THE MAYER-VIETORIS SEQUENCE FOR GENERALIZED HOMOLOGY THEORIES

RALF MEYER

ABSTRACT. We give a simple proof that generalized homology theories on categories of algebras have a long exact Mayer-Vietoris sequence computing the homology of a pull-back.

1. INTRODUCTION

Let $(F_n)_{n\in\mathbb{N}}$ be an abstract homology theory on a suitable categories of algebras, for instance, the category of C^*-algebras. That is, the $(F_n)_{n\in\mathbb{N}}$ are homotopy functors, and there are long exact sequences for all extensions of C^*-algebras. The *Mayer-Vietoris sequence* is a long exact sequence relating the homologies of the entries of a pull-back diagram. Recall that a commuting diagram

(1)
$$\begin{tikzcd} P \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & A_2 \arrow[d, "\alpha_2"] \\ A_1 \arrow[r, "\alpha_1"'] & B \end{tikzcd}$$

is called a *pull-back diagram* iff for all commuting diagrams

(2)
$$\begin{tikzcd} P' \arrow[dr, dotted] \arrow[drr, bend left, "\pi_2'"] \arrow[ddr, bend right, "\pi_1'"'] & & \\ & P \arrow[r, "\pi_2"'] \arrow[d, "\pi_1"] & A_2 \arrow[d, "\alpha_2"] \\ & A_1 \arrow[r, "\alpha_1"'] & B \end{tikzcd}$$

there is a unique morphism $P' \to P$ for which the diagram remains commutative. The pull-back diagram for $A_1 \to B \leftarrow A_2$ is unique up to isomorphism if it exists. In the category of C^*-algebras, the pull-back can be described explicitly:

(3)
$$P = \{(a_1, a_2) \in A_1 \times A_2 \mid \alpha_1(a_1) = \alpha_2(a_2)\},$$

and $\pi_j \colon P \to A_j$ is the projection onto the corresponding direct factor.

The *Mayer-Vietoris sequence* associated to a pull-back diagram is of the form

(4)
$$F_0(B) \xleftarrow{(\alpha_{1*},\ -\alpha_{2*})} F_0(A_1) \oplus F_0(A_2) \xleftarrow{\begin{pmatrix}\pi_{1*}\\ \pi_{2*}\end{pmatrix}} F_0(P) \xleftarrow{\delta}$$
$$F_1(B) \xleftarrow{(\alpha_{1*},\ -\alpha_{2*})} F_1(A_1) \oplus F_1(A_2) \xleftarrow{\begin{pmatrix}\pi_{1*}\\ \pi_{2*}\end{pmatrix}} F_1(P) \xleftarrow{\delta} F_2(B) \leftarrow \cdots$$

Date: November 20, 2000.
2000 *Mathematics Subject Classification.* 46L85, 46M20.
Key words and phrases. generalized homology theory, pull-back, exact sequence.

for suitable connecting maps δ. It is a result of Schochet [2] that this sequence is exact except possibly at $F_0(A_1) \oplus F_0(A_2)$ and $F_0(B)$ if α_1 or α_2 is surjective.

However, Schochet's proof is quite complicated. In fact, he only treats the special case where both α_1 and α_2 are surjective or one of them is a cofibration. We are going to present a very simple and elegant proof of the exactness of (4).

The Mayer-Vietoris sequence is an important tool for computations. Obviously, it is useful to determine (F_n) for commutative C^*-algebras and more generally for algebras that are obtained by gluing together simpler building blocks. Another less obvious application is to compute the homology of inductive or projective limits (see, for instance, [2] and [1]).

The basic ideas of our proof can be explained quickly. Suppose that α_2 is surjective. It is a general property of pull-backs that the kernels of α_2 and π_1 are canonically isomorphic. Let J be this kernel. We obtain a commuting diagram

$$(5) \qquad \begin{tikzcd} J \arrow[r, "j"] \arrow[d, equal] & P \arrow[r, "\pi_1"] \arrow[d, "\pi_2"] & A_1 \arrow[d, "\alpha_1"] \\ J \arrow[r, "\pi_2 \circ j"'] & A_2 \arrow[r, "\alpha_2"'] & B \end{tikzcd}$$

in which the rows are short exact sequences. Hence the rows yield long exact sequences of homology groups, which we view as chain complexes K_1 and K_2. The vertical arrows in (5) give rise to a chain map $f\colon K_1 \to K_2$. The crucial idea is to consider the *mapping cone* C_f. The complex C_f is closely related to (4). The difference is that C_f also contains contributions from J. However, since the vertical map $J \to J$ in (5) is an isomorphism, these contributions generate a *contractible* subcomplex. Therefore, C_f is chain homotopic to the complex (4). Since K_1 and K_2 are exact except possibly in degree 0, the mapping cone is exact except possibly in the two lowest degrees. Therefore, (4) is exact except possibly at $F_0(B)$ and $F_0(A_1) \oplus F_0(A_2)$.

The *only* thing we need to know about F is that it has natural long exact sequences for the extensions $J \to P \to A_1$ and $J \to A_2 \to B$ because the proof only manipulates these long exact sequences. Therefore, our proof works in very general circumstances.

2. Proof of the exactness of the Mayer-Vietoris sequence

First, we describe the setup of the proof. Let $\mathcal{C}$ be a category with a *zero object* Z. That is, for all $A \in \mathcal{C}$, there are *unique* morphisms $Z \to A$ and $A \to Z$. Hence for all objects A, B of $\mathcal{C}$ there is a distinguished morphism $A \to Z \to B$ called the *zero map*. For instance, $\mathcal{C}$ may be the category of C^*-algebras, of topological algebras, or of algebras without additional structure, or $\mathcal{C}$ may be the opposite category of the category of pointed topological spaces.

The *kernel* of a morphism $f\colon A \to B$ is the pull-back of $A \to B \leftarrow Z$, where Z is the zero object. That is, $\ker f$ comes with a canonical map $j\colon \ker f \to A$ and if $g\colon C \to A$ is a morphism for which $f \circ g$ is the zero map, then there is a unique morphism $g'\colon C \to \ker f$ with $g = j \circ g'$.

Let $(F_n)_{n \in \mathbb{N}}$ be a sequence of functors from $\mathcal{C}$ to some Abelian category, for instance the category of Abelian groups or its opposite category.

Let $\mathcal{P}$ be a class of morphisms in $\mathcal{C}$—called *admissible projections*—with the following property. Any admissible projection $f\colon A \to B$ has a kernel $j\colon J \to A$, and there is a *natural* long exact sequence

$$(6) \qquad \cdots \to F_2(B) \xrightarrow{\partial} F_1(J) \xrightarrow{j_*} F_1(A) \xrightarrow{f_*} F_1(B) \xrightarrow{\partial} F_0(J) \xrightarrow{j_*} F_0(A) \xrightarrow{f_*} F_0(B)$$

with suitable *connecting maps* $\partial\colon F_{n+1}(B) \to F_n(J)$ for all $n \in \mathbb{N}$. The naturality assumption is crucial. It insures that the diagram (7) below commutes.

For instance, if (F_n) is topological K-theory for Fréchet algebras [1], then $\mathcal{P}$ can be taken as the class of all quotient maps. If (F_n) is KK-theory for C^*-algebras, then $\mathcal{P}$ can be taken as the class of all quotient maps with a completely positive linear section. If (F_n) is periodic cyclic cohomology for, say, topological algebras, then $\mathcal{P}$ can be taken as the class of all quotient maps with a continuous linear section. In all these cases, the class $\mathcal{P}$ is closed under pull-backs by Remark 4. That is, $\alpha_2 \in \mathcal{P}$ implies $\pi_1 \in \mathcal{P}$ in a pull-back diagram (1).

Theorem 1. *Suppose that the morphisms $\alpha_2\colon A_2 \to B$ and $\pi_1\colon P \to A_1$ in a pull-back diagram are admissible projections. Then the Mayer-Vietoris sequence* (4) *is exact except possibly at $F_0(B)$ and $F_0(A_1) \oplus F_0(A_2)$.*

Of course, if $\mathcal{P}$ is closed under pull-backs, it suffices to assume that $\alpha_2 \in \mathcal{P}$.

Proof. Let $j\colon J \to P$ be a kernel of π_1. Consider the diagrams

$$\begin{tikzcd} J \arrow[r, "j"] \arrow[d, "0"] & P \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & A_2 \arrow[d, "\alpha_2"] \\ Z \arrow[r, "0"'] & A_1 \arrow[r, "\alpha_1"'] & B, \end{tikzcd} \qquad \begin{tikzcd} J \arrow[r, "\pi_2\circ j"] \arrow[d, "0"'] & A_2 \arrow[d, "\alpha_2"] \\ Z \arrow[r, "0"'] & B. \end{tikzcd}$$

Since the two squares in the first diagram are pull-backs, the second diagram is a pull-back as well. That is, the map $\pi_2 \circ j\colon J \to A_2$ is a kernel for α_2.

We use the following abbreviations to avoid cluttered diagrams. We omit brackets, writing F_nB, F_nP, F_nA_1, and F_nA_2 instead of $F_n(B)$, etc.. We use the same names π_1, π_2, α_1, α_2, j to denote the induced maps $F_nP \to F_nA_1$, etc..

Since π_1 and α_2 are admissible projections, (5) yields a commutative diagram

$$\begin{tikzcd} F_0A_1 \arrow[d, "\alpha_1"] & F_0P \arrow[l, "\pi_1"'] \arrow[d, "\pi_2"] & F_0J \arrow[l, "j"'] \arrow[d, equal] & F_1A_1 \arrow[l, "\partial"'] \arrow[d, "\alpha_1"] & F_1P \arrow[l, "\pi_1"'] \arrow[d, "\pi_2"] & F_1J \arrow[l, "j"'] \arrow[d, equal] & \cdots \arrow[l] \\ F_0B & F_0A_2 \arrow[l, "\alpha_2"] & F_0J \arrow[l, "\pi_2\circ j"] & F_1B \arrow[l, "\partial"] & F_1A_2 \arrow[l, "\alpha_2"] & F_1J \arrow[l, "\pi_2\circ j"] & \cdots \arrow[l] \end{tikzcd} \tag{7}$$

Consider the rows in (7) as chain complexes K_1, K_2 and the vertical arrows as a chain map $f\colon K_1 \to K_2$. Let C_f be its *mapping cone*. That is, C_f is the complex

$$F_0B \leftarrow F_0A_1 \oplus F_0A_2 \leftarrow F_0P \oplus F_0J \leftarrow F_0J \oplus F_1B \leftarrow F_1A_1 \oplus F_1A_2 \leftarrow \\ F_1P \oplus F_1J \leftarrow F_1J \oplus F_2B \leftarrow F_2A_1 \oplus F_2A_2 \leftarrow F_2P \oplus F_2J \leftarrow \cdots$$

with boundary maps

$$\begin{pmatrix} -\pi_1 & 0 \\ \pi_2 & \pi_2 j \end{pmatrix}\colon F_nP \oplus F_nJ \to F_nA_1 \oplus F_nA_2,$$
$$\begin{pmatrix} -j & 0 \\ \mathrm{id} & \partial \end{pmatrix}\colon F_nJ \oplus F_{n+1}B \to F_nP \oplus F_nJ,$$
$$\begin{pmatrix} -\partial & 0 \\ \alpha_1 & \alpha_2 \end{pmatrix}\colon F_nA_1 \oplus F_nA_2 \to F_{n-1}J \oplus F_nB.$$

Since α_2 and π_1 are admissible projections, the rows in (7) are exact. That is, $H_n(K_1) = H_n(K_2) = 0$ for all $n > 0$. The mapping cone long exact sequence

$$\cdots \to H_2(K_1) \to H_2(K_2) \to H_2(C_f) \to H_1(K_1) \to H_1(K_2) \to H_1(C_f) \to \cdots$$

implies that $H_n(C_f) = 0$ for all $n \geq 2$. That is, C_f is exact except possibly at F_0B and $F_0A_1 \oplus F_0A_2$. To obtain (4), it remains to get rid of the summands F_nJ in C_f. We do this by dividing out a contractible subcomplex.

Let $N \subseteq C_f$ be the direct sum of the summands $F_nJ \subseteq F_nJ \oplus F_{n+1}B$, $n \in \mathbb{N}$. The restriction of the boundary map of C_f to N is an isomorphism onto

$$N' := \{(x, y) \in F_nP \oplus F_nJ \mid x = -j(y)\}.$$

That is, $N \oplus N' \subseteq C_f$ is a contractible subcomplex. Hence the quotient complex $C' := C_f/(N + N')$ is chain homotopic to C_f, so that $H_n(C') = 0$ for all $n \geq 2$ as well. We may identify $(F_nP \oplus F_nJ)/N'$ with F_nP via the maps $F_nP \to F_nP \oplus F_nJ$, $x \mapsto (x, 0)$, and $F_nP \oplus F_nJ \to F_nP$, $(x, y) \mapsto x + j(y)$. Similarly, we identify $(F_nJ \oplus F_{n+1}B)/N \cong F_{n+1}B$ in the obvious way. Then the complex C' becomes

$$F_0B \leftarrow F_0A_1 \oplus F_0A_2 \leftarrow F_0P \leftarrow F_1B \leftarrow F_1A_1 \oplus F_1A_2 \leftarrow F_1P \leftarrow F_2B \leftarrow \cdots$$

with the boundary maps

$$\begin{aligned} \begin{pmatrix} -\pi_1 \\ \pi_2 \end{pmatrix} &: F_nP \to F_nA_1 \oplus F_nA_2, \\ j \circ \partial &: F_{n+1}B \to F_nP, \\ \begin{pmatrix} \alpha_1 & \alpha_2 \end{pmatrix} &: F_nA_1 \oplus F_nA_2 \to F_nB. \end{aligned}$$

Since $H_n(C') = 0$ for all $n \geq 2$, this sequence is exact except possibly at F_0B and $F_0A_1 \oplus F_0A_2$. Up to signs, the boundary maps above agree with those in (4). In order to get the right signs, we have to replace π_1 and α_2 by $-\pi_1$ and $-\alpha_2$ in (7). Since this change does not destroy the commutativity of (7), the above argument still goes through and yields the desired conclusion. □

3. Concluding remarks

Remark 2. The proof shows that the map δ in (4) is the composition

$$F_{n+1}(B) \xrightarrow{\partial} F_n(\ker \alpha_2) \xrightarrow{\cong} F_n(\ker \pi_1) \xrightarrow{j_*} F_n(P).$$

Remark 3. Theorem 1 holds equally well for cohomology theories because a contravariant functor into an Abelian category $\mathcal{A}$ can be viewed as a covariant functor to the opposite category of $\mathcal{A}$.

Remark 4. As a general rule, if α_2 has a cross-section with suitable properties, the same holds for π_1. That is, the class of all quotient maps with a "suitable" section is closed under pull-backs. To construct a section of π_1 from a section $\sigma\colon B \to A_2$ of α_2, we realize P as in (3). Then

$$\sigma'\colon A_1 \to P \subseteq A_1 \times A_2, \qquad a_1 \mapsto \big(a_1, \sigma \circ \alpha_1(a_1)\big),$$

is a section for π_1. Thus π_1 is surjective if α_2 is surjective. If σ is linear, bounded, completely positive, or equivariant with respect to a group action, the same holds for σ'. If $\mathcal{C}$ is a category of topological algebras, then π_1 is open if α_2 is open.

References

1. N. Christopher Phillips, *K-theory for Fréchet algebras*, Internat. J. Math. **2** (1991), no. 1, 77–129. MR **92e**:46143
2. Claude Schochet, *Topological methods for C^*-algebras. III. Axiomatic homology*, Pacific J. Math. **114** (1984), no. 2, 399–445. MR **86g**:46102

Mathematisches Institut, Westfälische Wilhelms-Universität Münster, Einsteinstr. 62, 48149 Münster, Germany

E-mail address: rameyer@math.uni-muenster.de

URL: http://www.math.uni-muenster.de/u/rameyer