as any open subgroup) and compact, which implies that the finite union $\bigcup_{\gamma\in\Omega} \gamma K$ is closed and compact.
We subsume: $\operatorname{supp}(\kappa') \cap \pi^{-1}(C)$ is the intersection of two closed sets, hence closed. Moreover, it is a subset of the compact set $\bigcup_{\gamma\in\Omega} \gamma K$. This proves its compactness, the openness is clear.
Now for the normalization: Take a $g \in G$, then we find a $\gamma \in \Gamma$ such that $UgK = U\gamma K$. We have

$$\int_U \kappa'(ug)\, \mathrm{d}\mu_U(u) = \mu_U(\gamma K \gamma^{-1} \cap U) \neq 0.$$

and we can normalize

$$\kappa(g) := \left(\int_U \kappa'(ug)\, \mathrm{d}\mu_U(u)\right)^{-1} \kappa'(g).$$

This function is still smooth (as it is fixed by K from the right) and fulfills both conditions. ☐

Again, let $P = MU$ be a parabolic subgroup of G. Then, as M normalizes U, both ${}_{U\backslash}\mathcal{D}(G)$ and $\mathcal{D}(U\backslash G)$ carry an M-G-bimodule structure.

Theorem 4.3.5 (**Identificaton with Jacquet Module**). *As M-G-bimodules, we have*

$${}_{U\backslash}\mathcal{D}(G) \cong \delta_P^{-1} \circledast \mathcal{D}(U\backslash G).$$

Proof. The proper identification is given by

$${}_{U\backslash}\mathcal{D}(G) \ni [\varphi] \longmapsto \left([x] \mapsto \int_U \varphi(ux)\, \mathrm{d}\mu_U(u)\right) \in \delta_P^{-1} \circledast \mathcal{D}(U\backslash G).$$

As one readily checks, this defines an M-G-bimodule homomorphism which is injective by Lemma 4.3.3. Now, let ψ be an element of $\delta_P^{-1} \circledast \mathcal{D}(U\backslash G)$. Then, if κ denotes a $U\backslash G$-cut-off function, we get a preimage

$$x \mapsto \kappa(x) \cdot \psi([x])$$

whence our mapping is surjective. ☐

As one would expect, there is an analogous statement if U acts from the right. We just have to take the inverse δ-twist:

Theorem 4.3.6. *As G-M-bimodules, we have*

$$\mathcal{D}(G)_{/U} \cong \mathcal{D}(G/U) \circledast \delta_P.$$

Proof. The strategy is completely analogous to the one used in the theorem above. $\square$

Observe that we did not use deep structure theory stemming from the fact that P is a parabolic subgroup. We simply used the properties that $U \subset G$ is closed and that M normalizes U. Therefore, we can state

Remark 4.3.7. Maintain the parabolic subgroup $P = MU \subset G$. Then the proof above is easily modified to yield

$$_{U\backslash}\mathcal{D}(P) \cong \delta_P^{-1} \circledast \mathcal{D}(U\backslash P)$$

and

$$\mathcal{D}(P)_{/U} \cong \mathcal{D}(P/U) \circledast \delta_P.$$

Recall that we have an M-M-equivariant homeomorphism $U\backslash P \cong M \cong P/U$. Hence, if we are only interested in the M-M-bimodule structure, the first equation says

$$_{U\backslash}\mathcal{D}(P) \cong \delta_P^{-1} \circledast \mathcal{D}(M)$$

and the second one says

$$\mathcal{D}(P)_{/U} \cong \mathcal{D}(M) \circledast \delta_P.$$

If $\mathcal{D}(G)$ is considered as a smooth G-representation (G acting from the left), it is clear from the definition that

$$r_P^G(\mathcal{D}(G)) = {}_{U\backslash}\mathcal{D}(G).$$

The general case is most easily seen via the characterization $r_P^G(V) \cong k \otimes_U V$ where V is a smooth G-representation and k is understood as a trivial right U-module. Then we have

$$r_P^G(V) = r_P^G(\mathcal{D}(G) \otimes_G V) \cong k \otimes_U \mathcal{D}(G) \otimes_G V \cong {}_{U\backslash}\mathcal{D}(G) \otimes_G V.$$

Taking into consideration the normalization, we end up with an M-module-isomorphism

$$\boldsymbol{r}_P^G(V) \;\cong\; \delta_P^{\frac{1}{2}} \circledast {}_{U\backslash}\mathcal{D}(G) \otimes_G V \;\cong\; \delta_P^{-\frac{1}{2}} \circledast \mathcal{D}(U\backslash G) \otimes_G V.$$

This characterization of the functor $\boldsymbol{r}_P^G$ is the main result of this section.

4.4 Parabolic Induction

Our task in this section is to establish an analogous result for parabolic induction. The argument is due to R. Meyer, in fact we adapt Theorem 4.10 in [Mey04] to our situation.

Let G be a reductive p-adic group and H a closed subgroup. As usual, we are interested in the k-valued representations, where k is a field in which p is a non-zero square. Fix a k-valued left Haar measure μ_H. For a smooth H-representation V consider the G-intertwiner

$$\begin{array}{rcl} {}_G\mathcal{D}(G)_H \circledast \delta_H^{-1} \otimes_H V & \xrightarrow{\eta_V} & \mathrm{ind}_H^G(V) \\ \varphi \otimes v & \longmapsto & \left(g \mapsto \displaystyle\int_H \varphi(g^{-1}h)hv \, \mathrm{d}\mu_H(h)\right). \end{array}$$

In fact, η_V can be understood as the V-component of a natural transformation

$$\eta\colon \mathcal{D}(G) \circledast \delta_H^{-1} \otimes_H \text{\textvisiblespace} \xrightarrow{\bullet} \mathrm{ind}_H^G(\text{\textvisiblespace}).$$

It is not hard to check that η is natural: Let $\tau : V \to W$ be an H-intertwiner, then

$$\begin{tikzcd} \varphi \otimes v \arrow[r, mapsto, "\eta_V"] \arrow[d, mapsto, "\mathcal{D}(G)\otimes\tau"'] \arrow[dr, phantom, "\circ"] & \left(g \mapsto \int_H \varphi(g^{-1}h)hv \, \mathrm{d}\mu_H(h)\right) \arrow[d, mapsto, "\mathrm{ind}(\tau)"] \\ \varphi \otimes \tau(v) \arrow[r, mapsto, "\eta_W"'] & \left(g \mapsto \tau\big(\int_H \varphi(g^{-1}h)hv \, \mathrm{d}\mu_H(h)\big) = \int_H \varphi(g^{-1}h)h\tau(v) \, \mathrm{d}\mu_H(h)\right) \end{tikzcd}$$

Now we can state:

Proposition 4.4.1. *Let X be a k-vector space. Then understand X as a representation of the trivial group 1. We have an isomorphism of G-modules*

$$\mathcal{D}(G) \otimes_k X \cong \mathrm{ind}_1^G(X).$$

Proof. The identification is given by

$$\Omega\colon \varphi \otimes x \longmapsto \big(g \mapsto \varphi(g^{-1})x\big).$$

Observe that Ω equals η_X for $H = 1$, if we choose the Haar measure on H such that the "whole group" has measure 1.
Ω is a G-module homomorphism. Let $\sum_{(\varphi,x)} \varphi \otimes x$ be in the kernel. Then take an open, compact subgroup K fixing all the φs and write

$$\sum_{(\varphi,x)} \varphi \otimes x = \sum_{(\varphi,x)} \sum_{g \in K\backslash G} \mathbf{1}_{Kg}\varphi(g) \otimes x = \sum_{g \in K\backslash G} \mathbf{1}_{Kg} \otimes \left(\sum_{(\varphi,x)} \varphi(g)x \right) = 0,$$

hence the map is injective. Surjectivity is even easier: If $f \in \mathrm{ind}(V)$, then $\sum_{x \in X} \mathbf{1}_{\Xi(x)} \otimes x$ is a pre-image under Ω, where

$$\Xi(x) = \begin{cases} \emptyset & \text{if } x = 0, \\ \{g \in G \mid f(g^{-1}) = x\} & \text{if } x \neq 0. \end{cases}$$

□

Let V be a smooth H-representation. We define an H-module homomorphism

$$\begin{aligned} \Theta\colon \mathcal{D}(H) \otimes_k kH \otimes_k V &\longrightarrow \mathcal{D}(H) \otimes_k V \\ \varphi \otimes \left(\sum_h \lambda_h h \right) \otimes v &\longmapsto \sum_h \delta_H^{-1}(h) \lambda_h \varphi(\_h^{-1}) \otimes v - \varphi \otimes \sum_h \lambda_h hv. \end{aligned}$$

We clearly have $\mathrm{im}(\Theta) = \langle \varphi(\_h^{-1})\delta_H^{-1}(h) \otimes v - \varphi \otimes hv \mid \varphi \in \mathcal{D}(H), h \in H, v \in V \rangle$. This gives

$$\mathrm{cok}(\Theta) \cong \mathcal{D}(H) \circledast \delta_H^{-1} \otimes_H V \cong V.$$

Lemma 4.4.2. *Let X be a vector space, then η induces an isomorphism of G-modules*

$$\eta_{\mathcal{D}(H)\otimes_k X} : \mathcal{D}(G) \circledast \delta_H^{-1} \otimes_H \mathcal{D}(H) \otimes_k X \xrightarrow{\sim} \mathrm{ind}_H^G\big(\mathcal{D}(H) \otimes_k X\big).$$

Proof. It is not hard to check that the following diagram commutes:

$$\begin{tikzcd} \mathcal{D}(G) \circledast \delta_H^{-1} \otimes_H \mathcal{D}(H) \otimes_k X \arrow[r, "\eta_{\mathcal{D}(H)\otimes_k X}"] \arrow[dd, equal] & \mathrm{ind}_H^G\big(\mathcal{D}(H) \otimes_k X\big) \arrow[d, equal] \\ & \mathrm{ind}_H^G\big(\mathrm{ind}_1^H(X)\big) \arrow[d, equal] \\ \mathcal{D}(G) \otimes_k X \arrow[r, equal] & \mathrm{ind}_1^G(X) \end{tikzcd}$$

We start at the lower left corner with $\varphi \otimes v$. Take a $K \in \mathscr{K}(H)$ that fixes φ from the right:

$$\begin{tikzcd}
\varphi \otimes e_K \otimes x \arrow[rr, mapsto, "\eta_{\mathcal{D}(H)\otimes_k X}"] & & \Big(g \mapsto \big(\int_H \varphi(g^{-1}h) e_K(h^{-1}\llcorner\!\lrcorner)\,\mathrm{d}\mu_H(h)\big) \otimes x\Big) \arrow[d, mapsto] \\
& & \Big(g \mapsto \Big[h' \mapsto \int_H \varphi(g^{-1}h) e_K(h^{-1}h'^{-1}) x \,\mathrm{d}\mu_H(h) = \varphi((h'g)^{-1})x\Big]\Big) \\
\varphi \otimes x \arrow[uu, mapsto] \arrow[rr, mapsto] & & \Big(g \mapsto \varphi(g^{-1})x\Big) \arrow[u, mapsto]
\end{tikzcd}$$

□

Now we are able to prove

Theorem 4.4.3 (Identification with Induction). *Let $H \subset G$ be a closed subgroup, V a smooth H-representation, then we have an isomorphism of G-modules*

$$\mathcal{D}(G) \circledast \delta_H^{-1} \otimes_H V \cong \mathrm{ind}_H^G(V).$$

Proof. We use the notation from above. We already know that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{D}(G) \circledast \delta_H^{-1} \otimes_H \big(\mathcal{D}(H) \otimes_k kH \otimes_k V\big) \arrow[r, "\mathcal{D}(G)\circledast\delta_H^{-1}\otimes_H\Theta"] \arrow[d, equal] & \mathcal{D}(G) \circledast \delta_H^{-1} \otimes_H \big(\mathcal{D}(H) \otimes_k V\big) \arrow[d, equal] \\
\mathrm{ind}_H^G\big(\mathcal{D}(H) \otimes_k kH \otimes_k V\big) \arrow[r, "\mathrm{ind}_H^G(\Theta)"'] & \mathrm{ind}_H^G\big(\mathcal{D}(H) \otimes_k V\big)
\end{tikzcd}$$

Hence

$$\mathrm{cok}\big(\mathcal{D}(G) \circledast \delta_H^{-1} \otimes_H (\Theta)\big) \cong \mathrm{cok}\big(\mathrm{ind}_H^G(\Theta)\big).$$

But both functors are right-exact, hence preserve cokernels. As the cokernel of Θ is just V, the result follows. □

We are interested in the case where $H = P = MU$ – a parabolic subgroup of a reductive p-adic group G. If V is a smooth M-representation, the parabolic induction $i_P^G(V)$ is given by inflating the M-action to a P-action on V and then inducing up to G. In our new language:

$$i_P^G(V) = \mathcal{D}(G) \circledast \delta_P^{-1} \otimes_P V$$

where we consider V as a P-module via $(mu) \cdot v = mv$.
This is not exactly what we want. We would like to relocate the inflation process away from V. This can be achieved like this:

$$\begin{aligned}\mathcal{D}(G) \circledast \delta_P^{-1} \otimes_P V = \mathcal{D}(G) \circledast \delta_P^{-1} \otimes_P \mathcal{D}(M) \otimes_M V &= \mathcal{D}(G) \circledast \delta_P^{-1} \otimes_P \mathcal{D}(P)_{/U} \circledast \delta_P^{-1} \otimes_M V \\ &= \mathcal{D}(G)_{/U} \circledast \delta_P^{-1} \otimes_M V.\end{aligned}$$

If we take into consideration the normalization, we can subsume

$$\begin{aligned} i_P^G(V) &= & \mathcal{D}(G)_{/U} \circledast \delta_P^{-1} \otimes_M V &\cong \mathcal{D}(G/U) \otimes_M V, \\ \boldsymbol{i}_P^G(V) &= & \mathcal{D}(G)_{/U} \circledast \delta_P^{-\frac{1}{2}} \otimes_M V &\cong \mathcal{D}(G/U) \circledast \delta_P^{\frac{1}{2}} \otimes_M V. \end{aligned}$$

4.5 Relating Twisted Products and Balanced Tensor Products

4.5.1 Generalizing Lemma 4.3.3 and Proposition 4.3.4

Let G be a closed subgroup of a reductive p-adic group. Consider an ℓ-space X which is countable at infinity and a proper ℓ-action $X \backsim G$. As usual, fix a left Haar measure μ_G on G with values in k, where k is a field in which p is a non-zero square.
We say that $X \backsim G$ **allows good decompositions** if for each $\varphi \in \mathcal{D}(X)$ we find a decomposition

$$X = \bigsqcup_{i \in I} \Psi_i$$

into open, compact subsets $\Psi_i \subset X$ such that

- For each $g \in G, i \in I$, the set $\Psi_i g$ equals Ψ_j for some $j \in I$,
- φ is constant on each Ψ_i,
- For each $i \in I$ we find an $x_i' \in X$ such that $\mu_G\big(\{g \in G | x_i' g \in \Psi_i\}\big) \neq 0$.

Observe that the first property induces a G-action on the index set I.

Define the subspace

$$\mathcal{D}(X)(G) = \langle \varphi - \delta_G(g) \cdot \varphi(\_g) \,|\, \varphi \in \mathcal{D}(X), g \in G \rangle \subset \mathcal{D}(X).$$

Now we can use the proof of Lemma 4.3.3 in a more general setting:

Lemma 4.5.1. *Consider a proper ℓ-action $X \backsim G$ that allows good decompositions. Then $\varphi \in \mathcal{D}(X)$ is contained in $\mathcal{D}(X)(G)$ precisely if*

$$\int_G \varphi(x\gamma) \, \mathrm{d}\mu_G(\gamma) = 0 \qquad \forall x \in X.$$

Proof. One direction is obvious: Each $\varphi \in \mathcal{D}(X)(G)$ is killed by this integration process.
For the other direction, take a φ that is killed by the integral. Fix a good decomposition $X = \bigsqcup_I \Psi_i$ with respect to φ. For $j \in I$, define the set $\langle j \rangle = \{i \in I \,|\, \Psi_i \cdot G = \Psi_j \cdot G\}$. Then I decomposes as

$$I = \bigsqcup_{j \in J} \langle j \rangle$$

for a suitable subset $J \subset I$. For $i \in I$ define the map

$$\xi_i \colon X \mapsto k \qquad x \mapsto \int_G \mathbf{1}_{\Psi_i}(x\gamma) \, \mathrm{d}\mu_G(\gamma).$$

For each $j \in J, i \in \langle j \rangle$, fix a $g_i \in G$ such that $\Psi_i = \Psi_j \cdot g_i$. This yields $\xi_i = \delta_G(g_i) \cdot \xi_j$:

$$\begin{aligned} \delta_G(g_i) \cdot \xi_j = \delta_G(g_i) \cdot \int_G \mathbf{1}_{\Psi_j}(x\gamma) \, \mathrm{d}\mu_G(\gamma) &= \delta_G(g_i) \cdot \int_G \mathbf{1}_{\Psi_i \cdot g_i^{-1}}(x\gamma) \, \mathrm{d}\mu_G(\gamma) \\ = \delta_G(g_i) \cdot \int_G \mathbf{1}_{\Psi_i}(x\gamma g_i) \, \mathrm{d}\mu_G(\gamma) &= \int_G \mathbf{1}_{\Psi_i}(x\gamma) \, \mathrm{d}\mu_G(\gamma) = \xi_i. \end{aligned}$$

Now fix a set of representatives $\{x_i\}_{i \in I} \subset X$ for the decomposition $X = \bigsqcup_I \Psi_i$. Write

$$\varphi = \sum_{i \in I} \varphi(x_i) \cdot \mathbf{1}_{\Psi_i} = \sum_{j \in J} \sum_{i \in \langle j \rangle} \varphi(x_i) \cdot \mathbf{1}_{\Psi_i}.$$

Therefore we have

$$0 = \int_G \varphi(x\gamma)\, \mathrm{d}\mu_G(\gamma) = \sum_{j\in J} \sum_{i\in\langle j\rangle} \varphi(x_i)\cdot \xi_i = \sum_{j\in J} \sum_{i\in\langle j\rangle} \varphi(x_i)\delta_G(g_i)\cdot \xi_j.$$

Observe that ξ_j is supported in $\Psi_j \cdot G$. Moreover, ξ_j does not vanish: Take a x'_j delivered by the third condition of the "good decomposition"-framework. Then we have

$$\xi_j(x'_j) = \mu_G\big(\{g \in G \,|\, x'_j g \in \Psi_j\}\big) \neq 0.$$

Hence the $\{\xi_j\}_{j\in J}$ are linearly independent in $\mathcal{D}(G)$ and we have

$$\sum_{i\in\langle j\rangle} \delta_G(g_i)\varphi(x_i) = 0$$

for all $j \in J$. We conclude

$$\varphi = \sum_{j\in J} \sum_{i\in\langle j\rangle} \mathbf{1}_{\Psi_i}\varphi(x_i) = \sum_{j\in J} \sum_{i\in\langle j\rangle} \big(\mathbf{1}_{\Psi_i} - \delta_G(g_i)\cdot \mathbf{1}_{\Psi_j}\big)\varphi(x_i) = \sum_{i,j} \mathbf{1}_{\Psi_i}\varphi(x_i) - \delta_G(g_i)\cdot \mathbf{1}_{\Psi_i}(\_\, g_i)\varphi(x_i).$$

□

We introduce a general principle: Let $X \curvearrowleft G$ be an ℓ-action. If the action is free and proper, the projection map into the orbit space

$$X \xrightarrow{\pi} X/G$$

makes up a principal G-bundle and X/G is Hausdorff (see [Ell00]).
Moreover, we have

Theorem 4.5.2. *X/G is an ℓ-space.*

Proof. As already said, X/G is Hausdorff. π is, as any bundle projection, an open and continuous mapping. Take $\xi \in X/G$ and fix a preimage $x \in X$ under π. First of all, x has an open, compact neighborhood U_x. $\pi(U_x)$ is then an open, compact neighborhood of ξ, proving local compactness. Now, let $\mathcal{O} \subset X/G$ be an open neighborhood of ξ. Then the open $\pi^{-1}(\mathcal{O})$ contains x, and we find an open, compact $U \subset \pi^{-1}(\mathcal{O})$ containing x.
Then $\mathcal{O} \supset \pi(U) \ni \xi$ and $\pi(U)$ is open and compact, whence X/G is totally disconnected. □

Now we are ready for a generalization of Proposition 4.3.4:

Theorem 4.5.3 (Cut-off Function). *Let $X \curvearrowleft G$ be a free and proper ℓ-action. Denote the projection $X \to X/G$ by the letter π. Then there is a cut-off function κ. This is a smooth function $X \to k$ such that*

(i) *For any compact $C \subset X/G$, $\pi^{-1}(C) \cap \operatorname{supp}(\kappa) \subset X$ is compact,*

(ii) *$\int_G \kappa(x\gamma)\, \mathrm{d}\mu_G(\gamma) = 1$ for all $x \in X$.*

Proof. Take an open cover $X/G = \bigcup_I U_i$ such that the U_i admit local cross-sections. Because X/G is an ℓ-space, we find a compatible decomposition

$$X/G = \bigsqcup_{\omega\in\Omega} \mathcal{O}_\omega$$

with all the $\mathcal{O}_\omega$ open and compact (see Lemma 2.1.12 (ii)).
Then we have $\pi^{-1}(\mathcal{O}_\omega) \cong \mathcal{O}_\omega \times G$ as an open subset of X. Let $K \in \mathscr{K}(G)$. Then $\pi^{-1}(\mathcal{O}_\omega)$ contains the open, compact subset $\mathcal{F}_\omega \cong \mathcal{O}_\omega \times K$.

Define the open set $\Xi = \bigcup_\Omega \mathcal{F}_\omega$ and the smooth map $\kappa' = \mathbf{1}_\Xi$.
Now let $C \subset X/G$ be compact. This implies

$$C \subset \bigcup_{\omega \in \Gamma} \mathcal{O}_\omega \quad \text{and} \quad C \cap \Big(\bigcup_{\omega \notin \Gamma} \mathcal{O}_\omega \Big) = \emptyset$$

for a finite subset $\Gamma \subset \Omega$. But then

$$\pi^{-1}(C) \subset \pi^{-1}\big(\bigcup_{\omega \in \Gamma} \mathcal{O}_\omega \big)$$

and

$$\pi^{-1}(C) \cap \Big(\bigcup_{\omega \notin \Gamma} \pi^{-1}(\mathcal{O}_\omega) \Big) = \pi^{-1}(C) \cap \Big(\bigcup_{\omega \notin \Gamma} \mathcal{F}_\omega \Big) = \emptyset.$$

Hence

$$\pi^{-1}(C) \cap \operatorname{supp}(\kappa') = \pi^{-1}(C) \cap \Big(\bigcup_{\omega \in \Gamma} \mathcal{F}_\omega \Big)$$

is the section of an open and closed and an open, closed and compact set, hence open and compact.

Now for the normalization: Any $x \in X$ induces a map

$$\xi_x \colon G \longrightarrow k \qquad \gamma \mapsto \kappa'(x \cdot \gamma).$$

As $x = (o, g) \in \mathcal{O}_\omega \times G$ for some $\omega \in \Omega, o \in \mathcal{O}_\omega$ and $g \in G$, ξ_x takes the form $\mathbf{1}_{g^{-1}K}$. We conclude that we can form the integral $\int_G \kappa'(\gamma x) \, \mathrm{d}\mu_G(\gamma)$ which equals $\mu_G(K) \neq 0$.
Define our final cut-off as

$$\kappa \colon X \longrightarrow k \qquad x \mapsto \mu_G(K)^{-1} \kappa'(x).$$

Obviously, κ is smooth and fulfills both conditions. $\square$

4.5.2 The Identification

Let X and G be as in the preceding section and consider an ℓ-action $X \curvearrowleft G$. Define the vector space $\mathcal{D}(X)_{/G} = \mathcal{D}(X)/\mathcal{D}(X)(G)$. Now we can prove

Theorem 4.5.4. *Assume that the action of G on X is free and proper and allows good decompositions. Then we have an isomorphism of vector spaces*

$$\mathcal{D}(X)_{/G} \cong \mathcal{D}(X/G).$$

Proof. Fix a left Haar measure μ_G on G. Then the identification is given by

$$[\varphi] \overset{\Theta}{\longmapsto} \left(xG \mapsto \int_G \varphi(x\gamma) \, \mathrm{d}\mu_G(\gamma) \right).$$

Let $[\varphi] \in \mathcal{D}(X)_{/G}$. Let us first show that $\Theta([\varphi])$ is smooth: Take a point $xG \in X/G$ at which we want to prove smoothness and fix a good decomposition $X = \bigsqcup_I \Psi_i$ with respect to φ. Then x is contained in some Ψ_i. Because the projection $\pi \colon X \to X/G$ is an open map, $\Psi_i G$ is open in X/G. We claim that $\Theta([\varphi])$ is constant on $\Psi_i G$. This is seen like this: Let $yG, y'G \in \Psi_i G \subset X/G$ with $y, y' \in \Psi_i$. Then $\Theta([\varphi])(yG) = \int_G \varphi(y\gamma) \, \mathrm{d}\mu_G(\gamma)$ and $\Theta([\varphi])(y'G) = \int_G \varphi(y'\gamma) \, \mathrm{d}\mu_G(\gamma)$. Because of the first condition of the "good decomposition"-framework, $\varphi(y\gamma) = \varphi(y'\gamma)$ for any $\gamma \in G$ and the two integrals coincide.
The statement that $\Theta([\varphi])$ is compactly supported is an easy consequence from the fact that π is continuous.
Injectivity follows from Lemma 4.5.1. Surjectivity is seen like this: Let $\psi \in \mathcal{D}(X/G)$. Denote by κ a cut-off function, then the assignment

$$\varphi \colon x \mapsto \kappa(x) \varphi([x])$$

gives rise to a preimage $[\varphi]$ under Θ. $\square$

This section is devoted to two important applications of Theorem 4.5.4. We need some preparation: From now on, consider two closed subgroups X and Y of some reductive p-adic group. The cartesian product $X \times Y$ is an ℓ-group and consequently $\mathcal{D}(X \times Y)$ is an X-Y-bimodule via

$$x\zeta y = \zeta(x^{-1}\,\llcorner\!\lrcorner\,,\,\llcorner\!\lrcorner\, y^{-1}).$$

It is not hard to prove the following isomorphism of X-Y-bimodules:

Proposition 4.5.5.

$$\mathcal{D}(X) \otimes_k \mathcal{D}(Y) \cong \mathcal{D}(X \times Y)$$

Proof. The identification is given by

$$\Theta\colon \varphi \otimes \psi \longmapsto \Big((x, y) \mapsto f(x) \cdot g(y) \Big) .$$

$\Theta(\varphi \otimes \psi)$ is clearly smooth, and compactness of the support is Tychonoff's theorem. Moreover, this process preserves the bimodule structure.
Injectivity is observed like this: Let $\sum_{(\varphi,\psi)} \varphi \otimes \psi$ be in the kernel. Take a K fixing φ, an L fixing ψ and write

$$\sum_{(\varphi,\psi)} \varphi \otimes \psi = \sum_{(\varphi,\psi)} \sum_{x,y} \mathbf{1}_{Kx} \cdot \varphi(x) \otimes \mathbf{1}_{Ly} \cdot \psi(y) = \sum_{x,y} \mathbf{1}_{Kx} \otimes \mathbf{1}_{Ly} \cdot \Big(\sum_{(\varphi,\psi)} \varphi(x) \cdot \psi(y) \Big) = 0.$$

Concerning surjectivity, take a $\zeta \in \mathcal{D}(X \times Y)$. Write $\operatorname{supp}(\zeta) = \bigcup_I S_i$ for open subsets $S_i \subset X \times Y$ on which ζ is constant. Note that any open set in $X \times Y$ can be written as a union $\bigcup_J (U_j, O_j)$ with open sets $U_j \subset X, O_j \subset Y$. Do this with all the S_i, then we end up with

$$\operatorname{supp}(\zeta) = \bigcup_B (U_\beta, O_\beta)$$

for some index set B. By compactness of support, we find a finite $\Gamma \subset B$ with

$$\operatorname{supp}(\zeta) = \bigcup_\Gamma (U_\gamma, O_\gamma).$$

Using the inclusion-exclusion principle, we can write

$$\zeta = \sum_\Gamma \zeta(U_\gamma, O_\gamma) \cdot \mathbf{1}_{(U_\gamma,O_\gamma)} - \sum_{\gamma \neq \gamma' \in \Gamma} \zeta(U_\gamma, O_\gamma) \cdot \mathbf{1}_{(U_\gamma \cap U_{\gamma'}, O_\gamma \cap O_{\gamma'})} + \ldots$$

But this provides us with a preimage

$$\sum_\Gamma \zeta(U_\gamma, O_\gamma) \cdot \mathbf{1}_{U_\gamma} \otimes \mathbf{1}_{O_\gamma} - \sum_{\gamma \neq \gamma' \in \Gamma} \zeta(U_\gamma, O_\gamma) \cdot \mathbf{1}_{U_\gamma \cap U_{\gamma'}} \otimes \mathbf{1}_{O_\gamma \cap O_{\gamma'}} + \ldots \qquad \square$$

In order to generalize this result, we need two more definitions:

Definition 4.5.6 (Twisted Product). Let X, Y be as above with closed subgroups $A \subset X, B \subset Y$. Moreover, assume that A and B (with the induced topology) are group-homeomorphic via $\sigma\colon A \xrightarrow{\sim} B$. Then we have an ℓ-group action

$$X \times Y \curvearrowleft A \quad \text{via} \quad (x, y) \bullet a = (xa, \sigma(a^{-1})y).$$

This action is continuous and we will call the quotient space by the name $X \underset{A=B}{\times} Y$. Observe that there is an obvious left X- and right Y-action on this space. Moreover, there is an obvious continuous projection

$$\pi : X \times Y \longrightarrow X \underset{A=B}{\times} Y$$

and we will write $[x, y]$ for the element $\pi(x, y)$.

These objects are in use in the theory of transformation groups, and we refer the reader to the first two chapters of [Bre72] for a more detailed treatment. We need

Remark 4.5.7. As the action of A on $X \times Y$ is free and proper,[1] we can use Theorem 4.5.2 to show that $X \underset{A=B}{\times} Y$ is an ℓ-space.

The next object we need is

Definition 4.5.8 (Balanced Tensor Product). Take a group isomorphism $\sigma : A \xrightarrow{\sim} B$, a right A-module S and a left B-module T. Then we can equip T with an A-action via $(\sum \lambda_a a)t \coloneqq \sum \lambda_a \sigma(a) t$. Call this left A-module T'. Then define

$$S \underset{A=B}{\otimes} T \coloneqq S \otimes_A T'.$$

The reader should take notice of the dependence of this definition on σ. Therefore, one should use this notion only if it is clear which homeomorphism is meant.

The first application of Theorem 4.5.4 is

Theorem 4.5.9. *Let X, Y be closed subgroups of some reductive p-adic group, $A \subset X$ and $B \subset Y$ closed subgroups with a group-homeomorphism $A \cong B$. Then there is an isomorphism of X-Y-bimodules*

$$\mathcal{D}(X) \circledast \delta_A^{-1} \underset{A=B}{\otimes} \mathcal{D}(Y) \;\cong\; \mathcal{D}(X \underset{A=B}{\times} Y).$$

Proof. We know that the action of A on $X \times Y$ is free and proper. We have to show that it allows good decompositions: Let $\varphi \in \mathcal{D}(X \times Y)$. Then we find $K \in \mathscr{K}(X)$ and $L \in \mathscr{K}(Y)$ such that $\varphi(Kx, yL) = \varphi(x, y)$ for any $x \in X, y \in Y$. Observe, that $\mu_A\big(A \cap x^{-1}Kx \cap \sigma^{-1}(yLy^{-1} \cap B)\big) \neq 0$ for $x \in X, y \in Y$. Therefore the decomposition

$$\bigsqcup_{\substack{x \in K\backslash X \\ y \in Y/L}} (Kx, yL)$$

is good with respect to φ.
Hence we have an isomorphism

$$\mathcal{D}(X \times Y)_{/A} \cong \mathcal{D}(X \times Y/A).$$

Because of Proposition 4.5.5 we can write the first space as $\mathcal{D}(X) \circledast \delta_A^{-1} \underset{A=B}{\otimes} \mathcal{D}(Y)$.
The second one equals $\mathcal{D}(X \underset{A=B}{\times} Y)$ by definition. We can write down an isomorphism explicitly:

$$\mathcal{D}(X) \circledast \delta_A^{-1} \underset{A=B}{\otimes} \mathcal{D}(Y) \longrightarrow \mathcal{D}(X \underset{A=B}{\times} Y) \qquad \varphi \otimes \psi \longmapsto \left([x, y] \mapsto \int_A \varphi(x\alpha) \cdot \psi(\sigma(\alpha^{-1})y)\, \mathrm{d}\mu_A(\alpha) \right)$$

This mapping is obviously equivariant with respect to the left X- and the right Y-action. $\square$

We come to the second example: Let G be a reductive p-adic group and $P = MU \subset G$ be a parabolic subgroup.

Proposition 4.5.10. *The set*

$$\Delta = \big\{ (mu, mu') \in P \times P \;\big|\; m \in M, u, u' \in U \big\}$$

is a closed subgroup of $P \times P$.

[1] This is part (a) of Lemma 3.1 in [Bil03].

Proof. That Δ is a subgroup of $P \times P$ follows easily from the fact that M normalizes U. Consider the map f given by the composition

$$P \times P \longrightarrow P \times P \longrightarrow P \qquad (x, y) \longmapsto (x, y^{-1}) \longmapsto xy^{-1}.$$

As both maps are continuous by the definition of a topological group, f is continuous. It is clear that Δ is the pre-image of U under f, hence closed. $\square$

Theorem 4.5.11. *The ℓ-action*

$$G \times G \curvearrowleft \Delta \qquad (g, g') \bullet (mu, mu') = (gmu, u'^{-1}m^{-1}g') \tag{4.1}$$

is free and proper and allows good decompositions.

Proof. It is obvious that the action is free. As Δ is a closed subgroup of $G \times G$, the action

$$G \times G \curvearrowleft \Delta \qquad (g, g') \bullet (mu, mu') = (gmu, g'mu').$$

is proper. Hence the map

$$\begin{array}{rcl} \big(G \times G\big) \times \Delta & \longrightarrow & \big(G \times G\big) \times \big(G \times G\big) \\ \big((g, g'), (mu, mu')\big) & \longmapsto & \big((g, g'), (gmu, g'mu')\big) \end{array}$$

is proper. Call it β. Consider the map

$$\big(G \times G\big) \times \Delta \xrightarrow{\alpha} \big(G \times G\big) \times \Delta \xrightarrow{\beta} \big(G \times G\big) \times \big(G \times G\big) \xrightarrow{\gamma} \big(G \times G\big) \times \big(G \times G\big)$$

where α sends $\big((g, g'), (mu, mu')\big)$ to $\big((g, g'^{-1}), (mu, mu')\big)$ and γ sends $\big((g, g'), (x, x')\big)$ to $\big((g, g'^{-1}), (x, x'^{-1})\big)$. These maps are both homeomorphisms, therefore $\gamma \circ \beta \circ \alpha$ is proper and this means that the action (4.1) is proper.
We show now that the action allows good decompositions: Fix a $\Pi \in \mathscr{K}(G \times G)$. Now let φ be an element of $\mathcal{D}(G \times G)$. Then there is an open, compact subgroup $K \subset G$ such that $(K, K) \subset \Pi$ and $\varphi(Kx, yK) = \varphi(x, y)$ for any two $x, y \in G$. Take as decomposition

$$G = \bigsqcup_{\substack{x \in K\backslash G \\ y \in G/K}} (Kx, yK).$$

Then we have

$$\Delta_{x,y} := \big\{\delta \in \Delta \,|\, (x, y)\delta \in (Kx, yK)\big\} = \Delta \cap \big(x^{-1}Kx \cap P, yKy^{-1} \cap P\big).$$

Now $(x^{-1}Kx, yKy^{-1})$ lies in $\mathscr{K}(G \times G)$, hence $\big(x^{-1}Kx \cap P, yKy^{-1} \cap P\big) \in \mathscr{K}(P \times P)$ and we conclude that $\mu_\Delta(\Delta_{x,y}) \neq 0$. $\square$

We have a nice characterization for such a left invariant Haar measure μ_Δ: Fix Haar measures μ_U and μ_M on U and M, then we can assign to any open, compact $A \subset \Delta$ the number

$$\int_U \int_U \int_M \mathbf{1}_A(mu, mu')\, \mathrm{d}\mu_M(m)\, \mathrm{d}\mu_U(u) \mathrm{d}\mu_U(u').$$

It is clear that this rule is left invariant. In order to see that it defines a regular Borel measure on Δ, one will have to use the fact that for any such A we find an open, compact subgroup $K \subset G$ which admits an Iwahori decomposition with respect to P and such that $(K \cap P, K \cap P) \subset A$. Moreover, we can immediately read off $\delta_\Delta(um, u'm) = \delta_P^2(m)$.
As the quotient of $G \times G$ by the Δ-action is $G/U \times_M U\backslash G$, this implies

Corollary 4.5.12.

$$\mathcal{D}(G \times G)_{/\Delta} \cong \mathcal{D}(G)_{/U} \circledast \delta_P^{-2} \otimes_M {}_{U\backslash}\mathcal{D}(G) \;\cong\; \mathcal{D}(G/U \times_M U\backslash G).$$

We will have to twist this result by a character "in the middle" as follows:

Remark 4.5.13. Observe that we have the following characterization:

$$\mathcal{D}(G/U \times_M U\backslash G) = \left\{\varphi\colon G \times G \longrightarrow k \;\middle|\; \substack{\varphi \text{ smooth and compactly supported modulo } \Delta, \\ \varphi(xmu, u'm^{-1}y) = \varphi(x,y)\ \forall x,y \in G, m \in M, u,u' \in U}\right\}$$

where the condition "compactly supported modulo Δ" means that $\overline{\mathrm{supp}(\varphi)}$ is compact when projected onto $G \times G/\Delta$. The reason that we do not have to include something like "smooth modulo Δ" is that the projection $G \times G \to G/U \times_M U\backslash G$ is open and continuous. Therefore, it is equivalent for φ to be smooth or to be smooth after projection.
Now, recall that $\mathcal{D}(G \times G)_{/\Delta}$ was defined as $\mathcal{D}(G \times G)/\mathcal{D}(G \times G)(\Delta)$ with $\mathcal{D}(G \times G)(\Delta) = \langle \varphi - \delta_\Delta(x)\varphi(\_x) \mid \varphi \in \mathcal{D}(G \times G), x \in \Delta\rangle$.
Understand the assignment $\chi\colon (mu, mu') \mapsto \delta_P^{-1}(m)$ as a character of Δ. Then, if we divide by $\langle \varphi - \chi(x)\delta_\Delta(x)\varphi(\_x) \mid \varphi \in \mathcal{D}(G \times G), x \in \Delta\rangle$ instead of $\mathcal{D}(G \times G)(\Delta)$, we get an isomorphism

$$\mathcal{D}(G)_{/U} \circledast \delta_P^{-1} \otimes_M {}_{U\backslash}\mathcal{D}(G) \;\cong\; \mathcal{D}^{\delta_P^{-1}}(G/U \times_M U\backslash G).$$

where

$$\mathcal{D}^{\delta_P^{-1}}(G/U \times_M U\backslash G) = \left\{\varphi\colon G \times G \longrightarrow k \;\middle|\; \substack{\varphi \text{ smooth and compactly supported modulo } \Delta, \\ \delta_P^{-1}(m)\varphi(xmu, u'm^{-1}y) = \varphi(x,y)\ \forall x,y \in G, m \in M, u,u' \in U}\right\}.$$

We can explicitly write down the isomorphism:

$$\begin{aligned} \mathcal{D}(G)_{/U} \circledast \delta_P^{-1} \otimes_M {}_{U\backslash}\mathcal{D}(G) \;&\longrightarrow\; \mathcal{D}^{\delta_P^{-1}}(G/U \times_M U\backslash G) \\ \varphi \otimes \psi \longmapsto \Bigg((x,y) \mapsto &\int_U \int_U \int_M \varphi(xum)\psi(u'm^{-1}y)\, \mathrm{d}\mu_M(m)\, \mathrm{d}\mu_U(u) \mathrm{d}\mu_U(u')\Bigg) \end{aligned}$$

4.6 Summary of Bimodule Identifications

We give a summary of the identifications we worked out:

$$\begin{aligned} {}_{U\backslash}\mathcal{D}(G) \;&\cong\; \delta_P^{-1} \circledast \mathcal{D}(U\backslash G) \\ \mathcal{D}(G)_{/U} \;&\cong\; \mathcal{D}(G/U) \circledast \delta_P^{+1} \\ {}_{U\backslash}\mathcal{D}(P) \;&\cong\; \delta_P^{-1} \circledast \mathcal{D}(U\backslash P) \\ \mathcal{D}(P)_{/U} \;&\cong\; \mathcal{D}(P/U) \circledast \delta_P^{+1} \\ \mathrm{ind}_H^G(V) \;&\cong\; \left(\mathcal{D}(G) \circledast \delta_H^{-1}\right) \otimes_H V \quad \text{for } V \in H\text{-Rep}_k \\ \left(\mathcal{D}(P) \circledast \delta_P^{-1}\right) \otimes_P V \;&\cong\; V \quad \text{for } V \in P\text{-Rep}_k \\ V \otimes_P \left(\mathcal{D}(P) \circledast \delta_P^{-1}\right) \;&\cong\; V \quad \text{for a smooth right } P\text{-module } V \end{aligned}$$

Therefore, we recover the basic functors of p-adic representation theory:

$$\begin{aligned} i_P^G(W) \;&=\; \mathcal{D}(G)_{/U} \circledast \delta_P^{-1} \otimes_M W \cong \mathcal{D}(G/U) \otimes_M W \\ \boldsymbol{i}_P^G(W) \;&=\; \mathcal{D}(G)_{/U} \circledast \delta_P^{-\frac{1}{2}} \otimes_M W \cong \mathcal{D}(G/U) \circledast \delta_P^{\frac{1}{2}} \otimes_M W \\ r_P^G(V) \;&=\; {}_{U\backslash}\mathcal{D}(G) \otimes_G V \cong \delta_P^{-1} \circledast \mathcal{D}(U\backslash G) \otimes_G V \\ \boldsymbol{r}_P^G(V) \;&=\; \delta_P^{\frac{1}{2}} \circledast {}_{U\backslash}\mathcal{D}(G) \otimes_G V \cong \delta_P^{-\frac{1}{2}} \circledast \mathcal{D}(U\backslash G) \otimes_G V \end{aligned}$$

with $V \in G\text{-Rep}_k, W \in H\text{-Rep}_k$.

Chapter 5

Frobenius Reciprocity Revisited

In this chapter, we are going to prove that the functor

$$M\text{-Rep}_k \longrightarrow G\text{-Rep}_k \qquad W \longmapsto \mathcal{D}(G)_{/U} \circledast \delta_P^{-1} \otimes_M W = \mathcal{D}(G/U) \otimes_M W$$

is right adjoint to the functor

$$G\text{-Rep}_k \longrightarrow M\text{-Rep}_k \qquad V \longmapsto {}_{U\backslash}\mathcal{D}(G) \otimes_G V = \delta_P^{-1} \circledast \mathcal{D}(U\backslash G) \otimes_G V.$$

We already know that the first functor equals parabolic induction[1] and the second one equals Jacquet restriction, therefore this chapter can be understood as another proof of Frobenius Reciprocity.
If the reader distrusts our proof in Section 4.4 that $\mathcal{D}(G)_{/U} \circledast \delta_P^{-1} \otimes_M W$ is isomorphic to $i_P^G(W)$, uniqueness of the adjoint functor gives another proof of this fact (at least if he is willing to accept Frobenius Reciprocity).

5.1 Establishing Unit and Counit Transformations

Observation 5.1.1. *Let V be a smooth G-representation, then*

$$i_P^G \circ r_P^G(V) = \mathcal{D}(G)_{/U} \circledast \delta_P^{-1} \otimes_M {}_{U\backslash}\mathcal{D}(G) \otimes_G V = \mathcal{D}^{\delta_P^{-1}}(G/U \times_M U\backslash G) \otimes_G V.$$

Proof. This is just Remark 4.5.13. □

Observation 5.1.2. *Let W be a smooth M-representation, then we have*

$$r_P^G \circ i_P^G(W) = {}_{U\backslash}\mathcal{D}(G) \otimes_G \mathcal{D}(G/U) \otimes_M W = {}_{U\backslash}\mathcal{D}(G/U) \otimes_M W.$$

We proceed like this: Define maps

$$\eta_0 \colon \mathcal{D}(G) \longrightarrow \mathcal{D}^{\delta_P^{-1}}(G/U \times_M U\backslash G) \qquad \varphi \longmapsto \left((x,y) \mapsto \int_U \varphi(xuy)\, \mathrm{d}\mu_U(u)\right)$$

and

$$\varepsilon_0 \colon {}_{U\backslash}\mathcal{D}(G/U) \longrightarrow \mathcal{D}(M) \qquad [\varphi] \longmapsto \varphi|M.$$

Proposition 5.1.3. *The maps η_0 and ε_0 are well-defined morphisms between G-G-bimodules (resp. between M-M-bimodules).*

[1] We use unnormalized Jacquet functors because this makes the calculations easier, but, contrary to the Geometric Lemma or Second Adjointness, this is arbitrary. In fact, unnormalized Frobenius Reciprocity is equivalent to normalized Frobenius Reciprocity.

Proof. We start with η_0.
First of all, the integral does exist because the map

$$U \hookrightarrow G \to G \qquad u \mapsto u \mapsto xuy$$

is proper and continuous for any $x, y \in G$, compare with our "Integration is possible"-paradigm. Let $\varphi \in \mathcal{D}(G)$. Then there is an open, compact subgroup $K \subset G$ such that $\varphi(KxK) = \varphi(x)$ for any $x \in G$. Consequently, $\eta_0(\varphi)(Kx, yK) = \eta_0(\varphi)(x, y)$ holds for any two $x, y \in G$, therefore $\eta_0(\varphi)$ is smooth.
Now we have to show that $\eta_0(\varphi)$ is compactly supported modulo Δ. For this, we fix a maximal open, compact subgroup K_0 such that the Iwasawa decomposition does hold: $G = UMK_0$. Moreover, fix a compact subset $\Xi \subset G$ that contains $\mathrm{supp}(\varphi)$. Then $\Xi_0 = \Xi \cdot K_0$ is compact and open in G. We claim that

$$\pi\big(\mathrm{supp}(\eta_0(\varphi))\big) \subset \pi(\Xi_0, K_0)$$

where π denotes the projection

$$\pi\colon G \times G \longrightarrow G/U \times_M U\backslash G.$$

Take $[x, k] \in G/U \times_M U\backslash G$. We can assume $k \in K_0$ since

$$G/U \times_M U\backslash G = G/U \times_M U\backslash UMK_0.$$

Now suppose $[x, k] \notin \pi(\Xi_0, K_0)$. This means that there is no $m \in M$ such that $xm \in \Xi_0 U$ and $m^{-1}k \in UK_0$. Assume $xuk \in \Xi_0$ for some $u \in U$, this yields $xu \in \Xi_0$, hence $x \in \Xi_0 U$, and $k \in K_0 \subset UK_0$ is clear. This is a contradiction. We conclude

$$[x, k] \notin \pi(\Xi_0, K_0) \;\Rightarrow\; \varphi(xuk) = 0 \;\forall u \in U \;\Rightarrow\; \eta_0(\varphi)(x, k) = \int_U \varphi(xuk)\,\mathrm{d}\mu_U(u) = 0.$$

We proved that $\eta_0(\varphi)$ is smooth and compactly supported modulo Δ. It is straightforward to see that $\eta_0(\varphi)$ lives in $\mathcal{D}^{\delta_P^{-1}}(G/U \times_M U\backslash G)$ and that the assignment η_0 is G-G-equivariant.

Now for ε_0: Consider the continuous map

$$z : M \hookrightarrow G \twoheadrightarrow G/U.$$

z is a closed mapping, what one easily deduces from $M \cong P/U$. Moreover,

$$z^{-1}(xU) = \begin{cases} \{g\} & \text{if } g \in xU \cap M \neq \emptyset, \\ \emptyset & \text{if } xU \cap M = \emptyset. \end{cases}$$

Therefore, the pre-image of a point is compact and we may conclude that z is proper. This gives a mapping

$$\mathcal{D}(G/U) \longrightarrow \mathcal{D}(M)$$

which clearly kills $\varphi - \varphi(u\llcorner\!\lrcorner)$. Therefore, it factors through ${}_{U\backslash}\mathcal{D}(G/U)$ and we recover ε_0. It is obvious that this map commutes with the M-M-action. $\square$

These maps give rise to the following natural transformations

$$\begin{array}{rcl} \eta\colon \mathrm{id}_{G\text{-}\mathrm{Rep}_k} & \stackrel{\bullet}{\longrightarrow} & i_P^G \circ r_P^G \\ \mathrm{id}(V) = V \cong \mathcal{D}(G) \otimes_G V & \stackrel{\eta_V}{\longrightarrow} & \mathcal{D}^{\delta_P^{-1}}(G/U \times_M U\backslash G) \otimes_G V \cong i_P^G \circ r_P^G(V) \\ \varphi \otimes v & \longmapsto & \eta_0(\varphi) \otimes v \end{array}$$

and

$$\begin{array}{rcl} \varepsilon\colon r_P^G \circ i_P^G & \stackrel{\bullet}{\longrightarrow} & \mathrm{id}_{M\text{-}\mathrm{Rep}_k} \\ r_P^G \circ i_P^G(W) \cong {}_{U\backslash}\mathcal{D}(G/U) \otimes_M W & \stackrel{\varepsilon_W}{\longrightarrow} & \mathcal{D}(M) \otimes_M W = \mathrm{id}(W) \\ [\varphi] \otimes w & \longmapsto & \varepsilon_0([\varphi]) \otimes w \end{array}$$

The map ε_0 induces a map

$$\mathcal{D}^{\delta_P^{-1}}(G/U \times_M U\backslash G)_{/U} \circledast \delta_P^{-1} \dashrightarrow \mathcal{D}(G/U)$$

which we want to analyze:

$$\begin{tikzcd}
\mathcal{D}^{\delta_P^{-1}}(G/U \times_M U\backslash G)_{/U} \circledast \delta_P^{-1} \arrow[r, dotted] \arrow[d, equal] & \mathcal{D}(G/U) \arrow[d, equal] \\
\mathcal{D}(G)_{/U} \circledast \delta_P^{-1} \otimes_M {}_{U\backslash}\mathcal{D}(G)_{/U} \circledast \delta_P^{-1} \arrow[d, equal] & \mathcal{D}(G)_{/U} \circledast \delta_P^{-1} \arrow[d, equal] \\
\mathcal{D}(G)_{/U} \circledast \delta_P^{-1} \otimes_M {}_{U\backslash}\mathcal{D}(G/U) \arrow[r, "\mathrm{id}\otimes\varepsilon_0"'] & \mathcal{D}(G)_{/U} \circledast \delta_P^{-1} \otimes_M \mathcal{D}(M)
\end{tikzcd}$$

In order to understand this dotted map, it suffices to chase around a pure tensor $[\varphi] \otimes [\psi]$ in $\mathcal{D}(G)_{/U} \circledast \delta_P^{-1} \otimes_M {}_{U\backslash}\mathcal{D}(G)_{/U} \circledast \delta_P^{-1}$:

$$\begin{tikzcd}
\Big((x,y) \mapsto \int_U \int_U \int_M \varphi(xum)\psi(u'm^{-1}y)\Big)_{/U} \arrow[r, dotted, mapsto] & \Big(gU \mapsto \int_U \int_M \varphi(gum) \int_U \psi(u'm^{-1})\Big) \\
{[\varphi] \otimes [\psi]} \arrow[u, mapsto] \arrow[d, mapsto] & \Big(g \mapsto \int_M \varphi(gm^{-1})\delta_P(m^{-1}) \int_U \psi(mu')\Big)_{/U} \arrow[u, mapsto] \\
{[\varphi] \otimes {}_{U\backslash}\Big(gU \mapsto \int_U \psi(gu')\Big)} \arrow[r, mapsto] & {[\varphi] \otimes \Big(m \mapsto \int_U \psi(mu')\Big)} \arrow[u, mapsto]
\end{tikzcd}$$

Therefore, the dotted map acts as

$$[\zeta] \longmapsto \Big(gU \mapsto \zeta(g,1)\Big) .$$

Or, equivalently, the dotted map sends $[\zeta]$ to the element $[\varphi] \in \mathcal{D}(G)_{/U} \circledast \delta_P^{-1}$ for which the identity

$$\int_U \varphi(gu)\, \mathrm{d}\mu_U(u) = \zeta(g,1)$$

holds for all $g \in G$.

With exactly the same reasoning we can work out the induced map

$${}_{U\backslash}\mathcal{D}^{\delta_P^{-1}}(G/U \times_M U\backslash G) \dashrightarrow \delta_P^{-1} \circledast \mathcal{D}(U\backslash G)$$

as the map that acts as follows:

$$[\zeta] \longmapsto \Big(Ug \mapsto \zeta(1,g)\Big)$$

Or, equivalently, it sends $[\zeta]$ to the element $[\varphi] \in {}_{U\backslash}\mathcal{D}(G)$ for which the identity

$$\int_U \varphi(ug)\, \mathrm{d}\mu_U(u) = \zeta(1,g)$$

holds for any $g \in G$. We are ready to prove

Theorem 5.1.4 (**Frobenius Reciprocity**). *η and ε are unit and counit transformations between the functors i_P^G and r_P^G, establishing a (right) adjointness relation between them.*

Proof. We have to prove the zig-zag equations

$$\mathrm{id}_{i_P^G} = i_P^G\varepsilon \circ \eta i_P^G \quad \text{and} \quad \mathrm{id}_{r_P^G} = \varepsilon r_P^G \circ r_P^G\eta.$$

We can prove the first identity if we can show that the composed G-M-module map

$$\begin{aligned}\mathcal{D}(G)_{/U} \circledast \delta_P^{-1} \;\cong\; \mathcal{D}(G) \otimes_G \mathcal{D}(G)_{/U} \circledast \delta_P^{-1} &\overset{\eta_0 \otimes \mathrm{id}}{\longrightarrow} \mathcal{D}(G)_{/U} \circledast \delta_P^{-1} \otimes_M {}_{U\backslash}\mathcal{D}(G) \otimes_G \mathcal{D}(G)_{/U} \circledast \delta_P^{-1} \\ \cong\; \mathcal{D}(G)_{/U} \circledast \delta_P^{-\frac{1}{2}} \otimes_M {}_{U\backslash}\mathcal{D}(G/U) &\overset{\mathrm{id} \otimes \varepsilon_0}{\longrightarrow} \mathcal{D}(G)_{/U} \circledast \delta_P^{-1} \otimes_M \mathcal{D}(M) \;\cong\; \mathcal{D}(G)_{/U} \circledast \delta_P^{-1}\end{aligned}$$

equals the identity. Using our dotted map, we can write this down as

$$\mathcal{D}(G)_{/U} \circledast \delta_P^{-1} \xrightarrow{\varepsilon_{0/U}} \mathcal{D}^{\delta_P^{-1}}(G/U \times_M U\backslash G)_{/U} \circledast \delta_P^{-1} \dashrightarrow \mathcal{D}(G)_{/U} \circledast \delta_P^{-1}.$$

And, in fact, $[\varphi]$ is mapped to

$$\left((x,y) \longmapsto \int_U \varphi(xuy)\, \mathrm{d}\mu_U(u)\right)_{/U}$$

and then to $[\psi]$ where $\int_U \psi(gu)\, \mathrm{d}\mu_U(u) = \int_U \varphi(gu1)\, \mathrm{d}\mu_U(u)$. Therefore $[\psi] = [\varphi]$ and we are done.

The second equation is done in exactly the same manner. We have to show that the composed M-G-module map

$$\begin{aligned}{}_{U\backslash}\mathcal{D}(G) \;\cong\; {}_{U\backslash}\mathcal{D}(G) \otimes_G \mathcal{D}(G) &\overset{\mathrm{id} \otimes \eta_0}{\longrightarrow} {}_{U\backslash}\mathcal{D}(G) \otimes_G \mathcal{D}(G)_{/U} \circledast \delta_P^{-1} \otimes_M {}_{U\backslash}\mathcal{D}(G) \\ \cong\; {}_{U\backslash}\mathcal{D}(G/U) \otimes_M {}_{U\backslash}\mathcal{D}(G) &\overset{\varepsilon_0 \otimes \mathrm{id}}{\longrightarrow} \mathcal{D}(M) \otimes_M {}_{U\backslash}\mathcal{D}(G) \cong {}_{U\backslash}\mathcal{D}(G)\end{aligned}$$

equals the identity. Now we can use the dashed map and write this as

$${}_{U\backslash}\mathcal{D}(G) \xrightarrow{{}_{U\backslash}\varepsilon_0} {}_{U\backslash}\mathcal{D}^{\delta_P^{-1}}(G/U \times_M U\backslash G) \dashrightarrow {}_{U\backslash}\mathcal{D}(G).$$

Again, $[\varphi]$ is mapped to

$${}_{U\backslash}\left((x,y) \longmapsto \int_U \varphi(xuy)\, \mathrm{d}\mu_U(u)\right)$$

and then to $[\psi]$ where $\int_U \psi(ug)\, \mathrm{d}\mu_U(u) = \int_U \varphi(1ug)\, \mathrm{d}\mu_U(u)$. Thus $[\psi] = [\varphi]$. $\square$

Chapter 6

The Geometric Lemma

Fix two standard parabolic subgroups $P = MU$ and $Q = NV$ within a reductive p-adic group G. If we have a smooth N-representation V, we can parabolically induce it up to G and then take the Jacquet restriction in order to obtain an M-representation. This gives a functor

$$\Gamma = \boldsymbol{r}_P^G \circ \boldsymbol{i}_Q^G \colon N\text{-Rep}_k \longrightarrow M\text{-Rep}_k,$$

and the Geometric Lemma provides us with a fairly good understanding of Γ. Recall the following definition:

Definition 6.0.5. Let $F, Q_i \colon \mathscr{A} \to \mathscr{B}$ be functors between abelian categories (where $1 \leq i \leq n$ for some $n \in \mathbb{N}$.) We say that F **has a finite filtration** by $\{Q_i\}$ (or, according to [BZ77], that F is glued from $\{Q_i\}$), if there are subfunctors $F_j \colon \mathscr{A} \to \mathscr{B}$ of F (where $0 \leq j \leq n$) such that there is a filtration

$$F_1(a) \hookrightarrow F_2(a) \hookrightarrow \ldots \hookrightarrow F_n(a) = F(a)$$

for each object a in $\mathscr{A}$ with

$$\forall j \geq 1 \ \exists i \text{ such that } F_j(a)/F_{j-1}(a) \cong G_i(a).$$

The Geometric Lemma tells us that Γ is glued from certain (nice) functors Γ_w where w runs through a set ${}_P\mathcal{W}_Q$ of representatives for $P\backslash G/Q$, as we will explain now.

6.1 Preparation

Our aim here is to understand certain sub- and quotient spaces of G. We begin with

Proposition 6.1.1. *Let $P = MU \subset G$ be a parabolic subgroup, denote its opposite by $\overline{P} = M\overline{U}$. As usual, understand P, $\overline{P}$ and $P \cdot \overline{P}$ equipped with the subspace topology. Then, there is a homeomorphism*

$$P \times_M \overline{P} \;\cong\; P \cdot \overline{P}$$

between ℓ-spaces, equivariant with respect to the left P- and right $\overline{P}$-multiplication.

Proof. $P \cdot \overline{P}$ is an ℓ-space since it is open in G (Proposition 4.10 (e) in [BT65]). P and $\overline{P}$ are closed in G, hence ℓ-spaces. M acts freely and properly on $P \times \overline{P}$, therefore the twisted product is an ℓ-space.
There is an obvious identification

$$P \cdot \overline{P} \ni p \cdot \overline{p} \xleftrightarrow{1:1} [p, \overline{p}] \in P \times_M \overline{P}.$$

Call this map ρ. We have to show that ρ is a homeomorphism.
By definition, the projection $P \times \overline{P} \to P \times_M \overline{P}$ is a topological quotient map. We want to show that this is also true for the multiplication

$$\mu' : P \times \overline{P} \longrightarrow P \cdot \overline{P}.$$

It is easy to see that μ' is continuous. We are done as soon as we can show that it is an open map. For this, fix a neighborhood basis $\{K_n\}_{n\in\mathbb{N}}$ for 1 in G such that each K_n admits an Iwahori decomposition with respect to P:

$$K_n = K_n^+ K_n^0 K_n^- \qquad K_n^+ = K_n \cap U,\ K_n^0 = K_n \cap M,\ K_n^- = K_n \cap \overline{U}.$$

(In $GL_n(F)$ we would take the congruence subgroups.) It suffices to show that the $\mu'(K_n \cap P, K_n \cap \overline{P})$ are open in $P \cdot \overline{P}$. But

$$\mu'(K_n \cap P, K_n \cap \overline{P}) = (K_n \cap P) \cdot (K_n \cap \overline{P}) = K_n^+ K_n^0 K_n^- = K_n$$

and K_n is open in G, hence open in $P \cdot \overline{P}$. This shows that μ' is open at the point $(1,1)$. Indeed, this is sufficient: For any other point $(p, \overline{p})$, $\big\{\big(p \cdot (K_n \cap P), (K_n \cap \overline{P}) \cdot \overline{p}\big)\big\}_{n\in\mathbb{N}}$ is a neighborhood basis and

$$\mu'\big(p \cdot (K_n \cap P), (K_n \cap \overline{P}) \cdot \overline{p}\big) = pK_n\overline{p}$$

is open. Thus μ' is an open mapping.
Therefore, we have a commutative diagram

$$\begin{tikzcd}
& P \times \overline{P} \arrow[dl, "\text{top. quotient map}"'] \arrow[dr, "\text{top. quotient map}"] & \\
P \times_M \overline{P} \arrow[rr, dotted, leftrightarrow, "\rho"'] & & P \cdot \overline{P}
\end{tikzcd}$$

proving that ρ is a homeomorphism. $\square$

We are interested in the following variation of this result: Let $P = MU, Q = NV$ be standard parabolic subgroups of G, w an element of ${}_P\mathcal{W}_Q$, then there is a homeomorphism

$$PwQ \cong P \underset{\substack{P \cap wQw^{-1} \\ = w^{-1}Pw \cap Q}}{\times} Q.$$

The homeomorphism between $P \cap wQw^{-1}$ and $w^{-1}Pw \cap Q$ is conjugation with w^{-1}.
We could prove this similar to the case $Q = \overline{P}, w = 1$ we investigated in the preceding proposition. In fact, [Cas95] (p. 11-12) states that multiplication gives an isomorphism of varieties (hence a homeomorphism)

$$P \times \{w\} \times \prod_{\substack{\alpha \in \Sigma^+ - \Sigma^+_\Omega \\ w^{-1}\alpha \notin \Sigma^+ - \Sigma^+_\Theta}} N_\alpha \xrightarrow{\sim} PwQ$$

where the N_α are certain subgroups of G. Moreover, there is a topological quotient map given by the projections[1]

$$Q \longrightarrow Q/N \cong V \longrightarrow (w^{-1}Uw \cap V)\backslash V \ \cong \prod_{\substack{\alpha \in \Sigma^+ - \Sigma^+_\Omega \\ w^{-1}\alpha \notin \Sigma^+ - \Sigma^+_\Theta}} N_\alpha.$$

We could now put this together into a similar proof. But we can get along with a rather different argument.

Lemma 6.1.2. *Let P, Q be standard parabolic subgroups of G, $w \in {}_P\mathcal{W}_Q$, then there is a P-Q-equivariant homeomorphism*

$$PwQ \cong P \underset{\substack{P \cap wQw^{-1} \\ = w^{-1}Pw \cap Q}}{\times} Q.$$

[1] Observe the misprint in [Cas95]: w is mixed up with w^{-1}.

Proof. Recall Corollary 1.6 from [BZ76]: If an ℓ-group (countable at infinity) acts transitively on an ℓ-space X, $x_0 \in X$ and H denotes the stability subgroup of x_0, then the natural map of $H\backslash G$ into X given by $Hg \mapsto g^{-1}x_0$ is a homeomorphism.
Bruhat decomposition gives a way of enumerating ${}_P\mathcal{W}_Q$ such that

$$\bigcup_{1\leq i\leq n-1} Pw_iQ \overset{\text{open}}{\subset} \bigcup_{1\leq i\leq n} Pw_iQ \qquad \text{for all } 1 < n \leq |{}_P\mathcal{W}_Q|. \tag{6.1}$$

The set $\bigcup_{1\leq i\leq n} Pw_iQ$ is open in G, hence an ℓ-space. Therefore, (6.1) shows that Pw_nQ is a closed subset of an ℓ-space, hence an ℓ-space on its own. The special case $n = 1$ is easy since this is the open coset, hence an ℓ-space.
We subsume: The conditions of Corollary 1.6 are met for the ℓ-action

$$P \times Q \curvearrowright PwQ \qquad (\tilde{p}, \tilde{q}) \bullet pwq = \tilde{p}pwq\tilde{q}^{-1}.$$

The stabilizer of the element $w \in PwQ$ is clearly $H = \{(x^{-1}, w^{-1}xw) \mid x \in P \cap wQw^{-1}\}$. This yields the second homeomorphism in

$$P \underset{\substack{P\cap wQw^{-1} \\ = w^{-1}Pw\cap Q}}{\times} Q \;\cong\; H\backslash(P \times Q) \;\cong\; PwQ$$

where the composition is indeed multiplication:

$$[p, q] \;\mapsto\; H \cdot (p^{-1}, q) \;\mapsto\; (p^{-1}, q)^{-1} \bullet w = pwq.$$

Therefore, equivariance is obvious. □

6.2 The Statement

Let $P = MU, Q = NV$ be as above. Then, according to Chapter 2.4, we have a finite set ${}_P\mathcal{W}_Q \subset G$ such that

$$G = \bigsqcup_{w\in {}_P\mathcal{W}_Q} PwQ$$

and ${}_P\mathcal{W}_Q$ can be enumerated such that

$$\bigcup_{1\leq i\leq n-1} Pw_iQ \overset{\text{open}}{\subset} \bigcup_{1\leq i\leq n} Pw_iQ \qquad \text{for all } 1 < n \leq |{}_P\mathcal{W}_Q|.$$

We cite

Proposition 6.2.1. *Let X be an ℓ-space, $Y \subset X$ open. Then we can extend any $\varphi \in \mathcal{D}(G)$ trivially on $X - Y$, what gives rise to the exact sequence*

$$0 \longrightarrow \mathcal{D}(Y) \longrightarrow \mathcal{D}(X) \longrightarrow \mathcal{D}(X - Y) \longrightarrow 0.$$

Proof. This is Proposition 2 in [Ber92]. That paper is concerned with the case $k = \mathbb{C}$, but the given proof holds for any field. □

Therefore, the filtration

$$Pw_1Q \overset{\text{open}}{\subset} \Big(Pw_1Q \cup Pw_2Q\Big) \overset{\text{open}}{\subset} \ldots \overset{\text{open}}{\subset} \Big(\bigcup_{1\leq i\leq |{}_P\mathcal{W}_Q|} Pw_iQ\Big) = G$$

gives the filtration

$$\mathcal{D}(Pw_1Q) \;\hookrightarrow\; \mathcal{D}(Pw_1Q \cup Pw_2Q) \;\hookrightarrow \ldots \hookrightarrow\; \mathcal{D}(G)$$

with P-Q-bimodule injections. The quotients of this filtration are

$$\mathcal{D}\Big(\bigcup_{1\le i\le(n-1)} Pw_iQ \;-\; \bigcup_{1\le i\le n} Pw_iQ\Big) = \mathcal{D}(Pw_nQ)$$

considered as P-Q-bimodules.

Observation 6.2.2. *Abbreviate $X = \bigcup_{i=1}^{n-1} w_i$ for some $n \le |\,_P\mathcal{W}_Q|$. Let S be a Q-representation, then the previous remarks give rise to a sequence*

$$0 \longrightarrow \mathcal{D}(PXQ)\otimes_Q S \longrightarrow \mathcal{D}(PXQ \sqcup Pw_nQ)\otimes_Q S \longrightarrow \mathcal{D}(Pw_nQ)\otimes_Q S \longrightarrow 0.$$

We claim that it is exact. As tensoring is a right exact process, we just have to prove that the map $\mathcal{D}(PXQ)\otimes_Q S \longrightarrow \mathcal{D}(PXQ\sqcup Pw_nQ)\otimes_Q S$ is injective. But this is easy to deduce from the fact that $PXQ\cdot Q \subset PXQ$ and the definition of the map.

Now take $S = W$ for some smooth N-representation W which we understand as a Q-representation by trivial inflation. If we take into account that taking U-Jacquet modules is exact (see Proposition 10 of [Ber92]), we end up with an exact sequence

$$0 \to {}_{U\backslash}\mathcal{D}\Big(\bigcup_{1\le i\le(n-1)} Pw_iQ\Big)_{/V}\otimes_N W \to {}_{U\backslash}\mathcal{D}\Big(\bigcup_{1\le i\le n} Pw_iQ\Big)_{/V}\otimes_N W \to {}_{U\backslash}\mathcal{D}\Big(Pw_nQ\Big)_{/V}\otimes_N W \to 0$$

for any $1 \le n \le |\,_P\mathcal{W}_Q|$. We subsume that the M-representation ${}_{U\backslash}\mathcal{D}(G)_{/V}\otimes_M W$ has a finite filtration by subrepresentations with quotients

$${}_{U\backslash}\mathcal{D}(PwQ)_{/V}\otimes_M W \qquad w \in {}_P\mathcal{W}_Q.$$

Now, recall (from Proposition 1.3.3 (c) of [Cas95]) that $M\cap wQw^{-1}$ is a parabolic subgroup of M with Levi decomposition

$$M\cap wQw^{-1} = \big(M\cap wNw^{-1}\big)\cdot\big(M\cap wVw^{-1}\big).$$

Similarly, we have a parabolic subgroup of N:

$$w^{-1}Pw\cap N = \big(w^{-1}Mw\cap N\big)\cdot\big(w^{-1}Uw\cap N\big)\subset N.$$

The Levi components $M\cap wNw^{-1}$ and $w^{-1}Mw\cap N$ are evidently isomorphic via w-conjugation. This induces an equivalence of categories

$$\hat{w} : \big(w^{-1}Mw\cap N\big)\text{-Rep}_k \longrightarrow \big(M\cap wNw^{-1}\big)\text{-Rep}_k.$$

Now we are in a position to prove

Lemma 6.2.3 (Geometric Lemma). *Let $P = MU$ and $Q = NV$ be standard parabolic subgroups of a reductive p-adic group G. Then the functor*

$$\Gamma = \boldsymbol{r}_P^G\circ\boldsymbol{i}_Q^G : N\text{-Rep}_k \longrightarrow M\text{-Rep}_k$$

has a finite filtration by subfunctors with quotients

$$\Gamma_w = \boldsymbol{i}^M_{M\cap wQw^{-1}}\circ\hat{w}\circ\boldsymbol{r}^N_{w^{-1}Pw\cap N} \qquad w\in {}_P\mathcal{W}_Q.$$

Proof. After what has been said, it remains to investigate the M-N-bimodule

$$\delta_P^{\frac{1}{2}} \circledast {}_{U\backslash}\mathcal{D}(PwQ)_{/V} \circledast \delta_Q^{-\frac{1}{2}}$$

$$= \delta_P^{\frac{1}{2}} \circledast {}_{U\backslash}\mathcal{D}(P) \circledast \delta_{P\cap wQw^{-1}}^{-1} \bigotimes_{\substack{P\cap wQw^{-1} \\ =w^{-1}Pw\cap Q}} \mathcal{D}(Q)_{/V} \circledast \delta_Q^{-\frac{1}{2}}$$

$$= \delta_P^{-\frac{1}{2}} \circledast \mathcal{D}(M) \circledast \delta_{P\cap wQw^{-1}}^{-1} \bigotimes_{\substack{P\cap wQw^{-1} \\ =w^{-1}Pw\cap Q}} \mathcal{D}(N) \circledast \delta_Q^{\frac{1}{2}}$$

$$= \delta_P^{-\frac{1}{2}} \circledast \mathcal{D}(M)_{/M\cap wVw^{-1}} \circledast \delta_{P\cap wQw^{-1}}^{-1} \bigotimes_{\substack{M\cap wNw^{-1} \\ =w^{-1}Mw\cap N}} {}_{w^{-1}Uw\cap N\backslash}\mathcal{D}(N) \circledast \delta_Q^{\frac{1}{2}}$$

$$= \mathcal{D}(M)_{/M\cap wVw^{-1}} \circledast \left(\delta_P^{\frac{1}{2}} \cdot \delta_{wQw^{-1}}^{\frac{1}{2}} \cdot \delta_{P\cap wQw^{-1}}^{-1}\right) \bigotimes_{\substack{M\cap wNw^{-1} \\ =w^{-1}Mw\cap N}} {}_{w^{-1}Uw\cap N\backslash}\mathcal{D}(N)$$

$$= \mathcal{D}(M)_{/M\cap wVw^{-1}} \circledast \delta_{M\cap wQw^{-1}}^{-\frac{1}{2}} \bigotimes_{\substack{M\cap wNw^{-1} \\ =w^{-1}Mw\cap N}} \delta_{w^{-1}Pw\cap N}^{\frac{1}{2}} \circledast {}_{w^{-1}Uw\cap N\backslash}\mathcal{D}(N)$$

The last equation sign does hold because of Theorem 2.5.4. This expression is just Γ_w. $\square$

6.2.1 An Example: $\mathrm{GL}_2(F)$

Assume $G = \mathrm{GL}_2(F)$ for a non-archimedean local field F. As parabolic subgroup we take the standard Borel subgroup

$$P = B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} = TU = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}.$$

Recall from Example 2.4.10 that the Weyl group $\mathcal{W}$ equals $S_2 \cong \mathbb{Z}/2\mathbb{Z}$ in this case. We can write down a set of representatives in G as

$$\mathcal{W} = \left\{ 1 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, w := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}.$$

Now we can describe $\boldsymbol{r}_B^G \circ \boldsymbol{i}_B^G(V)$ if V is a smooth M-representation:
Because of the identities $wBw^{-1} = wBw = wBw^{-1} = \overline{B}$ and $wTw^{-1} = wTw = w^{-1}Tw = T$, the occurring quotients in the filtration are $\Gamma_1(V) = \boldsymbol{i}_T^T \circ \boldsymbol{r}_T^T(V) = V$ and $\Gamma_w(V) = \boldsymbol{i}_T^T \circ \hat{w} \circ \boldsymbol{r}_T^T(V) = wV$ where we abbreviate wV for the representation (π, V) with $\pi(t)v = wtwv$.
In order to get the directions right, we must decide which set is open in G. Iwasawa's decomposition tells us that G/B is compact. Hence, if B would be open, we could conclude that G/B is finite. This is nonsense: For example, the rule

$$x \longmapsto \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \cdot B$$

gives an injective mapping $F \longrightarrow G/B$. Hence $BwB \subset G$ must be open and we can describe $\boldsymbol{r}_B^G \circ \boldsymbol{i}_B^G(V)$ as an extension of T-representations

$$0 \longrightarrow wV \longrightarrow \boldsymbol{r}_B^G \circ \boldsymbol{i}_B^G(V) \longrightarrow V \longrightarrow 0. \tag{6.2}$$

Now, consider the question of describing the space of **intertwining operators**, that is

$$\mathrm{Hom}_G(\boldsymbol{i}_B^G(V), \boldsymbol{i}_B^G(V'))$$

where V, V' are smooth irreducible T-representations. In this case, we can use Frobenius reciprocity and rewrite this space as

$$\mathrm{Hom}_M(\boldsymbol{r}_B^G(\boldsymbol{i}_B^G(V)), V').$$

Now, we can use the Geometric Lemma in order to study this space. If, for example, V equals V', we immediately get a non-zero element belonging to the projection in (6.2). The general picture is this

Theorem 6.2.4. *The space*

$$\mathrm{Hom}_G(\boldsymbol{i}_B^G(V), \boldsymbol{i}_B^G(V'))$$

is isomorphic to k if V' equals V or wV and vanishes otherwise.

Proof. This is very nicely explained in Section 9 of [BH06]. $\square$

Chapter 7

Second Adjointness

We saw that it is quite easy to prove Frobenius Reciprocity, that is, the fact that parabolic induction $\boldsymbol{i}_P^G$ is right adjoint to Jacquet restriction $\boldsymbol{r}_P^G$. As suggested by the representation theory of finite groups, there should be an adjointness relation in the other direction, too. In our setting, the situation is more subtle: In fact, $\boldsymbol{i}_P^G$ has a left adjoint: $\boldsymbol{r}_{\overline{P}}^G$ – Jacquet restriction with respect to the opposite parabolic $\overline{P} = M\overline{U}$. In contrast to Frobenius Reciprocity, this Second Adjointness property is a highly non-trivial result, obtained by Bernstein in 1986.[1]

7.1 Prerequisites

Suppose, we have two reductive p-adic groups G and H. Fix a field k in which p is a non-zero square. Let V be a G-representation and let M be a G-H-bimodule.

Definition 7.1.1. The space of (linear) G-intertwiners $\mathrm{Hom}_G(M, V)$ is a left H-module (hence an H-representation) via

$$(h\varphi)(m) := \varphi(mh).$$

Call its smooth part

$$\mathrm{Hom}_G^\infty(M, V)$$

and, for a subgroup $K \subset H$, the space of K-invariants

$$\mathrm{Hom}_G^K(M, V).$$

We have a first proposition:

Proposition 7.1.2. *Take a $K \in \mathscr{K}(G)$. If V is a smooth G-representation, we have an isomorphism of vector spaces*

$$\mathrm{Hom}_G^K(\mathcal{D}(G), V) \cong \mathrm{Hom}_G(\mathcal{D}(G/K), V) \cong V^K.$$

Proof. Step 1: Recall that we denote the normalizer of K in G by $\mathrm{N}_G(K)$. We want to establish the following isomorphism of G-$\mathrm{N}_G(K)$-bimodules

$$\mathcal{D}(G/K) \xrightarrow{\sim} \mathcal{D}(G)^K \qquad \varphi \longmapsto \Big(g \mapsto \varphi(gK)\Big)$$

where $\mathcal{D}(G)^K$ denotes the K-invariant space with respect to the *right* G-action.
This map is well defined, what follows from our general paradigm: $\pi\colon G \to G/K$ is a continuous

[1] The author is a bit confused about the date of this theorem. The first occurrence seems to be in the unpublished (and incomplete) draft [Ber87]. J.-F. Dat dates this paper to 1993. I chose 1986 because there is a nameless, undated and quite sketchy draft [Ber86] on the University of Chicago Web server, named Bernstein86.pdf, which seems to be the precursor of [Ber87] and contains the Second Adjointness Theorem.

and proper[2] G-$\mathrm{N}_G(K)$-equivariant map.
π is surjective, hence we get an injection

$$\pi^* : \mathcal{D}(G/K) \hookrightarrow \mathcal{D}(G).$$

Its image clearly lies in $\mathcal{D}(G)^K$.
We have to prove surjectivity: Take a $\psi \in \mathcal{D}(G)^K$, then we have a preimage

$$\overline{\psi} : G/K \to k \qquad gK \mapsto \psi(g).$$

Smoothness is trivial: As K is open, G/K is discrete. Compactness of support follows immediately from the continuousness of π. Therefore, $\overline{\psi} \in \mathcal{D}(G/K)$ and π^* provides us with the desired isomorphism.

Step 2: Our next observation is the following: Let M be a smooth G-G-bimodule, V a smooth G-representation and $K \in \mathscr{K}(G)$. Then

$$\mathrm{Hom}_G^K(M, V) \cong \mathrm{Hom}_G(M^K, V)$$

where M^K denotes the K-invariants regarding the *right* G-action. We have two maps

$$\begin{aligned} \imath : M^K &\longrightarrow M && \text{the injection,} \\ \pi : M &\longrightarrow M^K && m \longmapsto \int_K mk \, \mathrm{d}\mu_K(k). \end{aligned}$$

where μ_K is a Haar measure on K, normalized such that $\mu_K(K) = 1$. This gives two maps

$$\begin{tikzcd} \mathrm{Hom}_G^K(M, V) \arrow[rr, bend left, "\imath^*"] && \mathrm{Hom}_G(M^K, V) \arrow[ll, bend left, "\pi^*"] \end{tikzcd}$$

which are inverse to each other, as we now show: For the first direction, we clearly have

$$\imath^* \pi^* = (\pi \circ \imath)^* = \mathrm{id}^* = \mathrm{id}_{\mathrm{Hom}_G(M^K, V)}.$$

Moreover, we have

$$\rho \coloneqq \imath \circ \pi : M \ni m \longmapsto \int_K mk \, \mathrm{d}\mu_K(k) \in M,$$

hence

$$\pi^* \imath^* = \rho^* : \mathrm{Hom}_G^\infty(M, V) \longrightarrow \mathrm{Hom}_G^\infty(M, V)$$

what clearly amounts to the identity when restricted to the K-invariants.

Step 3: We want to prove $\mathrm{Hom}_G(\mathcal{D}(G)^K, V) \cong V^K$. Any element $\Theta \in \mathrm{Hom}_G(\mathcal{D}(G)^K, V)$ is completely determined by the value $v_0 \in V^K$ it answers to $\mathbf{1}_K$: Understand the symbol G/K as a set of representatives in G, then

$$\mathcal{D}(G)^K \ni \varphi = \sum_{g \in G/K} \lambda_g \cdot \mathbf{1}_{gK} \overset{\Theta}{\longmapsto} \sum_{g \in G/K} \lambda_g \cdot g v_0 \in V$$

where $\lambda_g = \varphi(g)$. Moreover, this rule produces (and allows only) one element in $\mathrm{Hom}_G(\mathcal{D}(G)^K, V)$ for any given $v_0 \in V^K$ that shall become the value at $\mathbf{1}_K$.
Therefore, the (linear) assignment $\Theta \mapsto \Theta(\mathbf{1}_K)$ is a bijection and we are done. $\square$

[2]See Chap. 3, §4, Cor. 2 of [Bou60].

7.2 Invariant Spaces of Jacquet Modules

For this section, fix a parabolic subgroup $P = MU \subset G$ and a Haar measure μ_G on G. Let K be an element of $\mathscr{K}_P(G)$ with Iwahori decomposition

$$K = K^+K^0K^- \;\text{ with }\; K^+ = K \cap U,\; K^0 = K \cap M,\; K^- = K \cap \overline{U}.$$

Moreover, let $\lambda \in G$ be strictly dominant with respect to P and K:

$$\lambda K^+\lambda^{-1} \supsetneq K^+ \supsetneq \lambda^{-1}K^+\lambda, \quad \lambda K^0\lambda^{-1} = K^0, \quad \lambda K^-\lambda^{-1} \subsetneq K^- \subsetneq \lambda^{-1}K^-\lambda$$

and

$$U = \bigcup_{i\in\mathbb{N}} \lambda^i K^+ \lambda^{-i}.$$

Abbreviate

$$K^{(i)} \coloneqq \lambda^i K \lambda^{-i}, \quad K_U^{(i)} \coloneqq \lambda^i K^+ \lambda^{-i}, \quad K_M^{(i)} \coloneqq \lambda^i K^0 \lambda^{-i}, \quad K_{\overline{U}}^{(i)} \coloneqq \lambda^i K^- \lambda^{-i}.$$

Fix a smooth G-representation V. Then we have a projective (and inductive) system of vector spaces

$$\tilde{\mathscr{S}}_\lambda = \left(\cdots \xrightarrow{t_{i-1}^i} V^{K^{(i)}} \xrightarrow{t_i^{i+1}} V^{K^{(i+1)}} \xrightarrow{t_{i+1}^{i+2}} \cdots \right)$$

where i runs through $\mathbb{Z}$ and the maps are given by

$$v \overset{t_i^{i+1}}{\longmapsto} \mu_G(K)^{-1} \cdot \int_{K^{(i+1)}} kv \, \mathrm{d}\mu_G(k).$$

For $i + 1 < j$ set

$$t_i^j = t_{j-1}^j \circ t_{j-2}^{j-1} \circ \cdots \circ t_i^{i+1}.$$

Proposition 7.2.1. *$\tilde{\mathscr{S}}_\lambda$ is isomorphic to the following system*

$$\mathscr{S}_\lambda = \left(\cdots \xrightarrow{t} V^K \xrightarrow{t} V^K \xrightarrow{t} \cdots \right)$$

where t is given by the rule

$$v \longmapsto \mu_G(K)^{-1} \cdot \int_K k\lambda^{-1}v \, \mathrm{d}\mu_G(k).$$

Proof. We should explain what we mean by "isomorphic": There are isomorphisms $(h_i)_{i\in\mathbb{Z}}$, such that all squares in

$$\begin{tikzcd}
\cdots \arrow[r, "t_{i-1}^i"] & V^{K^{(i)}} \arrow[r, "t_i^{i+1}"] \arrow[d, "h_i"] & V^{K^{(i+1)}} \arrow[r, "t_{i+1}^{i+2}"] \arrow[d, "h_{i+1}"] & V^{K^{(i+2)}} \arrow[r, "t_{i+2}^{i+3}"] \arrow[d, "h_{i+2}"] & \cdots \\
\cdots \arrow[r, "t"'] & V^K \arrow[r, "t"'] & V^K \arrow[r, "t"'] & V^K \arrow[r, "t"'] & \cdots
\end{tikzcd}$$

commute. Take

$$h_i \colon V^{K^{(i)}} \ni v \;\longmapsto\; \lambda^{-i}v \in V^K.$$

Since

$$\int_{K^{(i+1)}} kv \, \mathrm{d}\mu_G(k) = \int_G \mathbf{1}_{K^{(i+1)}}(\gamma)\, \gamma v \, \mathrm{d}\mu_G(\gamma)$$
$$= \int_G \mathbf{1}_K(\lambda^{-(i+1)}\gamma\lambda^{i+1})\, \gamma v \, \mathrm{d}\mu_G(\gamma) = \int_K \lambda^{i+1}k\lambda^{-(i+1)}v \, \mathrm{d}\mu_G(k),$$

we can easily verify that these squares commute. Take the square with upper map t_i^{i+1}, then

$$\begin{tikzcd} v \arrow[r, mapsto] \arrow[d, mapsto] & \mu_G(K)^{-1} \cdot \int_K \lambda^{i+1} k \lambda^{-(i+1)} v \, \mathrm{d}\mu_G(k) \arrow[d, mapsto] \\ \lambda^{-i} v \arrow[r, mapsto] & \mu_G(K)^{-1} \cdot \int_K k \lambda^{-(i+1)} v \, \mathrm{d}\mu_G(k) \end{tikzcd}$$

clearly commutes. ☐

This implies that the filtered limits of $\mathscr{S}_\lambda$ and $\tilde{\mathscr{S}}_\lambda$ coincide, if they exist. The same is then true about the colimits.

Lemma 7.2.2. *Let K be as above, $v \in \big(V_U\big)^{K_M}$ – the K_M-fixed space of the Jacquet module V_U. Then we find an $i \in \mathbb{Z}$ and a $\tilde{v} \in V^{K^{(i)}}$ such that $\tilde{v}$ is mapped to v under the natural projection*

$$\begin{aligned} \pi \colon V^{K^{(i)}} &\longrightarrow \big(V_U\big)^{K_M} \\ \tilde{v} &\longmapsto [\tilde{v}]. \end{aligned}$$

This is a direct consequence of the (strong) Jacquet Lemma (p. 65 in [Ber92]), but much easier to prove.

Proof. We have an exact sequence of vector spaces

$$0 \longrightarrow V(U) \longrightarrow V \longrightarrow V_U \longrightarrow 0.$$

Indeed, this is an exact sequence of smooth M-representations. Hence the reduced sequence

$$0 \longrightarrow \big(V(U)\big)^{K_M} \longrightarrow V^{K_M} \longrightarrow \big(V_U\big)^{K_M} \longrightarrow 0.$$

is exact. In particular, the map $V^{K_M} \to \big(V_U\big)^{K_M}$ is surjective. Take a preimage v' of v under this map. v' is G-smooth, hence we find an open, compact subgroup C of G that fixes v'. Because of Proposition 2.5.12 (iv) we find an $i \in \mathbb{Z}$ such that $K_{\overline{U}}^{(i)} \subset C \cap \overline{U}$. Hence $v' \in V^{K_{\overline{U}}^{(i)}}$. We find the desired $\tilde{v}$ as

$$\mu_G(K)^{-1} \cdot \int_{K^{(i)}} k v' \, \mathrm{d}\mu_G(g) \in V^{K^{(i)}}.$$

☐

Proposition 7.2.3. *For $v \in V^{K^{(i)}}$ and $i < j$ we have*

$$t_i^j(v) = \mu_G(K)^{-1} \cdot \int_{K^{(j)}} k v \, \mathrm{d}\mu_G(k).$$

Proof. We first prove the case $j = i + 2$. We have

$$t_i^{i+1}(v) = \mu_G(K)^{-1} \cdot \int_G \mathbf{1}_{K^{(i+1)}}(\gamma) \, \gamma v \, \mathrm{d}\mu_G(\gamma).$$

As $\gamma \in K^{(n)}$ can be written as $\gamma^+\gamma^0\gamma^-$ with $\gamma^+ \in K_U^{(n)}, \gamma^0 \in K_M^{(n)}$ and $\gamma^- \in K_{\overline{U}}^{(n)}$, we can write this as

$$t_i^{i+1}(v) = \mu_G(K)^{-1} \cdot \int_G \mathbf{1}_{K^{(i+1)}}(\gamma) \, \gamma^+ v \, \mathrm{d}\mu_G(\gamma)$$

because v is fixed by $\gamma^0 \in K_M^{(i+1)} = K_M^{(i)}$ and by $\gamma^- \in K_{\overline{U}}^{(i+1)} \subset K_{\overline{U}}^{(i)}$.
Hence, using Fubini's Theorem, we have

$$\begin{aligned} t_i^{i+2}(v) &= \mu_G(K)^{-2} \int_G \mathbf{1}_{K^{(i+2)}}(\delta) \, \delta \cdot \left(\int_G \mathbf{1}_{K^{(i+1)}}(\gamma) \, \gamma^+ v \, \mathrm{d}\mu_G(\gamma) \right) \mathrm{d}\mu_G(\delta) \\ &= \mu_G(K)^{-2} \int_G \left(\int_G \mathbf{1}_{K^{(i+2)}}(\delta) \mathbf{1}_{K^{(i+1)}}(\gamma) \, \delta\gamma^+ v \, \mathrm{d}\mu_G(\delta) \right) \mathrm{d}\mu_G(\gamma). \end{aligned}$$

The claim follows from change of variable $\delta \rightsquigarrow \delta \cdot (\gamma^+)^{-1}$ and the fact $(\gamma^+)^{-1} \in K^{(i+2)}$. It is clear how to proceed in order to produce an induction argument on i. $\square$

Our claim is now

Theorem 7.2.4. *As vector-spaces,*

$$\varinjlim \tilde{\mathscr{S}}_\lambda \cong \left(V_U\right)^{K_M}.$$

Proof. Let us recall the following characterization of the filtered colimit from Section 1.1:

$$\varinjlim \tilde{\mathscr{S}}_\lambda \ \cong \Big(\coprod_{i\in\mathbb{Z}} V^{K^{(i)}}\Big)_{/\sim}$$

where we understand $x \in V^{K^{(m)}}$ and $y \in V^{K^{(n)}}$ as equivalent precisely if there is an integer $c \geq \max(m,n)$ such that $t_m^c(x) = t_n^c(y) \in V^{K^{(c)}}$.

Now consider the map

$$\Theta\colon \coprod_{i\in\mathbb{Z}} V^{K^{(i)}} \longrightarrow \left(V_U\right)^{K_M}$$

where any $v \in V^{K^{(i)}} \subset V$ is mapped simply to its image $[v]$ under the projection onto the Jacquet module V_U. It is clear that $[v]$ is invariant under the action of K_M.
In order to see that Θ descends to a map from $\varinjlim \tilde{\mathscr{S}}_\lambda$ to $\left(V_U\right)^{K_M}$, it is sufficient to show that the statement

$$\Theta(v) = \Theta\big(t_i^{i+1}(v)\big) \qquad \text{for all } v \in V^{K^{(i)}}$$

holds for any $i \in \mathbb{Z}$. But this is easy: For such a v we find a finite subset $\mathcal{U} \subset K_U^{(i+1)}$ such that

$$t_i^{i+1}(v) = \mu_G(K)^{-1} \cdot \int_{K^{(i+1)}} kv \, \mathrm{d}\mu_G(k) = \sum_{u\in\mathcal{U}} \lambda_u \cdot uv$$

with $\sum_{\mathcal{U}} \lambda_u = 1$. Then, because $[v] = [uv]$ in V_U, we have

$$\Theta\big(t_i^{i+1}(v)\big) = \Theta\Big(\sum_{u\in\mathcal{U}} \lambda_u \cdot uv\Big) = \sum_{u\in\mathcal{U}} \lambda_u \cdot [v] = [v]$$

and therefore Θ induces a map $\varinjlim \tilde{\mathscr{S}}_\lambda \longrightarrow \left(V_U\right)^{K_M}$.

We claim that this map is an isomorphism. Regarding injectivity, we have to prove the following: Whenever $\Theta(x)$ equals $\Theta(y)$ with $x \in V^{K^{(m)}}$ and $y \in V^{K^{(n)}}$, we can conclude that $x \sim y$. In order to do this, recall that $[x] = [y]$ means that there is a finite subset $\Omega \subset U \times V$ such that

$$x = y + \sum_{(u,v)\in\Omega} (v - uv).$$

Then take a $c \in \mathbb{Z}$ big enough such that $u \in K^{(c)}$ for all $(u,v) \in \Omega$. We conclude

$$\int_{K^{(c)}} \gamma x \, \mathrm{d}\mu_G(\gamma) = \int_{K^{(c)}} \gamma y \, \mathrm{d}\mu_G(\gamma)$$

where the left hand side equals $t_m^c(x)$ and the right hand side equals $t_n^c(y)$, according to the proposition above. Surjectivity follows immediately from the last lemma. $\square$

We finish with the following observation:
If V happens to be a smooth G-G-bimodule, the system $\mathscr{S}$ is indeed a system of right G-modules (as the t_i^{i+1} intertwine with the right G-action). It is clear that the preceding theorem holds if we take the limit in the category of smooth G-representations and require the isomorphism

$$\varinjlim \tilde{\mathscr{S}}_\lambda \cong \left(V_U\right)^{K_M}$$

to be a right G-intertwiner.
Moreover, it was arbitrary that we took the left action first: If the symbols $V^{K^{(i)}}$ and $\left(V_U\right)^{K_M}$ are understood as (co)-invariants with respect to the right G-action, and if we redefine

$$t_i^{i+1}(v) = \mu_G(K)^{-1} \cdot \int_{K^{(i+1)}} vk \,\mathrm{d}\mu_G(k),$$

the preceding theorem holds as an isomorphism of left G-modules.

7.3 The Dual System

Consider a projective system

$$\mathscr{S} = \left(\cdots \xrightarrow{t_{i-1}^i} a_i \xrightarrow{t_i^{i+1}} a_{i+1} \xrightarrow{t_{i+1}^{i+2}} \cdots \right)$$

of G-representations, where i runs through $\mathbb{Z}$.
For any other G-representation x, we can form the system

$$\mathscr{S}^x = \left(\cdots \xleftarrow{\left(t_{i-1}^i\right)^*} \operatorname{Hom}_G(a_i, x) \xleftarrow{\left(t_i^{i+1}\right)^*} \operatorname{Hom}_G(a_{i+1}, x) \xleftarrow{\left(t_{i+1}^{i+2}\right)^*} \cdots \right).$$

$\mathscr{S}^x$ is an inductive system of vector spaces and admits the filtered limit[3]

$$\varprojlim \mathscr{S}^x \cong \operatorname{Hom}_G(\varinjlim \mathscr{S}, x).$$

In our setting, we have a quite detailed understanding of these t^*s:
Consider a smooth left G-module M and two subgroups $K \subset C \in \mathscr{K}(G)$.
Then we have the inclusion

$$\imath\colon {}^C M \hookrightarrow {}^K M$$

and the (normalized) projection

$$\pi\colon {}^K M \twoheadrightarrow {}^C M \qquad m \mapsto \mu_G(C)^{-1} \int_C \gamma m \,\mathrm{d}\mu_G(\gamma).$$

We will use the same notations if M is a right G-module, where we just have to redefine $\pi(m) = \mu_G(C)^{-1} \int_C m\gamma \,\mathrm{d}\mu_G(\gamma) \in M^C$. Now we have

Proposition 7.3.1. *Let $K \subset C$ be as above and take a smooth G-representation V, then*

(i) *The inclusion $\imath$ induces the commutative diagram*

$$\begin{tikzcd} \operatorname{Hom}_G^K(\mathcal{D}(G), V) \arrow[r, "\imath^*"] \arrow[d, equal] & \operatorname{Hom}_G^C(\mathcal{D}(G), V) \arrow[d, equal] \\ V^K \arrow[r, two heads, "\pi"'] & V^C \end{tikzcd}$$

[3]See V.4 in [ML98]. We give the (pedantic) remark that there the limits are taken in the category of sets, so we have to check that the isomorphism is linear, but this becomes obvious when writing down the identification mapping.

(ii) *The projection π induces the commutative diagram*

$$\begin{tikzcd}
\mathrm{Hom}_G^C(\mathcal{D}(G),V) \arrow[r, "\pi^*"] \arrow[d, equal] & \mathrm{Hom}_G^K(\mathcal{D}(G),V) \arrow[d, equal] \\
V^C \arrow[r, hook, "\imath"'] & V^K
\end{tikzcd}$$

Proof. We use the identification $\mathrm{Hom}_G^K(\mathcal{D}(G),V) \cong \mathrm{Hom}_G(\mathcal{D}(G)^K,V)$. We would like to emphasize that $\mathcal{D}(G)^K$ means the K-invariants of the *right* G-module $\mathcal{D}(G)$ while V^K means the K-invariants of the *left* G-module V.

For the first claim, start at the upper-left corner:

$$\begin{tikzcd}
\Theta \arrow[r, mapsto] \arrow[d, mapsto] & \Theta|\mathcal{D}(G)^C \arrow[d, mapsto] \\
\Theta(e_K) & \Theta(e_C)
\end{tikzcd}$$

Set $v_K = \Theta(e_K) \in V^K$. We have

$$\pi(v_K) = \mu_G(C)^{-1} \int_C \gamma v_K \,\mathrm{d}\mu_G(\gamma) = \frac{\mu_G(K)}{\mu_G(C)} \sum_{\gamma \in C/K} \gamma v_K.$$

On the other hand, write

$$\Theta(e_C) = \Theta\left(\sum_{\gamma\in C/K} \gamma e_K \frac{\mu_G(K)}{\mu_G(C)} \right) = \frac{\mu_G(K)}{\mu_G(C)} \sum_{\gamma\in C/K} \gamma\Theta(e_K) = \pi(v_K).$$

The second claim is similar: Consider

$$\begin{tikzcd}
\Theta \arrow[r, mapsto] \arrow[d, mapsto] & \big(\varphi \mapsto \Theta\big(\pi(\varphi)\big)\big) \arrow[d, mapsto] \\
\Theta(e_C) & \Theta\big(\pi(e_K)\big)
\end{tikzcd}$$

and

$$\pi(e_K)(x) = \mu_G(C)^{-1} \int_C e_K(x)\gamma \,\mathrm{d}\mu_G(\gamma) = \frac{\mu_G(K)}{\mu_G(C)} \sum_{\gamma\in K\backslash C} e_{K\gamma}(x) = e_C(x). \qquad \square$$

For our next result, we need the same setting as in the preceding section: Let $P = MU$ be a parabolic subgroup of G. Moreover, fix a $K \in \mathscr{K}_P(G)$ and an element $\lambda \in G$ that is strictly dominant with respect to P and K.
For a smooth G-representation V, recall the systems $\tilde{\mathscr{S}}^\lambda$ and $\mathscr{S}^\lambda$ from the preceding chapter. It is easy to see that the maps t_i^{i+1} factorize as follows:

$$\begin{tikzcd}
V^{K^{(i)}} \arrow[rr, "t_i^{i+1}"] \arrow[dr, hook, "\imath"'] & & V^{K^{(i+1)}} \\
& V^{K^{(i)}\cap K^{(i+1)}} \arrow[ur, two heads, "\pi"'] &
\end{tikzcd}$$

Now we are able to prove

Theorem 7.3.2. *Let $P = MU$, K, λ and V be as above, then*

$$\operatorname{Hom}_G^{K_M}(\mathcal{D}(G)_{/\overline{U}}, V) \cong \varprojlim \mathscr{S}^\lambda.$$

Proof. We will prove this for the system $\tilde{\mathscr{S}}^\lambda$, what amounts to the same.
Define the system

$$\mathscr{T} = \left(\cdots \xleftarrow{t_i^{i-1}} \mathcal{D}(G)^{K^{(i)}} \xleftarrow{t_{i+1}^i} \mathcal{D}(G)^{K^{(i+1)}} \xleftarrow{t_{i+2}^{i+1}} \cdots \right)$$

with

$$t_{i+1}^i(\varphi) = \mu_G(K)^{-1} \int_{K^{(i)}} \varphi\gamma \, \mathrm{d}\mu_G(\gamma).$$

Now λ^{-1} is strictly dominant with respect to $\overline{P}$ and K. The system $\mathscr{T}$ is precisely the system occurring in Theorem 7.2.4 if we replace P by $\overline{P}$, λ by λ^{-1} and V by $\mathcal{D}(G)$ and use the right G-action. This says

$$\varinjlim \mathscr{T} \cong \left(\mathcal{D}(G)_{/\overline{U}}\right)^{K_M}.$$

Therefore, we have

$$\operatorname{Hom}_G^{K_M}(\mathcal{D}(G)_{/\overline{U}}, V) \cong \operatorname{Hom}_G(\varinjlim \mathscr{T}, V) \cong \varprojlim \mathscr{T}^V$$

for the system

$$\mathscr{T}^V = \left(\cdots \xrightarrow{\left(t_i^{i-1}\right)^*} \operatorname{Hom}_G(\mathcal{D}(G)^{K^{(i)}}, V) \xrightarrow{\left(t_{i+1}^i\right)^*} \operatorname{Hom}_G(\mathcal{D}(G)^{K^{(i+1)}}, V) \xrightarrow{\left(t_{i+2}^{i+1}\right)^*} \cdots \right).$$

This looks clumsy, but in fact we know $\mathscr{T}^V$ very well:

$$\begin{tikzcd}
\operatorname{Hom}_G(\mathcal{D}(G)^{K^{(i)}}, V) \arrow[rr, "\left(t_{i+1}^i\right)^*"] \arrow[dr, "\pi^*"'] \arrow[ddd, equals] & & \operatorname{Hom}_G(\mathcal{D}(G)^{K^{(i+1)}}, V) \arrow[ddd, equals] \\
& \operatorname{Hom}_G(\mathcal{D}(G)^{K^{(i)} \cap K^{(i+1)}}, V) \arrow[ur, "\imath^*"'] \arrow[d, equals] & \\
& V^{K^{(i)} \cap K^{(i)}} \arrow[dr, "\pi"] & \\
V^{K^{(i)}} \arrow[ur, "\imath"] \arrow[rr, "t_i^{i+1}"'] & & V^{K^{(i+1)}}
\end{tikzcd}$$

where everything commutes. Hence $\mathscr{T}^V$ is isomorphic to $\tilde{\mathscr{S}}^\lambda \cong \mathscr{S}^\lambda$. □

7.4 Explicit Construction of the "Difficult" Unit

Let $P = MU$ be a parabolic subgroup of G. We start this fairly technical section with collecting the measures we need:

Proposition 7.4.1. *There exist Haar measures μ_U, μ_M, $\mu_{\overline{U}}$, μ_G and a left Haar measure $\mu_{\overline{P}}$ and a right Haar measure μ_P such that for any $K \in \mathscr{K}_P(G)$ with Iwahori factorization $K = K^+K^0K^-$ we have*

(i)
$$\mu_G(K) = \mu_U(K^+)\mu_M(K^0)\mu_{\overline{U}}(K^-)$$

(ii)
$$\mu_P(K^+K^0) = \mu_U(K^+)\mu_M(K^0) \quad \textit{and} \quad \mu_{\overline{P}}(K^0K^-) = \mu_M(K^0)\mu_{\overline{U}}(K^-).$$

Proof. First of all, fix the Haar measures μ_U on U, μ_M on M and $\mu_{\overline{U}}$ on $\overline{U}$ arbitrarily. The other ones are delivered by Proposition 2.5.1. Observe that they are indeed k-valued, as μ_U, μ_M and $\mu_{\overline{U}}$ are k-valued. Then we have

$$\mu_G(K) = \int_G \mathbf{1}_K(\gamma)\, \mathrm{d}\mu_G(\gamma) = \int_U \int_M \int_{\overline{U}} \mathbf{1}_K(um\overline{u}) \cdot \delta_P(m)^{-1}\, \mathrm{d}\mu_U(u)\, \mathrm{d}\mu_M(m)\, \mathrm{d}\mu_{\overline{U}}(\overline{u})$$
$$= \int_U \mathbf{1}_{K^+}(u)\, \mathrm{d}\mu_U(u) \int_M \mathbf{1}_{K^0}(m)\, \mathrm{d}\mu_M(m) \int_{\overline{U}} \mathbf{1}_{K^+}(\overline{u})\, \mathrm{d}\mu_{\overline{U}}(\overline{u}) = \mu_U(K^+)\mu_M(K^0)\mu_{\overline{U}}(K^-)$$

because the delta factor is trivial on compact subgroups. The second claim is similar, where we get two left Haar measures μ_P and $\mu_{\overline{P}}$. But as delta factors are trivial on compact subgroups, we destroy nothing if we replace μ_P by the right Haar measure

$$P \supset X \longmapsto \int_P \delta_P^{-1}(p) \cdot \mathbf{1}_X(p)\, \mathrm{d}\mu_P(p). \qquad \square$$

When dealing with more than one group, it may not always be clear which measure we use for the normalization. In order to avoid ambiguity, we write for example

$$e_K^G = \mu_G(K)^{-1} \cdot \mathbf{1}_K \quad \text{or} \quad e_K^H = \mu_H(K)^{-1} \cdot \mathbf{1}_K.$$

Lemma 7.4.2. *With the choice of Haar measures given by the preceding proposition, we have:*

(i) *Under the isomorphism*
$$\mathcal{D}(P)_{/U} \cong \mathcal{D}(M) \circledast \delta_P^{+1},$$
the element $[e^P_{K^0K^+}]$ *corresponds to* $e^M_{K^0}$.

(ii) *Under the isomorphism*
$$\mathcal{D}(\overline{P}) \otimes_M \mathcal{D}(P) \cong \mathcal{D}(\overline{P} \times_M P) \cong \mathcal{D}(\overline{P}P),$$
the element $e^{\overline{P}}_{K^-K^0} \otimes e^P_{K^0K^+}$ *corresponds to* e^G_K.

Proof. (i): $[e^P_{K^0K^+}]$ is mapped to

$$\int_U e^P_{K^0K^+}(\_u)\, \mathrm{d}\mu_U(u) = \mu_P(K^0K^+)^{-1} \int_U \mathbf{1}_{K^0K^+}(\_u)\, \mathrm{d}\mu_U(u) = \frac{\mu_U(K^+)}{\mu_P(K^0K^+)} \mathbf{1}_{K^0}(\_) = e^M_{K^0}(\_).$$

(ii) The element $\mathbf{1}_{K^-K^0} \otimes \mathbf{1}_{K^0K^+}$ is mapped to the function that we temporarily denote by Θ:

$$\overline{P}P = \overline{U}MU \ni \overline{u}mu \quad \longmapsto \quad \Theta(\overline{u}mu) := \int_M \mathbf{1}_{K^-K^0}(\overline{u}m\nu)\mathbf{1}_{K^0K^+}(\nu^{-1}u)\, \mathrm{d}\mu_M(\nu)$$

Θ vanishes outside of K:

$$\begin{array}{rcl} \overline{u} \notin K^- & \Rightarrow & \nexists \nu \in M \text{ s. t. } \overline{u}m\nu \in K^-K^0 \\ u \notin K^+ & \Rightarrow & \nexists \nu \in M \text{ s. t. } \nu^{-1}u \in K^0K^+ \\ m \notin K^0 & \Rightarrow & \left(\nu^{-1}u \in K^0K^+ \Rightarrow \overline{u}m\nu \notin K^-K^0\right) \end{array}$$

So, let $\overline{u}mu \in K^-K^0K^+$. We then have

$$\Theta(\overline{u}mu) = \int_M \mathbf{1}_{K^-K^0}(\overline{u}m\nu)\mathbf{1}_{K^0K^+}(\nu^{-1}u)\, \mathrm{d}\mu_M(\nu) = \int_M \mathbf{1}_{K^0}(\nu)\, \mathrm{d}\mu_M(\nu) = \mu_M(K^0).$$

This says $\Theta = \mu_M(K^0) \cdot \mathbf{1}_K$ and the claim follows from the proposition. $\square$

These results enable us to explain the occurrence of an M-intertwining injection

$$W \hookrightarrow \boldsymbol{r}^G_{\overline{P}} \circ \boldsymbol{i}^G_P(W) \tag{7.1}$$

predicted by the Geometric Lemma (as $\overline{P} \cdot 1 \cdot P$ is the unique open coset in $\overline{P}\backslash G/P$).

Recall from Proposition 6.2.1 that trivial continuation gives a $\overline{P}$-P-bimodule injection

$$\mathcal{D}(\overline{P}P) \hookrightarrow \mathcal{D}(G).$$

Using our various identifications, we have

$$\mathcal{D}(M) \;\cong\; \delta_{\overline{P}}^{-1} \circledast \mathcal{D}(M) \otimes_M \mathcal{D}(M) \circledast \delta_P \;\cong\; {}_{\overline{U}\backslash}\mathcal{D}(\overline{P}) \otimes_M \mathcal{D}(P)_{/U} \;\cong\; {}_{\overline{U}\backslash}\mathcal{D}(\overline{P}P)_{/U}$$

and

$${}_{\overline{U}\backslash}\mathcal{D}(G)_{/U} \;\cong\; {}_{\overline{U}\backslash}\mathcal{D}(G) \otimes_G \mathcal{D}(G)_{/U},$$

where the vanishing of the delta-factors is due to the identity $\delta_P = \delta_{\overline{P}}^{-1}$. Because taking Jacquet-modules is exact, this provides us with an M-M-bimodule injection[4]

$$\eta' \colon \mathcal{D}(M) \;\hookrightarrow\; {}_{\overline{U}\backslash}\mathcal{D}(G) \otimes_G \mathcal{D}(G)_{/U}$$

which, adding the normalizing delta-factors via Proposition 4.1.2, gives rise to (7.1).

Observation 7.4.3. *If $K \in \mathscr{K}_P(G)$ with Iwahori decomposition $K = K^+K^0K^-$, η' maps $e^M_{K^0}$ to $[e^G_K] \otimes [e^G_K]$.*

Proof.

$$e^M_{K^0} \cong e^M_{K^0} \otimes e^M_{K^0} \cong [e^{\overline{P}}_{K^-K^0}] \otimes [e^P_{K^0K^+}] \cong [e^G_K]$$

is mapped by η' to

$$[e^G_K] \cong [e^G_K] \otimes [e^G_K].$$

∎

7.5 Second Adjointness Theorem

Let $P = MU$ be a parabolic subgroup of a reductive p-adic group G. We just established a map

$$\eta' \colon \mathcal{D}(M) \;\hookrightarrow\; {}_{\overline{U}\backslash}\mathcal{D}(G) \otimes_G \mathcal{D}(G)_{/U}.$$

For a smooth M-representation W and a smooth G-representation V, η' gives rise to a map

$$\eta'_{V,W} \colon \operatorname{Hom}_G(\mathcal{D}(G)_{/U} \otimes_M W, V) \;\longrightarrow\; \operatorname{Hom}_M(W, V_{\overline{U}}) \cong \operatorname{Hom}_M(\mathcal{D}(M) \otimes_M W, {}_{\overline{U}\backslash}\mathcal{D}(G) \otimes_G V)$$

via

$$\Theta \;\longmapsto\; \Big(\varphi \otimes_M w \mapsto [\alpha] \otimes_G \Theta\big([\beta] \otimes_M w\big)\Big)$$

where $[\alpha] \otimes [\beta] \in {}_{\overline{U}\backslash}\mathcal{D}(G) \otimes_G \mathcal{D}(G)_{/U}$ is an image of φ under η'.

One way of proving Second Adjointness is showing that the $\eta'_{V,W}$ are isomorphisms (as Bernstein and Kazhdan do). We will follow another way.

[4]Injectivity is not clear as tensoring is not necessarily left exact. But it is easy to see by hand and we explained (and used) this fact already in the preparation for the Geometric Lemma, see Observation 6.2.2. But, after all, we do not need injectivity in the sequel.

Have a look at this:

$$\begin{tikzcd}
\mathrm{Hom}_G\Big(\mathcal{D}(G)_{/U} \otimes_M \mathrm{Hom}^\infty_G(\mathrm{D}(G)_{/U}, V), V\Big) \arrow[r] & \mathrm{Hom}_M\Big(\mathrm{Hom}^\infty_G(\mathcal{D}(G)_{/U}, V), {}_{\overline{U}\backslash}\mathcal{D}(G) \otimes_G V\Big) \\
\mathrm{Hom}_M\Big(\mathrm{Hom}^\infty_G(\mathcal{D}(G)_{/U}, V), \mathrm{Hom}_G(\mathcal{D}(G)_{/U}, V)\Big) \arrow[u] &
\end{tikzcd}$$

where the horizontal map is the appropriate component of η' and the vertical map comes from the Hom-Tensor-adjunction.
This suggests that there is a distinguished M-map

$$\digamma\colon\ \mathrm{Hom}^\infty_G\big(\mathcal{D}(G)_{/U}, V\big) \ \longrightarrow\ V_{\overline{U}} = {}_{\overline{U}\backslash}\mathcal{D}(G) \otimes_G V$$

coming from the injection mapping in the lower-left Hom-set.

Observation 7.5.1. *If $\digamma$ is an isomorphism, Second Adjointness holds.*

Proof. Recall that Second Adjointness means there are natural isomorphisms

$$\mathrm{Hom}_G(\boldsymbol{i}^G_P(W), V) \ \cong\ \mathrm{Hom}_G(W, \boldsymbol{r}^G_{\overline{P}}(V))$$

for any smooth G-representation V and any smooth M-representation W. Our characterization of $\boldsymbol{i}^G_P$ makes it obvious (using the Hom-Tensor-adjunction for bimodules, see Theorem 5.10 in [Hun80]) that there must exist a right adjoint:

$$\mathrm{Hom}_G\big(\mathcal{D}(G)_{/U} \circledast \delta_P^{-\frac{1}{2}} \otimes_M W, V\big) \ \cong\ \mathrm{Hom}_M\big(W, \mathrm{Hom}_G(\mathcal{D}(G)_{/U} \circledast \delta_P^{-\frac{1}{2}}, V)\big)$$

where, according to Proposition 3.1.5, we can replace the right hand side by

$$\mathrm{Hom}_M\big(W, \mathrm{Hom}^\infty_G(\mathcal{D}(G)_{/U} \circledast \delta_P^{-\frac{1}{2}}, V)\big).$$

Therefore the functor

$$\mathrm{Hom}^\infty_G(\mathcal{D}(G)_{/U} \circledast \delta_P^{-\frac{1}{2}}, \llcorner\!\lrcorner) = \delta_P^{-\frac{1}{2}} \circledast \mathrm{Hom}^\infty_G(\mathcal{D}(G)_{/U}, \llcorner\!\lrcorner)$$

is a right adjoint for $\boldsymbol{i}^G_P$. It remains to provide an M-isomorphism

$$\delta_P^{-\frac{1}{2}} \circledast \mathrm{Hom}^\infty_G(\mathcal{D}(G)_{/U}, V) \ \cong\ \delta_{\overline{P}}^{\frac{1}{2}} \circledast V_{\overline{U}}.$$

As $\delta_P = \delta_{\overline{P}}^{-1}$, we can ignore the delta-factors. ☐

Now, how shall we proceed in order to prove that $\digamma$ is an isomorphism? Observe, that $K^0 = K \cap M$ gets arbitrary small in M as K runs through $\mathscr{K}_P(G)$. Thus, as a consequence of Lemma 3.5.2, we are done as soon as we can prove that for any such K the induced map

$$\digamma_K\colon\ \mathrm{Hom}^{K^0}_G\big(\mathcal{D}(G)_{/U}, V\big) \ \longrightarrow\ \big(V_{/\overline{U}}\big)^{K^0}$$

is an isomorphism of vector spaces.

We state the following crucial result without proof (see Appendix A for a more detailed treatment). It can be proved only if $k = \mathbb{C}$.

Theorem 7.5.2 (**(Weak) Stabilization Theorem).** *For any smooth G-representation V and any $K \in \mathscr{K}(G)$, the map $V^K \xrightarrow{t} V^K$ is weakly stable.*

We come to the main result:

Theorem 7.5.3. *Weak Stabilization implies Second Adjointness.*

Proof. For $K \in \mathscr{K}_P(G)$, fix $\lambda \in G$ strictly dominant with respect to $\overline{P}$ and K.
Observation 7.4.3 tells us that $\Theta \in \mathrm{Hom}_G^{K^0}\big(\mathcal{D}(G)_{/U}, V\big)$ is mapped to ${}_{\overline{U}}\big[\Theta\big([e^G_{K(i)}]_U\big)\big]$ by $\digamma_K$. For any $i \in \mathbb{Z}$, have a look at

$$\begin{tikzcd}
\mathrm{Hom}_G^{K^0}\big(\mathcal{D}(G)_{/U}, V\big) \arrow[r, dotted, "\digamma_K"] \arrow[d, "\text{Projection from } \varprojlim"'] & \big(V_{\overline{U}}\big)^{K^0} \\
\mathrm{Hom}_G\big(\mathcal{D}(G)^{K(i)}, V\big) \arrow[r, "\text{Isomorphism from Prop. 7.1.2}"'] & V^{K(i)} \arrow[u, "\text{Injection into } \varinjlim"']
\end{tikzcd}$$

This diagram commutes:

$$\begin{tikzcd}
\Theta \arrow[r, mapsto, dotted] \arrow[d, mapsto] & {}_{\overline{U}}\big[\Theta\big([e^G_{K(i)}]_U\big)\big] \\
\Big(\varphi \longmapsto \Theta\big([\varphi]_U\big)\Big) \arrow[r, mapsto] & \Theta\big([e_{K(i)}]_U\big) \arrow[u, mapsto]
\end{tikzcd}$$

Therefore, $\digamma_K$ equals the natural map

$$\varprojlim \mathscr{S}^\lambda \longrightarrow \varinjlim \mathscr{S}^\lambda$$

and the result follows from Lemma 1.2.5. $\square$

7.6 Implications from the Second Adjointness Theorem

In this section, we want to give a brief discussion of some results that can be established using the Second Adjointness Theorem in a crucial way. As this theorem is only known to be true in the case $k = \mathbb{C}$, we will restrict ourselves to this case.
The most obvious implication is:

Proposition 7.6.1. *Let $P = MU \subset G$ be as in the preceding section, then the functor $\boldsymbol{r}_P^G$ is continuous.*

Proof. This is a well-known alternative characterization for a right adjoint functor. $\square$

There is the very important concept of cuspidal representations. These representations can be understood as the building blocks of the category G-Rep.

Definition 7.6.2. A smooth G-representation is called **quasi-cuspidal** if $\boldsymbol{r}_P^G = 0$ holds for all $P \neq G$.

It is clear how the above proposition implies

Corollary 7.6.3. *Products of quasi-cuspidal representations are quasi-cuspidal.*

Using this, we can work out the following remark.

Remark 7.6.4. Fix an open, compact subgroup $K \subset G$.
Theorem 16 in [Ber92] tells us that for any quasi-cuspidal representation V we find a compact modulo center set Ω_V such that every matrix coefficient is supported in Ω_V. A **matrix coefficient** is a mapping of the form

$$m_{\xi,v} \colon G \longrightarrow \mathbb{C} \qquad g \longmapsto \langle \xi, gv \rangle$$

where $\xi \in V', v \in V$ are K-invariant vectors.
Now, let V_i be a sequence of quasi-cuspidal representations of G. We see that the matrix coefficients are bounded above in the following sense: The representation $W = \prod_i V_i$ is cuspidal, hence all the Ω_{V_i} are contained in the compact modulo center set Ω_W. For the connection between this fact and the Uniform Admissibility Theorem we refer the reader to section III of [Ber92].

Using another well-known fact from general category theory, we have

Proposition 7.6.5. *The functor* $\boldsymbol{i}_P^G$ *maps projective objects to projective objects.*

Definition 7.6.6. An object Π is called a **projective generator** if the Hom-functor

$$X \longmapsto \mathrm{Hom}_G(\Pi, X)$$

is exact and faithful.

We cite lemma 22 from [Ber92]:

Lemma 7.6.7. *Let $\mathcal{C}$ be an abelian category with arbitrary direct sums. If there exists a finitely generated projective generator Π, then $\mathcal{C}$ is isomorphic to the category of right modules over* $\mathrm{End}_{\mathcal{C}}(\Pi)$.

Luckily, parabolic induction preserves faithfulness as well:

Proposition 7.6.8. $\boldsymbol{i}_P^G$ *maps projective generators to projective generators.*

Proof. See proposition 34 in [Ber92]. □

We now cite two theorems which exploit this fact:

7.6.1 Trace Paley-Wiener Theorem

Suppose, we have an irreducible, quasi-cuspidal representation V of M, where M is the Levi factor of a parabolic subgroup $P = MU \subset G$.
We need the following notation:

Definition 7.6.9. Denote by G° the subgroup generated by all compact subgroups. Then an **unramified character** is a character $\chi\colon G \longrightarrow \mathbb{C}$ such that $\chi|G^\circ$ is trivial.

We cite moreover from [Ber92] that the set of unramified characters is isomorphic to $\left(\mathbb{C}^\times\right)^l$ for some $l \in \mathbb{N}$.

Now V gives rise to a family of G-representations

$$\{V_\chi\}_\chi \qquad \text{with } V_\chi = \boldsymbol{i}_M^G(\chi \circledast V)$$

where χ runs through all unramified characters of M.
Accordingly, an element φ in the Hecke algebra $\mathcal{H}(G)$ induces a family of maps $(f_{\varphi,\chi})_\chi$ defined by

$$f_{\varphi,\chi}\colon V_\chi \longrightarrow V_\chi \qquad v \longmapsto \int_G \varphi(\gamma)\gamma v \, \mathrm{d}\mu_G(\gamma).$$

Such a family $(f_{\varphi,\chi})_\chi$ fulfills the following conditions:

- $f_{\varphi,\chi}$, considered as a function on χ, is a regular function,
- There is an open, compact subgroup $K \subset G$ such that

$$\int_K \varphi(\gamma)\gamma f_{\varphi,\chi}(v) \, \mathrm{d}\mu_G(\gamma) = f_{\varphi,\chi}(v) = f_{\varphi,\chi}\left(\int_K \varphi(\gamma)\gamma v \, \mathrm{d}\mu_G(\gamma)\right),$$

- For any G-map $\tau\colon V_\chi \longrightarrow V_{\chi'}$ we have $f_{\varphi,\chi'} \circ \tau = \tau \circ f_{\varphi,\chi}$.

The main result is now

Theorem 7.6.10. *Any family of maps*

$$(h_\chi)_\chi \qquad h_\chi\colon V_\chi \longrightarrow V_\chi$$

that fulfills the three conditions above equals $(f_{\varphi,\chi})_\chi$ for some $\varphi \in \mathcal{H}(G)$.

Unfortunately, the proof uses the powerful structure considerations by Bernstein which we failed to introduce. Therefore we can only assure the reader that the crucial ingredient is proposition 7.6.8. The proof can be found in III.5.2 of [Ber92]. For a more detailed treatment see Chapter A.5 in [DKV84] or the article [BDK86].

7.6.2 Cohomological Duality

As usual, let G be a reductive p-adic group and V a smooth G-representation. We can understand V as a left $\mathcal{H}(G)$-module and hence form the groups

$$\operatorname{Ext}^i(V) := \operatorname{Ext}^i\big(V, \mathcal{H}(G)\big).$$

This construction has some interesting properties:

Observation 7.6.11. *If V is irreducible, $\operatorname{Ext}^i(V)$ vanishes for all but one i. As $\operatorname{Ext}^i(V)$ carries a right G-module structure, we can define an assignment*

$$D : G\text{-Rep} \longrightarrow G\text{-Rep}$$

as follows: To any irreducible G-representation, take the unique non-vanishing $\operatorname{Ext}^i(V)$ and convert it into a left G-module in the standard way. This module is called $D(V)$.

Proposition 7.6.12. *If V is irreducible, $D(V)$ is irreducible.*

The main result is

Theorem 7.6.13. *The mapping D is a duality map: $D(D(V)) = V$, and it defines a bijection on the set of equivalence classes of irreducible smooth G-representations.*

Additionally, this map intertwines with the induction and restriction in the following way:

Lemma 7.6.14. *Keep the notations from above, then*

- $D \circ \boldsymbol{i}_P^G = \boldsymbol{i}_{\overline{P}}^G \circ D$;
- $D \circ \boldsymbol{r}_P^G = \boldsymbol{r}_{\overline{P}}^G \circ D$.

Unfortunately, again we are unable to prove anything because of our lack in the general decomposition theory. Again, we ask the reader to believe that the Second Adjointness Theorem plays a crucial role and that a more comprehensive treatment can be found in [Ber92] (see chapter IV.5).

7.7 In Search of a Counit

Recall our strategy from Chapter 5 to prove Frobenius Reciprocity: We gave natural transformations

$$\begin{aligned} \eta\colon \operatorname{id}_{G\text{-Rep}_k} &\xrightarrow{\bullet} \boldsymbol{i}_P^G \circ \boldsymbol{r}_P^G && \text{– the unit,} \\ \varepsilon\colon \boldsymbol{r}_P^G \circ \boldsymbol{i}_P^G &\xrightarrow{\bullet} \operatorname{id}_{M\text{-Rep}_k} && \text{– the counit.} \end{aligned}$$

As they fulfill the so-called zig-zag equations

$$\mathrm{id}_{\boldsymbol{i}_P^G} = \boldsymbol{i}_P^G \varepsilon \circ \eta \boldsymbol{i}_P^G \quad \text{and} \quad \mathrm{id}_{\boldsymbol{r}_P^G} = \varepsilon \boldsymbol{r}_P^G \circ \boldsymbol{r}_P^G \eta,$$

we conclude that $\boldsymbol{i}_P^G$ is right adjoint to $\boldsymbol{r}_P^G$.
Consequently, if we were able to give natural transformations

$$\eta' \colon \mathrm{id}_{M\text{-}\mathrm{Rep}_k} \xrightarrow{\bullet} \boldsymbol{r}_{\overline{P}}^G \circ \boldsymbol{i}_P^G$$
$$\varepsilon' \colon \boldsymbol{i}_P^G \circ \boldsymbol{r}_{\overline{P}}^G \xrightarrow{\bullet} \mathrm{id}_{G\text{-}\mathrm{Rep}_k}$$

which fulfill the associated zig-zag equations, this would immediately imply Second Adjointness. Observe, that we already know η' very well from Section 7.4.
Therefore, as the transformations η, ε and η' (understood as bimodule mappings) seem to be very natural and "canonical", it is tempting to look out for a suitable G-G-mapping

$$\mathcal{D}(G)_{/U} \otimes_M {}_{\overline{U}\backslash}\mathcal{D}(G) \longrightarrow \mathcal{D}(G)$$

giving rise to ε'. As M acts freely, properly and with good decompositions on the ℓ-space $G/U \times \overline{U}\backslash G$, it is easy to show that the first symbol can be replaced by $\mathcal{D}(G/U \times_M \overline{U}\backslash G)$.
Despite serious efforts, the author is unable to construct such a map or even to point out where the difficulty lies and where something like a stabilization theorem[5] could come into play.

The author would like to thank David Kazhdan for sending him of some of his notes, where he constructs such a map and proves the zig-zag equations. The author looks forward to a prospective publication by Kazhdan.

[5]One could say that stability is the real reason why Second Adjointness holds.

Appendix A

Stabilization Theorem

Here, we will say a few words about the stabilization property used in a crucial manner during the proof of Second Adjointness.
Let G be a reductive p-adic group and take as ground field $k = \mathbb{C}$.

Theorem A.0.1 (**Weak Stabilization Theorem**). *For any parabolic subgroup $P = MU$ of G and any open, compact subgroup $K \subset G$ which admits an Iwahori factorization with respect to P there exists a $\lambda \in \Lambda^{++}$ (strictly dominant with respect to P and K) such that the linear mapping*

$$\begin{aligned} t\colon V^K &\longrightarrow V^K \\ v &\longmapsto \int_K k\lambda v \,\mathrm{d}\mu_G(k) \end{aligned}$$

is weakly stable.

Theorem A.0.2 (**Stabilization Theorem**). *Replace the word "weakly" in the above theorem by "eventually".*

Clearly, Stabilization implies Weak Stabilization.

Observation A.0.3. *Let V be smooth and admissible. Then, due to dimension reasons,* $\ker(t^n)$ *and* $\operatorname{im}(t^n)$ *stabilize in V^K. This says that stability holds for all admissible representations. Note that every irreducible smooth representation is admissible (see Theorem* 12 *in* [Ber92]).

The Stabilization Theorem was proved by Bernstein in [Ber87] or [Ber92] for all smooth representations of reductive p-adic groups. He proceeds like this:
Let V be a t-stable $\mathbb{C}$-vector space, V' be t'-stable. Then $V \oplus V'$ is $t \oplus t'$-stable. On the other hand, if $f\colon V \longrightarrow V'$ is a linear map that commutes with the t- and t'-action, $\ker(f)$ is a t-stable vector space and $\operatorname{coker}(f)$ is a t'-stable vector space.
Put into other words: We can understand vector spaces with an endomorphism as $\mathbb{C}[x]$-modules. Then the subcategory of x-stable modules is closed under taking sums, kernels and cokernels.

As a next step, Bernstein considers some smooth M-representation W and realizes $\boldsymbol{i}_P^G(W)$ as a cokernel in the following way:

$$\prod_\alpha \Pi \longrightarrow \prod_\beta \Pi \longrightarrow \boldsymbol{i}_P^G(W) \longrightarrow 0$$

where $\Pi = \boldsymbol{i}_P^G(\Pi(D))$ for a projective generator $\Pi(D)$ for D-Rep. Here D-Rep is the category of representations with Jordan-Hölder components in a cuspidal component D. For all these notions see Chapter II of [Ber92]. It is possible to prove then that both $\prod_\alpha \Pi$ and $\prod_\beta \Pi$ are t^c-stable for some number c. One can conclude that induced representations are eventually stable.

Next, Bernstein realizes an arbitrary G-representation V as a kernel in the following way:

$$0 \longrightarrow V \longrightarrow \bigoplus_{I_1} \boldsymbol{i}_{P_i}^G(W_i) \longrightarrow \bigoplus_{I_2} \boldsymbol{i}_{P_i}^G(W_i)$$

where the P_i are parabolic subgroups and the W_i are certain smooth (cuspidal) representations of the Levi part of W_i. This shows that V fulfills the Stabilization property, this means that each V^K is t^{c_K}-stable for numbers $c_K \in \mathbb{N}$. Indeed, Bernstein shows that the numbers c_K are bounded above by numbers $c(G,K)$ that solely depend on G and K and *not* on the particular representation.

The author regrets that he cannot present more then this very rough sketch of the proof. He would like to emphasize that the important steps (representing $\boldsymbol{i}_P^G(W)$ as a cokernel and representing V as a kernel) use the full force of Bernstein's decomposition of G-Rep (as described in [Ber92]) and rely crucially on the choice of base field $k = \mathbb{C}$.

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