DIMENSIONAL REGULARIZATION

RALF MEYER

ABSTRACT. We explain some mathematical ideas underlying dimensional regularization, which is an important ingredient in the BPHZ renormalization scheme. Our discussion is limited to the case of scalar field theories.

1. INTRODUCTION

The problem of perturbative renormalization theory is to give a meaning to certain divergent integrals arising from Feynman diagrams. The analytical difficulty is to regularize the occurring divergent integrals, associating to each Feynman diagram some finite value. The physical difficulty is to do this in such a way that the outcome has a physical meaning. The contributions of the various Feynman diagrams are not independent of each other: If a graph contains divergent subgraphs, then the way they have been regularized must be taken into account.

In this talk we will concentrate on the analytical part of the problem. We will explain how to associate a finite value to a Feynman diagram. This value is physically correct only if the diagram contains no divergent subgraphs. The combinatorics needed for general diagrams will be explained in other lectures. The procedure we use to regularize an integral is called *dimensional regularization*. It is used in the BPHZ renormalization scheme.

The basic idea behind dimensional regularization is that renormalization is much easier in low dimensions. Therefore, we try to write down the divergent integrals that we have to regularize in such a way that the dimension of the physical space-time becomes an external parameter that can be varied. Then we boldly allow the dimension D to be an arbitrary complex number. As expected, our integral becomes convergent for $\operatorname{Re} D \ll 0$ and defines an analytic function there. This function can be continued meromorphically to all of $\mathbb{C}$. However, it may have a pole at the physical dimension d of space-time. Finally, the regularized value is obtained by *minimal subtraction*: We subtract the pole part in the Laurent expansion around d and evaluate the remaining function at d. The pole part is called the *counterterm* and has to be recorded because it is needed when dealing with divergent subgraphs.

Although the meaning of physics in a D-dimensional space-time with $D \in \mathbb{C}$ is rather opaque, dimensional regularization is used by physicists because of its simplicity and good invariance properties. This scheme is always gauge and Lorentz invariant. These properties will not become apparent in the discussion below, however, because we will limit ourselves to scalar theories, thus excluding gauge theories. The reason for this restriction is that a scalar field theory can automatically be formulated in all non-negative integer dimensions. Hence we have sufficiently

This note is based on a talk the author held at the 2002 workshop "Theory of Renormalization and Regularization" in Hesselberg, which was sponsored by the Volkswagen Stiftung.

many dimensions to make a guided guess about the correct formulas for the D-dimensional integral for $D \in \mathbb{C}$. Since the spin is special to four-dimensional space-time, it is unclear how to formulate a non-scalar theory in other dimensions. Physicists, of course, know how to do this. Mathematicians, however, may find the case of scalar field theories already challenging enough.

We have prepared the lecture following Pavel Etingof (see [1]). However, our presentation differs significantly from his. In particular, we have a more powerful method for the meromorphic extension.

2. A ROUGH OUTLINE OF THE METHOD

We briefly describe how a physicist might carry out dimensional regularization. Let V be d-dimensional space-time. Consider a Feynman diagram with m external legs and corresponding momenta $q_1, \dots, q_m \in V$, and with $n-m$ loops ($n \geq m$). The amplitude of this diagram can be written as

$$I_f(q_1, \dots, q_m) := \int_{V^{n-m}} f(q_1, \dots, q_n)\, dq_{m+1} \dots dq_n,$$

where the function f is of the form

$$f(q) = \frac{P(q)}{\prod_j (l_j(q)^2 + m_j^2)}$$

with a certain polynomial P, and a number of linear forms l_j on V^n and masses $m_j > 0$. The rational function f has no poles, but does not decay fast enough at infinity, so that the integral diverges.

Pretending that the integral exists, we rewrite each factor in the denominator. The simple rule $\int_0^\infty \exp(-at)\, dt = a^{-1}$ yields

$$I_f = \int_{t_j \geq 0} \exp\left(-\sum_j t_j m_j^2\right) \int_{V^{n-m}} P(q) \exp\left(-\sum_j t_j l_j(q)^2\right) dq\, dt.$$

More precisely, since masses have a physical dimension, $\exp(-m_j^2)$ is meaningless. We should first introduce a *mass scale* Λ and apply the above rule to $a = (l_j(q)^2 + m_j^2)/\Lambda^2$ instead. Thus the result of the regularization depends on the choice of this mass scale. In the following, we will simplify our notation by assuming that all masses are measured with respect to some scale and hence dimensionless.

For fixed $t = (t_j)$, the Gaussian integral over q above can be computed explicitly. To write down the result, we assume that the field theory is scalar. For physical reasons, the function f is Lorentz invariant, which means, constant on orbits of the Lorentz group action on V^{n-m} because f has scalar values. Therefore, the external momenta do not enter directly, but only through their inner product matrix $C := (q_i \cdot q_j)_{i,j \leq m}$. The integration over V^{n-m} gives us a factor $(\det B(t))^{-d/2}$, where $B(t)$ is the restriction of the bilinear form $\sum_j t_j l_j^2(q)$ to V^{n-m}. Altogether we obtain

$$I_f = \int_{t_j \geq 0} \exp\bigl(-\sum_j t_j m_j^2\bigr) \psi(t, C, d) (\det B(t))^{-d/2}\, dt$$

with some polynomial function ψ that comes from the polynomial P. In this formula, the dimension d of space-time has become an external parameter which may be replaced by any complex number D. Now mathematics provides the following results: First, for $\operatorname{Re} D \ll 0$ the integral $I_f(D)$ exists; secondly, this function of D has a meromorphic extension to $\mathbb{C}$.

Once we have a meromorphic function $I_f(D)$, the regularized value of the integral is obtained by *minimal subtraction* at the physical dimension $D = d$. We consider the Laurent series

$$I_f(D) = \sum_{n\in\mathbb{Z}} a_n \cdot (D-d)^n$$

around d. Then the *counterterm* is the function

$$I_f^{\mathrm{counter}} := \sum_{n=-1}^{-\infty} a_n \cdot (D-d)^n,$$

the *regularized value* is

$$I_f^{\mathrm{reg}} := a_0.$$

If our graph has divergent subgraphs, then we should subtract appropriate terms involving the counterterms for the divergent subgraphs, according to the BPHZ renormalization scheme.

3. The D-dimensional integral

In this section, we construct the D-dimensional integral for Schwartz functions. In the next section, we will show how to extend this construction to more general functions.

We will always work with the Euclidean model of space-time. The final results can, of course, be translated to Minkowski space-time by a Wick rotation. However, intermediate results depend on the positivity of the space-time metric. We let V be Euclidean space-time. We let β be the positive definite metric on V. We denote the physical dimension of space-time by d, and use D for the dimension when viewed as a variable. In the Euclidean picture, the Lorentz group is replaced by the orthogonal group $\mathrm{O}(d)$.

3.1. **Coordinate-free reformulation of the integral.** If W is a finite dimensional vector space, we write $\mathrm{S}^2 W^*$ for the vector space of symmetric bilinear forms on V, and $\overline{\mathrm{S}}^2_+ W^*$ and $\mathrm{S}^2_+ W^*$ for the subsets of positive semi-definite and positive definite bilinear forms on W. Thus

$$\mathrm{S}^2_+ W^* \subseteq \overline{\mathrm{S}}^2_+ W^* \subseteq \mathrm{S}^2 W^*.$$

We write $\mathcal{S}(\overline{\mathrm{S}}^2_+ W^*)$ and $\mathcal{S}(\mathrm{S}^2 W^*)$ for the spaces of Schwartz functions on $\overline{\mathrm{S}}^2_+ W^*$ and $\mathrm{S}^2 W^*$. By definition, a function on $\overline{\mathrm{S}}^2_+ W^*$ is a Schwartz function iff it is the restriction of a Schwartz function on $\mathrm{S}^2 W^*$.

Let

$$E := \mathbb{R}^n, \qquad F := \mathbb{R}^m \subseteq \mathbb{R}^n.$$

Both E and F are vector spaces equipped with fixed volume forms—so that we can integrate functions on them—and carry no further structure. We view an n-tuple $q := (q_1, \dots, q_n)$ with $q_j \in V$ as an element of the vector space $\mathrm{Hom}(E,V)$. The m-tuple $(q_1, \dots, q_m)$ is nothing but the restriction $q|_F \in \mathrm{Hom}(F,V)$. The inner product matrix with entries $q_i \cdot q_j$ is identified with the bilinear form $q^*(\beta) \in \overline{\mathrm{S}}^2_+ E^*$ obtained by pulling back $\beta \in \mathrm{S}^2_+ V^*$ along q. Points $q_1, q_2 \in \mathrm{Hom}(E,V)$ lie on the same $\mathrm{O}(d)$-orbit iff $q_1^*\beta = q_2^*\beta$. Therefore, Lorentz invariance implies that the function f only depends on $q^*(\beta) \in \overline{\mathrm{S}}^2_+ E^*$. We suppose that f is already given as a function defined on all of $\overline{\mathrm{S}}^2_+ E^*$. Then we can rewrite our (still divergent) integral as

$$I^d_f(k^*\beta) = \int_{\{q \in \mathrm{Hom}(E,V) \mid x|_F = k\}} f(q^*\beta)\,dq$$

for $k \in \mathrm{Hom}(F,V)$. The right hand side only depends on $k^*\beta$, not on k itself. Notice that in the above formula, d may already be any non-negative integer! However, the above formula only determines $I^d_f(C)$ if $\operatorname{rank} C \leq d$, and it only depends on the values of f on form of rank d.

3.2. **Extrapolation to complex dimensions.** To guess a formula for the D-dimensional integral, we plug special functions into I^d. Let

$$\phi_B(A) := \exp(-\operatorname{tr}(AB)) \qquad \forall B \in \mathrm{S}^2_+ E.$$

Notice that this is a Schwartz function on $\overline{\mathrm{S}}^2_+ E^*$ by positivity. For the time being, the integral $I^d(\phi_B)(C)$ is defined only for $C \in \overline{\mathrm{S}}^2_+ F^*$ with rank $C \leq d$ because other C cannot be written as $k^*\beta$ for $k \in \mathrm{Hom}(F,V)$. We have

(1) $$I^d(\phi_B)(C) = \pi^{(n-m)d/2} \exp(-\operatorname{tr}(C \cdot B^{F^*})) \cdot (\det B_{F^\perp})^{-d/2}.$$

In this formula, $B_{F^\perp}$ is the restriction of B to $F^\perp = (E/F)^* \subseteq E^*$, and $B^{F^*} \in \mathrm{S}^2_+ F$ is the bilinear form on F^* that is *canonically* associated to B. That is, we restrict B to the orthogonal complement $(F^\perp)^{\perp_B}$ of $F^\perp \subseteq E^*$ with respect to the inner product B and identify

$$F^* \cong E^*/F^\perp \cong (F^\perp)^{\perp_B}.$$

If we split $E \cong F \oplus G$ arbitrarily and describe B by a 2×2-block matrix (B_{ij}) with respect to the resulting decomposition $E^* = F^* \oplus G^*$, then we have

(2) $$B_{F^\perp} = B_{22}, \qquad B^{F^*} = B_{11} - B_{12} B_{22}^{-1} B_{21}.$$

If the decomposition of E diagonalizes B, we simply get $B^{F^*} = B_{11}$. To verify (1), choose a basis in which B is diagonal and observe that the resulting integral is Gaussian.

In the remainder of this section, we will establish the following:

Theorem 1. *There are unique distributions $I^D|_C \in \mathcal{S}(\overline{\mathrm{S}}^2_+ E^*)'$ that satisfy*

(3) $$I^D|_C(\phi_B) = \pi^{(n-m)D/2} \exp(-\operatorname{tr}(C \cdot B^{F^*})) \cdot (\det B_{F^\perp})^{-D/2}$$

for all $B \in \mathrm{S}^2_+ E$, $C \in \overline{\mathrm{S}}^2_+ F^*$, and $D \in \mathbb{C}$. In addition, these distributions piece together to a continuous linear map

$$I = (I^D)_{D\in\mathbb{C}} \colon \mathcal{S}(\overline{\mathrm{S}}^2_+ E^*) \to \mathcal{O}(\mathbb{C}, \mathcal{S}(\overline{\mathrm{S}}^2_+ F^*)).$$

Definition 2. The operator I^D is called *D-dimensional integral with parameters*. If $F = \{0\}$, it is called *D-dimensional integral*.

3.3. **Construction of the D-dimensional integral with parameters.** We present two descriptions for a distribution $I^D|_C$ satisfying (3). The first one only works for $\operatorname{Re} D > n-1$. The second one works for $\operatorname{Re} D < 2$ and is used to extend the first description to all $D \in \mathbb{C}$.

For $l \in \mathbb{N}$, $x \in \mathbb{C}$, define

$$\Gamma_l(x) := (2\pi)^{l(l-1)/4} \prod_{j=0}^{l-1} \Gamma\Big(x - \frac{j}{2}\Big).$$

For $\operatorname{Re} D > n-1$ and $C \in \mathrm{S}^2_+ F^*$, let

$$\rho^D(A,C) := \pi^{(n-m)D/2} \frac{\Gamma_m(D/2)}{\Gamma_n(D/2)} \cdot \frac{(\det A)^{(D-n-1)/2}}{(\det C)^{(D-m-1)/2}} \cdot \delta(A_F - C).$$

This is a well-defined distribution because the function $(\det A)^{-1}$ is locally integrable on $\mathrm{S}^2 F^*$.

Lemma 3. *Let* $\operatorname{Re} D > n-1$, $C \in \mathrm{S}^2_+ F^*$, *and* $B \in \mathrm{S}^2_+ E$. *Then*

$$\int_{\mathrm{S}^2_+ E^*} \rho^D(A,C)\phi_B(A)\, dA = \pi^{(n-m)D/2} \exp(-\operatorname{tr}(C \cdot B^{F^*})) \cdot (\det B_{F^\perp})^{-D/2}.$$

Proof. We follow the standard route of analysis on symmetric cones (see [2]). Both sides of the equation are basis independent, they only use the subspace $F \subseteq E$, the volume forms on E and F, and the positive definite bilinear forms C and B. Although we will write down the computations in an invariant form, the simplest way to verify it uses a basis (x_i) for E that is as compatible as possible with the given structure. Let (x_i^*) be the corresponding dual basis for E^*. We can view bilinear forms as matrices and hence compose them using these bases. We choose the basis (x_i) so that: $x_1, \dots, x_m$ span F; the volume forms on E and F are equal to $x_1 \wedge \cdots \wedge x_m$ and $x_1 \wedge \cdots \wedge x_n$, respectively; the positive definite bilinear forms C and B take the normal forms

$$C = c \cdot 1_m, \qquad B = \begin{pmatrix} b^{F^*} \cdot 1_m & 0 \\ 0 & b_{F^\perp} \cdot 1_{n-m} \end{pmatrix}$$

with $c, b^{F^*}, b_{F^\perp} > 0$. In fact, we must have

$$c = (\det C)^{1/m}, \qquad b^{F^*} = (\det B^{F^*})^{1/m}, \qquad b_{F^\perp} = (\det B_{F^\perp})^{1/(n-m)}.$$

We let

$$G := F^{\perp_B} = \operatorname{span}\{x_{m+1}, \dots, x_n\} \subseteq E, \qquad G^* = F^\perp \subseteq E^*.$$

We represent an element A of $\mathrm{S}^2_+ E^*$ as a block matrix (A_{ij}) with respect to the decomposition $E = F \oplus G$. If A is positive definite, so is A_{11}. Hence A_{11} is invertible and we can define

$$Y := A_{21}A_{11}^{-1}, \qquad Y^* := A_{11}^{-1}A_{12}, \qquad X := A_{22} - A_{21}A_{11}^{-1}A_{12}.$$

The computation

$$\begin{pmatrix} 1 & 0 \\ Y & 1 \end{pmatrix} \cdot \begin{pmatrix} A_{11} & 0 \\ 0 & X \end{pmatrix} \cdot \begin{pmatrix} 1 & Y^* \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

shows that $A > 0$ iff $A_{11} > 0$ and $X > 0$ and that $\det A = \det A_{11} \cdot \det X$. We apply the change of variables $A \mapsto (A_{11}, X, A_{21})$, which identifies

$$\mathrm{S}^2_+ E^* \cong \mathrm{S}^2_+ F^* \times \mathrm{S}^2_+ F^\perp \times \operatorname{Hom}(F^*, F^\perp).$$

Its Jacobian has determinant 1 everywhere. Simplifying first the δ-function and then the Gaussian integral for A_{21}, we obtain:

$$\begin{aligned}
&\int_{S^2_+ E^*} \rho^D(A,C)\phi_B(A)\,dA \\
&\qquad = \pi^{(n-m)D/2}\frac{\Gamma_m(D/2)}{\Gamma_n(D/2)}(\det C)^{-(n-m)/2}\exp(-\operatorname{tr}(B^{F^*}C))\times \\
&\qquad\qquad \int_{S^2_+ F^\perp}(\det X)^{(D-n-1)/2}\exp(-\operatorname{tr} B_{F^\perp}X)\times \\
&\qquad\qquad \int_{\operatorname{Hom}(F^*,F^\perp)}\exp(-\operatorname{tr} B_{F^\perp}A_{21}C^{-1}A_{21}^t)\,dA_{21}\,dX \\
&\qquad = \pi^{(n-m)D/2}\frac{\Gamma_m(D/2)}{\Gamma_n(D/2)}(2\pi)^{m(n-m)/2}\exp(-\operatorname{tr}(B^{F^*}C))\times \\
&\qquad\qquad \int_{S^2_+ F^\perp}(\det B_{F^\perp})^{m/2}(\det X)^{(D-n-1)/2}\exp(-\operatorname{tr} B_{F^\perp}X)\,dX.
\end{aligned}$$

Since the only structure that our spaces carry is a volume form, the remaining integral over $S^2_+ F^\perp$ can only depend on the determinant of $B_{F^\perp}$. A scaling argument shows that it must be a constant multiple of $(\det B_{F^\perp})^{-D/2}$. To determine that constant, we split $G = G_1 \oplus G_2$ and use the same decomposition of $S^2_+ G$ as above. This reduces the computation for G to the simpler computations for G_1 and G_2. Eventually, one is led to the case where G has dimension 1. In that case, the integral clearly gives rise to some Γ-function. The constants in the definition of $\rho(A,C)$ are chosen so as to cancel these constant factors. ☐

Thus we may define $I^D|_C := \rho^D(\cdot,C)$ for $\operatorname{Re} D > n-1$ and $C \in S^2_+ F^*$. For $\operatorname{Re} D \in \mathbb{N}$, we already know what $I^D|_C$ has to do, and we know that it is supported on bilinear forms of rank $\leq D$. For $D \leq n-1$, this set has Lebesgue measure zero. This is why we cannot describe $I^D|_C$ so easily in that region.

Next we consider the Fourier transform of $I^D|_C$. By analytic continuation, formula (3) holds for all $B \in S^2 E \otimes \mathbb{C}$ with $\operatorname{Re} B > 0$. For $\operatorname{Re} B \searrow 0$, we get

$$I^D|_C(\phi_{iB}) = \pi^{(n-m)D/2}\exp(-\operatorname{tr}(iC\cdot B^{F^*}))\cdot \lim_{J\searrow 0}(\det(J+iB_{F^\perp}))^{-D/2} \tag{4}$$

for all $B \in S^2 E$. Notice that the bilinear form B^{F^*} is still defined for almost all B by (2). Even more, since $B \mapsto B^{-1}\cdot \det B$ is a polynomial function, the function $B \mapsto B^{F^*}$ is a rational function with denominator $\det B_{F^\perp}$.

Since the functions $\phi_{iB}(A)$ for $B \in S^2 E$ are exactly the characters of the group $S^2 E^*$, (4) describes the Fourier transform of $I^D|_C$. Since a distribution is determined uniquely by its Fourier transform, equation (4) shows that there is at most one distribution satisfying (3). For $\operatorname{Re} D < 2$, the limit in (4) exists even as a locally integrable function on $S^2 E$ and has temperate growth for $B \to \infty$. Hence it defines an element of the distribution space $\mathcal{S}(S^2 E)'$. Its Fourier transform is an element of $\mathcal{S}(S^2 E^*)'$, that is, a tempered distribution on $S^2 E^*$. This yields a useful formula for $I^D|_C$ if $\operatorname{Re} D < 2$.

For the time being, we are more interested in the region $\operatorname{Re} D > n-1$, where $\rho^D(\cdot,C)$ is a tempered distribution for all $C \in S^2_+ F^*$. Hence the limit in (4) must exist in $\mathcal{S}(S^2 E)'$ and be equal to the Fourier transform $\hat\rho^D(B,C)$ of $\rho^D(\cdot,C)$. Let

$$P(B) := \pi^{-(n-m)}\det B_{F^\perp},$$

this is a polynomial function on $\mathrm{S}^2 E$. We have

$$\hat{\rho}^{D-2}(B,C) = P(B)\cdot\hat{\rho}^{D}(B,C).$$

Consequently,

$$\rho^{D-2k}(A,C) = \partial(P)^k \rho^D(A,C) \qquad \forall k\in\mathbb{N}, D\in\mathbb{C}, \operatorname{Re} D > n+2k, \tag{5}$$

where $\partial(P)$ is the differential operator adjoint to P, acting on the variable A. Since $\partial(P)$ is a continuous operator on $\mathcal{S}(\overline{\mathrm{S}}^2_+ E^*)$, the distributions $\partial(P)^k\rho^D(\cdot,C)$ are well-defined for all $k\in\mathbb{N}$, $\operatorname{Re} D > n$. This provides a holomorphic extension of $I^D|_C$ to $D\in\mathbb{C}$. Evidently, the distributions $I^D|_C$ for $D\in\mathbb{C}$ piece together to a continuous linear operator

$$I|_C\colon\ \mathcal{S}(\overline{\mathrm{S}}^2_+ E^*) \to \mathcal{O}(\mathbb{C}) \qquad \text{for all } C\in \mathrm{S}^2_+ F^*.$$

If $C\in\overline{\mathrm{S}}^2_+ F^*\setminus \mathrm{S}^2_+ F^*$, we can still use the above construction in the following way. Let $K\subseteq F$ be the nullspace of C, so that C gives rise to a positive definite bilinear form C_* on the quotient space F/K. Since $\overline{\mathrm{S}}^2_+(E/K)^*\subseteq \overline{\mathrm{S}}^2_+ E^*$, we obtain a continuous linear map

$$I|_C\colon\ \mathcal{S}(\overline{\mathrm{S}}^2_+ E^*) \xrightarrow{\text{restr.}} \mathcal{S}(\overline{\mathrm{S}}^2_+(E/K)^*) \xrightarrow{I|_{C_*}} \mathcal{O}(\mathbb{C}),$$

which satisfies (3). The uniqueness of $I|_C$ again follows from the computation of the Fourier transform.

Summing up, we have constructed for each $C\in\overline{\mathrm{S}}^2_+ F^*$ a continuous linear map $I|_C\colon\ \mathcal{S}(\overline{\mathrm{S}}^2_+ E^*)\to\mathcal{O}(\mathbb{C})$. Theorem 1 asserts, in addition, that these maps piece together to a map with values in $\mathcal{O}(\mathbb{C},\mathcal{S}(\overline{\mathrm{S}}^2_+ F^*))$. Since we will not use this fact and since its proof is rather complicated, we omit it.

4. Extension to functions without rapid decay

So far, we have learned how to perform a D-dimensional integral for Schwartz functions, with arbitrary D. However, the functions on $\mathrm{S}^2 E^*$ that arise from Feynman diagrams are not Schwartz functions. They are rational functions on $\mathrm{S}^2 E^*$ without singularities in $\mathrm{S}^2_+ E^*$. Our first claim is that the D-dimensional integral of such a function makes sense for $\operatorname{Re} D \ll 0$. Our second claim is that the resulting analytic function has a meromorphic continuation to $\mathbb{C}$. Actually, we do not quite need rational functions. We will specify larger classes of functions for which these results hold. In the following, we assume that $n>m$. Otherwise, the map I^D is just the identity map, and there is nothing to prove.

First, we examine the set $\overline{\mathrm{S}}^2_+ E^*$ more closely. We fix an inner product on E, so that the trace of a bilinear form is defined. The trace $\operatorname{tr} A$ is positive for $A\in\overline{\mathrm{S}}^2_+ E^*$, and we can estimate the operator norm (that is, largest eigenvalue) by $\|A\|\le\operatorname{tr} A$. This shows that the set $(\overline{\mathrm{S}}^2_+ E^*)_1$ of positive bilinear forms with trace 1 is compact. Furthermore, we have a homeomorphism

$$\overline{\mathrm{S}}^2_+ E^*\setminus\{0\}\cong(\overline{\mathrm{S}}^2_+ E^*)_1\times\mathbb{R}^*_+, \qquad A\mapsto (A/\operatorname{tr} A, \operatorname{tr} A).$$

Definition 4. Let $f\colon\ \overline{\mathrm{S}}^2_+ E^*\to\mathbb{C}$ be a smooth function and let $a\in\mathbb{R}$. For a multi-index α, let ∂^α be the corresponding constant coefficient differential operator on $\mathrm{S}^2 E^*$. We say that f has *order* $\le a$ iff

$$\partial^\alpha f(A) = O\big((\operatorname{tr} A)^{a+|\alpha|}\big) \qquad \text{for } \operatorname{tr} A\to\infty \text{ and all } \alpha.$$

Let $\mathcal{S}^{(a)}(\overline{\mathrm{S}}^2_+ E^*)$ be the space of functions of order $\leq a$, equipped with its canonical topology.

Clearly, a function on $\overline{\mathrm{S}}^2_+ E^*$ is Schwartz iff it has order a for all $a \in \mathbb{R}$. It is straightforward to see that a rational function of total degree a without singularities in $\overline{\mathrm{S}}^2_+ E^*$ has order $\leq a$. Hence the following theorem applies to the functions that arise from Feynman diagrams.

Theorem 5. *If $f\colon \overline{\mathrm{S}}^2_+ E^* \to \mathbb{C}$ has order $< \infty$, then $I^D(f)$ exists for $\operatorname{Re} D \ll 0$. More precisely, for all $a \in \mathbb{R}$, the distribution $I^D|_C$ extends continuously to $\mathcal{S}^{(a)}(\overline{\mathrm{S}}^2_+ E^*)$ if $\operatorname{Re} D < 2 - 2(a+n^2)/(n-m)$, $C \in \overline{\mathrm{S}}^2_+ F^*$.*

Proof. Let f be a function on $\overline{\mathrm{S}}^2_+ E^*$ of order $\leq a$. Let $\partial(P)$ be the constant coefficient differential operator that figures in (5). Since it has order $n-m$, we have $\partial(P)^k f = O(\operatorname{tr} A)^{a-(n-m)k}$ for all $k \in \mathbb{N}$, and the same estimate for all derivatives of $\partial(P)^k f$. A measurable function on $\overline{\mathrm{S}}^2_+ E^*$ is integrable if it is $O(\operatorname{tr} A)^{-b}$ for some $b > n^2$. As a result, the function $\partial(P)^k f$ and all its derivatives are integrable provided $k > (a+n^2)/(n-m)$. We can extend f to a function $\overline{f}\colon \mathrm{S}^2 E^* \to \mathbb{C}$ with the same decay property. Since the Fourier transform turns differentiation into multiplication by polynomials, we conclude that the Fourier transform of $\partial(P)^k \overline{f}$ has rapid decay (however, we claim nothing about the derivatives of this function!). Hence it can be paired with the Fourier transform of $I^D|_C$ for $\operatorname{Re} D < 2$, which is a locally integrable function with temparate growth by (4). By (5), this defines $I^{D-2k}|_C(f)$ for $\operatorname{Re} D < 2$. As a result, $I^D|_C$ extends continuously to $\mathcal{S}^{(a)}(\overline{\mathrm{S}}^2_+ E^*)$ if $\operatorname{Re} D < 2 - 2(a+n^2)/(n-m)$. □

Definition 6. Let $(f_l)_{l \in \mathbb{Z}}$ be a family of smooth functions on $(\overline{\mathrm{S}}^2_+ E^*)_1$ and let $f\colon \overline{\mathrm{S}}^2_+ E^* \to C$ be a smooth function. Let $\chi\colon \overline{\mathrm{S}}^2_+ E^* \to [0,1]$ be a cutoff function that vanishes for $\operatorname{tr} A \ll 1$ and is identically 1 for $\operatorname{tr} A \gg 1$. We say that

$$\sum_{l \in \mathbb{Z}} f_l(A/\operatorname{tr} A) \cdot (\operatorname{tr} A)^l$$

is an *asymptotic expansion* for f iff $f_l = 0$ for $l \gg 0$ and

$$f(A) - \sum_{l > a} f_l(A/\operatorname{tr} A) \cdot (\operatorname{tr} A)^l \cdot \chi(A) \in \mathcal{S}^{(a)}(\overline{\mathrm{S}}^2_+ E^*)$$

for all $a \in \mathbb{R}$. We call f *symbolic* iff it has such an asymptotic expansion.

The above condition is reminiscent of regularity conditions for symbols of pseudodifferential operators. A symbolic function has order $< \infty$. A Schwartz function is symbolic with asymptotic expansion $f \sim 0$. It is easy to see that a rational function without singularities in $\overline{\mathrm{S}}^2_+ E^*$ is symbolic. Hence the following theorem applies to functions from Feynman diagrams.

Theorem 7. *Let $f\colon \overline{\mathrm{S}}^2_+ E^* \to \mathbb{C}$ be a symbolic function and let $C \in \overline{\mathrm{S}}^2_+ F^*$. Then $I^D|_C(f)$, which is defined for $\operatorname{Re} D \ll 0$, extends to a meromorphic function on $\mathbb{C}$ with only simple poles, which must be in $\mathbb{Z} \cdot 2/(n-m)$.*

Proof. We need the equivariance of I^D with respect to dilations. We define actions of $\mathbb{R}^*_+$ on $\mathcal{S}(\overline{\mathrm{S}}^2_+ E^*)$ and $\mathcal{S}(\overline{\mathrm{S}}^2_+ F^*)$ by $\sigma_t f(B) := f(tB)$. It follows from the defining property (3) that

$$I^D(\sigma_t f) = t^{(n-m)D/2}\sigma_t I^D(f) \qquad \forall D \in \mathbb{C}, f \in \mathcal{S}(\overline{\mathrm{S}}^2_+ E^*).$$

We consider the integrated form of this action to, say, compactly supported distributions h on $\mathbb{R}^*_+$. Using the Laplace transform

$$\hat{h}(s) := \int_0^\infty h(t)\cdot t^s \,\frac{dt}{t}$$

we have

$$I^D(\sigma_h f) = \hat{h}(D\cdot(n-m)/2)\cdot\sigma_h\big(I^D(f)\big).$$

Consider now the action of σ_h on the asymptotic expansion of f! Let

$$f(A) \sim \sum f_l(A/\operatorname{tr}A)\cdot(\operatorname{tr}A)^l \qquad \text{for } A\to\infty.$$

Suppose $f_l = 0$ for $l > a$, so that f has order $\leq a$. Clearly, σ_t multiplies the term $f_l(A/\operatorname{tr}A)(\operatorname{tr}A)^l$ by t^l. As a result, $\sigma_h f$ is again symbolic with asymptotic expansion

$$\sigma_h f \sim \sum \hat{h}(l)\cdot f_l(A/\operatorname{tr}A)\cdot(\operatorname{tr}A)^l.$$

Fix $s\in\mathbb{R}$. Theorem 5 yields $b\in\mathbb{Z}$ such that $I^D(f)$ exists for $\operatorname{Re}D < s$ and f of order $\leq b$. Let h be the compactly supported distribution on $\mathbb{R}^*_+$ with

$$\hat{h}(z) = \prod_{l\in\mathbb{Z}\cap]b,a]}(z-l).$$

Then $\sigma_h f$ has order $\leq b$ because the higher terms in the asymptotic expansion are multiplied by $\hat{h}(l) = 0$. Hence $I^D(\sigma_h f)$ exists for $\operatorname{Re}D < s$. The inverse Laplace transform of $1/h$ is a distribution $\tilde{h}$ on $\mathbb{R}^*_+$ with support contained in $[1,\infty[$. The standard properties of the Laplace transform yield $\sigma_{\tilde{h}}\sigma_h = \sigma_{h*\tilde{h}} = \mathrm{id}$. Using also the equivariance of I^D, we obtain

$$I^D(f) = \hat{h}(D\cdot(n-m)/2)^{-1}\cdot\sigma_{\tilde{h}}\big(I^D(\sigma_h f)\big)$$

for $\operatorname{Re}D \ll 0$. However, the right hand side continues to makes sense for $\operatorname{Re}D < s$. Hence $I^D|_C(f)$ is a meromorphic function for $\operatorname{Re}D < s$ whose poles are simple and contained in the set $\mathbb{Z}\cdot 2/(n-m)$. Since s is arbitrary, the theorem is proved. □

The above construction only yields simple poles. This is because we disregarded divergent subgraphs. The counterterms they produce already depend on D themselves. Therefore, they yield terms of the form $I^D(f_D)$, where f_D may have a pole at d. In this way, higher poles may be created.

REFERENCES

[1] Pavel Etingof. A note on dimensional regularization. In *Quantum Fields and Strings: A Course for Mathematicians*, American Mathematical Society, 2000.

[2] Jacques Fauraut and Adam Korányi. *Analysis on Symmetric Cones*. Oxford University Press, 1994.

MATHEMATISCHES INSTITUT, WESTFÄLISCHE WILHELMS-UNIVERSITÄT MÜNSTER, EINSTEINSTRASSE 62, 48149 MÜNSTER, GERMANY

E-mail address: rameyer@math.uni-muenster.de

URL: http://www.math.uni-muenster.de/u/rameyer