HOMOLOGICAL ALGEBRA FOR SCHWARTZ ALGEBRAS

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Abstract. Let G be a reductive group over a non-Archimedean local field. For two tempered smooth representations, it makes no difference for the Ext-groups whether we work in the category of tempered smooth representations of G or of all smooth representations of G. Similar results hold for certain discrete groups. We explain the basic ideas from functional analysis and geometric group theory that are needed to state this result correctly and prove it.

1. Introduction

These notes are based on my lecture at the conference "Symmetries in Algebra and Number Theory" in Göttingen in October 2008, where I discussed results of [5, 8, 9]. Since details are available in these articles, our presentation will sometimes be informal and limited to the most basic ideas.

First I briefly introduce some categories of representations studied in representation theory, before focussing on the categories of smooth and tempered smooth representations of reductive p-adic groups.

It was observed for such groups that it makes no difference for homological algebra in which of these two categories we work: both $\mathrm{Ext}^*_G(V, W)$ and $\mathrm{Tor}^G_*(V, W)$ agree in both worlds if V and W are tempered smooth representations. Even more is true: the derived category of tempered smooth representations is a full subcategory of the derived category of smooth representations. All this can be deduced from the exactness of a single chain complex, namely, the chain complex that computes $\mathrm{Tor}^{\mathrm{C}^\infty_\mathrm{c}(G)}_*\bigl(\mathcal{S}(G), \mathcal{S}(G)\bigr)$ for the Schwartz algebra $\mathcal{S}(G)$ of G, where $\mathrm{C}^\infty_\mathrm{c}(G)$ denotes the Hecke algebra of G.

The result as stated above is false, however: to get a correct statement we must incorporate some functional analysis into our homological algebra in order to replace tensor products by complete tensor products. I explain this for the Schwartz algebra $\mathcal{S}(\mathbb{Z})$ for the group of integers, where the same problem appears. After the necessary excursion into bornological vector spaces and homological algebra for them, we can correctly state our main result.

Let A be a dense subalgebra of a bornological algebra B. By density, a map between two B-modules is B-linear once it is A-linear. Hence the category of B-modules is always a full subcategory of the category of A-modules. This becomes false when we pass to derived categories. But there are many cases where the canonical functor from the derived category of B-modules to the derived category of A-modules is fully faithful. I first met this phenomenon along the way in [7] and studied it more systematically in [6]. The same phenomenon has been studied under different names in slightly different contexts by other authors, as kindly pointed out to me by Alexei Yu. Pirkovskii and Henning Krause. The first instance I know is the notion of *absolute localisation* in Joseph L. Taylor's work on the functional calculus for several commuting operators on a Banach space ([14]); this notion is formulated for continuous homomorphisms of topological algebras. Another instance is the notion of a *homological epimorphism* introduced by Werner Geigle

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and Helmut Lenzing in [2]; this notion is formulated for homomorphisms of rings and has been applied to the representation theory of finite-dimensional algebras. Amnon Neeman and Andrew Ranicki call such homomorphisms *stably flat* and use them in connection with algebraic K-theory ([10], see also [4]). The same name is used by Alexei Yu. Pirkovskii in [11]. I call such maps *isocohomological* because they preserve cohomology.

The proof that the embedding $\mathrm{C}_\mathrm{c}^\infty(G) \to \mathcal{S}(G)$ of the Hecke algebra into the Schwartz algebra of a reductive p-adic group is isocohomological is based on an idea from geometric group theory. To make this point, I also discuss a similar result for discrete groups from [8], which deals with the group ring $\mathbb{C}[G]$ of a finitely generated discrete group G and a Schwartz algebra $\mathcal{S}(G)$ defined by weighted ℓ_1-estimates. It turns out that the chain complex whose contractibility is crucial for an isocohomological embedding $\mathbb{C}[G] \to \mathcal{S}(G)$ for a discrete group G is a coarse invariant of G. This leads to a recipe for contracting it, based on the notion of a *combing* from geometric group theory.

Reductive p-adic groups have such a combing because they act cocompactly on a CAT(0)-space – their affine Bruhat–Tits building. This led me to prove the result for reductive p-adic groups. After I established results on isocohomological embeddings for Abelian groups in [7], I wanted to extend them to reductive p-adic groups such as $\mathrm{Sl}_n(\mathbb{Q}_p)$ but found this difficult. Therefore, I first worked out a similar problem for discrete groups in [8], in a way that ought to generalise to reductive p-adic groups; then I carried out this generalisation in [9].

In these notes I only sketch some rough ideas of the proof of the result for discrete groups. After this sketch of a proof, I turn to some applications. The first is a vanishing result for certain Ext and Tor-groups for square-integrable representations. Together with results of Peter Schneider and Ulrich Stuhler from [12], this provides a combinatorial formula for the formal dimension of a square-integrable representation, which implies that these dimensions are quantised. As a consequence, the number of square-integrable irreducible representations that contain a U-invariant vector for a compact open subgroup U of G grows at most linearly in $\mathrm{vol}(U)^{-1}$.

2. Categories of representations

Here we discuss some classes of representations that have been studied in representation theory. Later on, we will focus on smooth representations and tempered smooth representations.

The first class of representations to be studied were the finite-dimensional ones. Infinite-dimensional representations came into focus because of quantum mechanics, which required understanding unitary representations of certain Lie groups on Hilbert spaces.

For any locally compact group G, we know one basic example of a unitary representation: the regular representation on the Hilbert space $L_2(G)$, given by $g \cdot f(x) := f(g^{-1}x)$ for all $g, x \in G$, $f \in L_2(G)$. If G is a compact group, then any irreducible representation of G is contained in the regular representation. For non-compact groups, we must restrict to unitary representations – unitarity is automatic for compact groups – and weaken our notion of containment: already for the Abelian group $\mathbb{R}$, irreducible representations are only *weakly* contained in the regular representation. (A unitary representation π of G on a Hilbert space $\mathcal{H}$ is weakly contained in another unitary representation ρ of G if its matrix coefficients $g \mapsto \langle \pi_g \vec{v}, \vec{w} \rangle$ for $\vec{v}, \vec{w} \in \mathcal{H}$ can be approximated locally uniformly by linear combinations of matrix coefficients of ρ.)

All non-compact semi-simple Lie groups have unitary representations that are not weakly contained in the regular representations – the simplest example is the trivial representation of $\mathrm{Sl}(2,\mathbb{R})$. Unitary representations that are weakly contained in the regular representation are called *tempered* unitary representations. The systematic study of the representation theory of Lie groups showed that tempered unitary representations are much easier to classify than general unitary representations – we still lack a complete description of the latter for general semi-simple Lie groups, while the tempered unitary representations are, in principle, classified.

Lie algebra methods are one of the major tools for studying finite-dimensional representations of Lie groups. These do not directly apply to unitary representations on Hilbert space because the Lie algebra is only represented by unbounded operators. But they do apply nicely once we pass to a suitable subset of smooth vectors. Many results in representation theory are established first in this category of smooth representations and then translated to unitary Hilbert space representations.

This lecture mainly deals with reductive p-adic groups instead of Lie groups. A *reductive p-adic group* is a reductive linear algebraic group over a non-Archimedean local field. Their representation theory is remarkably similar to the representation theory of Lie groups. For the following, it suffices to think of basic examples of reductive p-adic groups such as the special linear groups $\mathrm{Sl}_n(\mathbb{Q}_p)$ or $\mathrm{Sl}_n(\mathbb{F}_q[\![t^{-1},t])$ over the fields $\mathbb{Q}_p$ of p-adic integers or over a local function field $\mathbb{F}_q[\![t^{-1},t]$. All groups we consider are locally compact and totally disconnected, that is, their topology has a basis consisting of subsets that are both compact and open. Moreover, the unit element has a neighbourhood basis of compact open subgroups.

Definition 2.1. A representation of a locally compact, totally disconnected group (on a $\mathbb{C}$-vector space) is called *smooth* if each vector is fixed by some open subgroup. Let $\mathrm{Rep}(G)$ denote the category of smooth representations of G.

Example 2.2. Let $G = \mathrm{Sl}_2(\mathbb{Q}_p)$ and let V be the space of locally constant functions on the projective line $\mathbb{P}^1\mathbb{Q}_p$, equipped with the induced action of G. This is a smooth representation of G. The constant function is G-invariant, so that V contains a subrepresentation isomorphic to the trivial representation. The quotient $V/\mathbb{C}\cdot 1$ is a tempered, irreducible smooth representation called the *Steinberg representation* of G. The representation V itself is not tempered because the trivial representation of G is not tempered.

The definition of tempered smooth representations is more subtle. It may seem natural to require the existence of an invariant inner product such that the Hilbert space completion carries a tempered unitary representation. But it is better not to require tempered smooth representations to be unitary at all.

One reason for this is the following *desideratum*: the category of tempered smooth representations should be closed under extensions. In particular, a smooth representation of finite length (that is, one with a finite Jordan–Hölder series) should be tempered if all its irreducible subquotients are tempered. But the property of being unitary is not closed under extensions.

Example 2.3. Represent the group $\mathbb{Z}$ on $\mathbb{C}^2$ by unipotent matrices: $n \mapsto \left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right)$. This representation is a non-trivial extension of the trivial representation by itself and should therefore be tempered because the trivial representation of $\mathbb{Z}$ is a tempered unitary representation. But a unipotent matrix is not unitary for any inner product.

The correct definition of tempered smooth representations uses Schwartz algebras. Smooth representations of a group G may be viewed as non-degenerate modules over the convolution algebra $\mathrm{C}_\mathrm{c}^\infty(G)$ of smooth, compactly supported functions on G. For totally disconnected groups, "smooth" means "locally constant," and $\mathrm{C}_\mathrm{c}^\infty(G)$ is also called the *Hecke algebra* of G. For Lie groups, the above statement is only literally correct if we incorporate some functional analysis into our definitions

– we should study smooth representations and $\mathrm{C}_{\mathrm{c}}^{\infty}(G)$-modules in the category of bornological vector spaces (see [5]).

Tempered representations are defined as modules over a certain completion of $\mathrm{C}_{\mathrm{c}}^{\infty}(G)$, generically called *Schwartz algebra*. The cases $G = \mathbb{Z}$ and $G = \mathbb{R}$ are most familiar: here the Schwartz algebra $\mathcal{S}(G)$ is the convolution algebra of rapidly decreasing functions of Laurent Schwartz. Recall that the Fourier transform provides algebra isomorphisms between $\mathcal{S}(\mathbb{Z})$ and the algebra $\mathrm{C}^{\infty}(\hat{\mathbb{Z}})$ of smooth functions on the Pontrjagin dual $\hat{Z} \cong \mathbb{T}$ of $\mathbb{Z}$ with pointwise multiplication, and between $\mathcal{S}(\mathbb{R})$ with convolution and $\mathcal{S}(\mathbb{R})$ with pointwise product. We will soon use $\mathcal{S}(\mathbb{Z})$ to motivate our results for reductive p-adic groups.

The definition of the Schwartz algebra $\mathcal{S}(G)$ for a reductive p-adic group G is due to Harish–Chandra and involves two ingredients: uniform smoothness and rapid decay. A function on G is *uniformly smooth* if it is U-invariant on the left and on the right for some compact-open subgroup U in G, so that it descends to a function on the double coset space $G /\!/ U$. A uniformly smooth function f on G has *rapid decay* if and only if $f \cdot (\ell + 1)^k \in L_2(G)$ for all $k \in \mathbb{N}$, where ℓ is an appropriate length function on G; for $G = \mathrm{Sl}_n(\mathbb{Q}_p)$, we may take

$$\ell(g) := \log_p \max\{\|g\|_\infty, \|g^{-1}\|_\infty\}$$

where $\|g\|_\infty$ denotes the maximum of the p-adic norms of the matrix entries of g. It is non-trivial that convolutions of uniformly smooth functions of rapid decay are well-defined and have rapid decay.

The above description of $\mathcal{S}(G)$ by L_2-estimtes is due to Marie-France Vignéras [15]. Harish–Chandra defines $\mathcal{S}(G)$ using a weighted supremum norm instead. For uniformly smooth functions, an L_2-estimate is equivalent to suitable weighted L_p-estimates for $p \in [0, \infty]$ because the set $G /\!/ U$ of double cosets – unlike G itself or G/U – has polynomial growth with respect to the length function ℓ.

Roughly speaking, a tempered smooth representation of G is a module over the Schwartz algebra $\mathcal{S}(G)$. But we must modify this definition because as stated above, the derived category of $\mathcal{S}(G)$ does *not* embed into the derived category of $\mathrm{C}_{\mathrm{c}}^{\infty}(G)$.

Before we discuss this problem, we briefly define Schwartz algebras for finitely generated discrete groups. If ℓ is a word-length function on such a group G, we let $f \in \mathcal{S}(G)$ if $f \cdot (\ell + 1)^k \in \ell_1(G)$ for all $k \in \mathbb{N}$, that is,

$$\sum_{g \in G} |f(g)|(\ell(g) + 1)^k < \infty.$$

We use this definition although modules over $\mathcal{S}(G)$ have little to do with tempered unitary representations in general; they are more closely related to uniformly bounded Banach space representations. For groups of rapid decay, the Jolissaint algebra [3] is an interesting alternative to $\mathcal{S}(G)$ that is closely related to tempered unitary representations. But our main results are false for the Jolissaint algebra, that is, its derived category does *not* embed into the derived category of the group ring.

3. Why do we need bornological modules?

Our main result asserts that the derived category of tempered smooth representations of a reductive p-adic group is a full subcategory of the category of all its smooth representations. In particular, if V and W are both tempered smooth representations, then it makes no difference for $\mathrm{Ext}^*_G(V, W)$ and $\mathrm{Tor}^*_G(V, W)$ in which of the two categories of smooth representations we work.

In this section, we explain why this theorem is false and how to rectify it. The issue is that we cannot work in a purely algebraic setting; we must complete tensor products, forcing us to incorporate some functional analysis into our setup.

The problem is the same for reductive p-adic groups and Abelian groups and already appears for the group $G = \mathbb{Z}$. Hence we study this very simple case here. Let $A := \mathbb{C}[\mathbb{Z}]$ and $B := \mathcal{S}(\mathbb{Z})$. The basic input for all homological computations with A-modules is a free A-bimodule resolution of A: once we know such a resolution, we can get free resolutions for all left or right A-modules and compute derived functors.

The algebra A is isomorphic to the algebra $\mathbb{C}[t, t^{-1}]$ of Laurent polynomials. There is a very small free A-bimodule resolution of A,

$$0 \to A \otimes A \xrightarrow{d} A \otimes A \xrightarrow{m} A \to 0,$$

where $d(f \otimes g) := tf \otimes g - f \otimes tg$ and $m(f \otimes g) = f \cdot g$ for $f, g \in \mathbb{C}[t, t^{-1}]$. It is routine to check that the above chain complex is exact.

The basic question is whether it remains exact when we replace A by the completion B. The resulting chain complex

$$0 \to B \otimes B \xrightarrow{d} B \otimes B \xrightarrow{m} B \to 0 \tag{3.1}$$

computes $\operatorname{Tor}^*_A(B, B)$ and should be exact if $\operatorname{Tor}^*_A(B, B) \cong \operatorname{Tor}^*_B(B, B)$. Conversely, if this single chain complex is exact, then $\operatorname{Ext}^n_A(V, W) \cong \operatorname{Ext}^n_B(V, W)$ for all B-modules V and W, similarly for Tor, and the canonical functor from the derived category of B to the derived category of A is fully faithful (see [6]).

Unfortunately, the chain complex in (3.1) is not exact. Clearly, d is injective and m is surjective, but $\ker m$ is bigger than the range of d. To remedy this, we have to complete our tensor products and replace $B \otimes B = \mathcal{S}(\mathbb{Z}) \otimes \mathcal{S}(\mathbb{Z})$ by $B \mathbin{\hat{\otimes}} B := \mathcal{S}(\mathbb{Z} \times \mathbb{Z})$. This is isomorphic by the Fourier transform to $\mathrm{C}^\infty(\mathbb{T}^2)$, and the completed chain complex

$$0 \to \mathrm{C}^\infty(\mathbb{T}^2) \xrightarrow{d} \mathrm{C}^\infty(\mathbb{T}^2) \xrightarrow{m} \mathrm{C}^\infty(\mathbb{T}) \to 0$$

is exact. Here $df(x, y) := (x - y)f(x, y)$ and $mf(x) = f(x, x)$.

Thus we need to modify our setup and consider a category of modules where the algebraic tensor product is replaced by a completed tensor product with $B \mathbin{\hat{\otimes}} B = \mathcal{S}(\mathbb{Z} \times \mathbb{Z})$.

The most familiar choice is the category of complete locally convex topological vector spaces with the complete projective topological tensor product. While this may still work well enough for the example $\mathbb{Z}$, we get serious problems for the Schwartz algebra $\mathcal{S}(G)$ of a reductive p-adic group G because the product in $\mathcal{S}(G)$ is not jointly continuous. To accomodate this, we may follow [1] and use the complete inductive topological tensor product, which is designed to be universal for separately continuous bilinear maps. But this tensor product behaves rather badly for general locally convex topological vector spaces – even associativity is unclear. Everything works fine if we also restrict attention, say, to the category of nuclear LF-spaces. This is enough to cover $\mathcal{S}(G)$ and the modules we need.

There is, however, a better way to combine functional analysis with homological algebra, where the problems with completed tensor products mentioned above disappear. In other words, these problems are mere artefacts produced by unsuitable definitions. Instead of complete locally convex topological vector spaces, we should use complete convex bornological vector spaces. A bornology on a vector space is a collection of bounded subsets with certain properties – thus bounded subsets replace open subsets in bornological analysis. Correspondingly, boundedness replaces continuity for linear and bilinear maps. Any complete convex bornological vector space is an inductive limit of Banach spaces in a natural way.

The category of complete bornological vector spaces has very good algebraic properties. For instance, the complete projective tensor product in this category and the internal Hom functor are related by the familiar adjointness isomorphism

$$\operatorname{Hom}\bigl(X, \operatorname{Hom}(Y, Z)\bigr) \cong \operatorname{Hom}(X \hat{\otimes} Y, Z).$$

Any vector space carries a canonical bornology, called the *fine bornology*. Its bounded subsets are the bounded subsets of finite-dimensional subspaces. This defines a fully faithful, fully exact embedding of the category of vector spaces into the category of complete bornological vector spaces. Furthermore, the embedding is symmetric monoidal, that is, compatible with (complete) tensor products. Roughly speaking, nothing happens when we equip a vector space with the fine bornology.

Topological vector spaces also carry canonical bornologies. The most useful choice is the precompact bornology, consisting of all precompact subsets. This defines a fully faithful, fully exact, symmetric monoidal embedding of the category of Fréchet spaces into the category of complete bornological vector spaces. The precompact bornology also defines such an embedding on the category of nuclear LF-spaces. On this category, the complete projective bornological tensor product agrees with the complete inductive topological tensor product – exactly what we need. Thus $\mathcal{S}(G) \hat{\otimes} \mathcal{S}(G) \cong \mathcal{S}(G \times G)$ for any reductive p-adic group G.

We now modify our definitions to allow smooth representations on (complete, convex) bornological vector spaces; from now on, all bornological vector spaces are requird complete and convex. A group representation of a locally compact, totally disconnected group G on a bornological vector space V is called *smooth* if, for each bounded subset S, there is a compact-open subgroup U of G with $u \cdot \vec{v} = \vec{v}$ for all $u \in U$, $\vec{v} \in S$. The category of smooth representations of G on bornological vector spaces is isomorphic to the category of essential bornological $\mathrm{C}_\mathrm{c}^\infty(G)$-modules (a bornological A-module is essential if the multiplication map $A \hat{\otimes} M \to M$ is a bornological quotient map).

The *tempered smooth representations* of G are the essential $\mathcal{S}(G)$-modules, where we equip $\mathcal{S}(G)$ with the obvious bornology: a subset of $\mathcal{S}(G)$ is bounded if its elements are uniformly smooth and of uniformly rapid decay.

When doing homological algebra in this setting, it is better to replace exactness by a stronger condition, namely, the existence of a bounded contracting homotopy. Otherwise, we would get a complicated derived category even for the trivial algebra $\mathbb{C}$ because there are non-trivial extensions of bornological vector spaces. To get rid of these complications, we do *relative homological algebra* with respect to the category of bornological vector spaces. Formally, this means that we turn categories of bornological vector spaces and modules into *exact categories* in the sense of Quillen. These exact categories have enough injective and enough projective objects, so that homological algebra works as usual. In particular, we can form derived categories. The main point to remember is that resolutions are required to have a bounded contracting homotopy.

We can now state our main theorem:

Theorem 3.2. *The embedding $\mathrm{C}_\mathrm{c}^\infty(G) \to \mathcal{S}(G)$ induces a fully faithful functor between the derived categories of non-degenerate bornological modules over $\mathrm{C}_\mathrm{c}^\infty(G)$ and $\mathcal{S}(G)$. In particular, if both V and W are essential $\mathcal{S}(G)$-modules, then*

$$\operatorname{Ext}^n_{\mathrm{C}_\mathrm{c}^\infty(G)}(V, W) \cong \operatorname{Ext}^n_{\mathcal{S}(G)}(V, W) \quad \textit{and} \quad \operatorname{Tor}_n^{\mathrm{C}_\mathrm{c}^\infty(G)}(V, W) \cong \operatorname{Tor}_n^{\mathcal{S}(G)}(V, W).$$

Various equivalent conditions for such an embedding of derived categories are given in [6]. The one that is practical to check is the following:

Theorem 3.3. *Let A and B be bornological algebras and let $f\colon A \to B$ be an essential bounded algebra homomorphism; that is, B is essential as an A-bimodule.*

Let $P_\bullet \to A$ be a projective A-bimodule resolution of A. The functor between the derived categories of essential bornological modules over A and B induced by f is fully faithful if and only if $B \hat{\otimes}_A P_\bullet \hat{\otimes}_A B \to B$ has a bounded contracting homotopy.

We also call f *isocohomological* if this is the case. The argument above shows that the embedding $\mathbb{C}[\mathbb{Z}] \to \mathcal{S}(\mathbb{Z})$ is isocohomological. Our main theorem states that the embedding $\mathrm{C}_\mathrm{c}^\infty(G) \to \mathcal{S}(G)$ for a reductive p-adic group is isocohomological.

The proof of Theorem 3.3 is not hard. Only the sufficiency of the condition is relevant here. The main point is that if $B \hat{\otimes}_A P_\bullet \hat{\otimes}_A B \to B$ has a bounded contracting homotopy, then it is a projective B-bimodule resolution of B. Hence all derived functors for B can be computed using this bimodule resolutions. When we compare them with similar computations for A, we get the same answer because the two resolutions are so closely related.

4. Some ideas from the proof of the main theorem

We have now stated our main theorem correctly and established an analogous result for the discrete group $\mathbb{Z}$. To find a proof for reductive p-adic groups, we identify a geometric property for discrete groups related to non-negative curvature that ensures that the embedding $\mathbb{C}[G] \to \mathcal{S}(G)$ is isocohomological.

What does "geometric" mean here? Any finitely generated group G becomes a metric space with respect to a word-length function. This metric is not unique, but it is unique up to quasi-isometry. Moreover, it frequently happens that a group G is quasi-isometric to a nice geometric object like a smooth manifold. In general, if G acts cocompactly, properly, and by isometries on a metric space X, then G and X are quasi-isometric. For instance, the fundamental group of a Riemannian manifold is quasi-isometric to the universal covering of this manifold.

Any reductive p-adic group G acts cocompactly and properly on a nice geometric space – its affine Bruhat–Tits building. It is known that such buildings have non-positive curvature – formally, they have the CAT(0)-property that triangles in them are thinner than comparison triangles in flat Euclidean space. Hence a geometric non-positive curvature condition covers reductive p-adic groups as well.

It is shown in [8] that the question whether or not the embedding $\mathbb{C}[G] \to \mathcal{S}(G)$ is isocohomological for a discrete group G depends only on the quasi-isometry type of G. This can be formalised as follows: to any metric space we may associate a certain chain complex that is a quasi-isometry invariant, and for the underlying metric space of a finitely generated discrete group G, this chain complex has a bounded contracting homotopy if and only if the embedding $\mathbb{C}[G] \to \mathcal{S}(G)$ is isocohomological.

The chain complex in question is a certain completion of the reduced bar complex on X. More precisely, we take the reduced chain complex $C_\bullet(X)$ of the simplicial set with X^{n+1} as its set of n-simplices, and face and degeneracy maps deleting or adding one of the entries. The completion involves chains with controlled support – there is $R > 0$ such that $f(x_0, \dots, x_n)$ vanishes if $d(x_i, x_j) > R$ for some i, j – and with rapid decay in the sense that the function $(\ell(x_0)+1)^k f(x_0, \dots, x_n)$ on X^{n+1} is absolutely summable for each $k \in \mathbb{N}$. For a discrete group G, this chain complex has a bounded contracting homotopy if and only if the embedding $\mathbb{C}[G] \to \mathcal{S}(G)$ is isocohomological. Thus our task is to find a contracting homotopy for the uncompleted chain complex $C_\bullet(X)$ that is bounded with respect to the appropriate bornology and therefore extends to the completion.

The reduced bar complex above is huge and may therefore appear impractical for cohomology computations. Nevertheless, it is ideal for our current purpose because of its good functoriality properties. By design, the bar construction makes sense for any discrete set X, and any map $f\colon X \to Y$ induces a chain map. If $f, g\colon X \to Y$

are two maps, then the induced chain maps are chain homotopic – simply because the bar complex is contractible. But there is, in fact, an explicit and simple formula for a chain homotopy between the chain maps induced by f and g.

In particular, this chain homotopy for the identity map and a constant map provides a contracting homotopy for $C_\bullet(X)$. This is just the standard contracting homotopy of the reduced bar complex $C_\bullet(X)$, and it is unbounded for the relevant bornology. To get a bounded contracting homotopy, we must contract our group more gently: we need a sequence of maps $f_n\colon X \to X$ that, roughly speaking, move each $x \in X$ to the base point 0 in small steps; typically, we just let $f_n(x)$ be points on a quasi-geodesic in X from x to 0; "moving in small steps" means that there is $S > 0$ with $d\bigl(f_n(x), f_{n+1}(x)\bigr) < R$ for all $n \in \mathbb{N}$, $x \in X$.

Given such a sequence of maps (f_n), we sum up the canonical chain homotopies between the chain maps induced by the f_n and hope that the sum remains bounded. This requires further geometric conditions. Since our homotopy must preserve controlled supports, we need that $f_n(x)$ and $f_n(y)$ remain close for all $n \in \mathbb{N}$ if x and y are close; more precisely, for each $R > 0$ there is $S > 0$ such that if $d(x, y) < R$, then $d\bigl(f_n(x), f_n(y)\bigr) < S$ for all $n \in \mathbb{N}$. A sequence of maps (f_n) with the two properties described above is called a (synchronous) *combing* on G. In addition, to preserve the rapid decay condition, we need that the number of $n \in \mathbb{N}$ with $f_n(x) \neq f_{n+1}(x)$ grows at most polynomially in $\ell(x) = d(x, 0)$; if the points $f_n(x)$ follow a quasi-geodesic, then this number automatically grows linearly (polynomial growth rules out some groups that only admit more complicated combings where each element follows a huge detour). We can now formulate the main result of [8]:

Theorem 4.1. *Let G be a finitely generated discrete group that has a combing of polynomial growth. Then the embedding $\mathbb{C}[G] \to \mathcal{S}(G)$ is isocohomological.*

In a Riemannian metric, geodesics remain close to each other if the curvature is non-positive. For instance, this happens in flat $\mathbb{R}^n$. Since the group $\mathbb{Z}^n$ with word-length metric is quasi-isometric to $\mathbb{R}^n$, the group $\mathbb{Z}^n$ has a combing as well. This explains why the embedding $\mathbb{C}[\mathbb{Z}^n] \to \mathcal{S}(\mathbb{Z}^n)$ is isocohomological. Another class of groups that admit combings are hyperbolic groups, fundamental groups of non-positively curved manifolds, and cocompact lattices in Lie groups – the latter act cocompactly and properly on non-positively curved Riemannian manifolds.

For discrete groups, categories of representations are usually not well-behaved, so that Ext- and Tor-groups for them are usually hard to compute. The most interesting case of Theorem 4.1 seems to concern the trivial representation. The groups $\mathrm{Ext}^*_{\mathbb{C}[G]}(\mathbb{C}, \mathbb{C})$ are group cohomology, while $\mathrm{Ext}^*_{\mathcal{S}(G)}(\mathbb{C}, \mathbb{C})$ are group cohomology with polynomial growth; that is, we take the standard bar complex computing group cohomology and take the cohomology of the subcomplex of cochains of polynomial growth. Theorem 4.1 implies that both chain complexes are homotopy equivalent if G has a combing of polynomial growth. In particular, every class in the group cohomology is represented by a cocycle of polynomial growth.

Any reductive p-adic group acts cocompactly on its affine Bruhat–Tits building, which is another example of a non-positively curved space – formally, a $\mathrm{CAT}(0)$-space. Hence reductive p-adic groups have combings of linear growth. This suggested to me that Theorem 4.1 should remain true for reductive p-adic groups. In fact, I checked this for $\mathrm{Sl}_2(\mathbb{Q}_p)$ by hand and added a remark to this extent in the introduction of [8]. This intrigued Peter Schneider who had tried to prove such a statement but hit the problem described in §3 and concluded that the statement was false.

As expected, the proof of Theorem 4.1 carries over to reductive p-adic groups. But two new problems appear that still requires a significant amount of additional work.

One issue is that the trivial representation of a reductive p-adic group is not tempered. In the discrete case, the trivial representation is always a module over $\mathcal{S}(G)$ because the latter is defined using ℓ_1-estimates. This allows some simplification in the chain complexes to be considered. The chain complex described involves chains on X^n that are compactly supported in all but one direction. To treat reductive p-adic groups, we must allow functions that are compactly supported in all but *two* directions, and that satisfy a certain growth condition in the other two directions.

A second issue is the uniform smoothness in the definition of the Schwartz algebra. The contracting homotopy must be constructed more carefully in order to preserve this property, and the argument requires results of Bruhat and Tits about stabilisers of points in the building.

5. Some applications

In this section, we let G be a semi-simple p-adic group such as $\mathrm{Sl}_n(\mathbb{Q}_p)$ or $\mathrm{Sl}_n(\mathbb{F}_q[\![t^{-1},t]\!])$. An extension to reductive groups such as $\mathrm{Gl}_n(\mathbb{Q}_p)$ or $\mathrm{Gl}_n(\mathbb{F}_q[\![t^{-1},t]\!])$ is possible but requires more notation, which I do not want to introduce here.

What can we learn from our main result that the embedding $\mathrm{C}_\mathrm{c}^\infty(G) \to \mathcal{S}(G)$ is isocohomological? First of all, this implies that the subcategory of tempered smooth representations is closed under extensions in the category of all smooth representations because Ext^1 is the same in both categories. We already observed this important property of the category of tempered smooth representations in §2. Example 2.3 shows that the categories of unitary representations or of uniformly bounded Banach space representations are not closed under extensions. Hence the embedding $\mathbb{C}[\mathbb{Z}] \to \ell_1(\mathbb{Z})$ is not isocohomological.

Another important general consequence is that $\mathcal{S}(G)$ has a projective bimodule resolution of finite length – the same length as for $\mathrm{C}_\mathrm{c}^\infty(G)$. This means that derived functors for $\mathcal{S}(G)$ vanish above some dimension.

In principle, this projective bimodule resolution can be used to compute the Hochschild and cyclic homology of $\mathcal{S}(G)$. While such a computation for $\mathrm{C}_\mathrm{c}^\infty(G)$ is feasible, the case of $\mathcal{S}(G)$ is considerably more complicated because it requires careful estimates about the growth of the length function on conjugacy classes. It is known that $\mathrm{C}_\mathrm{c}^\infty(G)$ and $\mathcal{S}(G)$ have isomorphic periodic cyclic homology ([13]), but it seems very hard to prove this using the isocohomological embedding. There is no general theorem to this effect.

Square-integrable representations are special tempered representations that are isolated among tempered representations. That is, they are both projective and injective in the category of tempered smooth representations. Hence $\mathrm{Ext}^n_{\mathcal{S}(G)}(V,W)$ vanishes for $n \neq 0$ if V or W is square-integrable and the other one tempered. Our main theorem yields the same for $\mathrm{Ext}^n_{\mathrm{C}_\mathrm{c}^\infty(G)}(V,W)$. For instance, this applies if V is the Steinberg representation of $\mathrm{Sl}_2(\mathbb{Q}_p)$ (see Example 2.2). This vanishing result is remarkable because, by its very definition, the Steinberg representation V has a non-trivial extension by the trivial representation $\mathbb{C}$, so that $\mathrm{Ext}^1_{\mathrm{C}_\mathrm{c}^\infty(G)}(V,\mathbb{C}) \neq 0$.

Vanishing results for square-integrable representations are particularly important because these are the atoms of the Plancherel measure on the set of irreducible representations of G. Moreover, since $\mathrm{Ext}^n_{\mathrm{C}_\mathrm{c}^\infty(G)}(V,W)$ vanishes unless V and W have the same central character, the support of the function $V \mapsto \dim \mathrm{Ext}^n_{\mathrm{C}_\mathrm{c}^\infty(G)}(V,W)$ is always a finite set. Hence this function vanishes almost everywhere if W is tempered and $n \neq 0$. In contrast, the function $V \mapsto \dim \mathrm{Ext}^1_{\mathrm{C}_\mathrm{c}^\infty(\mathrm{Sl}_2 \mathbb{Q}_p)}(V,\mathbb{C})$ does not vanish almost everywhere because it is non-zero at the Steinberg representation, which is an atom of the Plancherel measure.

Another interesting application is the quantisation of formal dimensions of square-integrable representations:

Theorem 5.1. *There is $\alpha > 0$ such that the formal dimension of any irreducible square-integrable representation of G belongs to $\alpha \cdot \mathbb{N}_{\geq 1}$.*

This implies a bound on the number of irreducible square-integrable representations that contain U-fixed vectors for some compact-open subgroup U: any such representation is contained in $L_2(G/U)$, which has formal dimension $\mathrm{vol}(U)^{-1}$. Since formal dimensions are additive and positive, we conclude that there are at most $1/(\alpha \operatorname{vol} U)$ square-integrable representations that contain U-fixed vectors.

How is our main result related to formal dimensions? The following conceptual explanation is taken from [9]. The ring $\mathrm{C}_\mathrm{c}^\infty(G)$ is a regular Noetherian ring, that is, any finitely generated $\mathrm{C}_\mathrm{c}^\infty(G)$-module has a resolution of finite length by finitely generated projective modules. Hence any finitely generated $\mathrm{C}_\mathrm{c}^\infty(G)$-module determines a class in the algebraic K-theory of $\mathrm{C}_\mathrm{c}^\infty(G)$: take the Euler characteristic $\sum_{n=0}^\infty (-1)^n [P_n]$ of a finite type projective resolution $P_\bullet$. Since the formal dimension of representations defines a linear map $\mathrm{K}_0(\mathrm{C}_\mathrm{c}^\infty G) \to \mathbb{R}$, this yields a notion of formal dimension for all finitely generated $\mathrm{C}_\mathrm{c}^\infty(G)$-modules.

There seems to be no general formula for such a resolution for a general finitely generated module. But if we restrict to representations of finite length, then Peter Schneider and Ulrich Stuhler construct an explicit projective resolution in [12]. This can be used to compute the formal dimension mentioned above and shows that it is quantised. But it is not clear whether this new notion of formal dimension agrees with the usual one that is based on traces on group von Neumann algebras.

This is exactly where our main theorem is needed. It implies that a projective $\mathrm{C}_\mathrm{c}^\infty(G)$-module resolution of an $\mathcal{S}(G)$-module remains a resolution when we base change to $\mathcal{S}(G)$. If we start with a square-integrable representation V, then V is a projective $\mathcal{S}(G)$-module. Hence the resolution $\mathcal{S}(G) \hat{\otimes}_{\mathrm{C}_\mathrm{c}^\infty(G)} P_\bullet$ of V must split, so that $[V] = \sum_{n=0}^\infty (-1)^n [\mathcal{S}(G) \hat{\otimes}_{\mathrm{C}_\mathrm{c}^\infty(G)} P_n]$ in $\mathrm{K}_0(\mathcal{S}G)$. This implies that the combinatorial formal dimension computed using the class of V in $\mathrm{K}_0(\mathrm{C}_\mathrm{c}^\infty G)$ agrees with the usual formal dimension.

6. Conclusion

We have seen that the derived category of tempered smooth representations of a reductive p-adic group G is a full subcategory of the derived category of all smooth representations G. To achieve this, we had to incorporate some functional analysis into our categories of modules. The proof of this result is inspired by the proof of a similar result for discrete groups that uses ideas from geometric group theory.

This comparison result is useful in two ways: it shows that the modules over the Schwartz algebra have reasonably simple projective resolutions because this happens over the Hecke algebra. And it shows that Ext and Tor vanish for tempered representations if one of them is square-integrable. We have also seen one consequence – the quantisation of formal dimensions of square-integrable representations – whose statement does not involve any homological algebra.

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