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Continuity of joint spectra

prepared by

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Contents

1 Introduction and overview of results

We are interested in the continuity of the map

$$\Sigma\colon \mathcal{T} \to \mathcal{C}(\mathbb{C}^n), \qquad t \mapsto \sigma_j(A_t).$$

$\mathcal{T}$ is a topological space. $\mathcal{C}(\mathbb{C}^n)$ are the non-empty closed subsets of $\mathbb{C}^n$. A_t are families of pairwise strongly commuting normal operators on Hilbert spaces $\mathcal{H}_t$. σ_j denotes the joint spectrum.

As a motivation consider quantum mechanics. Here, observables are described by self-adjoint operators on a given Hilbert space. Several commuting observables can be measured simultaneously. The possible measurement outcomes are tuples of real numbers. These will make up the corresponding joint spectrum. For many applications it is too difficult to calculate these spectra exactly. Physicists will then resort to approximations of the given operators. Let us say $A_t \xrightarrow{t \to t_0} A$ in some sense, where A is the family of operators of interest. If one now wants to obtain results about the spectrum of A by studying the spectra of the A_t it is in general necessary that the spectra change continuously with t. So we ask ourselves: When exactly is this the case? Of course, the answer depends on the topology on $\mathcal{C}(\mathbb{C}^n)$.

We will start by considering families of pairwise commuting normal bounded operators in Section 2. Using the framework of C^*-algebras there is a straightforward generalization from the spectrum of a single normal operator to the case of families of pairwise commuting normal operators. We present the construction of this joint spectrum and show some useful properties in Section 2.2.

Before we can talk about the continuity of Σ we have to specify a topology on $\mathcal{C}(\mathbb{C}^n)$. In the case of bounded operators the joint spectrum will be compact. Therefore, it is enough to consider the non-empty compact subsets $\mathcal{K}(\mathbb{C}^n)$ of $\mathbb{C}^n$. The Hausdorff metric introduces a measure of distance between compact subsets. The corresponding topology makes $\mathcal{K}(\mathbb{C}^n)$ a complete metric space. This is discussed in Section 2.3.

We will further introduce the rather abstract concept of continuous fields of C^*-algebras in Section 2.4. It turns out that for bounded operators the following are equivalent (Theorem 2.5.1 and Corollary 2.5.2):

(i) Σ is continuous in the Hausdorff-metric.

(ii) The map $\mathcal{T} \ni t \mapsto \|p(A_t)\| \in [0,\, \infty)$ is continuous for all polynomials p.

(iii) The map $\mathcal{T} \ni t \mapsto \|\phi(A_t)\| \in [0,\, \infty)$ is continuous for all continuous functions $\phi\colon \mathbb{C}^n \to \mathbb{C}$.

(iv) There is a continuous field of unital C^*-algebras generated by the A_t such that the elements of A_t are continuous sections.

This is a direct generalization of results obtained by Siegfried Beckus in his dissertation on the 'Spectral approximation of aperiodic Schrödinger operators' [1] (2017). He considered the case of single normal operators and their spectra. The implication that self-adjoint continuous sections of continuous fields of C^*-algebras have continuous spectra had already been shown by Kaplanski in 1951 [7]. Beckus seems to be the first to show the converse. He also introduced the formulation in terms of continuous norm maps for certain sets of functions as in statements (*ii*) and (*iii*).

There are situations where continuous fields of C^*-algebras naturally arise, e.g., from fields of groupoids [1]. This yields a direct application of the result above. We will not discuss this direction here, though.

In the second part of the thesis we consider the case of families of pairwise strongly commuting normal unbounded operators (Section 3). We start by introducing the necessary definitions and results such as spectral measures and integrals, as well as a version of the Spectral Theorem and the (Borel) functional calculus (Sections 3.1, 3.2 and 3.3). Now, we need to consider topologies on $\mathcal{C}(\mathbb{C}^n)$. Here, the Hausdorff metric is not well-defined. Instead we work with the Vietoris and the Fell topology, which are possible generalizations (Section 3.4).

Maps between subsets of $\mathbb{C}^n$ can induce maps between the corresponding spaces of closed subsets. E.g., for $\phi\colon \mathbb{C}^n \to \mathbb{C}^n$ define $\overline{\phi}\colon \mathcal{C}(\mathbb{C}^n) \to \mathcal{C}(\mathbb{C}^n),\ K \mapsto \overline{\phi(K)}$. Here, the bar denotes the closure in $\mathbb{C}^n$. For different situations we study the connection between the continuity of the underlying maps and the Fell or Vietoris continuity of the induced maps. We introduce a bounded transform b. It will bijectively transform unbounded operators into bounded ones via the functional calculus of

$$b\colon \mathbb{C} \to \mathbb{C}, \qquad b(z) := \frac{z}{\sqrt{1 + |z|^2}}.$$

This map conserves the relevant properties. E.g., normal, self-adjoint or positive operators will be mapped to normal, self-adjoint or positive bounded operators, respectively (Proposition 3.3.1). Also, two operators strongly commute if and only if their bounded transforms commute (Definition 3.3.2 and Lemma 3.3.3).

Our results for the case of unbounded operators are the following (Theorem 3.5.5):

(i) The Vietoris continuity of Σ implies the Hausdorff continuity of the spectra of the corresponding bounded transforms.

(ii) The Hausdorff continuity of the spectra of the bounded transforms implies the Fell continuity of Σ.

(iii) The result for the bounded case applies to the bounded transforms.

(iv) Σ is Vietoris continuous if and only if the map $\mathcal{T} \ni t \mapsto \|\phi(A_t)\| \in [0, \infty)$ is continuous for all bounded continuous functions $\phi \in C_b(\mathbb{C}^n)$.

As the Fell topology is coarser than the Vietoris topology this result is not entirely satisfactory. Let B be the open unit ball in $\mathbb{C}$. We will define the β-topology on $\mathcal{C}(\mathbb{C}^n)$ by transporting the Hausdorff topology from a subset of $\mathcal{K}\big(\overline{B^n}\big)$ (Section 3.6). This yields (Proposition 3.6.6)

(v) Σ is β-continuous if and only if the spectra of the bounded transforms are Hausdorff continuous.

The β-topology is strictly finer than the Fell topology and strictly coarser than the Vietoris topology (Proposition 3.6.8). It is constructed using the specific choice of the bounded transform b. We take a brief look at possible alternative constructions using the Cayley transform for self-adjoint operators and the resolvent (Section 3.7). This leads to the γ-topology on $\mathcal{C}(\mathbb{C}^n)$. In particular, stronger topologies with an analogous property to (v) would be of interest. We could not settle whether the β- or γ-topology are the optimal choice in this regard. Depending on the specific situation one or the other notion of continuity may be preferable.

In the last part of the thesis we briefly introduce multiplier algebras (Section 4). In this context Woronowicz defined the concept of unbounded elements affiliated with C^*-algebras [11]. Roughly, an unbounded operator a on a C^*-algebra $\mathfrak{A}$ is affiliated with $\mathfrak{A}$ if its bounded transform is a multiplier of $\mathfrak{A}$. We extend this notion to continuous fields of C^*-algebras. The relation to the continuity of Σ is explored.

If the bounded transforms of the families of operators A_t generate a continuous field of C^*-algebras, then $(A_t)_{t\in\mathcal{T}}$ is in general not affiliated with this field (Proposition 4.2.2). However, if the C_0-functions of A_t generate a continuous field of C^*-algebras in a certain sense, then $(A_t)_{t\in\mathcal{T}}$ is affiliated with this field (Proposition 4.2.3). The existence of this field can be characterized by the ζ-continuity of Σ (Lemma 4.2.7). The ζ-topology is yet another topology on $\mathcal{C}(\mathbb{C}^n)$. On $\mathcal{C}(\mathbb{R})$ it coincides with the γ-topology (Corollary 4.2.8).

In the end, we consider constant fields of C^*-algebras. For the constant field generated by the compact operators $\mathbb{K}(\mathcal{H})$ the affiliation relation does not yield a useful notion of continuity: The continuity of Σ is not implied with respect to any topology on $\mathcal{C}(\mathbb{C}^n)$

where limits are unique (Proposition 4.2.13). For the constant field generated by the bounded operators $\mathbb{B}(\mathcal{H})$ the affiliation relation implies the continuity of the bounded transforms in norm. This is too strong in the sense that it cannot be captured by the continuity of Σ in any topology on $\mathcal{C}(\mathbb{C}^n)$ (Proposition 4.2.17).

2 Bounded commuting normal operators

In this section we define the joint spectrum for several commuting normal bounded operators. We introduce the Hausdorff metric and give a characterization of the continuity of joint spectra for the case of bounded operators.

2.1 Basic C^*-algebra theory

We begin by introducing the necessary basics of C^*-algebra theory. These definitions, theorems and the corresponding proofs can be found in introductory textbooks on C^*-algebras. See for example the books by Davidson [4] or Dixmier [5].

Definition 2.1.1. A *Banach algebra* $(\mathfrak{A}, \|\cdot\|)$ is a complete, normed $\mathbb{C}$-algebra whose norm is submultiplicative:

$$\forall a, b \in \mathfrak{A}\colon \|ab\| \leq \|a\|\|b\|.$$

Definition 2.1.2. A *Banach* **-algebra* is a Banach algebra $\mathfrak{A}$ with an involution $^*\colon \mathfrak{A} \to \mathfrak{A},\ a \mapsto a^*$, satisfying

(i) $\forall a, b \in \mathfrak{A}\colon (a+b)^* = a^* + b^*$,

(ii) $\forall a \in \mathfrak{A}\ \forall \lambda \in \mathbb{C}\colon (\lambda a)^* = \lambda^* a^*$ where λ^* is the complex conjugate of λ,

(iii) $\forall a, b \in \mathfrak{A}\colon (ab)^* = b^* a^*$,

(iv) $\forall a \in \mathfrak{A}\colon (a^*)^* = a$,

(v) $\forall a \in \mathfrak{A}\colon \|a^*\| = \|a\|$.

For $a \in \mathfrak{A}$ we call a^* the *adjoint* of a.

Definition 2.1.3. A C^**-algebra* $\mathfrak{A}$ is a Banach *-algebra where the C^*-identity holds:

$$\forall a \in \mathfrak{A}\colon \|a^* a\| = \|a\|^2.$$

For a *unital* C^**-algebra* we further require the existence of a neutral element with respect to multiplication $1 \in \mathfrak{A}$.

Remark 2.1.4. For C^*-algebras the involution is automatically isometric by the C^*-identity and submultiplicativity of the norm. We also have $1^* = 1$ and $\|1\| = 1$ (unless $\dim \mathfrak{A} = 0$).

Definition 2.1.5. Let X be a topological space. We write

$$C(X) := \{f : X \to \mathbb{C} \mid f \text{ continuous}\}$$

for the continuous functions on X.

Example 2.1.6. Let X be a compact Hausdorff space. Then, $C(X)$ is a unital, commutative C^*-algebra under pointwise operations. The norm is the supremum norm

$$\|\cdot\|_\infty : C(X) \to [0, \infty), \qquad \|f\|_\infty := \sup_{x \in X} |f(x)|,$$

and the involution is pointwise complex conjugation.

We denote the algebra of bounded linear operators on a Hilbert space $\mathcal{H}$ by $\mathbb{B}(\mathcal{H})$.

Example 2.1.7. Let $\mathcal{H}$ be a Hilbert space and let $\mathfrak{A} \subseteq \mathbb{B}(\mathcal{H})$ be an (operator) norm-closed subalgebra of the bounded linear operators on $\mathcal{H}$. Then, $\mathfrak{A}$ is a C^*-algebra. The involution corresponds to the adjoint with respect to the inner product on $\mathcal{H}$. (All Hilbert spaces are assumed to be complex.)

Remark 2.1.8. The Gelfand-Naimark Theorem states that these are the general cases: Any C^*-algebra is isomorphic to a closed subalgebra of $\mathbb{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. Any unital, commutative C^*-algebra is isomorphic to $C(X)$ for some compact Hausdorff space X.

An isomorphism of C^*-algebras is a linear, multiplicative, isometric bijection that preserves the involution (and is then automatically continuous).

From here on let $\mathfrak{A}$ be a unital C^*-algebra.

Remark 2.1.9. This is no serious restriction as we can isometrically embed any C^*-algebra into a unital one. There are more things to be said about non-unital (in particular commutative) C^*-algebras, but we do not consider them here.

Definition 2.1.10. In analogy to $\mathbb{B}(\mathcal{H})$ we say that $a \in \mathfrak{A}$ is

- *normal* if $a^*a = aa^*$,
- *self-adjoint* if $a^* = a$,
- *unitary* if $a^*a = 1 = aa^*$.

Definition 2.1.11. Let $a \in \mathfrak{A}$. We regard $\mathbb{C}$ as a subset of $\mathfrak{A}$ via $\lambda \mapsto \lambda 1$. The *resolvent set* of a is

$$\rho(a) := \{\lambda \in \mathbb{C} \mid (a - \lambda) \text{ is invertible in } \mathfrak{A}\},$$

that is, the set of complex numbers λ for which the *resolvent function* $R_\lambda(a) := (a-\lambda)^{-1}$ exists. The *spectrum* of a is

$$\sigma(a) := \mathbb{C}\backslash\rho(a).$$

The *spectral radius* of a is

$$r(a) := \sup|\sigma(a)| = \sup\{|\lambda| \mid \lambda \in \sigma(a)\}.$$

Theorem 2.1.12. *Let $a \in \mathfrak{A}$. Then, $\sigma(a) \subseteq \mathbb{C}$ is non-empty and compact. If a is normal, then*

$$\|a\| = r(a).$$

Definition 2.1.13. Let $A \subseteq \mathfrak{A}$. We write $C_1^*(A) \subseteq \mathfrak{A}$ for the smallest unital C^*-algebra that contains A. In the case of $A = \{a\},\ a \in \mathfrak{A}$, we write $C_1^*(a) := C_1^*(\{a\})$.

2.2 Joint spectrum

In the following, n will always denote a natural number, $n \in \mathbb{N} = \{1, 2, 3, \ldots\}$. From here on we mainly consider C^*-subalgebras of $\mathbb{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. Most things will make sense in more general settings, though.

Throughout this section let

$$A = (a_k)_{k=1}^n \subseteq \mathbb{B}(\mathcal{H})$$

be a family of normal and pairwise commuting bounded operators. We will define a joint spectrum of A.

Due to B. Fuglede we have the following theorem [6].

Theorem 2.2.1. *Let $a, b \in \mathbb{B}(\mathcal{H})$, let a be normal and let $ab = ba$. Then, $a^*b = ba^*$.*

Thus, in the set $A \cup A^* \subseteq \mathbb{B}(\mathcal{H})$ any two operators commute and the following definition makes sense.

Definition 2.2.2. *(Polynomial functional calculus)* We write $\mathcal{P}_n$ for the set of polynomials in n variables $(z_1, \ldots, z_n) \in \mathbb{C}^n$ and their conjugates $(z_1^*, \ldots, z_n^*)$. For $p \in \mathcal{P}_n$ define $p(A) \in \mathbb{B}(\mathcal{H})$ as the corresponding operator obtained by formally replacing $z_k \mapsto a_k$, $z_k^* \mapsto a_k^*$.

By Theorem 2.2.1

$$\mathfrak{A} := C_1^*(A) = \overline{\{p(A) \mid p \in \mathcal{P}_n\}}^{\|\cdot\|} \tag{2.1}$$

is a unital and commutative C^*-algebra.

Definition 2.2.3. The *spectrum* $\hat{\mathfrak{A}}$ of a commutative, unital C^*-algebra $\mathfrak{A}$ is the set of characters:

$$\hat{\mathfrak{A}} := \{\chi\colon \mathfrak{A} \to \mathbb{C} \mid \chi \text{ unital } {}^*\text{-algebra homomorphism}\}.$$

The *Gelfand transform* of an element $a \in \mathfrak{A}$ is the map

$$\hat{a}\colon \hat{\mathfrak{A}} \to \mathbb{C}, \qquad \chi \mapsto \hat{a}(\chi) := \chi(a).$$

Remark 2.2.4. Characters on C^*-algebras have norm one, in particular they are continuous.

Theorem 2.2.5. *By equipping $\hat{\mathfrak{A}}$ with the coarsest topology that makes all Gelfand transforms $\hat{a}$ continuous it becomes a non-empty, compact Hausdorff space.*

Equip $C(\hat{\mathfrak{A}})$ with the supremum norm. Then we have the commutative Gelfand-Naimark Theorem.

Theorem 2.2.6. *The Gelfand transform $\mathfrak{A} \to C(\hat{\mathfrak{A}})$, $a \mapsto \hat{a}$, is an isomorphism of C^*-algebras.*

Recall that we consider the specific case of $\mathfrak{A} = C_1^*(A)$ where $A = (a_k)_{k=1}^n \subseteq \mathbb{B}(\mathcal{H})$ is a family of normal and pairwise commuting operators.

Theorem 2.2.7. *The map*

$$E_A\colon \hat{\mathfrak{A}} \to \mathbb{C}^n, \qquad \chi \mapsto (\chi(a_k))_{k=1}^n,$$

is a homeomorphism onto a non-empty, compact subset of $\mathbb{C}^n$.

Proof. The Gelfand transforms $\hat{a}_k$ are continuous by definition of the topology on $\hat{\mathfrak{A}}$. Then, $E_A = (\hat{a}_1, \dots, \hat{a}_n)$ is continuous because it is continuous in every component.

Let $\chi_1, \chi_2 \in \hat{\mathfrak{A}}$ and let $E_A(\chi_1) = E_A(\chi_2)$. So, for all $k \in \{1, \dots, n\}$: $\chi_1(a_k) = \chi_2(a_k)$. Thus, for all $p \in \mathcal{P}_n$:

$$\chi_1(p(A)) = p((\chi_1(a_k))_{k=1}^n) = p((\chi_2(a_k))_{k=1}^n) = \chi_2(p(A))$$

because characters are *-algebra homomorphisms. The subset $\{p(A) \mid p \in \mathcal{P}_n\} \subseteq C_1^*(A)$ is dense (cf. equation (2.1)). Hence, $\chi_1 = \chi_2$ because characters are continuous. Therefore, E_A is injective.

Accordingly, E_A is a continuous bijection onto its image $\operatorname{Im} E_A = E_A(\hat{\mathfrak{A}}) \subseteq \mathbb{C}^n$. The domain $\hat{\mathfrak{A}}$ is compact and non-empty by Theorem 2.2.5. The image is Hausdorff as a subset of $\mathbb{C}^n$. Thus, E_A is a homeomorphism onto its image. □

Definition 2.2.8. We call

$$\sigma_j(A) := \operatorname{Im}(E_A) = \Big\{ (\hat{a}_i(\chi))_{i=1}^n \mid \chi \in \widehat{C_1^*(A)} \Big\} \subseteq \mathbb{C}^n$$

the *joint spectrum* of A.

Corollary 2.2.9. *The joint spectrum of A with the induced topology from $\mathbb{C}^n$ is homeomorphic to $\hat{\mathfrak{A}}$ by Theorem* 2.2.7. *In particular, $\sigma_j(A)$ is compact and non-empty. $\mathfrak{A} = C_1^*(A)$ and $C(\sigma_j(A))$ are isomorphic as C^*-algebras by the commutative Gelfand-Naimark Theorem (Theorem* 2.2.6*).*

There are more connections and analogies to the spectra of single normal operators. In particular, our notion of the joint spectrum reduces to the usual spectrum in the case of single operators:

Theorem 2.2.10. *Let $\mathfrak{A}$ be a commutative, unital C^*-algebra and $a \in \mathfrak{A}$. Then*

$$\operatorname{Im}(\hat{a}) = \sigma(a).$$

Remark 2.2.11. The spectrum of $a \in \mathfrak{A}$ is independent of $\mathfrak{A}$ in the sense that it is the same when we consider a as an element of $C_1^*(a) \subseteq \mathfrak{A}$.

Corollary 2.2.12. *Let $a \in \mathbb{B}(\mathcal{H})$ be normal. Then*

$$\sigma(a) = \Big\{ \hat{a}(\chi) \mid \chi \in \widehat{C_1^*(a)} \Big\} = \sigma_j((a)).$$

Corollary 2.2.13. *Let $A = (a_k)_{k=1}^n \subseteq \mathbb{B}(\mathcal{H})$ be a family of normal and pairwise commuting bounded operators. Then*

$$\sigma_j(A) \subseteq \prod_{k=1}^n \sigma(a_k).$$

Lemma 2.2.14. *Let $p \in \mathcal{P}_n$. Then $p(\sigma_j(A)) = \sigma(p(A))$.*

Proof. Calculate

$$\begin{aligned} p(\sigma_j(A)) &= p\Big(E_A(\hat{\mathfrak{A}})\Big) = p\Big(\Big\{(\hat{a}_k(\chi))_{k=1}^n \mid \chi \in \hat{\mathfrak{A}}\Big\}\Big) \\ &= \Big\{ p((\chi(a_k))_{k=1}^n) \mid \chi \in \hat{\mathfrak{A}} \Big\} = \Big\{ \chi(p(A)) \mid \chi \in \hat{\mathfrak{A}} \Big\} \\ &= \Big\{ \widehat{p(A)}(\chi) \mid \chi \in \hat{\mathfrak{A}} \Big\} = \sigma(p(A)). \end{aligned}$$

In the last equation of the second line we used that the characters χ are *-algebra homomorphisms. □

Corollary 2.2.15. *The spectra of the a_k can be recovered as the axis projections of the joint spectrum. Let $p_k \colon \mathbb{C}^n \to \mathbb{C},\ (z_1, \dots, z_n) \mapsto z_k$. Then*

$$\sigma(a_k) = p_k(\sigma_j(A)).$$

Proof. We have $p_k \in \mathcal{P}_n$ and $p_k(A) = a_k$. □

We have a generalization of the continuous functional calculus (CFC) to several variables.

Definition 2.2.16. Let $\phi \colon \mathbb{C}^n \to \mathbb{C}$ be continuous. Define $\phi(A) \in \mathfrak{A}$ to be the operator obtained from the inverse Gelfand transform of the map $\phi \circ E_A \in C(\hat{\mathfrak{A}})$.

Remark 2.2.17. Only the values of ϕ on the joint spectrum of A are relevant for $\phi(A)$. Also, $\phi(A) \in \mathfrak{A}$ by definition and $\mathfrak{A}$ is commutative. In particular, $\phi(A)$ is normal. For polynomials $p \in \mathcal{P}_n$ the naive definition for $p(A)$ that we used up to now and the one obtained from the continuous functional calculus coincide. Denoting the latter with '$\mathrm{CFC}_A(p)$' the reason is

$$\widehat{p(A)}(\chi) = p(\chi(a_1), \dots, \chi(a_n)) = p \circ E_A(\chi) = \widehat{\mathrm{CFC}_A(p)}(\chi) \quad \text{for all} \quad \chi \in \hat{\mathfrak{A}}.$$

Lemma 2.2.18. *Let $\phi \colon \sigma_j(A) \to \mathbb{C}$ be continuous. Then*

$$\sigma(\phi(A)) = \phi(\sigma_j(A)).$$

Proof. We have $\sigma(\phi(A)) = \sigma(\phi \circ E_A) = \sigma(\phi)$ by Corollary 2.2.9 and the definition of $\phi(A)$. The multiplication in $C(\sigma_j(A))$ is pointwise. Then, for $\lambda \in \mathbb{C}$ we have that $(\phi - \lambda) \in C(\sigma_j(A))$ is invertible if and only if $0 \notin \operatorname{Im}(\phi - \lambda)$. Thus, $\lambda \in \sigma(\phi)$ if and only if $\lambda \in \operatorname{Im} \phi = \phi(\sigma_j(A))$. □

Example 2.2.19. Let $A = (a_k)_{k=1}^n \subseteq \mathbb{B}(\mathcal{H})$ be as above and write the joint spectrum as

$$\sigma_j(A) = \left\{ \lambda^{(\chi)} := (\chi(a_1), \dots, \chi(a_n)) \in \mathbb{C}^n \;\middle|\; \chi \in \hat{\mathfrak{A}} \right\}.$$

Let $\mu := (\mu_k)_{k=1}^m \subseteq \mathbb{C}$ be finitely many complex numbers. Identify $\mu_k \mathrel{\widehat{=}} \mu_k \cdot 1 \in \mathbb{B}(\mathcal{H})$ as bounded, normal operators that commute with everything in $\mathbb{B}(\mathcal{H})$. We have the normal and commuting set of bounded operators $B := (a_1, \dots, a_n, \mu_1, \dots, \mu_m)$. The assertion $\mathfrak{A} = C_1^*(A) = C_1^*(B) =: \mathfrak{B}$ holds. So, $\hat{\mathfrak{A}} = \hat{\mathfrak{B}}$. Characters are unital, therefore

$$\begin{aligned}
\sigma_j(A \cup \mu) &= \sigma_j(B) = E_B(\hat{\mathfrak{B}}) \\
&= \Big\{ (\chi(a_1), \dots, \chi(a_n), \chi(\mu_1), \dots, \chi(\mu_m)) \in \mathbb{C}^{n+m} \;\Big|\; \chi \in \hat{\mathfrak{A}} \Big\} \\
&= \Big\{ \big(\lambda^{(\chi)}, \mu\big) \in \mathbb{C}^{n+m} \;\Big|\; \chi \in \hat{\mathfrak{A}} \Big\} \\
&= \sigma_j(A) \times \{\mu\}.
\end{aligned}$$

In the first line we interpret μ as a subset of $\mathbb{C} \subseteq \mathbb{B}(\mathcal{H})$, in the last line as an element of $\mathbb{C}^m$.

Example 2.2.20. Let $\mathcal{H}$ be finite-dimensional. That is, the a_k are complex, normal, commuting $N \times N$ matrices. Then, the a_k are simultaneously diagonalizable. In the common eigenbasis we can write

$$a_k = \operatorname{diag}(\lambda_k^{(1)}, \dots, \lambda_k^{(N)}) \in \mathbb{C}^{N \times N},$$

where $\lambda_k^{(1)}, \dots, \lambda_k^{(N)}$ are the eigenvalues of a_k. Identify

$$\mathbb{C}^{N \times N} \ni \operatorname{diag}(\lambda_k^{(1)}, \dots, \lambda_k^{(N)}) \mapsto (\lambda_k^{(1)}, \dots, \lambda_k^{(N)}) \in \mathbb{C}^N.$$

The C^*-algebra $\mathfrak{A} = C_1^*(A)$ generated by the a_k is isomorphic to a C^*-subalgebra of $\mathbb{C}^N$ with entrywise multiplication and conjugation. The norm is the sup-norm

$$\|(z_1, \dots, z_N)\|_\infty = \max_{k \in \{1, \dots, N\}} |z_k|.$$

Characters on this space are in particular linear maps $\mathbb{C}^N \to \mathbb{C}$. Thus, $\hat{\mathfrak{A}}$ can be viewed as a subset of $\mathbb{C}^N$ in the sense

$$\hat{\mathfrak{A}} \ni \chi \mathrel{\widehat{=}} (\chi_1, \dots, \chi_N) \in \mathbb{C}^N, \qquad \chi(z) = \chi_1 \cdot z_1 + \dots + \chi_N \cdot z_N \in \mathbb{C}.$$

Characters further need to be unital. As $1 = (1, \dots, 1) \in \mathbb{C}^N$ is the multiplicative unit of $\mathfrak{A}$ we have

$$\forall \chi \in \hat{\mathfrak{A}} \colon 1 = \chi(1) = \chi_1 + \dots + \chi_N.$$

Finally, characters need to be multiplicative. Let $e^{(k)} = (0, \dots, 1, \dots, 0) \in \mathbb{C}^N$ be the k^{th} unit vector: $e_j^{(k)} = \delta_{kj}$. Then, for all $k \in \{1, \dots, N\}$ and every character $\chi \in \hat{\mathfrak{A}}$

$$\chi_k = \chi(e^{(k)}) = \chi(e^{(k)} e^{(k)}) = \chi(e^{(k)})^2 = \chi_k^2.$$

Thus, $\chi_k \in \{0, 1\}$ and $\hat{\mathfrak{A}} \subseteq \{e^{(k)} \mid k \in \{1, \ldots, N\}\}$. In fact, $\hat{\mathfrak{A}} = \{e^{(k)} \mid k \in \{1, \ldots, N\}\}$ because all $e^{(k)}$ are multiplicative. Now,

$$E_A(e^{(k)}) = \left(e^{(k)}(a_1), \ldots, e^{(k)}(a_n)\right) = \left(\lambda_1^{(k)}, \ldots, \lambda_n^{(k)}\right) \in \mathbb{C}^n.$$

This yields the joint spectrum

$$\begin{aligned}\sigma_j(A) &= \left\{\left(\lambda_1^{(k)}, \ldots, \lambda_n^{(k)}\right) \;\middle|\; k \in \{1, \ldots, N\}\right\} \\ &\subset \left\{\left(\lambda_1^{(k_1)}, \ldots, \lambda_n^{(k_n)}\right) \;\middle|\; k_1, \ldots, k_n \in \{1, \ldots, N\}\right\} = \prod_{k=1}^{n} \sigma(a_k).\end{aligned}$$

Here, the joint spectrum corresponds to the 'diagonal' in the product of the individual spectra. The *joint eigenvalues* $\lambda^{(k)} := (\lambda_1^{(k)}, \ldots, \lambda_n^{(k)})$ correspond to the same k^{th} common eigenspace. In the case of self-adjoint a_k we can interpret them as observables in the context of quantum mechanics. Commuting observables can be measured simultaneously. Then, the joint eigenvalues correspond to the possible joint measurement outcomes.

2.3 Hausdorff topology

To talk about the continuity of joint spectra, which are compact subsets of $\mathbb{C}^n$, we need a topology on that space.

Definition 2.3.1. Let (X, d) be a complete metric space. We define the space of non-empty compact subsets

$$\mathcal{K}(X) := \{K \subseteq X \mid K \text{ compact}, \ K \neq \emptyset\}.$$

The *Hausdorff topology* on $\mathcal{K}(X)$ is induced by the *Hausdorff metric* (see Figure 2.1 for a visualization):

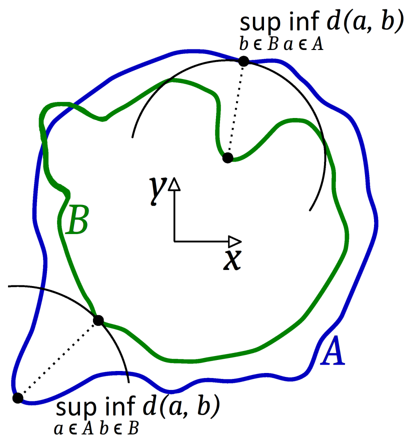

Figure 2.1: Visualization of the Hausdorff distance between two compact subsets of $\mathbb{R}^2$ [8].

$$d_H \colon \mathcal{K}(X)^2 \to [0, \infty), \qquad d_H(A, B) := \max\left\{\sup_{a\in A}\inf_{b\in B} d(a, b), \ \sup_{b\in B}\inf_{a\in A} d(a, b)\right\}.$$

Remark 2.3.2. The assumption of X being complete is not necessary for the Hausdorff metric to be well defined. We will only consider closed subspaces of $\mathbb{C}^n$, though.

Lemma 2.3.3 (Thm. II-5 in [3, p. 41])**.** *Let* $(X,\, d)$ *be a complete metric space. Then,* $(\mathcal{K}(X),\, d_H)$ *is a complete metric space.*

Lemma 2.3.4 (Thm. II-6 in [3, p. 41])**.** *A basis for the Hausdorff topology on* $\mathcal{K}(X)$ *is given by sets of the form*

$$\mathcal{U}_X(F,\ (O_k)_{k=1}^m) := \{K \in \mathcal{K}(X) \mid K \cap F = \emptyset,\ \forall k \in \{1, \ldots, m\} \colon K \cap O_k \neq \emptyset\}$$

for $F \subseteq X$ *closed and* $O_k \subseteq X$ *open for all* k*. That is, every open subset of* $\mathcal{K}(X)$ *is a union of these types of sets.*

The following lemma is similar to Proposition 2.2.3 in [1, p. 27]. In this version we can drop the assumption of ϕ being closed, though. This is important because, e.g., polynomials in several variables are not necessarily closed maps.

Lemma 2.3.5. *Let* $X,\ Y$ *be complete metric spaces. Let* $\phi \colon X \to Y$ *be continuous. Write* $\widetilde{\phi} \colon \mathcal{K}(X) \to \mathcal{K}(Y),\ K \mapsto \phi(K)$*, for the induced map on the corresponding spaces of compact subsets equipped with the Hausdorff topology. Then,* $\widetilde{\phi}$ *is continuous.*

Proof. For all $K \in \mathcal{K}(X) \colon \widetilde{\phi}(K) = \phi(K) \in \mathcal{K}(Y)$ because ϕ is continuous. Thus, $\widetilde{\phi}$ is well defined. Use the characterization of Lemma 2.3.4 for the Hausdorff topology. The rest of the proof is similar to the proof of Proposition 2.2.3 in [1, p. 27].

We show continuity at every point. Let $K_0 \in \mathcal{K}(X)$ and let $W \subseteq \mathcal{K}(Y)$ be an open neighborhood of $\phi(K_0)$. W.l.o.g. $W = \mathcal{U}_Y(F,\, (O_k)_{k=1}^m)$ for a closed set $F \subseteq Y$ and open sets $O_k \subseteq Y$. Define

$$V := \mathcal{U}_X(\phi^{-1}(F),\, (\phi^{-1}(O_k))_{k=1}^m) \subseteq \mathcal{K}(X).$$

$\phi^{-1}(F) \subseteq X$ is closed and the $\phi^{-1}(O_k) \subseteq X$ are open because ϕ is continuous. Thus, V is open. Let $A \subseteq X,\, B \subseteq Y$ be any sets. Then,

$$\phi(A) \cap B = \emptyset \ \Leftrightarrow\ \forall a \in A \colon \phi(a) \notin B \ \Leftrightarrow\ \forall a \in A \colon a \notin \phi^{-1}(B) \ \Leftrightarrow\ A \cap \phi^{-1}(B) = \emptyset.$$

This equivalence yields

$$\begin{aligned} K \in V \ &\Leftrightarrow\ K \cap \phi^{-1}(F) = \emptyset,\ \forall k \colon K \cap \phi^{-1}(O_k) \neq \emptyset \\ &\Leftrightarrow\ \phi(K) \cap F = \emptyset,\ \forall k \colon \phi(K) \cap O_k \neq \emptyset \\ &\Leftrightarrow\ \phi(K) \in \mathcal{U}_Y(F,\, (O_k)_{k=1}^m) = W. \end{aligned}$$

In particular, $\widetilde{\phi}(V) \subseteq W$ and $K_0 \in V$ because $\phi(K_0) \in W$ by assumption. Then, $\widetilde{\phi}$ is continuous as it is continuous at every point $K_0 \in \mathcal{K}(X)$. □

Corollary 2.3.6. *For the norm map* $|\cdot|\colon \mathbb{C} \to [0, \infty)$ *and polynomials* $p \in \mathcal{P}_n$ *the induced maps* $\widetilde{|\cdot|}$ *and* $\widetilde{p}$ *are continuous.*

Lemma 2.3.7. *The map*

$$\sup\colon \mathcal{K}(\mathbb{R}) \to \mathbb{R}, \qquad K \mapsto \sup K,$$

is continuous.

Proof. For non-empty, compact sets $K \in \mathcal{K}(\mathbb{R})$ the supremum is indeed a real number. So, sup is well defined. Again use the characterization of Lemma 2.3.4 for the Hausdorff topology. The proof is now the same as the proof of Proposition 2.2.4 in [1, p. 28].

Let $K_0 \in \mathcal{K}(\mathbb{R})$, let $s := \sup K_0 \in \mathbb{R}$ and let $\epsilon > 0$. $s \in K_0$ because K_0 is closed. Define

$$F := [s+\epsilon, \infty) \qquad \text{and} \qquad O := (s-\epsilon, \infty).$$

Then, $F \subseteq \mathbb{R}$ is closed, $O \subseteq \mathbb{R}$ is open, $K_0 \cap F = \emptyset$ and $\mathcal{K}_0 \cap O \neq \emptyset$. Thus, $V := \mathcal{U}_{\mathbb{R}}(F, (O)) \subseteq \mathcal{K}(\mathbb{R})$ is an open neighborhood of K_0. Let $K \in V$. Then, $K \cap F = \emptyset$ and $K \cap O \neq \emptyset$. This means $\sup K < s + \epsilon$ and $\sup K > s - \epsilon$. That is,

$$|\sup K_0 - \sup K| < \epsilon$$

and sup is continuous at K_0. Therefore, sup is continuous as K_0 was arbitrary. □

2.4 Continuous fields of C^*-algebras

We will relate the continuity of joint spectra to continuous fields of C^*-algebras.

Definition 2.4.1. Let $\mathcal{T}$ be a topological space and $(\mathfrak{C}_t)_{t\in\mathcal{T}}$ a family of unital C^*-algebras. Let $\Gamma \subseteq \mathfrak{C} := \prod_{t\in\mathcal{T}} \mathfrak{C}_t$. Then, $((\mathfrak{C}_t)_{t\in\mathcal{T}}, \Gamma)$ is called a *continuous field of unital C^*-algebras* if the following assertions hold.

(B1) Γ is a unital *-subalgebra of $\mathfrak{C}$ (for the pointwise involution and multiplication).

(B2) For all $t \in \mathcal{T}$ the set $\Gamma_t := \{a_t \mid (a_t)_{t\in\mathcal{T}} \in \Gamma\}$ is dense in $\mathfrak{C}_t$.

(B3) For all $(a_t)_{t\in\mathcal{T}} \in \Gamma$ the map $\mathcal{T} \ni t \mapsto \|a_t\| \in [0, \infty)$ is continuous.

(B4) A section $(a_t)_{t\in\mathcal{T}} \in \mathfrak{C}$ is an element of Γ if and only if for all $t_0 \in \mathcal{T}$ and for all $\epsilon > 0$ there exists $(b_t)_{t\in\mathcal{T}} \in \Gamma$ and a neighborhood $U \subseteq \mathcal{T}$ of t_0 such that for all $t \in U\colon \|a_t - b_t\| < \epsilon$.

The elements of Γ are called the *continuous sections* of $\mathfrak{C}$.[1] A continuous section $(a_t)_{t\in\mathcal{T}}$ is called *normal* if $a_t \in \mathfrak{C}_t$ is normal for every $t \in \mathcal{T}$.

Remark 2.4.2. This definition also makes sense for non-unital C^*-algebras. Then, Γ is just a *-subalgebra of $\mathfrak{C}$. In this case we speak of *continuous fields of C^*-algebras.* The following lemma holds in the non-unital case as well.

Lemma 2.4.3 (Prop. 2.7.6 in [1, p. 54], Prop. 10.2.3 in [5, p. 216])**.** *Let $\mathcal{T}$ be a topological space and let $(\mathfrak{C}_t)_{t\in\mathcal{T}}$ be a family of unital C^*-algebras. Let $\Lambda \subseteq \mathfrak{C}$ satisfy* (B1), (B2) *and* (B3) (*replace Γ by Λ*)*. Then, there exists a unique subset $\Gamma \subseteq \mathfrak{C}$ with $\Lambda \subseteq \Gamma$ such that $((\mathfrak{C}_t)_{t\in\mathcal{T}}, \Gamma)$ is a continuous field of unital C^*-algebras.*

We call such a set Λ a *generating family.*

For single operators we have the following two results.

Lemma 2.4.4 (Prop. 2.7.8 in [1, p. 54], Prop. 10.3.3 in [5, p. 219])**.** *Let $((\mathfrak{C}_t)_{t\in\mathcal{T}}, \Gamma)$ be a continuous field of unital C^*-algebras and let $(a_t)_{t\in\mathcal{T}} \in \Gamma$ be a continuous, normal section. Let $\phi \in C(\mathbb{C})$. Then $(\phi(a_t))_{t\in\mathcal{T}} \in \Gamma$ is also a continuous, normal section.*

Theorem 2.4.5 (Thm. 2.7.9 in [1, p. 55])**.** *Let $((\mathfrak{C}_t)_{t\in\mathcal{T}}, \Gamma)$ be a continuous field of unital C^*-algebras and let $(a_t)_{t\in\mathcal{T}} \in \Gamma$ be a continuous, normal section. Then, the map*

$$\mathcal{T} \ni t \mapsto \sigma(a_t) \in \mathcal{K}(\mathbb{C})$$

is continuous with respect to the Hausdorff metric on $\mathcal{K}(\mathbb{C})$.

We can generalize to families of operators.

Definition 2.4.6. Let $\mathcal{T}$ be a topological space and for every $t \in \mathcal{T}$ let $\mathcal{H}_t$ be a Hilbert space. Then, we call $(\mathcal{H}_t)_{t\in\mathcal{T}}$ a *field of Hilbert spaces* (over $\mathcal{T}$).

Definition 2.4.7. Let $\mathcal{T}$ be a topological space and let $(\mathcal{H}_t)_{t\in\mathcal{T}}$ be a field of Hilbert spaces. For every $t \in \mathcal{T}$ let $A_t \subseteq \mathbb{B}(\mathcal{H}_t)$ be a subset of $n \in \mathbb{N}$ operators. We call $(A_t)_{t\in\mathcal{T}}$ a *field of families of bounded operators.* The field is said to be *normal* or *commuting* whenever all A_t consist of normal or pairwise commuting elements, respectively. If $n = 1$ we speak of *fields of bounded operators.*

Theorem 2.4.8. *Let $\mathcal{T} \neq \emptyset$ be a topological space and let $(\mathcal{H}_t)_{t\in\mathcal{T}}$ be a field of Hilbert spaces. Let $n \in \mathbb{N}$. For $t \in \mathcal{T}$ let $A_t := (a_{t,k})_{k=1}^n \subseteq \mathbb{B}(\mathcal{H}_t)$ be such that $(A_t)_{t\in\mathcal{T}}$ is a normal, commuting field of families of bounded operators. Let $((\mathfrak{C}_t)_{t\in\mathcal{T}}, \Gamma)$ be a continuous field of unital C^*-algebras such that*

[1]In the literature they are sometimes called *continuous vector fields.*

(a) for all $t \in \mathcal{T}\colon \mathfrak{C}_t \subseteq \mathbb{B}(\mathcal{H}_t)$ *is a* C^**-subalgebra with* $A_t \subseteq \mathfrak{C}_t$*,*

(b) for all $k \in \{1, \ldots, n\}\colon (a_{t,k})_{t\in\mathcal{T}} \in \Gamma$ *is a continuous section.*

Then, the following assertions hold.

(i) For all $\phi \in C(\mathbb{C}^n)\colon (\phi(A_t))_{t\in\mathcal{T}} \in \Gamma$ *is a continuous section.*

(ii) The map $\Sigma\colon \mathcal{T} \to \mathcal{K}(\mathbb{C}^n)$, $t \mapsto \sigma_j(A_t)$*, is continuous with respect to the Hausdorff metric on* $\mathcal{K}(\mathbb{C}^n)$*.*

Proof. (i) (Adapted from [1, p. 54], Prop. 2.7.7, 2.7.8): Γ is a unital *-subalgebra by (B1): Γ is closed under finite algebraic operations (sums, products, involutions, scaling by complex numbers). Also $(1)_{t\in\mathcal{T}} \in \Gamma$ is the multiplicative unit and $(a_{t,k})_{t\in\mathcal{T}} \in \Gamma$ by assumption. Therefore, for every polynomial $p \in \mathcal{P}_n$ the section $(p(A_t))_{t\in\mathcal{T}} \in \Gamma$ is continuous.

To show $(\phi(A_t))_{t\in\mathcal{T}} \in \Gamma$ we use property (B4). Let $t_0 \in \mathcal{T}$ and let $\epsilon > 0$. For every $k \in \{1, \ldots, n\}$ the map $\mathcal{T} \ni t \mapsto \|a_{k,t}\| \in [0, \infty)$ is continuous by (B3). Then, there exists an open neighborhood $U \subseteq \mathcal{T}$ of t_0 such that $s_k := \sup_{t\in U} \|a_{k,t}\| < \infty$. Consequently, $s := \max_{k\in\{1,\ldots,n\}} s_k < \infty$. For all $k \in \{1, \ldots, n\}$ and for all $t \in U\colon \sigma(a_{t,k}) \subseteq B_s(0) := \{z \in \mathbb{C} \mid |z| \le s\}$. Therefore, for all $t \in U\colon \sigma_j(A_t) \subseteq B_s(0)^n \subseteq \mathbb{C}^n$ by Corollary 2.2.13. $B_s(0)^n$ is compact as a product of compact sets. Thus, the set of polynomials $\mathcal{P}_n$ restricted to $B_s(0)^n$ is dense in $C(B_s(0)^n)$ with respect to the supremum norm. Then, there exists $p \in \mathcal{P}_n$ such that

$$\forall t \in U\colon \|\phi(A_t) - p(A_t)\| \overset{\text{Cor. 2.2.9}}{=} \left\|\phi|_{\sigma_j(A_t)} - p|_{\sigma_j(A_t)}\right\|_\infty \le \left\|(\phi - p)|_{B_s(0)^n}\right\|_\infty < \epsilon.$$

Hence, $(\phi(A_t))_{t\in\mathcal{T}} \in \Gamma$ because $(p(A_t))_{t\in\mathcal{T}} \in \Gamma$.

(ii) (Adapted from [1, p. 55], Prop. 2.7.9): Σ is well defined by Corollary 2.2.9. We show continuity at every point. Let $t_0 \in \mathcal{T}$ and let $W \subseteq \mathcal{K}(\mathbb{C}^n)$ be an open neighborhood of $\Sigma(t_0)$. W.l.o.g. $W = \mathcal{U}_{\mathbb{C}^n}(F, (O_k)_{k=1}^m)$ for a closed subset $F \subseteq \mathbb{C}^n$ and open subsets $O_k \subseteq \mathbb{C}^n$ for $k \in \{1, \ldots, m\}$ by Lemma 2.3.4. We construct an open neighborhood $U \subseteq \mathcal{T}$ of t_0 with $\Sigma(U) \subseteq W$.

The assertion $\sigma_j(A_{t_0}) \cap F = \emptyset$ holds by assumption. Then, there exists a continuous function $\phi\colon \mathbb{C}^n \to [0, 1]$ with

$$\phi|_F \equiv 1, \qquad \phi|_{\sigma_j(A_{t_0})} \equiv 0$$

by Urysohn's Lemma. In particular, $\phi(A_{t_0}) = 0 = \|\phi(A_{t_0})\|$. The section $(\phi(A_t))_{t\in\mathcal{T}} \in \Gamma$ is normal and continuous by (i). Then, $\mathcal{T} \ni t \mapsto \|\phi(A_t)\| \in [0, \infty)$ is continuous by (B3). Thus, there exists an open neighborhood $U_F \subseteq \mathcal{T}$ of t_0 with

$\{\|\phi(A_t)\|\}_{t\in\mathcal{U}_F} \subseteq [0,\, 1/2)$. Suppose that there exists $t \in U_F$ such that $\sigma_j(A_t) \cap F \neq \emptyset$. Then, there exists $\lambda \in \sigma_j(A_t) \cap F$ and

$$1/2 > \|\phi(A_t)\| = \left\|\phi|_{\sigma_j(A_t)}\right\|_\infty \geq \left\|\phi|_{\{\lambda\}}\right\|_\infty = 1.$$

This is a contradiction. Therefore, for all $t \in U_F\colon \sigma_j(A_t) \cap F = \emptyset$.

Let $k \in \{1, \ldots, m\}$. The assertion $\sigma_j(A_{t_0}) \cap O_k \neq \emptyset$ holds by assumption. Let $z_0 \in \sigma_j(A_{t_0}) \cap O_k$ and let $\epsilon > 0$ be small enough such that

$$B_\epsilon(z_0) := \{z \in \mathbb{C}^n \,|\|z - z_0\| < \epsilon\} \subseteq O_k.$$

Then, $\sigma_j(A_{t_0}) \cap B_\epsilon(z_0) \neq \emptyset$. There exists a continuous function $\phi\colon \mathbb{C}^n \to [0,\, 1]$ with

$$\phi(z_0) = 1, \qquad \phi|_{\mathbb{C}\setminus B_\epsilon(z_0)} \equiv 0$$

by Urysohn's Lemma. In particular, $(\phi(A_t))_{t\in\mathcal{T}} \in \Gamma$ is a continuous, normal section by (i). Also,

$$\|\phi(A_{t_0})\| = \left\|\phi|_{\sigma_j(A_{t_0})}\right\|_\infty = 1$$

because $z_0 \in \sigma_j(A_{t_0})$. The map $\mathcal{T} \ni t \mapsto \|\phi(A_t)\| \in [0,\, \infty)$ is again continuous by (B3). Therefore, there exists an open neighborhood $U_{O,k} \subseteq \mathcal{T}$ of t_0 with $\{\|\phi(A_t)\|\}_{t\in U_{O,k}} \subseteq (1/2,\, 1]$. Thus, for every $t \in U_{O,k}$ there exists $z_t \in \sigma_j(A_t)$ with $\phi(z_t) > 1/2$. Then, $z_t \in B_\epsilon(z_0)$ by definition of ϕ. In particular, for every $t \in U_{O,k}$:

$$\sigma_j(A_t) \cap O_k \supseteq \sigma_j(A_t) \cap B_\epsilon(z_0) \neq \emptyset.$$

Define the open set

$$U := U_F \cap \left(\bigcap_{k=1}^m U_{O,k}\right) \subseteq \mathcal{T}.$$

$t_0 \in U$ so U is an open neighborhood of t_0. Further, for all $t \in U$ and $k \in \{1, \ldots, m\}\colon \sigma_j(A_t) \cap F = \emptyset$ and $\sigma_j(A_t) \cap O_k \neq \emptyset$. This means, for all $t \in U\colon \sigma_j(A_t) \in \mathcal{U}_{\mathbb{C}^n}(F,\, (O_k)_{k=1}^m) = W$ or $\Sigma(U) \subseteq W$. Thus, Σ is continuous because it is continuous at every point. $\square$

2.5 Characterizing the continuity of joint spectra

The following theorem is a generalization of Theorem 2.7.12 in [1, p. 56]. The proof is conceptually similar.

Theorem 2.5.1. *Let $\mathcal{T} \neq \emptyset$ be a topological space and let $(\mathcal{H}_t)_{t\in\mathcal{T}}$ be a field of Hilbert spaces. Let $n \in \mathbb{N}$. For $t \in \mathcal{T}$ let $A_t := (a_{t,k})_{k=1}^n \subseteq \mathbb{B}(\mathcal{H}_t)$ be such that $(A_t)_{t\in\mathcal{T}}$ is a normal, commuting field of families of bounded operators. Then, the following are equivalent.*

(i) The map $\Sigma\colon \mathcal{T} \to \mathcal{K}(\mathbb{C}^n)$, $t \mapsto \sigma_j(A_t)$, is continuous with respect to the Hausdorff metric on $\mathcal{K}(\mathbb{C}^n)$.

(ii) For all $p \in \mathcal{P}_n$ the maps $\mathcal{T} \ni t \mapsto \|p(A_t)\| \in [0, \infty)$ are continuous.

(iii) There exists a continuous field of unital C^*-algebras $((\mathfrak{C}_t)_{t\in\mathcal{T}}, \Gamma)$ satisfying

(a) for all $t \in \mathcal{T}$: $\mathfrak{C}_t \subseteq \mathbb{B}(\mathcal{H}_t)$ is a C^*-subalgebra with $A_t \subseteq \mathfrak{C}_t$,

(b) for all $k \in \{1, \dots, n\}$: $(a_{t,k})_{t\in\mathcal{T}} \in \Gamma$ is a continuous section.

Proof. *(i)* $\Rightarrow$ *(ii)*: Let $p \in \mathcal{P}_n$. For all $t \in \mathcal{T}$: $p(A_t) \in \mathbb{B}(\mathcal{H}_t)$ is normal and bounded. In particular, these operators have compact spectra. Then,

$$\begin{aligned}\mathcal{T} \ni t &\overset{\Sigma}{\mapsto} \sigma_j(A_t) \overset{\widetilde{p}}{\mapsto} p(\sigma_j(A_t)) \overset{\text{Lemma 2.2.14}}{=} \sigma(p(A_t)) \overset{\widetilde{|\cdot|}}{\mapsto} |\sigma(p(A_t))| \\ &\overset{\sup}{\longmapsto} \sup|\sigma(p(A_t))| \overset{\text{Thm. 2.1.12}}{=} \|p(A_t)\| \in [0, \infty)\end{aligned}$$

is continuous as a composition of continuous maps: Σ is continuous by assumption, $\widetilde{p}$ and $\widetilde{|\cdot|}$ are continuous by Corollary 2.3.6 and sup is continuous by Lemma 2.3.7.

(ii) $\Rightarrow$ *(iii)*: Let $\mathfrak{C}_t = C_1^*(A_t) \supseteq A_t$ and $\mathfrak{C} = \prod_{t\in\mathcal{T}} \mathfrak{C}_t$. We show that the set

$$\Lambda := \{(p(A_t))_{t\in\mathcal{T}} \mid p \in \mathcal{P}_n\} \subseteq \mathfrak{C}$$

is a generating set for a continuous field of unital C^*-algebras.

The constant polynomial $p \equiv 1$ generates the unit element $(1)_{t\in\mathcal{T}} \in \Lambda$. Sums, products, conjugates and multiples by complex numbers of polynomials are again polynomials. Thus, Λ is a unital *-subalgebra of $\mathfrak{C}$ and (B1) holds. For all $t \in \mathcal{T}$ the set

$$\Lambda_t := \{a_t \mid (a_t)_{t\in\mathcal{T}} \in \Lambda\} = \{p(A_t) \mid p \in \mathcal{P}_n\}$$

is dense in $\mathfrak{C}_t = C_1^*(A_t)$ by definition (cf. Equation (2.1)) so (B2) holds. (B3) holds by assumption. Therefore, Λ is a generating family and there exists a unique subset $\Gamma \subseteq \mathfrak{C}$ with $\Lambda \subseteq \Gamma$ such that $((\mathfrak{C}_t)_{t\in\mathcal{T}}, \Gamma)$ is a continuous field of unital C^*-algebras by Lemma 2.4.3.

(a) is satisfied by definition. Let $k \in \{1, \dots, n\}$ and let $p_k(z_1, \dots, z_n) := z_k$. Then, $p_k \in \mathcal{P}_n$ and

$$(a_{t,k})_{t\in\mathcal{T}} = (p_k(A_t))_{t\in\mathcal{T}} \in \Lambda \subseteq \Gamma$$

is a continuous section. Hence, (b) holds.

$(iii) \Rightarrow (i)$: This is the content of Theorem 2.4.8. □

Corollary 2.5.2. *In the setting of Theorem* 2.5.1 (i), (ii) *and* (iii) *are also equivalent to the following.*

(iv) There exists a subset $\Gamma \subseteq \prod_{t\in\mathcal{T}} C_1^*(A_t)$ *such that* $((C_1^*(A_t))_{t\in\mathcal{T}}, \Gamma)$ *is a continuous field of unital* C^**-algebras with generating family*

$$\Lambda = \{(p(A_t))_{t\in\mathcal{T}} \mid p \in \mathcal{P}_n\} \subseteq \Gamma.$$

(v) For all $\phi \in C(\mathbb{C}^n)$ *the map* $\mathcal{T} \ni t \mapsto \|\phi(A_t)\| \in [0, \infty)$ *is continuous.*

Proof. The equivalence of (i), (ii), (iii) and (iv) follows from the proof of the above theorem.

Assume (iii). Then, for all $\phi \in C(\mathbb{C}^n)$: $(\phi(A_t))_{t\in\mathcal{T}} \in \Gamma$ is a continuous normal section by Theorem 2.4.8. Therefore, (v) holds by the definition of continuous fields of C^*-algebras (B3). (v) implies (ii) because polynomials are continuous. □

Example 2.5.3. Consider the case $n = 1$. Let $\mathcal{T} = \mathcal{N}(\mathcal{H})$ be the subset of normal operators of $\mathbb{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. Then, $(a)_{a\in\mathcal{N}(\mathcal{H})}$ defines a normal field of bounded operators on the constant field of Hilbert spaces $(\mathcal{H})_{a\in\mathcal{N}(\mathcal{H})}$. For all $p \in \mathcal{P}_n$ the map

$$\mathcal{N}(\mathcal{H}) \ni a \mapsto \|p(a)\| \in [0, \infty)$$

is continuous because adjoining, scaling, products, sums and the norm map are continuous on $\mathbb{B}(\mathcal{H})$. Therefore, the map

$$\mathcal{N}(\mathcal{H}) \ni a \mapsto \sigma(a) \in \mathcal{K}(\mathbb{C})$$

is continuous. In particular, if $(a_n)_{n\in\mathbb{N}} \subseteq \mathcal{N}(\mathcal{H})$ is a sequence of normal operators that converges in operator norm: $a_n \xrightarrow{n\to\infty} a \in \mathcal{N}(\mathcal{H})$, then $\sigma(a_n) \xrightarrow{n\to\infty} \sigma(a)$ in the Hausdorff metric.

Remark 2.5.4. In applications one might like to study the spectrum of a given normal (in physics often self-adjoint) operator $a \in \mathcal{N}(\mathcal{H})$. If this problem is too hard one can resort to an approximation by a sequence $(a_n)_{n\in\mathbb{N}}$ and consider some finite n. For this to be 'reasonable' the spectra $\sigma(a_n)$ need to converge to $\sigma(a)$ in some sense. But often convergence of the a_n in operator norm is too much of a restriction. As

Beckus shows in [1] one can use statement *(ii)* from the above theorem to define a topology on $\mathbb{B}(\mathcal{H})$. It is constructed in such a way that $\sigma(a_n)$ converges in the Hausdorff metric if and only if a_n converges in this topology. He then shows that this topology is coarser than the topology induced by the operator norm. But in general it is neither finer nor coarser than the strong operator topology. In particular, strong convergence of a_n is in general not sufficient for the convergence of $\sigma(a_n)$.

Example 2.5.5. Let $(\mathcal{T},\, d)$ be a compact metric space and let $K \subseteq \mathbb{C}$ be compact and non-empty. Consider the Hilbert space $L^2(K, \mathrm{d}x)$. Any continuous function f on K acts as a normal, bounded operator $\mathrm{Mult}(f)$ on this Hilbert space via pointwise multiplication. The norm is $\|\mathrm{Mult}(f)\| = \|f\|_\infty = \max_{z \in K} |f(z)|$.

Let

$$F\colon \mathcal{T} \times K \to \mathbb{C}^n, \qquad (t,\, z) \mapsto F(t, z) := (f_1(t, z), \dots, f_n(t, z)),$$

be a continuous map. This yields the normal, commuting field of families of bounded operators

$$(A_t)_{t \in \mathcal{T}} := ((\mathrm{Mult}(f_k(t, \cdot)))_{k=1}^n)_{t \in \mathcal{T}}$$

on the constant field of Hilbert spaces $(L^2(K, \mathrm{d}x))_{t \in \mathcal{T}}$.

Claim: For every polynomial $p \in \mathcal{P}_n$ the map

$$\mathcal{T} \ni t \mapsto \|p(A_t)\| = \|p \circ F(t, \cdot)\|_\infty \in [0,\, \infty)$$

is uniformly continuous.

Proof. $p \circ F$ is uniformly continuous because it is continuous and $\mathcal{T} \times K$ is a compact metric space. Let $\epsilon > 0$. Then, there exists a $\delta > 0$ such that for all $t_1, t_2 \in \mathcal{T}$ and $z_1, z_2 \in K$

$$d(t_1, t_2) + |z_1 - z_2| < \delta \;\Rightarrow\; |p \circ F(t_1, z_1) - p \circ F(t_1, z_2)| < \epsilon.$$

Apply this for '$z = z_1 = z_2$'. Then, $d(t_1, t_2) < \delta$ implies

$$\begin{aligned} \big|\|p \circ F(t_1, \cdot)\|_\infty - \|p \circ F(t_2, \cdot)\|_\infty\big| &\leq \|p \circ F(t_1, \cdot) - p \circ F(t_2, \cdot)\|_\infty \\ &= \max_{z \in K} |p \circ F(t_1, z) - p \circ F(t_2, z)| < \epsilon. \end{aligned}$$

□

From Theorem 2.5.1 it follows that

$$\mathcal{T} \ni t \mapsto \sigma_j(A_t) = \sigma_j((\mathrm{Mult}(f_k(t,\cdot)))_{k=1}^n) \in \mathcal{K}(\mathbb{C}^n)$$

is continuous. Furthermore, $(C_1^*((\mathrm{Mult}(f_k(t,\cdot)))_{k=1}^n))_{t\in\mathcal{T}}$ yields a continuous field of unital C^*-algebras, where the continuous sections Γ can be constructed as described in Corollary 2.5.2.

For every $t \in \mathcal{T}$ these C^*-algebras are (isomorphic to) C^*-subalgebras of $C(K)$. The characters on $C(K)$ are exactly the evaluation maps at fixed points: For $z \in K$ and $f \in C(K)$ define $\chi_z(f) := f(z)$. Then, $\widehat{C(K)} = \{\chi_z \mid z \in K\}$.[2] Thus,

$$\begin{aligned}\sigma_j(A_t) &= \{(\chi_z(f_1(t,\cdot)),\dots,\chi_z(f_k(t,\cdot))) \mid z \in K\}\\ &= \{F(t,z) \mid z \in K\}\\ &= \mathrm{Im}(F(t,\cdot)).\end{aligned}$$

For the special case $n = 1$ an example for the continuous change of spectra is depicted in Figure 2.2.

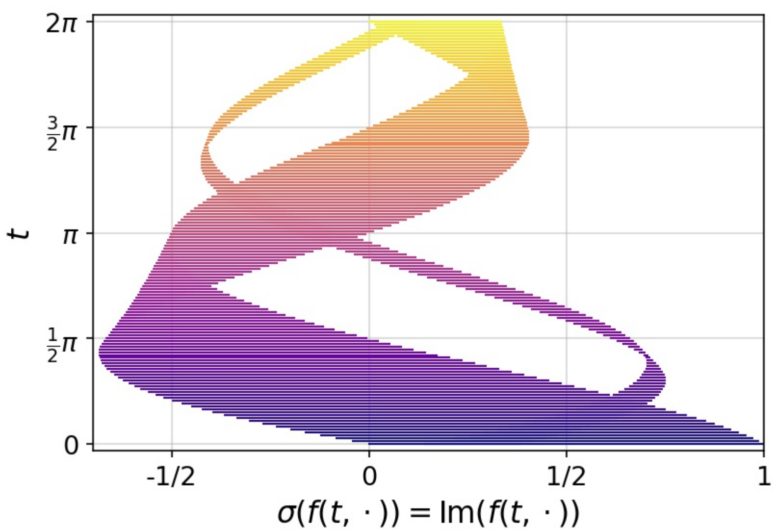

Figure 2.2: For the case of $n = 1$ and $f(t,z) := \pi\sin(z-t)/(t+\pi)$ the spectra of $\mathrm{Mult}(f(t,\cdot))$ as operators on $L^2([0,\ \pi/2] \cup [11/4,\ 3],\ \mathrm{d}x)$ are depicted for $t \in [0,\ 2\pi]$.

[2] $\widehat{C(X)} \cong X$ for any compact Hausdorff space X in this way.

3 Unbounded strongly commuting normal operators

We aim to generalize our results to the case of unbounded operators. The next three subsections introduce the necessary notions and results that we need to work with unbounded operators. They are mainly based on the textbook by Schmüdgen [9]. We omit the proofs.

3.1 Unbounded operators

Definition 3.1.1. An *unbounded operator* a on a Hilbert space $\mathcal{H}$ is a linear, not necessarily continuous map

$$a\colon \mathcal{D}(a) \to \mathcal{H}.$$

Here, the linear subspace $\mathcal{D}(a) \subseteq \mathcal{H}$ is called the *domain* of a. The set of unbounded operators on a given Hilbert space $\mathcal{H}$ is denoted by $\mathbb{L}(\mathcal{H})$. Two unbounded operators $a_1, a_2 \in \mathbb{L}(\mathcal{H})$ are said to be equal ($a_1 = a_2$) if

(i) the domains are equal: $\mathcal{D}(a_1) = \mathcal{D}(a_2)$,

(ii) they coincide on the common domain: $\forall x \in \mathcal{D}(a_1)\colon a_1(x) = a_2(x)$.

The following definitions are such that they reduce to the previously introduced notions in the special case of bounded operators.
If the domain of an unbounded operator is dense in the corresponding Hilbert space, we speak of *densely defined* unbounded operators. In this case we can define the *adjoint operator*.

Lemma 3.1.2. *Let $a \in \mathbb{L}(\mathcal{H})$ be densely defined. Then, there is a well defined unbounded operator $a^* \in \mathbb{L}(\mathcal{H})$ with domain*

$$\mathcal{D}(a^*) = \{y \in \mathcal{H} \mid \exists u \in \mathcal{H} :\ \forall x \in \mathcal{D}(a)\ \langle a(x), y\rangle = \langle x, u\rangle\}$$

that satisfies

$$\forall x \in \mathcal{D}(a)\ \forall y \in \mathcal{D}(a^*)\colon \langle a(x), y\rangle = \langle x, a^*(y)\rangle.$$

a^* is called the adjoint operator *of a.*[3]

Remark 3.1.3. The adjoint need not be densely defined.

Definition 3.1.4. Let $a \in \mathbb{L}(\mathcal{H})$ be densely defined. Then a is called *self-adjoint* if $a^* = a$.

[3] $\langle\,\cdot\,,\,\cdot\,\rangle$ denotes the inner product on $\mathcal{H}$. It is taken to be linear in the second argument.

Remark 3.1.5. Note that

$$\forall x, y \in \mathcal{D}(a)\colon \langle a(x), y\rangle = \langle x, a(y)\rangle$$

is not a sufficient condition for a to be self-adjoint. Similarly, $a^*a = aa^*$ is not sufficient for a to be normal if we want similar properties as in the bounded case.

Definition 3.1.6. An unbounded operator $a \in \mathbb{L}(\mathcal{H})$ is called *closed* if its *graph*

$$\mathcal{G}(a) := \{(x, a(x)) \mid x \in \mathcal{D}(a)\}$$

is a closed subset of the direct sum $\mathcal{H} \oplus \mathcal{H}$. For non-closed operators it is possible that the closure of the graph $\overline{\mathcal{G}(a)}$ defines a valid unbounded operator. Then, this operator is denoted $\overline{a}$ and called the *closure* of a.

Remark 3.1.7. For unbounded operators $a, b \in \mathbb{L}(\mathcal{H})$ the notion $a \subseteq b$ is meant as an inclusion of the graphs: $\mathcal{G}(a) \subseteq \mathcal{G}(b)$.

Definition 3.1.8. An unbounded, densely defined operator $a \in \mathbb{L}(\mathcal{H})$ is called *normal* if

(i) a is closed: $\bar{a} = a$,

(ii) $a^*a = aa^*$.

This is equivalent to requiring $\mathcal{D}(a) = \mathcal{D}(a^*)$ and

$$\forall x \in \mathcal{D}(a)\colon \|a(x)\| = \|a^*(x)\|.$$

Remark 3.1.9. The adjoint of an unbounded, densely defined operator is closed. In particular, self-adjoint operators are closed and therefore also normal.

Remark 3.1.10. Let $a \in \mathbb{L}(\mathcal{H})$ be injective. Define an operator $a^{-1} \in \mathbb{L}(\mathcal{H})$ by $\mathcal{D}(a^{-1}) := \operatorname{Im}(a)$ and for $x \in \mathcal{D}(a)\colon a^{-1}(a(x)) := x$. Denote by I the identity operator on $\mathcal{H}$. Then,

$$aa^{-1},\ a^{-1}a \subseteq I.$$

a^{-1} is the *inverse* of a in $\mathbb{L}(\mathcal{H})$.

We regard the complex numbers as a subset of $\mathbb{B}(\mathcal{H}) \subseteq \mathbb{L}(\mathcal{H})$ via $\lambda \mapsto \lambda \cdot I$.

Definition 3.1.11. Let $a \in \mathbb{L}(\mathcal{H})$ be closed. The *resolvent set* of a is

$$\rho(a) := \{\lambda \in \mathbb{C} \mid (a - \lambda) \text{ has a bounded, everywhere on } \mathcal{H} \text{ defined inverse}\},$$

that is, the set of complex numbers λ for which the *resolvent function* $R_\lambda(a) := (a-\lambda)^{-1}$ exists in $\mathbb{B}(\mathcal{H})$. The *spectrum* of a is

$$\sigma(a) := \mathbb{C}\backslash\rho(a).$$

Lemma 3.1.12. *The spectra of closed unbounded operators are closed subsets of* $\mathbb{C}$.

3.2 Spectral measures and integrals

The Spectral Theorem for unbounded, normal operators can be formulated in terms of spectral measures.

Definition 3.2.1. Let $\mathcal{A}$ be a σ-algebra on a set Ω. A *spectral measure* on $\mathcal{A}$ is a map E from $\mathcal{A}$ into the orthogonal projections on a given Hilbert space $\mathcal{H}$ such that

(i) $E(\Omega) = I$,

(ii) E is countably additive: For any sequence $(M_m)_{m\in\mathbb{N}} \subseteq \mathcal{A}$ of pairwise disjoint sets $M_m \subseteq \Omega$ whose union is in $\mathcal{A}$ we have

$$E\left(\bigcup_{m=1}^{\infty} M_m\right) = \sum_{m=1}^{\infty} E(M_m).$$

The infinite sum of orthogonal projections is required to converge strongly.

Remark 3.2.2. Given point (i), point (ii) is equivalent to

(iii) $\forall x \in \mathcal{H}\colon E_x(\cdot) := \langle x, E(\cdot)x\rangle$ defines a (scalar) measure on Ω.

$E(\emptyset) = 0$ and spectral projections are also finitely additive.

Lemma 3.2.3. *Let E be a spectral projection on the σ-algebra $\mathcal{A}$. Let $M_1, M_2 \in \mathcal{A}$. Then,*

$$E(M_1)E(M_2) = E(M_1 \cap M_2).$$

If $M_1 \subseteq M_2$, then $E(M_1) \leq E(M_2)$.

From here on we will only consider the case where $\Omega \subseteq \mathbb{C}^n$ and $\mathcal{A} = \mathcal{B}(\Omega)$ is the Borel σ-algebra on Ω.

Definition 3.2.4. The *support* of a spectral measure E is the complement of all open sets whose measure is the zero projection

$$\operatorname{supp}(E) := \Omega \setminus \left(\bigcup_{\substack{N \subseteq \Omega \text{ open} \\ E(N)=0}} N \right).$$

Similar to Lebesgue integration one can define spectral integrals of E-a.e. finite Borel measurable functions f:

$$\mathbb{I}(f) \equiv \int_\Omega f(t)\,\mathrm{d}E(t) \equiv \int_\Omega f\,\mathrm{d}E \in \mathbb{L}(\mathcal{H}).$$

For measurable sets $M \subseteq \Omega$ and their characteristic functions χ_M one defines

$$\mathbb{I}(\chi_M) := E(M)$$

and then proceeds as in the case of usual (scalar) measures. Special care has to be taken for the case of unbounded functions. We omit further details on the construction and state some properties.[4]

Proposition 3.2.5. *Let $\Omega \subseteq \mathbb{C}^n$ and let E be a spectral measure on $\mathcal{B}(\Omega)$ with values in the orthogonal projections of a Hilbert space $\mathcal{H}$. Let f, g be Borel measurable E-a.e. finite functions on Ω. Let $\alpha, \beta \in \mathbb{C}$ and let $x \in \mathcal{D}(\mathbb{I}(f))$, $y \in \mathcal{D}(\mathbb{I}(g))$. Then*

(i) $\mathcal{D}(\mathbb{I}(f)) = \left\{ v \in \mathcal{H} \mid \int_\Omega |f(t)|^2\,\mathrm{d}\langle v, E(t)v\rangle < \infty \right\}$,

(ii) $\langle \mathbb{I}(f)x, \mathbb{I}(g)y\rangle = \int_\Omega f(t)^* g(t)\,\mathrm{d}\langle x, E(t)y\rangle$,

(iii) $\mathbb{I}(f^*) = \mathbb{I}(f)^*$,

(iv) $\mathbb{I}(\alpha f + \beta g) = \overline{\alpha\mathbb{I}(f) + \beta\mathbb{I}(g)}$,

(v) $\mathbb{I}(fg) = \overline{\mathbb{I}(f)\mathbb{I}(g)}$,

(vi) $\mathbb{I}(f)$ *is normal, in particular closed and* $\mathbb{I}(f)^*\mathbb{I}(f) = \mathbb{I}(f^* f)$,

(vii) f *E-a.e. real-valued (non-negative)* $\Rightarrow \mathbb{I}(f)$ *is self-adjoint (and positive)*,

(viii) $f = g$ *E-a.e.* $\Rightarrow \mathbb{I}(f) = \mathbb{I}(g)$,

(ix) f *E-a.e. bounded* $\Rightarrow \mathbb{I}(f) \in \mathbb{B}(\mathcal{H})$ *and* $\|\mathbb{I}(f)\| = \|f\|_\infty$,

[4]For the purpose of this thesis one can also read the following proposition as 'there exists a map $\mathbb{I}\colon \{E\text{-a.e. finite Borel functions}\} \to \mathbb{L}(\mathcal{H})$ with the listed properties'.

(x) *f E-a.e. non zero* $\Leftrightarrow \mathbb{I}(f)$ *invertible. In this case* $\mathbb{I}(f)^{-1} = \mathbb{I}(1/f)$,

(xi) *The spectrum of* $\mathbb{I}(f)$ *is the essential range of* f:
$\sigma(\mathbb{I}(f)) = \{\lambda \in \mathbb{C} \mid \forall \epsilon > 0\colon E(\{t \in \Omega \mid |f(t) - \lambda| < \epsilon\}) \neq 0\}$,

(xii) *For* p *a polynomial in one complex variable and its conjugate:* $p(\mathbb{I}(f)) = \mathbb{I}(p \circ f)$.

Remark 3.2.6. $\langle x, E(t)y\rangle$ can be rewritten as a linear combination of four scalar measures using the polarization formula.

3.3 Spectral Theorem

We can transform unbounded operators into bounded ones.

Proposition 3.3.1. *Let* $a \in \mathbb{L}(\mathcal{H})$ *be densely defined and closed. Then,*

$$b(a) := a(I + a^*a)^{-1}$$

is a bounded linear operator defined on $\mathcal{H}$ *with norm less than or equal to one.* $b(a)$ *is called the* bounded transform *of* a. *If* a *is normal, self-adjoint or positive, then so is* $b(a)$.

In the finite-dimensional and bounded cases there are nice spectral theorems for several commuting normal operators. To get analogous results in the unbounded case we need the correct notion of commutativity.

Definition 3.3.2. We say that two unbounded, normal operators $a_1, a_2 \in \mathbb{L}(\mathcal{H})$ *strongly commute* if their bounded transforms commute: $b(a_1)b(a_2) = b(a_2)b(a_1)$.

Lemma 3.3.3. *For two unbounded, normal operators* $a_1, a_2 \in \mathbb{L}(\mathcal{H})$ *consider the following statements:*

(i) a_1 *and* a_2 *strongly commute.*

(ii) $\exists \lambda \in \rho(a_1)\colon R_\lambda(a_1)a_2 \subseteq a_2 R_\lambda(a_1)$.

(iii) $\forall \lambda \in \rho(a_1)\colon R_\lambda(a_1)a_2 \subseteq a_2 R_\lambda(a_1)$.

(iv) $\exists \lambda \in \rho(a_1)\ \exists \mu \in \rho(a_2)\colon R_\lambda(a_1)R_\mu(a_2) = R_\mu(a_2)R_\lambda(a_1)$.

(v) *Their resolvents commute:*
$\forall \lambda \in \rho(a_1)\ \forall \mu \in \rho(a_2)\colon R_\lambda(a_1)R_\mu(a_2) = R_\mu(a_2)R_\lambda(a_1)$.

(vi) $a_1 a_2 \subseteq a_2 a_1$.

(vii) They commute: $a_1 a_2 = a_2 a_1$.

If $\rho(a_1) \neq \emptyset$ *then* $(i), (ii)$ *and* (iii) *are equivalent. If* $\rho(a_1) \neq \emptyset$ *and* $\rho(a_2) \neq \emptyset$ *then* $(i), (iv)$ *and* (v) *are equivalent. If* a_1 *is bounded then* (i) *and* (vi) *are equivalent. If* a_1 *and* a_2 *are both bounded then* $(i), (vi)$ *and* (vii) *are equivalent.*

Theorem 3.3.4 (Spectral Theorem)**.** *Let* $A = (a_k)_{k=1}^n \subseteq \mathbb{L}(\mathcal{H})$ *be* $n \in \mathbb{N}$ *pairwise strongly commuting normal operators. Then, there exists a unique spectral measure* E *on the Borel* σ*-algebra* $\mathcal{B}(\mathbb{C}^n)$ *such that for all* $k \in \{1, \ldots, n\}$

$$a_k = \int_{\mathbb{C}^n} z_k \, \mathrm{d}E(z_1, \ldots, z_n).$$

In the setting of the above theorem we can define a functional calculus.

Lemma 3.3.5. *Let* E *be the spectral measure for* A *and let* f *be an* E*-a.e. finite Borel function on* $\mathbb{C}^n$*. Define*

$$f(A) = f(a_1, \ldots, a_n) := \mathbb{I}(f) = \int_{\mathbb{C}^n} f(z) \, \mathrm{d}E(z).$$

$f(A)$ *commutes strongly with all* a_k *and also with any other Borel function of* A.

Definition 3.3.6. In the setting of the Spectral Theorem 3.3.4 we define the *joint spectrum* of A

$$\sigma_j(A) := \operatorname{supp}(E).$$

This is a closed, non-empty subset of $\mathbb{C}^n$.

Lemma 3.3.7. *For every continuous function* $f \in C(\sigma_j(A))$

$$\sigma(f(A)) = \overline{f(\sigma_j(A))}.$$

Remark 3.3.8. In the case of a single operator $A = (a_1)$ the joint spectrum coincides with the usual spectrum of a_1: $\sigma_j((a_1)) = \sigma(a_1)$.

Remark 3.3.9. For $A = (a_k)_{k=1}^n \subseteq \mathbb{L}(\mathcal{H})$ as in the Spectral Theorem

$$\sigma_j(A) \subseteq \prod_{k=1}^n \sigma(a_k).$$

$\sigma_j(A)$ consists of the *joint approximate eigenvalues* of A: $\lambda = (\lambda_1, \ldots, \lambda_n) \in \sigma_j(A)$ if and only if there exists a sequence of unit vectors $(x_m)_{m\in\mathbb{N}} \subseteq \bigcap_{k=1}^n \mathcal{D}(a_k)$ such that

$$\lim_{m\to\infty} (a_k(x_m) - \lambda_k \cdot x_m) = 0 \quad \text{for all } k \in \{1, \dots, n\}.$$

Remark 3.3.10. Let $K \subseteq \mathbb{C}^n$ be a Borel set such that $\sigma_j(A) \subseteq K$. Then

$$a_k = \int_K z_k \, \mathrm{d}E(z).$$

We show that this notion of the joint spectrum and the functional calculus reduce to the definitions for the bounded case from Section 2.2.

Lemma 3.3.11. *Let $A = (a_k)_{k=1}^n \subseteq \mathbb{B}(\mathcal{H}) \subseteq \mathbb{L}(\mathcal{H})$ be pairwise commuting normal bounded operators. Then,*

(i) the functional calculus as defined in Lemma 3.3.5 *coincides with the continuous functional calculus for all $f \in C(\mathbb{C}^n)$,*

(ii) the joint spectrum of A in the sense of unbounded operators coincides with the joint spectrum as defined for bounded operators in Section 2.2.

Proof. Let E be the spectral measure for A. Let $\sigma_u = \operatorname{supp}(E)$ be the joint spectrum of A as in the above definition. Let σ_b be the joint spectrum of A as defined in Section 2.2 for bounded operators. $\sigma_b \subseteq \mathbb{C}^n$ is compact and non-empty by Corollary 2.2.9. σ_u is compact and non-empty by Remark 3.3.9.

Recall that we denote the set of complex polynomials in n variables and their conjugates by $\mathcal{P}_n$. Let $p \in \mathcal{P}_n$. The assertion $p(A) = \mathrm{CFC}(p)$ holds by Remark 2.2.17. Here, the left hand side denotes the formal replacement $(z_1, \dots, z_n) \mapsto (a_1, \dots, a_n)$ in the argument of p. The right hand side denotes the operator obtained by the continuous functional calculus. Similarly, $p(A) = \mathbb{I}(p)$ by Proposition 3.2.5 ((*iii*), (*iv*), (*v*)) because bounded operators are closed.

Let χ_u, χ_b and χ_{ub} denote the characteristic functions for σ_u, σ_b and $\sigma_u \cup \sigma_b$, respectively. Let $f \in C(\mathbb{C}^n)$. Then

$$\mathbb{I}(f) = \mathbb{I}(\chi_u f) = \mathbb{I}(\chi_{ub} f) \quad \text{and} \quad \mathrm{CFC}(f) = \mathrm{CFC}(\chi_b f) = \mathrm{CFC}(\chi_{ub} f).$$

Now, there exists a sequence $(p_m)_{m\in\mathbb{N}} \subseteq \mathcal{P}_n$ such that $\chi_{ub} p_m \xrightarrow{m\to\infty} \chi_{ub} f$ uniformly. This implies

$$\mathrm{CFC}(f) = \lim_{m\to\infty} \mathrm{CFC}(\chi_{ub} p_m) = \lim_{m\to\infty} p_m(A) = \lim_{m\to\infty} \mathbb{I}(\chi_{ub} p_m) = \mathbb{I}(f).$$

For the first equality we use that the inverse Gelfand transform is continuous. For the last equality we use that the map $\mathbb{I}$ is continuous on bounded functions by Proposition 3.2.5 (ix). This shows (i).

Suppose there exists $z \in \sigma_u$ such that $z \notin \sigma_b$. Then, there exists a continuous function $f\colon C(\mathbb{C}^n) \to [0, 1]$ such that $f(z) = 1$ and $f|_{\sigma_b} \equiv 0$ by Urysohn's Lemma. Therefore, $0 = \mathrm{CFC}(\chi_b f) = \mathbb{I}(\chi_u f)$. Then, $1 = \|\chi_u f\|_\infty = \|\mathbb{I}(\chi_u f)\| = 0$ by Proposition 3.2.5 (ix). This is a contradiction.
Suppose there exists $z \in \sigma_b$ such that $z \notin \sigma_u$. Then, there exists a continuous function $f\colon C(\mathbb{C}^n) \to [0, 1]$ such that $f(z) = 1$ and $f|_{\sigma_u} \equiv 0$. Therefore, $0 = \mathbb{I}(\chi_u f) = \mathrm{CFC}(\chi_b f)$. This is a contradiction because the Gelfand transform is an isomorphism and $f|_{\sigma_b} \not\equiv 0$.
Hence, $\sigma_b = \sigma_u$. This shows (ii). □

3.4 Vietoris and Fell topology

Let $B := \{\lambda \in \mathbb{C} \mid |\lambda| < 1\} \subseteq \mathbb{C}$ be the open unit ball. We define the map

$$b : \mathbb{C}^n \to B^n, \qquad (z_1, \ldots, z_n) = z \mapsto (b_1(z), \ldots, b_n(z)),$$

where for $k \in \{1, \ldots, n\}$

$$b_k(z) = \frac{z_k}{\sqrt{1 + z_k^* z_k}}.$$

This is a homeomorphism when considering the induced topology on B^n. The inverse is given by

$$b^{-1} : B^n \to \mathbb{C}^n, \qquad (u_1, \ldots, u_n) = u \mapsto (b_1^{-1}(u), \ldots, b_n^{-1}(u)),$$

where for $k \in \{1, \ldots, n\}$

$$b_k^{-1}(u) = \frac{u_k}{\sqrt{1 - u_k^* u_k}}.$$

Definition 3.4.1. For a family of normal and pairwise strongly commuting unbounded operators $A = (a_k)_{k=1}^n \subseteq \mathbb{L}(\mathcal{H})$ the family of normal bounded operators

$$(b_k)_{k=1}^n = B_A := b(A) := (b_k(A))_{k=1}^n \subseteq \mathbb{B}(\mathcal{H})$$

is called the *bounded transform* of A.

The joint spectrum for unbounded operators is closed but in general not compact.

Definition 3.4.2. Let X be a Hausdorff space. We define the space of closed, non-empty subsets of X

$$\mathcal{C}(X) := \{K \subseteq X \mid K \text{ closed}, \ K \neq \emptyset\}.$$

Now the Hausdorff metric is not well-defined on $\mathcal{C}(X)$. The distance between two closed subsets may be formally infinite. However, the characterization of the basis elements for the Hausdorff topology is adaptable.

Definition 3.4.3 ([1])**.** Let X be a Hausdorff space. Sets of the form

$$\mathcal{U}_X(F,\ (O_k)_{k=1}^m) := \{K \in \mathcal{C}(X) \mid K \cap F = \emptyset,\ \forall k \in \{1, \dots, m\} \colon K \cap O_k \neq \emptyset\}$$

for $F \subseteq X$ closed and $O_k \subseteq X$ open for all k define a basis for a topology on $\mathcal{C}(X)$. This is the *Vietoris topology* on $\mathcal{C}(X)$. If we require the sets F to be compact the above sets $\mathcal{U}_X$ define a basis for the *Fell topology* on $\mathcal{C}(X)$.

Remark 3.4.4 ([1]). For complete metric spaces the Hausdorff topology on $\mathcal{K}(X) \subseteq \mathcal{C}(X)$ corresponds to the subspace topology induced by the Vietoris topology. For compact spaces the Vietoris and Fell topology coincide. For compact metric spaces all three topologies coincide. In general, the Fell topology is coarser than the Vietoris topology.

In the following, we will call maps *Hausdorff*, *Vietoris* or *Fell continuous* if they are continuous with respect to these topologies on the domain and/or the codomain depending on the context.

We generalize Lemma 2.3.5 in different ways.

Lemma 3.4.5. *Let X, Y be locally compact Hausdorff spaces. Let $\phi \colon X \to Y$ be a continuous and closed map. Define*

$$\widetilde{\phi} \colon \mathcal{C}(X) \to \mathcal{C}(Y), \qquad K \mapsto \phi(K).$$

Then, $\widetilde{\phi}$ is Vietoris continuous. If ϕ is also a proper map, then $\widetilde{\phi}$ is Fell continuous.

Proof. $\widetilde{\phi}$ is well-defined because ϕ is a closed map. For the case of Vietoris continuity the rest of the proof is the same as for Lemma 2.3.5. This proof can also be found in [1] (Proposition 2.2.3).
For the case of Fell continuity the proof is again the same. The appearing set $\mathcal{U}_X(\phi^{-1}(F), (\phi^{-1}(O_k))_{k=1}^m)$ is Fell-open in $\mathcal{C}(X)$ for compact $F \subseteq Y$ and open $O_k \subseteq Y$ because ϕ is proper. $\square$

Corollary 3.4.6. *The maps* $b\colon C^n \to B^n$ *and* $b^{-1}\colon B^n \to \mathbb{C}^n$ *are continuous and proper because they are homeomorphisms. Thus, the corresponding maps* $\widetilde{b}$ *and* $\widetilde{b^{-1}}$ *are Vietoris and Fell continuous.*

In the case that ϕ is only continuous we need to take the closure of the image to get a well-defined map.

Lemma 3.4.7 (Prop. II.7 in [10])**.** *Let* X, Y *be Hausdorff spaces, let* Y *be normal and let* $\phi\colon X \to Y$ *be continuous. Define*

$$\overline{\phi}\colon \mathcal{C}(X) \to \mathcal{C}(Y), \qquad K \mapsto \overline{\phi(K)}.$$

Then, $\overline{\phi}$ *is Vietoris continuous.*

Corollary 3.4.8. *View* b *as a map* $b\colon \mathbb{C}^n \to \mathbb{C}^n$*. Then, the corresponding map* $\overline{b}$ *is Vietoris continuous.*

Lemma 3.4.9. *Let* X, Y *be Hausdorff spaces such that* $Y \subseteq X$ *is an open subset. Consider the subspace topology on* Y*. Then, the restriction map*

$$R\colon \mathcal{C}(X) \to \mathcal{C}(Y), \qquad K \mapsto K \cap Y,$$

is Fell-continuous.

Proof. R is well-defined by the definition of the subspace topology on Y. Let $K_0 \in \mathcal{C}(X)$ and let $W \subseteq \mathcal{C}(Y)$ be an open neighborhood of $R(K_0) = K_0 \cap Y$. W.l.o.g. $W = \mathcal{U}_Y(F, (O_k)_{k=1}^m)$ for a compact subset $F \subseteq Y$ and open subsets $O_k \subseteq Y$. Now, $F \subseteq X$ is compact because the inclusion $Y \hookrightarrow X$ is continuous. For every k: $O_k \subseteq X$ is open because $Y \subseteq X$ is open. This yields the Fell-open set $V := \mathcal{U}_X(F, (O_k)_{k=1}^m) \subseteq \mathcal{C}(X)$. Let $K \in \mathcal{C}(X)$. Then,

$$\begin{aligned} K \in V \;\Leftrightarrow\;& R(K) \cap F = K \cap Y \cap F = K \cap F = \emptyset \\ & \text{and for all } k\colon R(K) \cap O_k = K \cap Y \cap O_k = K \cap O_k \neq \emptyset \\ \Leftrightarrow\;& R(K) \in W. \end{aligned}$$

Therefore, $K_0 \in V$ and V is an open neighborhood of K_0. Also, $R(V) \subseteq W$. Thus, R is continuous as it is continuous at every point. ☐

Corollary 3.4.10. *The map*

$$R\colon \mathcal{C}(\mathbb{C}^n) \to \mathcal{C}(B^n), \qquad K \mapsto K \cap B^n,$$

is Fell continuous.

3.5 Bounded transform

In the proof of the Spectral Theorem in [9, pp. 101 sq.] some more useful relations are proven. In the following, closures are meant with respect to the topology on $\mathbb{C}^n$.

Lemma 3.5.1. *Let $A = (a_k)_{k=1}^n \subseteq \mathbb{L}(\mathcal{H})$ be a family of pairwise strongly commuting normal operators. Let E be the spectral measure for A and let F be the spectral measure for $b(A)$. Then*

(i) $\operatorname{supp} F \subseteq \overline{B^n}$,

(ii) $F(B^n) = I$, $F(\mathbb{C}^n \backslash B^n) = 0$,

(iii) $E = F \circ b$.

For the last equality we view b as a map from $\mathcal{B}(\mathbb{C}^n)$ into itself via $M \to b(M)$.

The bounded transform is one possibility to obtain bounded operators from unbounded ones. The following theorem shows how the corresponding spectra are related.

Theorem 3.5.2. *Let $A = (a_k)_{k=1}^n \subseteq \mathbb{L}(\mathcal{H})$ be a family of normal and pairwise strongly commuting unbounded operators. Then*

(i) $b(\sigma_j(A)) = \sigma_j(b(A)) \cap B^n$,

(ii) $\sigma_j(b(A)) = \overline{b(\sigma_j(A))}$.

Proof. Let E be the spectral measure for A and let F be the spectral measure for $b(A)$. $E = F \circ b$ by Lemma 3.5.1. In fact, this is how E is constructed. We calculate

$$\begin{aligned} b^{-1}(\operatorname{supp}(F) \cap B^n) &= b^{-1}\left(\left(\mathbb{C}^n \backslash \bigcup_{\substack{F(M)=0,\\ M\subseteq\mathbb{C}^n \text{ open}}} M\right) \cap B^n\right) \\ &= b^{-1}\left(B^n \backslash \bigcup_{\substack{F(M)=0,\\ M\subseteq\mathbb{C}^n \text{ open}}} (M \cap B^n)\right) \\ &= b^{-1}(B^n) \backslash \bigcup_{\substack{F(M\cap B^n)=0,\\ M\cap B^n\subseteq B^n \text{ open}}} b^{-1}(M \cap B^n) \\ &= \mathbb{C}^n \backslash \bigcup_{\substack{F(N)=0,\\ N\subseteq B^n \text{ open}}} b^{-1}(N) \end{aligned}$$

$$
\begin{aligned}
&= \mathbb{C}^n \backslash \bigcup_{\substack{E(b^{-1}(N))=0,\\ b^{-1}(N)\subseteq \mathbb{C}^n \text{ open}}} b^{-1}(N) \\
&= \mathbb{C}^n \backslash \bigcup_{\substack{E(P)=0,\\ P\subseteq \mathbb{C}^n \text{ open}}} P \\
&= \operatorname{supp}(E).
\end{aligned}
$$

In the first line we expand the definition for the support of a spectral measure. The second line is obtained by set theoretic manipulations. For the third line we use that b is bijective, the definition of the induced topology on subspaces and the fact that $F(M) = 0$ is equivalent to $F(M \cap B^n) = 0$ for any open set $M \subseteq \mathbb{C}^n$:

$$
\begin{aligned}
&F(M) = 0 \;\Rightarrow\; F(M \cap B^n) \overset{\text{Lemma 3.2.3}}{=} F(M)F(B^n) = 0, \\
&F(M \cap B^n) = 0 \;\Rightarrow\; F(M) = F((M \cap B^n) \cup (M\backslash B^n)) = F(M \cap B^n) + F(M\backslash B^n) \\
&\qquad\qquad\overset{\text{Lemma 3.2.3}}{\leq} F(\mathbb{C}^n\backslash B^n) \overset{\text{Lemma 3.5.1}}{=} 0 \;\Rightarrow\; F(M) = 0.
\end{aligned}
$$

Here, we used the finite additivity of spectral measures and that $F(M) \geq 0$ as an orthogonal projection.

For line four we again use the definition of the subspace topology. In line five we insert the relation between E and F. For line six we use that b is a homeomorphism. The last line holds by definition.

Applying the map b to both sides we obtain

$$
b(\operatorname{supp}(E)) = \operatorname{supp}(F) \cap B^n. \tag{3.1}
$$

Now, inserting the two relations $\operatorname{supp}(F) = \sigma_j(b(A))$ and $\operatorname{supp}(E) = \sigma_j(A)$ we obtain the first equality.

Starting from Equation (3.1)

$$
\overline{b(\operatorname{supp}(E))} = \overline{\operatorname{supp}(F) \cap B^n} \subseteq \overline{\operatorname{supp}(F) \cap \overline{B^n}} = \overline{\operatorname{supp}(F)} = \operatorname{supp}(F).
$$

We used that $\operatorname{supp}(F) \subseteq \overline{B^n}$ by Lemma 3.5.1 and that $\operatorname{supp}(F)$ is closed by definition. Now let $x \in \operatorname{supp}(F)$ and let $U \subseteq \mathbb{C}^n$ be open such that $x \in U$. Then $F(U) \neq 0$ because '$F(U) = 0$' would contradict $x \in \operatorname{supp}(F)$. Consequently,

$$
F(U \cap B^n) \overset{\text{Lemma 3.2.3}}{=} F(U)F(B^n) \overset{\text{Lemma 3.5.1}}{=} F(U) \neq 0.
$$

Suppose $(U \cap B^n) \cap \operatorname{supp}(F) = \emptyset$. Then, $U \cap B^n \subseteq \mathbb{C}\backslash \operatorname{supp}(F)$ and $F(U \cap B^n) \leq$

$F(\mathbb{C}\setminus \operatorname{supp}(F)) = 0$ by Lemma 3.2.3. This means $F(U \cap B^n) = 0$, which is a contradiction. Therefore, $U \cap (\operatorname{supp}(F) \cap B^n) \neq \emptyset$. This implies $x \in \overline{\operatorname{supp}(F) \cap B^n}$ because U was arbitrary. Thus,

$$\operatorname{supp}(F) \subseteq \overline{\operatorname{supp}(F) \cap B^n} = \overline{b(\operatorname{supp}(E))}.$$

We have shown that $\operatorname{supp}(F) = \overline{b(\operatorname{supp}(E))}$. Again, inserting the relations $\operatorname{supp}(F) = \sigma_j(b(A))$ and $\operatorname{supp}(E) = \sigma_j(A)$ we obtain the second equality. □

We will use this result to relate the continuity of the joint spectra for unbounded operators to the continuity of the joint spectra of the corresponding bounded transforms.
In analogy to the bounded case we have the following notion.

Definition 3.5.3. Let $\mathcal{T}$ be a topological space and let $(\mathcal{H}_t)_{t\in\mathcal{T}}$ be a field of Hilbert spaces. For every $t \in \mathcal{T}$ let $A_t \subseteq \mathbb{L}(\mathcal{H}_t)$ be a subset of $n \in \mathbb{N}$ unbounded operators. We call $(A_t)_{t\in\mathcal{T}}$ a *field of families of unbounded operators*. The field is said to be *normal* or *strongly commuting* whenever all A_t consist of normal or pairwise strongly commuting elements, respectively.

Lemma 3.5.4. *Let $\mathcal{T}$ be a topological space and let $(\mathcal{H}_t)_{t\in\mathcal{T}}$ be a field of Hilbert spaces. Let $n \in \mathbb{N}$. For $t \in \mathcal{T}$ let $A_t := (a_{t,k})_{k=1}^n \subseteq \mathbb{L}(\mathcal{H}_t)$ be such that $(A_t)_{t\in\mathcal{T}}$ is a normal, strongly commuting field of families of unbounded operators. Define*

$$B_t := (b_{t,k})_{k=1}^n := b(A_t).$$

Then, $(B_t)_{t\in\mathcal{T}}$ is called the bounded transform *of $(A_t)_{t\in\mathcal{T}}$. $(B_t)_{t\in\mathcal{T}}$ is a normal commuting field of families of bounded operators on $(\mathcal{H}_t)_{t\in\mathcal{T}}$.*

Proof. For every $t \in \mathcal{T}$ the elements of B_t are normal and bounded by Proposition 3.3.1. They pairwise commute by definition of the strong commutativity for the elements of A_t. □

We generalize Theorem 2.5.1 to the case of unbounded operators.

Theorem 3.5.5. *Let $\mathcal{T} \neq \emptyset$ be a topological space and let $(\mathcal{H}_t)_{t\in\mathcal{T}}$ be a field of Hilbert spaces. Let $n \in \mathbb{N}$. For $t \in \mathcal{T}$ let $A_t := (a_{t,k})_{k=1}^n \subseteq \mathbb{L}(\mathcal{H}_t)$ be such that $(A_t)_{t\in\mathcal{T}}$ is a normal strongly commuting field of families of unbounded operators. Let $(B_t)_{t\in\mathcal{T}}$ be its bounded transform. Define the maps*

$$\Sigma\colon \mathcal{T} \to \mathcal{C}(\mathbb{C}^n),\ t \mapsto \sigma_j(A_t), \qquad \textit{and} \qquad \Sigma_b\colon \mathcal{T} \to \mathcal{K}(\mathbb{C}^n),\ t \mapsto \sigma_j(B_t).$$

The following assertions hold.

(i) Theorem 2.5.1 *applies to the normal commuting field of families of bounded operators* $(B_t)_{t\in\mathcal{T}}$.

(ii) Σ *is Vietoris continuous if and only if the norm map*

$$\mathcal{T} \ni t \mapsto \|\phi(A_t)\| \in [0, \infty)$$

is continuous for every bounded continuous function $\phi\colon \mathbb{C}^n \to \mathbb{C}$.

(iii) If Σ *is Vietoris continuous, then* Σ_b *is Hausdorff continuous.*

(iv) If Σ_b *is Hausdorff continuous, then* Σ *is Fell continuous.*

Proof. (i): This holds by Lemma 3.5.4.

(ii), '$\Rightarrow$': This part is similar to the proof of '$(i) \Rightarrow (ii)$' for Theorem 2.5.1. Let $f \in C(\mathbb{C}^n)$ be bounded. Then, the norm map is continuous as a composition of continuous maps:

$$\begin{aligned} \mathcal{T} \ni t \overset{\Sigma}{\mapsto} \sigma_j(A_t) \overset{\overline{\phi}}{\mapsto} \overline{\phi(\sigma_j(A_t))} &\overset{\text{Lemma 3.3.7}}{=} \sigma(\phi(A_t)) \overset{\widetilde{|\cdot|}}{\mapsto} |\sigma(\phi(A_t))| \\ \overset{\sup}{\longmapsto} \sup|\sigma(\phi(A_t))| &\overset{\text{Thm. 2.1.12}}{=} \|\phi(A_t)\| \in [0, \infty). \end{aligned}$$

Σ is Vietoris continuous by assumption. $\overline{\phi}$ is Vietoris continuous by Lemma 3.4.7. For every $t \in \mathcal{T}$: $\phi(A_t)$ is a bounded normal operator because ϕ is bounded (cf. Proposition 3.2.5 (vi), (ix)). Therefore, $\sigma(\phi(A_t))$ is compact. So, $\overline{\phi}$ is also continuous with respect to the Hausdorff topology on the codomain $\mathcal{K}(\mathbb{C})$ (cf. Remark 3.4.4). $\widetilde{|\cdot|}$ is Hausdorff continuous by Corollary 2.3.6 and sup is Hausdorff continuous by Lemma 2.3.7.

(ii), '$\Leftarrow$': Consider the proof of Theorem 2.4.8 (ii). Replace '$\mathcal{K}(\mathbb{C}^n)$' by '$\mathcal{C}(\mathbb{C}^n)$' and interpret $\mathcal{U}_{\mathbb{C}^n}(F, (O_k)_{k=1}^m)$ as the basis elements of the Vietoris topology instead of the Hausdorff topology. Finally, use that the appearing maps ϕ are continuous and bounded. Their corresponding norm maps are continuous by assumption. We do not restate the proof.

(iii): Write Σ_b as a composition of maps

$$\mathcal{T} \ni t \overset{\Sigma}{\mapsto} \sigma_j(A_t) \overset{\overline{b}}{\mapsto} \overline{b(\sigma_j(A_t))} \overset{\text{Thm. 3.5.2}}{=} \sigma_j(b(A_t)) = \sigma_j(B_t) \in \mathcal{C}(\mathbb{C}^n).$$

Σ is Vietoris continuous by assumption. $\overline{b}$ is Vietoris continuous by Corollary 3.4.8. This shows Σ_b to be Vietoris continuous. The Hausdorff topology corresponds to the subspace topology on $\mathcal{K}(\mathbb{C}^n) \subseteq \mathcal{C}(\mathbb{C}^n)$ induced by the Vietoris topology (cf. Remark 3.4.4). Thus, Σ_b is also Hausdorff continuous because $\mathrm{Im}(\Sigma_b) \subseteq \mathcal{K}(\mathbb{C}^n)$.

(*iv*): Write Σ as a composition of maps

$$\mathcal{T} \ni t \overset{\Sigma_b}{\longmapsto} \sigma_j(B_t) \overset{R}{\mapsto} \sigma_j(B_t) \cap B^n \overset{\widetilde{b^{-1}}}{\longmapsto} b^{-1}(\sigma_j(b(A_t) \cap B^n)) \overset{\text{Thm. 3.5.2}}{=} \sigma_j(A_t) \in \mathcal{C}(\mathbb{C}^n).$$

Σ_b is Hausdorff continuous by assumption. The inclusion of its image, a subset of $\mathcal{K}(\mathbb{C}^n)$, into the domain of R, $\mathcal{C}(\mathbb{C}^n)$ with the Fell topology, is also continuous: The Hausdorff topology on $\mathcal{K}(\mathbb{C}^n)$ is the subspace topology induced by the Vietoris topology on $\mathcal{C}(\mathbb{C}^n)$ by Remark 3.4.4. The Fell topology is, in turn, coarser than the Vietoris topology by the same remark. R is Fell continuous by Corollary 3.4.10. $\widetilde{b^{-1}}$ is Fell continuous by Corollary 3.4.6. This shows Σ to be Fell continuous. □

The reverse implications of the statements (*iii*) and (*iv*) are in general not true. In the following lemma we construct two corresponding counterexamples.

Lemma 3.5.6. *In the setting of the above theorem we have:*

(i) The Hausdorff continuity of Σ_b does not imply the Vietoris continuity of Σ.

(ii) The Fell continuity of Σ does not imply the Hausdorff continuity of Σ_b.

Proof. Let $S\colon \mathcal{C}(B^n) \to \mathcal{C}(\mathbb{C}^n),\ K \mapsto \overline{K}$. In the proof of the above theorem we have seen that

$$\Sigma_b = S \circ \widetilde{b} \circ \Sigma \quad \text{and} \quad \Sigma = \widetilde{b^{-1}} \circ R \circ \Sigma_b.$$

$\widetilde{b}$ and $\widetilde{b^{-1}}$ are Vietoris continuous by Corollary 3.4.6. They are also inverse to each other. That is, $\widetilde{b}\colon \mathcal{C}(\mathbb{C}^n) \to \mathcal{C}(B^n)$ is a homeomorphism with respect to the Vietoris topology. The same is true with respect to the Fell topology by the same corollary.

Consider the case $n = 1$. Every non-empty, closed subset of $\mathbb{C}$ can be realised as the spectrum of a normal, unbounded operator on some Hilbert space [9].[5] So, for every map $\mathcal{T} \ni t \mapsto K_t \in \mathcal{C}(B)$ there exists a normal field of unbounded operators $(a_t)_{t\in\mathcal{T}}$ on some field of Hilbert spaces such that $\Sigma(t) = \widetilde{b^{-1}}(K_t)$ for all $t \in \mathcal{T}$.
We construct such a map that is not Vietoris continuous but where $t \mapsto S(K_t) \in \mathcal{K}(\mathbb{C})$ is Hausdorff continuous. Then,

$$t \mapsto \Sigma_b(t) = S \circ \widetilde{b} \circ \widetilde{b^{-1}}(K_t) = S(K_t)$$

[5] We prove a more general version of this statement in Lemma 3.6.7.

is Hausdorff continuous, but Σ is not Vietoris continuous. This shows (i).

Let $\mathcal{T} = [1/2, 1]$ and define $K_t := \{0\} \cup [t, 1) \in \mathcal{C}(B)$. Consider the open set

$$W := \mathcal{U}_B([1/2, 1), (B)) \subseteq \mathcal{C}(B).$$

This is a Vietoris-open neighborhood of $K_1 = \{0\} \in \mathcal{C}(B)$ because $K_1 \cap [1/2, 1) = \emptyset$ and $K_1 \cap B \neq \emptyset$. Let $V \subseteq \mathcal{T}$ be an open neighborhood of $1 \in \mathcal{T}$. Then, there is an $\epsilon \in (0, 1/2)$ such that $(1-\epsilon) \in V$. We have $K_{(1-\epsilon)} = \{0\} \cup [1-\epsilon, 1) \ni (1-\epsilon)$. Therefore, $K_{(1-\epsilon)} \cap [1/2, 1) \neq \emptyset$ and $\{K_t \mid t \in V\} \not\subseteq W$. Thus, $t \mapsto K_t$ is not Vietoris continuous because it is not continuous at $t = 1$.
Now consider $t \mapsto S(K_t) = \{0\} \cup [t, 1] \in \mathcal{K}(\mathbb{C})$. Let $t_0 \in \mathcal{T}$ and let $\epsilon > 0$. If $t \in \mathcal{T}$ with $|t - t_0| < \delta := \epsilon$, then

$$\begin{aligned} d_H(S(K_t),\, S(K_{t_0})) &= \max\left\{ \sup_{x \in \{0\} \cup [t,1]} \inf_{y \in \{0\} \cup [t_0,1]} |x-y|,\ \sup_{y \in \{0\} \cup [t_0,1]} \inf_{x \in \{0\} \cup [t,1]} |x-y| \right\} \\ &= |t - t_0| < \epsilon. \end{aligned}$$

Thus, $t \mapsto S(K_t)$ is Hausdorff continuous.

Similarly, if $t \mapsto K_t \in \mathcal{K}(\mathbb{C})$, then there exists a normal field of unbounded operators $(a_t)_{t \in \mathcal{T}}$ on some field of Hilbert spaces such that $\Sigma(t) = \widetilde{b^{-1}} \circ R(K_t)$ for all $t \in \mathcal{T}$. We construct such a map that is not Hausdorff continuous with the property $S \circ R(K_t) = K_t$ and such that $t \mapsto R(K_t)$ is Fell continuous. Then

$$t \mapsto \Sigma_b(t) = S \circ \widetilde{b} \circ \widetilde{b^{-1}} \circ R(K_t) = K_t$$

is not Hausdorff continuous, but Σ is Fell continuous. This shows (ii).

Let $\mathcal{T} = [1/2, 1]$ and consider

$$t \mapsto K_t := \begin{cases} \{0\} \cup [t, 1] & \text{for } 1/2 \leq t < 1, \\ \{0\} & \text{for } t = 1 \end{cases} \in \mathcal{C}(\mathbb{C}).$$

Set $\epsilon = 1/2 > 0$ and let $\delta > 0$. Then, $(1 - \delta/2) \in \mathcal{T}$ and $|1 - (1 - \delta/2)| < \delta$. But

$$d_H(K_1,\, K_{(1-\delta/2)}) = 1 > \epsilon.$$

Thus, $t \mapsto K_t$ is not Hausdorff continuous because it is not continuous at $t = 1$.

Now consider $t \mapsto R(K_t) = \{0\} \cup [t, 1)$. We already showed that $t \mapsto \{0\} \cup [t, 1] \in \mathcal{K}(\mathbb{C})$ is Hausdorff continuous. The latter is also Vietoris and hence Fell continuous when viewed as a map into $\mathcal{C}(\mathbb{C})$ by Remark 3.4.4. Then

$$\mathcal{T} \ni t \mapsto R(\{0\} \cup [t,1]) = \{0\} \cup [t,1) = R(K_t) \in \mathcal{C}(B)$$

is Fell continuous because R is Fell continuous by Corollary 3.4.10. □

3.6 β-topology

We would like a topology on $\mathcal{C}(\mathbb{C}^n)$ such that

$$\begin{aligned} &\Sigma\colon \mathcal{T} \to \mathcal{C}(\mathbb{C}^n),\ t \mapsto \sigma_j(A_t),\ \text{is continuous} \\ \Leftrightarrow\ &\Sigma_b\colon \mathcal{T} \to \mathcal{K}(\mathbb{C}^n),\ t \mapsto \sigma_j(b(A_t)),\ \text{is Hausdorff-continuous.} \end{aligned} \tag{3.2}$$

We use Theorem 3.5.2 to transport the Hausdorff topology from $\mathcal{K}\big(\overline{B^n}\big) \subseteq \mathcal{K}(\mathbb{C}^n)$ to $\mathcal{C}(\mathbb{C}^n)$. Define

$$D := \left\{K \in \mathcal{K}\big(\overline{B^n}\big) \;\middle|\; \overline{K \cap B^n} = K\right\} \subseteq \mathcal{K}\big(\overline{B^n}\big)\,.$$

Lemma 3.6.1. *The map*

$$R\colon D \to \mathcal{C}(B^n), \qquad K \mapsto K \cap B^n,$$

is bijective with inverse

$$S\colon \mathcal{C}(B^n) \to D, \qquad L \mapsto \overline{L}.$$

The closure is with respect to the topology on $\mathbb{C}^n$.

Proof. Let $L \in \mathcal{C}(B^n)$. Then, there exists $L' \in \mathcal{C}(\mathbb{C}^n)$ such that $L = L' \cap B^n$. Now,

$$\overline{L' \cap B^n} \cap B^n \subseteq \overline{L'} \cap B^n = L' \cap B^n \subseteq \overline{L' \cap B^n} \cap B^n.$$

Hence, $\overline{L' \cap B^n} \cap B^n = L' \cap B^n$ or equivalently $\overline{L} \cap B^n = L$. Therefore, $\overline{\overline{L} \cap B^n} = \overline{L}$ and $\overline{L} \in D$. In particular, $L = R\big(\overline{L}\big)$ and R is surjective.

Let $K_1, K_2 \in D$ and let $R(K_1) = R(K_2)$. Then,

$$K_1 = \overline{K_1 \cap B^n} = \overline{R(K_1)} = \overline{R(K_2)} = \overline{K_2 \cap B^n} = K_2$$

and R is injective.

Let $L \in \mathcal{C}(B^n)$. The assertion $L = R\left(\overline{L}\right)$ holds as shown above. Thus,

$$R^{-1}(L) = R^{-1}\big(R(\overline{L})\big) = \overline{L} = S(L)$$

and $R^{-1} = S$. □

Corollary 3.6.2. *The map*

$$\beta\colon \mathcal{C}(\mathbb{C}^n) \to D, \qquad M \mapsto S \circ b(M),$$

is a bijection with inverse

$$\beta^{-1}\colon D \to \mathcal{C}(\mathbb{C}^n), \qquad K \mapsto b^{-1} \circ R(K).$$

Remark 3.6.3. The topology on $D \subseteq \mathcal{K}\big(\overline{B^n}\big)$ is the subspace topology. On $\mathcal{K}\big(\overline{B^n}\big)$ we consider the Hausdorff topology, which coincides with the Fell and Vietoris topology by Remark 3.4.4.

Definition 3.6.4. The *β-topology* on $\mathcal{C}(\mathbb{C}^n)$ is the topology obtained by declaring the map β to be a homeomorphism. The open sets are exactly all preimages of the open sets in D under β.

Remark 3.6.5. A basis for the β-topology is given by

$$\begin{aligned}\mathcal{U}_\beta(F,\ (O_k)_{k=1}^m) &:= \{\beta^{-1}(K) \mid K \in D,\ K \cap F = \emptyset,\ \forall k \in \{1,\dots,m\}\colon K \cap O_k \neq \emptyset\} \\ &\overset{K=\beta(M)}{=} \{M \in \mathcal{C}(\mathbb{C}^n) \mid \overline{b(M)} \cap F = \emptyset,\ \forall k \in \{1,\dots,m\}\colon \overline{b(M)} \cap O_k \neq \emptyset\}.\end{aligned}$$

Here, $F \subseteq \overline{B^n}$ is closed and the $O_k \subseteq \overline{B^n}$ are open. However, we get the same basis sets for $F \subseteq \mathbb{C}^n$ closed and the $O_k \subseteq \mathbb{C}^n$ open.

Proposition 3.6.6. *Let $\mathcal{T} \neq \emptyset$ be a topological space and let $(\mathcal{H}_t)_{t\in\mathcal{T}}$ be a field of Hilbert spaces. Let $n \in \mathbb{N}$. For $t \in \mathcal{T}$ let $A_t := (a_{t,k})_{k=1}^n \subseteq \mathbb{L}(\mathcal{H}_t)$ be such that $(A_t)_{t\in\mathcal{T}}$ is a normal strongly commuting field of families of unbounded operators. Let $(B_t)_{t\in\mathcal{T}}$ be its bounded transform. Define the maps*

$$\Sigma\colon \mathcal{T} \to \mathcal{C}(\mathbb{C}^n),\ t \mapsto \sigma_j(A_t), \qquad \textit{and} \qquad \Sigma_b\colon \mathcal{T} \to \mathcal{K}(\mathbb{C}^n),\ t \mapsto \sigma_j(B_t).$$

Then, Σ is β-continuous if and only if Σ_b is Hausdorff continuous.

Proof. For all $t \in \mathcal{T}$ we have $\sigma_j(B_t) = \beta(\sigma_j(A_t))$ by Theorem 3.5.2. Hence, $\Sigma_b = \beta\circ\Sigma$. Now use that β is a homeomorphism by definition of the β-topology. □

Our previous results indicate that the β-topology is coarser than the Vietoris topology and finer than the Fell topology. We prove this by using the following lemma.

Lemma 3.6.7. *For every $K \in \mathcal{C}(\mathbb{C}^n)$ there is a family $A := (a_k)_{k=1}^n \subseteq \mathbb{L}(\ell^2(\mathbb{N}))$ of normal pairwise strongly commuting unbounded operators such that $K = \sigma_j(A)$.*

Proof. (Inspired by Example 2.2 in [9]) There exists a sequence $(\lambda_m)_{m\in\mathbb{N}} \subseteq \mathbb{C}^n$ such that $\overline{\{\lambda_m\}_{m\in\mathbb{N}}} = K$ because K is separable as a closed, non-empty subset of $\mathbb{C}^n$. For $m \in \mathbb{N}$ write $\lambda_m = (\lambda_m^{(1)}, \dots, \lambda_m^{(n)}) \in \mathbb{C}^n$. For $k \in \{1, \dots, n\}$ let a_k be the multiplication operator corresponding to the sequence $(\lambda_m^{(k)})_{m\in\mathbb{N}} \subseteq \mathbb{C}$

$$a_k \colon \mathcal{D}(a_k) \to \ell^2(\mathbb{N}), \qquad x = (x_m)_{m\in\mathbb{N}} \mapsto a_k(x), \qquad (a_k(x))_m := \lambda_m^{(k)} x_m \ \text{ for } \ m \in \mathbb{N},$$

$$\mathcal{D}(a_k) = \left\{ x \in \ell^2(\mathbb{N}) \ \middle| \ \sum_{m=1}^{\infty} |(a_k(x))_m|^2 < \infty \right\}.$$

The adjoint operators are given by

$$(a_k^*(x))_m = \left(\lambda_m^{(k)}\right)^* x_m, \ \text{ for all } \ x \in \mathcal{D}(a_k^*) = \mathcal{D}(a_k) \text{ and } \ m \in \mathbb{N}.$$

We have $\|a_k^*(x)\| = \|a_k(x)\|$ for all $x \in \mathcal{D}(a_k)$. Also, the a_k are closed because $(a_k^*)^* = a_k$. Hence, all a_k are normal. For every $k \in \{1, \dots, n\}$:

$$((I + a_k^* a_k)(x))_m = \left(1 + \left|\lambda_m^{(k)}\right|^2\right) x_m, \ \text{ for all } \ x \in \mathcal{D}(a_k^* a_k) \ \text{ and } \ m \in \mathbb{N}$$

$$\Rightarrow (b(a_k)x)_m = \frac{\lambda_m^{(k)} x_m}{1 + \left|\lambda_m^{(k)}\right|^2}, \ \text{ for all } \ x \in \ell^2(\mathbb{N}) \ \text{ and } \ m \in \mathbb{N}.$$

The bounded transforms $b(a_k)$ again act by pointwise multiplication with some sequence. In particular, the bounded transforms pairwise commute. Hence, the a_k pairwise strongly commute.

$\sigma_j(A)$ consists exactly of the joint approximate eigenvalues of A by Remark 3.3.9.

'$K \subseteq \sigma_j(A)$': For $l \in \mathbb{N}$ let $e_l \in \ell^2(\mathbb{N})$ be the unit vector with entries $(e_l)_m = \delta_{lm}$. For all $k \in \{1, \dots, n\}$ and all $l \in \mathbb{N}$:

$$(a_k(e_l))_m = \lambda_m^{(k)} \delta_{lm} = \lambda_l^{(k)} (e_l)_m \ \text{ for all } \ m \in \mathbb{N} \implies a_k(e_l) = \lambda_l^{(k)} e_l.$$

So, for every $l \in \mathbb{N}$: λ_l is a joint eigenvalue of A. Therefore,

$$K = \overline{\{\lambda_m\}_{m\in\mathbb{N}}} \subseteq \overline{\sigma_j(A)} = \sigma_j(A).$$

'$\sigma_j(A) \subseteq K$': Let $\mu = (\mu_1, \dots, \mu_k) \in \sigma_j(A) \subseteq \mathbb{C}^n$. Denote the euclidean norm on $\mathbb{C}^n$ by $\|\cdot\|_2$. There exists a sequence of unit vectors $(x^{(l)})_{l\in\mathbb{N}}$, $x^{(l)} = (x_m^{(l)})_{m\in\mathbb{N}} \in \bigcap_{k=1}^n \mathcal{D}(a_k)$, such that for all $k \in \{1, \dots, n\}$

$$\begin{aligned}
&\lim_{l\to\infty} \left(a_k\left(x^{(l)}\right) - \mu_k x^{(l)}\right) = 0 \ \text{ in } \ \ell^2(\mathbb{N}).\\
\Rightarrow\ &\inf_{m\in\mathbb{N}} \|\lambda_m - \mu\|_2^2 = \inf_{m\in\mathbb{N}} \sum_{k=1}^n \left|\lambda_m^{(k)} - \mu_k\right|^2\\
&\le n \inf_{m\in\mathbb{N}} \underbrace{\max_{k\in\{1,\dots,n\}} \left|\lambda_m^{(k)} - \mu_k\right|^2}_{\text{attained at } k=k_m} \underbrace{\sum_{m'=1}^\infty \left|x_{m'}^{(l)}\right|^2}_{=1\ \forall l\in\mathbb{N}}\\
&= n \sum_{m'=1}^\infty \inf_{m\in\mathbb{N}} \left|\lambda_m^{(k_m)} - \mu_{k_m}\right|^2 \left|x_{m'}^{(l)}\right|^2\\
&\le n \sum_{m=1}^\infty \left|\lambda_m^{(k_m)} - \mu_{k_m}\right|^2 \left|x_m^{(l)}\right|^2\\
&\le n \sum_{m=1}^\infty \sum_{k=1}^n \left|\lambda_m^{(k)} - \mu_k\right|^2 \left|x_m^{(l)}\right|^2\\
&= n \sum_{k=1}^n \left\|a_k\left(x^{(l)}\right) - \mu_k x^{(l)}\right\|^2 \xrightarrow{l\to\infty} 0.
\end{aligned}$$

Hence, $\inf_{m\in\mathbb{N}} \|\lambda_m - \mu\|_2^2 = 0$ and there is a sequence $(m_j)_{j\in\mathbb{N}} \subseteq \mathbb{N}$ satisfying

$$\lim_{j\to\infty} \left\|\lambda_{m_j} - \mu\right\|_2^2 = \inf_{m\in\mathbb{N}} \|\lambda_m - \mu\|_2^2 = 0.$$

Thus, $\mu \in \overline{\{\lambda_m\}_{m\in\mathbb{N}}} = K$ and $\sigma_j(A) \subseteq K$. □

Proposition 3.6.8. *On $\mathcal{C}(\mathbb{C}^n)$ the β-topology is strictly coarser than the Vietoris topology and strictly finer than the Fell topology.*

Proof. Consider $\mathcal{T} = \mathcal{C}(\mathbb{C}^n)$ with the Vietoris topology. For every $t \in \mathcal{T}$ there is a family $A_t \subseteq \mathbb{L}(\ell^2(\mathbb{N}))$ of normal pairwise strongly commuting unbounded operators with $\sigma_j(A_t) = t$ by the previous lemma. The map

$$\Sigma\colon \mathcal{T} \to \mathcal{C}(\mathbb{C}^n), \qquad t \mapsto \sigma_j(A_t) = t,$$

is Vietoris continuous.[6] Hence, the corresponding map $\Sigma_b = \beta \circ \Sigma$ is Hausdorff continuous by Theorem 3.5.5. Then, Σ is β-continuous by Proposition 3.6.6. That is, for every β-open subset $M \subseteq \mathcal{C}(\mathbb{C}^n)$ its preimage $\Sigma^{-1}(M) = M \subseteq \mathcal{T}$ is Vietoris-open.

[6] The 'specifically stated' topologies refer to $\mathcal{C}(\mathbb{C}^n)$. $\mathcal{T}$ is considered with the Vietoris topology.

Therefore, the β-topology is coarser than the Vietoris topology.

Now consider $\mathcal{T} = \mathcal{C}(\mathbb{C}^n)$ with the β-topology. Then, the map Σ is β-continuous.[7] Hence, the corresponding map Σ_b is Hausdorff continuous by Proposition 3.6.6. Thus, Σ is Fell continuous by Theorem 3.5.5. That is, for every Fell-open subset $M \subseteq \mathcal{C}(\mathbb{C}^n)$ its preimage $\Sigma^{-1}(M) = M \subseteq \mathcal{T}$ is β-open. Therefore, the β-topology is finer than the Fell topology.

The β-topology cannot coincide with the Vietoris or the Fell topology by Lemma 3.5.6 and Proposition 3.6.6: The Hausdorff continuity of Σ_b implies the β- but not the Vietoris continuity of Σ. The β- but not the Fell continuity of Σ implies the Hausdorff continuity of Σ_b. Thus, the β-topology is strictly coarser than the Vietoris and strictly finer than the Fell topology. □

Remark 3.6.9. The only property of the β-topology needed for the above proof is that Σ is β-continuous if and only if Σ_b is Hausdorff continuous.

3.7 Cayley transform and resolvent

The β-topology on $\mathcal{C}(\mathbb{C}^n)$ depends on our choice of the map b for the bounded transform. Of course, there are other ways to transform unbounded operators into bounded ones. We ask ourselves if and how the corresponding topologies on $\mathcal{C}(\mathbb{C}^n)$ differ.

We consider the *Cayley transform* (see, e.g., Chapter 13.1 in [9]) and restrict to the case of self-adjoint operators. It can be defined by the functional calculus of the homeomorphism

$$c\colon \mathbb{R} \to \mathbb{S}^1 \backslash \{1\}, \qquad x \mapsto \frac{x-i}{x+i}.$$

For self-adjoint operators $a \in \mathbb{L}(\mathcal{H})$ the operator $c(a) \in \mathbb{B}(\mathcal{H})$ is unitary, in particular bounded. The inverse Cayley transform is given by

$$c^{-1}\colon \mathbb{S}^1 \backslash \{1\} \to \mathbb{R}, \qquad u \mapsto i\,\frac{1+u}{1-u}.$$

In analogy to equation (3.2) we would like to have a topology on $\mathcal{C}(\mathbb{R}^n)$ such that

$$\begin{aligned} &\Sigma\colon \mathcal{T} \to \mathcal{C}(\mathbb{R}^n),\ t \mapsto \sigma_j(A_t),\ \text{ is continuous} \\ \Leftrightarrow\ &\Sigma_c\colon \mathcal{T} \to \mathcal{K}(\mathbb{T}^n),\ t \mapsto \sigma_j(c(A_t)),\ \text{ is Hausdorff-continuous.} \end{aligned}$$

[7] The specifically stated topologies again refer to $\mathcal{C}(\mathbb{C}^n)$. $\mathcal{T}$ is now considered with the β-topology.

Here, $A_t = (a_{t,k})_{k=1}^n \subseteq \mathbb{L}(\mathcal{H})$ is a self-adjoint strongly commuting field of families of unbounded operators. The joint spectra are subsets of $\mathbb{R}^n$ because the spectra of single self-adjoint operators are subsets of $\mathbb{R}$ and by Remark 3.3.9. Similarly, the families of unitaries $c(A_t) = (c(a_{t,k}))_{k=1}^n \subseteq \mathbb{B}(\mathcal{H})$ have joint spectra in the n-torus $\mathbb{T}^n = (\mathbb{S}^1)^n$. They pairwise commute because they are bounded and pairwise strongly commute by Lemma 3.3.5.

We now *assume* the existence of at least one such topology on $\mathcal{C}(\mathbb{R}^n)$ and call it the γ-topology. It has to be strictly coarser than the Vietoris topology and strictly finer than the Fell topology by the same arguments as presented in the proof of Proposition 3.6.8 and Remark 3.6.9.

Lemma 3.7.1. *On $\mathcal{C}(\mathbb{R})$ the γ-topology is not finer than the β-topology.*

Proof. We use some properties of unbounded multiplication operators (cf. Example 3.8 and 5.3 in [9]).

Let $\mathcal{T} = [0, 2\pi]$. For $t \in \mathcal{T}$ consider the Hilbert spaces $\mathcal{H}_t = L^2([t, t+\pi], \mathrm{d}x)$ and let $a_t = \mathrm{Mult}(-\cot(\cdot/2)) \in \mathbb{L}(\mathcal{H}_t)$. The domains are $D_t = \{f \in \mathcal{H}_t \mid f(\cdot)\cot(\cdot/2) \in \mathcal{H}_t\}$. This defines a field of unbounded self-adjoint operators $(a_t)_{t\in\mathcal{T}}$. We have

$$c(a_t) = \mathrm{Mult}(c(-\cot(\cdot/2)) = \mathrm{Mult}(\exp(i\cdot)).$$

The spectrum of a multiplication operator corresponds to its essential range. In this case

$$\sigma(c(a_t)) = \left\{e^{i\lambda} \mid \lambda \in [t,\, t+\pi]\right\}.$$

The map $\Sigma_c\colon \mathcal{T} \ni t \mapsto \sigma(c(a_t)) \in \mathcal{K}(\mathbb{C})$ is Hausdorff continuous: Let $t_0 \in \mathcal{T}$ and let $\epsilon > 0$. Let $|t - t_0| < \delta := \epsilon$. Then,

$$d_H(\sigma(c(a_t)),\ \sigma(c(a_{t_0}))) = \sup_{\lambda\in[t,\, t+\pi]}\ \inf_{\lambda_0\in[t_0,\, t_0+\pi]} \left|e^{i\lambda} - e^{i\lambda_0}\right| \le \epsilon.$$

Thus, the map $\Sigma\colon \mathcal{T} \ni t \mapsto \sigma(a_t) \in \mathcal{C}(\mathbb{R})$ is γ-continuous by definition.

Now consider the bounded transform

$$b(a_t) = \mathrm{Mult}(b(-\cot(\cdot/2)) = \mathrm{Mult}\left(\frac{-\cot(\cdot/2)}{\sqrt{1+\cot(\cdot/2)^2}}\right).$$

The spectra are

$$\sigma(b(a_t)) = \begin{cases} \left[\frac{-\cot(t/2)}{\sqrt{1+\cot(t/2)^2}},\ \frac{-\cot((t+\pi)/2)}{\sqrt{1+\cot((t+\pi)/2)^2}}\right], & t \leq \pi \\ \left[-1,\ \frac{-\cot((t-\pi)/2)}{\sqrt{1+\cot((t-\pi)/2)^2}}\right] \cup \left[\frac{-\cot(t/2)}{\sqrt{1+\cot(t/2)^2}},\ 1\right], & t > \pi. \end{cases}$$

The map $\Sigma_b \colon \mathcal{T} \ni t \mapsto \sigma(b(a_t)) \in \mathcal{K}(\mathbb{C})$ is not Hausdorff continuous because it is not Hausdorff continuous at $t = \pi$: Consider the Hausdorff-open neighborhood $W = \mathcal{U}_\mathbb{C}([-1, -1/2], (\mathbb{C}))$ of $\Sigma_b(\pi)$. For every $t \in \mathcal{T}$ with $t > \pi : \Sigma_b(t) \notin W$.
Therefore, $\Sigma \colon \mathcal{T} \ni t \mapsto \sigma(a_t) \in \mathcal{C}(\mathbb{C})$ is not β-continuous. The same holds when restricting the codomain to $\mathcal{C}(\mathbb{R})$ using the subspace topology. □

Remark 3.7.2. The map Σ does not 'look' continuous at $t = \pi$ the same way $\sigma(b(a_t))$ is not Hausdorff continuous at $t = \pi$. This discontinuity is of a type not seen by the Fell topology. As we just showed the same holds for the γ-topology in this example. The Vietoris and β-topology, however do detect it. This indicates that the γ-topology might not be so useful, e.g., in the light of applications. Of course we did not prove that the converse cannot happen. Also, there may be situations where it makes sense to regard 'spectrum appearing at infinity' as continuous. For a visualization see Figure 3.1.

As a second possibility we briefly consider the resolvent. It also transforms unbounded operators into bounded ones. Here, we have the additional complication that we need some $\lambda \in \mathbb{C}$ such that for all $t \in \mathcal{T} \colon \lambda \in \rho(a_t)$ or equivalently $\bigcap_{t \in \mathcal{T}} \rho(a_t) \neq \emptyset$. In the case of multiple strongly commuting operators this has to hold 'in every entry'. That is, $\forall k \in \{1, \ldots, n\}$ we need $\bigcap_{t \in \mathcal{T}} \rho(a_{t,k}) \neq \emptyset$.

We again restrict to the case of self-adjoint operators. Then, e.g., $\lambda = -i$ works. In this case the resolvent can be obtained from the functional calculus of

$$R_{-i} \colon \mathbb{R} \to \mathbb{C}, \qquad x \mapsto \frac{1}{x+i}.$$

Constant shifts and multiplication by constants are Hausdorff continuous:

$$\forall K_1, K_2 \in \mathcal{K}(\mathbb{C})\ \forall s, r \in \mathbb{C} \colon$$
$$d_H(K_1 + s,\ K_2 + s) = d_H(K_1,\ K_2) = d_H(r \cdot K_1,\ r \cdot K_2) \cdot |r|^{-1}.$$

Let $R'_{s,r} := rR_{-i} + s$. Then, for any field of strongly commuting self-adjoint unbounded

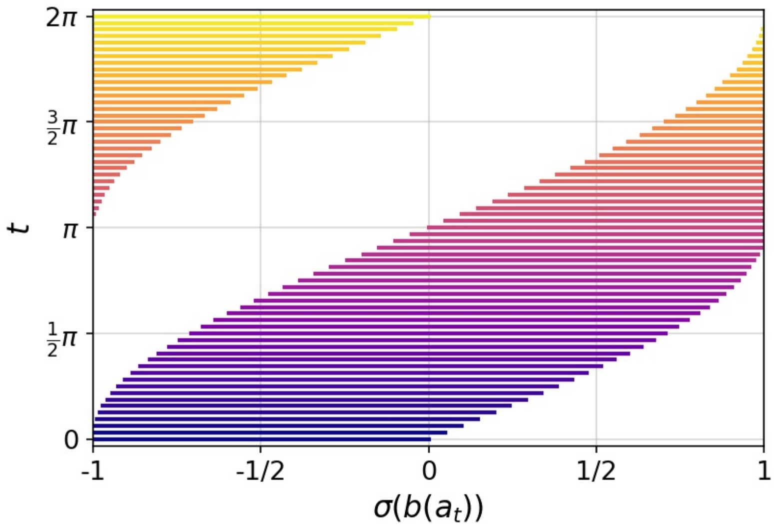

Figure 3.1: The spectra of $b(a_t)$ for $t \in [0, 2\pi]$ are depicted. At $t = \pi$ there is a Hausdorff discontinuity: The spectrum suddenly picks up a contribution at -1. The spectra of a_t qualitatively look the same, but the range is $(-\infty, \infty)$ instead of $[-1, 1]$. In the latter case the Fell topology does not see this discontinuity. For the case of the Cayley transform one can think of $[-1, 1] \mapsto [0, 2\pi]$ as the phases of the spectra of $c(a_t)$ (of course then the change of the boundaries is actually linear). This effectively identifies the points -1 and 1 in the image above leading to a γ-continuous change.

operators $(a_t)_{t\in\mathcal{T}}$:

$$\begin{aligned} &\mathcal{T} \ni t \mapsto \sigma(R_{-i}(a_t)) \in \mathcal{K}(\mathbb{C}) \text{ is Hausdorff continuous} \\ \Leftrightarrow\ &\mathcal{T} \ni t \mapsto \sigma(R'_{s,r}(a_t)) \in \mathcal{K}(\mathbb{C}) \text{ is Hausdorff continuous.} \end{aligned}$$

Choose $s = 1$ and $r = -2i$. For all $x \in \mathbb{R}$:

$$R'_{1,-2i}(x) = \frac{-2i}{x+i} + 1 = \frac{x-i}{x+i} = c(x).$$

Therefore, using the resolvent yields the same 'problem' as with the Cayley transform. It can happen that: $(a_t)_{t\in\mathcal{T}}$ is a field of unbounded self-adjoint operators; $\mathcal{T} \ni t \mapsto \sigma(R_{-i}(a_t)) \in \mathcal{K}(\mathbb{C})$ is Hausdorff continuous; $\mathcal{T} \ni t \mapsto \sigma(a_t) \in \mathcal{C}(\mathbb{R})$ is neither Vietoris nor β-continuous and also does not 'look' continuous.

We presume that out of the three possibilities considered the bounded transform might be most suitable to describe the continuity of spectra of unbounded operators through bounded ones. It also has the advantage that it works for all normal operators. As mentioned before there may be situations dealing with self-adjoint operators where the γ-topology is preferable.

4 Unbounded affiliated elements

In the last section one of the approaches for dealing with the spectra of unbounded operators was to investigate under what conditions certain bounded functions of these operators form continuous fields of C^*-algebras. Now we consider an alternative approach: unbounded elements affiliated with C^*-algebras and fields of C^*-algebras.

4.1 Multiplier algebra and affiliation relation

Definition 4.1.1 ([11])**.** Let $\mathfrak{A}$ be a C^*-algebra. In particular, $\mathfrak{A}$ is a Banach space. By $\mathbb{B}(\mathfrak{A})$ we denote the algebra of bounded linear operators on $\mathfrak{A}$. Let $a \in \mathbb{B}(\mathfrak{A})$. If there exists an element $b \in \mathbb{B}(\mathfrak{A})$ such that

$$\forall x, y \in \mathfrak{A} \colon x^* a y = (bx)^* y,$$

then we call b the *adjoint* of a and write $b = a^*$.

Remark 4.1.2. These adjoints are the adjoint operators with respect to the $\mathfrak{A}$-valued inner product $\langle x,\, y\rangle = x^* y$ on $\mathfrak{A}$. Not every bounded operator on $\mathfrak{A}$ has an adjoint.

Definition 4.1.3 ([11])**.** The set of all bounded operators on a C^*-algebra $\mathfrak{A}$ for which the adjoint exists is called the *multiplier algebra* of $\mathfrak{A}$:

$$M(\mathfrak{A}) := \{a \in \mathbb{B}(\mathfrak{A}) \mid a^* \text{ exists}\}.$$

The elements of $M(\mathfrak{A})$ are called *multipliers* of $\mathfrak{A}$. We endow $M(\mathfrak{A})$ with the natural algebraic operations. The considered norm is

$$\|\cdot\| \colon M(\mathfrak{A}) \to [0,\, \infty), \qquad \|a\| := \sup_{x \in \mathfrak{A},\, \|x\|=1} \|ax\|.$$

This makes $M(\mathfrak{A})$ a unital C^*-algebra. The identity operator on $\mathfrak{A}$ is denoted by $I \in M(\mathfrak{A})$.

Lemma 4.1.4 ([11])**.** *Let $\mathfrak{A}$ be a C^*-algebra. Then*

(i) $\forall a \in M(\mathfrak{A})\ \forall x, y \in \mathfrak{A} \colon (ax)y = a(xy)$*,*

(ii) $\mathfrak{A} \subseteq M(\mathfrak{A})$ *as left multiplication operators,*

(iii) $\mathfrak{A}$ *is unital* $\Leftrightarrow M(\mathfrak{A}) = \mathfrak{A}$*,*

(iv) $\mathfrak{A} \subseteq M(\mathfrak{A})$ *is a closed, two-sided ideal.*

Definition 4.1.5 ([2])**.** The *strict topology* τ_s on $M(\mathfrak{A})$ is induced by the families of seminorms $(a \mapsto \|ax\|)_{x\in\mathfrak{A}}$ and $(a \mapsto \|a^*x\|)_{x\in\mathfrak{A}}$. A net $(a_\lambda)_{\lambda\in\Lambda} \subseteq M(\mathfrak{A})$ converges in the strict topology to $a \in M(\mathfrak{A})$ if and only if

$$\forall x \in \mathfrak{A}\colon \|a_\lambda x - ax\|,\ \|a_\lambda^* x - a^* x\| \xrightarrow{\lambda\in\Lambda} 0.$$

Lemma 4.1.6. *If $M(\mathfrak{A}) = \mathfrak{A}$, then the strict topology coincides with the norm topology on $\mathfrak{A}$.*

Proof. Let $(a_\lambda)_{\lambda\in\Lambda} \subseteq \mathfrak{A}$ be a net. Convergence in norm implies convergence in the strict topology. For the reverse implication set $x = I \in \mathfrak{A}$ in the definition above. □

Lemma 4.1.7 ([2])**.** *$(M(\mathfrak{A}), \tau_s)$ is a topological vector space. $\mathfrak{A} \subseteq M(\mathfrak{A})$ is dense in the strict topology and $(M(\mathfrak{A}), \tau_s)$ is complete.*

Example 4.1.8 ([11])**.** Let X be a locally compact Hausdorff space. Let $C_0(X)$ be the C^*-algebra of continuous functions on X vanishing at infinity. Then, $M(C_0(X)) = C_b(X)$ is the C^*-algebra of continuous bounded functions on X.

Let $\mathbb{K}(\mathcal{H})$ be the C^*-algebra of compact operators on a Hilbert space $\mathcal{H}$. Then, $M(\mathbb{K}(\mathcal{H})) = \mathbb{B}(\mathcal{H})$.

We give the definition for the affiliation relation introduced by Woronowicz [11]. Heuristically, an unbounded linear operator on $\mathfrak{A}$ is affiliated with $\mathfrak{A}$ if its bounded transform is a multiplier of $\mathfrak{A}$.

Definition 4.1.9. Let $\mathfrak{A}$ be a C^*-algebra. By $\mathbb{L}(\mathfrak{A})$ we denote the set of linear operators on $\mathfrak{A}$. Analogous to the Hilbert space setting a potentially unbounded operator $a \in \mathbb{L}(\mathfrak{A})$ is only defined on a linear subspace, its domain $D(a) \subseteq \mathfrak{A}$. The operator a is *closed* if its graph is closed.

Definition 4.1.10 (Def. 1.1 in [11])**.** Let $\mathfrak{A}$ be a C^*-algebra and let $a \in \mathbb{L}(\mathfrak{A})$ be densely defined. a is *affiliated* with $\mathfrak{A}$ – we write $a\,\eta\,\mathfrak{A}$ – if there exists a multiplier $b \in M(\mathfrak{A})$ such that $\|b\| \leq 1$ and

$$\forall x, y \in \mathfrak{A}\colon (x \in D(a),\ ax = y) \ \Leftrightarrow\ (\exists z \in \mathfrak{A}\colon x = (I - b^*b)^{\frac{1}{2}} z,\ y = bz).$$

Lemma 4.1.11 ([11])**.** *Let $a\,\eta\,\mathfrak{A}$. Then,*

(i) a is a closed operator,

(ii) the multiplier b is uniquely determined by a and vice versa,

(iii) $D(a) \subseteq \mathfrak{A}$ *is a right ideal,*

(iv) $\forall x \in D(a)\ \forall y \in \mathfrak{A}\colon a(xy) = (ax)y$,

(v) $\mathfrak{A}$ *unital implies* $a \in \mathfrak{A}$.

Proposition 4.1.12 (Example 1 in [11])**.** *The multiplier algebra consists exactly of the bounded elements affiliated with* $\mathfrak{A}$*:*

$$M(\mathfrak{A}) = \{a \in \mathbb{L}(\mathfrak{A}) \mid a\,\eta\,\mathfrak{A},\ a \in \mathbb{B}(\mathfrak{A})\}.$$

For $a \in M(\mathfrak{A})$ *the corresponding multiplier* b *as in Definition* 4.1.10 *is given by the bounded transform:*

$$b = b(a) = a(I + a^*a)^{-\frac{1}{2}}.$$

Example 4.1.13 ([11])**.** Let X be a locally compact Hausdorff space. Let $C_0(X)$ be the C^*-algebra of continuous functions on X vanishing at infinity. The set of elements affiliated with $C_0(X)$ is $C(X)$, the continuous functions on X.

Let $\mathbb{K}(\mathcal{H})$ be the C^*-algebra of compact operators on a Hilbert space $\mathcal{H}$. The set of elements affiliated with $\mathbb{K}(\mathcal{H})$ is the set of closed operators on $\mathcal{H}$.

Proposition 4.1.14 (Example 4 in [11])**.** *Let* $\mathfrak{A}$ *be a* C^**-subalgebra of* $\mathbb{B}(\mathcal{H})$ *for some Hilbert space* $\mathcal{H}$*. Let* $a \in \mathbb{L}(\mathcal{H})$ *be a closed operator. Then,*

$$a\,\eta\,\mathfrak{A} \;\Leftrightarrow\; \big(b(a) = a(I + a^*a)^{-\frac{1}{2}} \in M(\mathfrak{A}),\ (I + a^*a)^{-\frac{1}{2}}\mathfrak{A} \subseteq \mathfrak{A} \text{ is dense}\big).$$

Lemma 4.1.15 (Thm. 1.4 in [11])**.** *Let* $a\,\eta\,\mathfrak{A}$*. There exists an operator* $a^*\,\eta\,\mathfrak{A}$ *such that*

$$\forall x, y \in \mathfrak{A}\colon (x \in D(a^*),\ y = a^*x) \;\Leftrightarrow\; (\forall z \in D(a)\colon x^*(az) = y^*z).$$

It satisfies $b(a^*) = b(a)^*$ *and* $(a^*)^* = a$*. This* **-operation reduces to the usual definitions in special cases, e.g.,* $a \in M(\mathfrak{A})$*,* $a \in \mathbb{L}(\mathcal{H})$ *closed or* $a \in C(X)$*.*

4.2 Affiliation relation for continuous fields of C^*-algebras

Definition 4.2.1. Let $\mathcal{T}$ be a topological space. Let $\mathfrak{C} = ((\mathfrak{C}_t)_{t\in\mathcal{T}}, \Gamma)$ be a continuous field of C^*-algebras. For every $t \in \mathcal{T}$ let $a_t \in \mathbb{L}(\mathfrak{C}_t)$ be densely defined. We say that $a = (a_t)_{t\in\mathcal{T}}$ is *affiliated* with $\mathfrak{C}$ – we write $a\,\eta\,\mathfrak{C}$ – if

(i) for every $t \in \mathcal{T}$: $a_t\,\eta\,\mathfrak{C}_t$,

(ii) for every $x = (x_t)_{t\in\mathcal{T}} \in \Gamma$:
$b(a)x = (b(a_t)x_t)_{t\in\mathcal{T}} \in \Gamma$ and $b(a)^*x = (b(a_t)^*x_t)_{t\in\mathcal{T}} \in \Gamma$.

Let $(\mathcal{H}_t)_{t\in\mathcal{T}}$ be a field of Hilbert spaces. Let $n \in \mathbb{N}$. For $t \in \mathcal{T}$ let $A_t = (a_{t,k})_{k=1}^n \subseteq \mathbb{L}(\mathcal{H}_t)$ be such that $A = (A_t)_{t\in\mathcal{T}}$ is a normal strongly commuting field of families of unbounded operators. We say that A is *affiliated* with $\mathfrak{C}$ – we write $A\ \eta\ \mathfrak{C}$ – if for every $k \in \{1,\dots,n\}$: $a_k = (a_{t,k})_{t\in\mathcal{T}}$ is affiliated with $\mathfrak{C}$.

We study for which continuous fields of C^*-algebras the affiliation relation yields a useful notion for the continuity of the spectra.

Let $\mathcal{T}$ be a topological space. Let $(\mathcal{H}_t)_{t\in\mathcal{T}}$ be a field of Hilbert spaces. Let $n \in \mathbb{N}$. For $t \in \mathcal{T}$ let $A_t = (a_{t,k})_{k=1}^n \subseteq \mathbb{L}(\mathcal{H}_t)$ be such that $A = (A_t)_{t\in\mathcal{T}}$ is a normal strongly commuting field of families of unbounded operators.

Proposition 4.2.2. *Assume that the bounded transform $b(A) = (b(A_t))_{t\in\mathcal{T}}$ of A generates a continuous field of unital C^*-algebras $\mathfrak{C} = ((\mathfrak{C}_t)_{t\in\mathcal{T}}, \Gamma)$ in the sense of Theorem* 2.5.1 *and Corollary* 2.5.2*. Then, in general A is not affiliated with $\mathfrak{C}$.*

Proof. Consider the case $n = 1$. Assume there is a $t \in \mathcal{T}$ such that $a_t := a_{t,1}$ is not bounded. a_t is a closed linear operator on $\mathcal{H}_t$. Its bounded transform $b_t := b(a_t)$ is an element of $\mathfrak{C}_t = C_1^*(b(A_t))$. For every $z \in \mathbb{C}$: $(1 + z^*z)^{-\frac{1}{2}} = (1 - b(z)^*b(z))^{\frac{1}{2}}$. Therefore,

$$(1 + a_t^*a_t)^{-\frac{1}{2}} = (1 - b_t^*b_t)^{\frac{1}{2}} \in \mathfrak{C}_t.$$

Suppose, there is a sequence $(x_m)_{m\in\mathbb{N}} \subseteq \mathfrak{C}_t$ with

$$\lim_{m\to\infty} (1 + a_t^*a_t)^{-\frac{1}{2}} x_m = b_t \in \mathfrak{C}_t$$

For every $m \in \mathbb{N}$ there is a continuous function $f_m \in C(\sigma_j(b(A_t))) = C(\sigma(b_t))$ with $f_m(b_t) = x_m$. Then, the equation above translates to an equation in $C(\sigma(b_t))$:

$$\lim_{m\to\infty} \sup_{z\in\sigma(b_t)} \left| f_m(z)\sqrt{1 - |z|^2} - z \right| = 0.$$

However, there is a $\lambda \in \sigma(b_t)$ with norm 1 because $\sigma(a_t) \subseteq \mathbb{C}$ is unbounded (cf. Theorem 3.5.2). Hence,

$$\lim_{m\to\infty} \sup_{z\in\sigma(b_t)} \left| f_m(z)\sqrt{1 - |z|^2} - z \right| \geq \lim_{m\to\infty} \left| f_m(\lambda)\sqrt{1 - |\lambda|^2} - \lambda \right| = 1 > 0.$$

This is a contradiction. Therefore, $(1 + a_t^* a_t)^{-\frac{1}{2}} \mathfrak{C}_t$ is not dense in $\mathfrak{C}_t$. This means that a_t is not affiliated with $\mathfrak{C}_t$ by Proposition 4.1.14. Then, A is not affiliated with $\mathfrak{C}$. □

The bounded transform of A generates a continuous field of unital C^*-algebras if and only if the spectrum map $\Sigma\colon \mathcal{T} \to \mathcal{C}(\mathbb{C}^n),\ t \mapsto \sigma_j(A_t)$, is β-continuous. So, the β-continuity of Σ does not imply that A is affiliated with the continuous field of C^*-algebras generated by the bounded transform of A.

Now, we consider a different field of C^*-algebras. For $t \in \mathcal{T}$ we denote the C^*-algebra obtained from the functional calculus of all C_0-functions on the joint spectrum of A_t by $C_0(A_t)$. This C^*-algebra is isomorphic to $C_0(\sigma_j(A_t))$ by Proposition 3.2.5. For every $t \in \mathcal{T}$ set $\mathfrak{C}_t := C_0(A_t)$. Define

$$\Lambda := \{(\phi(A_t))_{t\in\mathcal{T}} \mid \phi \in C_0(\mathbb{C}^n)\} \subseteq \prod_{t\in\mathcal{T}} \mathfrak{C}_t.$$

Λ is a *-subalgebra of $\prod_{t\in\mathcal{T}} \mathfrak{C}_t$. For every $t \in \mathcal{T}$ the set

$$\Lambda_t = \{x_t \mid (x_t)_{t\in\mathcal{T}} \in \Lambda\} = \{\phi(A_t) \mid \phi \in C_0(\mathbb{C})\} = \mathfrak{C}_t$$

is dense in $\mathfrak{C}_t$. Hence, Λ satisfies (B1) and (B2) from Definition 2.4.1. Therefore, there exists a continuous field of C^*-algebras $\mathfrak{C} := ((\mathfrak{C}_t)_{t\in\mathcal{T}}, \Gamma)$ with generating family $\Lambda \subseteq \Gamma$ if and only if the map

$$\mathcal{T} \ni t \mapsto \|\phi(A_t)\| \in [0, \infty)$$

is continuous for every $\phi \in C_0(\mathbb{C}^n)$ (cf. Lemma 2.4.3). We call this condition '(C)'.

Proposition 4.2.3. *Assume* (C). *Then* $A \,\eta\, \mathfrak{C}$.

Proof. Let $k \in \{1, \ldots, n\}$. For every $t \in \mathcal{T}$: $a_{t,k}$ is a closed linear operator on $\mathcal{H}_t$. It can be obtained from the functional calculus of the map

$$p_k\colon \sigma_j(A_t) \to \mathbb{C}, \qquad z \mapsto z_k.$$

p_k is affiliated with $C_0(\sigma_j(A_t))$ because it is continuous (cf. Example 4.1.13). Hence, $a_{t,k}$ is affiliated with $\mathfrak{C}_t \cong C_0(\sigma_j(A_t))$.

Let $(x_t)_{t\in\mathcal{T}} \in \Gamma$ be a continuous section of $\mathfrak{C}$. Let $t_0 \in \mathcal{T}$ and let $\epsilon > 0$. There exists an open neighborhood $U \subseteq \mathcal{T}$ of t_0 and a continuous section $(y_t)_{t\in\mathcal{T}} \in \Lambda$ such that $\|x_t - y_t\| < \epsilon$ for every $t \in U$ (cf. Proposition 10.2.2 in [5]). There is a C_0-function ϕ with $y_t = \phi(A_t)$ for every $t \in \mathcal{T}$ by definition of Λ. The map $(b \circ p_k) \cdot \phi$ is also a

C_0-function because b is bounded. Thus,

$$(b(a_{t,k})y_t)_{t\in\mathcal{T}} = (((b \circ p_k) \cdot \phi)(A_t))_{t\in\mathcal{T}} \in \Lambda \subseteq \Gamma$$

is a continuous section. Considering $C_0(A_t)$ as a C^*-subalgebra of $\mathbb{B}(\mathcal{H}_t)$ we have that

$$\|b(a_{t,k})x_t - b(a_{t,k})y_t\| \leq \|b(a_{t,k})\|\|x_t - y_t\| < \epsilon$$

for every $t \in U$. Hence, $(b(a_{t,k})x_t)_{t\in\mathcal{T}} \in \Gamma$ is a continuous section by property (B4) of continuous fields of C^*-algebras. Similarly, $(b(a_{t,k})^*x_t)_{t\in\mathcal{T}} \in \Gamma$ is a continuous section. Therefore, $(a_{t,k})_{t\in\mathcal{T}}\ \eta\ \mathfrak{C}$ for every k and $A\ \eta\ \mathfrak{C}$ holds. □

In the case of a single self-adjoint operator condition (C), i.e., that Λ defines a generating family, is related to the γ-topology.

Proposition 4.2.4. *Consider the case $n = 1$ and write $a_t := a_{t,1}$. If $A = (a_t)_{t\in\mathcal{T}}$ is a self-adjoint field, then condition* (C) *holds if and only if the spectra of the a_t are γ-continuous.*

Proof. Condition (C) – that the continuous field of C^*-algebras $\mathfrak{C}$ exists – is equivalent to the map

$$\mathcal{T} \ni t \mapsto \|\phi(A_t)\| \in [0, \infty)$$

being continuous for every $\phi \in C_0(\mathbb{C})$. The condition that the spectrum map

$$\Sigma : \mathcal{T} \to \mathcal{C}(\mathbb{R}), \qquad t \mapsto \sigma(a_t),$$

is γ-continuous is equivalent to

$$\Sigma_R : \mathcal{T} \to \mathcal{K}(\mathbb{C}), \qquad t \mapsto \sigma(R_{-i}(a_t)),$$

being Hausdorff continuous (cf. Section 3.7). Here, R_{-i} is the resolvent function for $-i$.

Let Σ be γ-continuous. Let $\phi \in C_0(\mathbb{C})$. Define a function ψ via

$$\psi(x) := \begin{cases} \phi\big(\frac{1}{x} - i\big) & \text{for } x \in \mathbb{C}\backslash\{0\}, \\ 0 & \text{for } x = 0. \end{cases}$$

ψ is continuous. Also, $\psi \circ R_{-i} = \phi$. Then, the norm map

$$\begin{aligned}\mathcal{T} \ni t &\xmapsto{\Sigma_R} \sigma(R_{-i}(a_t)) \xmapsto{\overline{\psi}} \overline{\psi(\sigma(R_{-i}(a_t)))} = \sigma(\psi \circ R_{-i}(a_t)) = \sigma(\phi(a_t)) \\ &\xmapsto{\sup \circ \widetilde{|\cdot|}} \sup |\sigma(\phi(a_t))| = \|\phi(a_t)\| \in [0, \infty)\end{aligned}$$

is continuous as a composition of continuous maps and (C) holds.

Assume (C). $R_{-i} \in C_0(\mathbb{R})$. Therefore, the section $(R_{-i}(a_t))_{t\in\mathcal{T}} \in \Gamma$ is continuous. Then, Σ_R is Hausdorff continuous by Theorem 2.4.8. So, Σ is γ-continuous. □

For $n > 1$ the resolvent functions are not C_0-functions on $\mathbb{R}^n$. Therefore, for families of operators, which also may not be self-adjoint, a different topology on $\mathcal{C}(\mathbb{C}^n)$ needs to be considered.

Definition 4.2.5. Let $\mathcal{T}$ be a topological space. For $t \in \mathcal{T}$ let $K_t \in \mathcal{C}(\mathbb{C}^n)$. We call the map $\mathcal{T} \ni t \mapsto K_t \in \mathcal{C}(\mathbb{C}^n)$ *ζ-continuous* if $\mathcal{T} \ni t \mapsto \overline{\phi(K_t)} \in \mathcal{K}(\mathbb{C})$ is Hausdorff continuous for every $\phi \in C_0(\mathbb{C}^n)$.

Remark 4.2.6. The ζ-continuity corresponds to a topology (the *ζ-topology*) on $\mathcal{C}(\mathbb{C}^n)$. It can be defined as the coarsest topology on $\mathcal{C}(\mathbb{C}^n)$ that makes the maps

$$\overline{\phi}\colon \mathcal{C}(\mathbb{C}^n) \to \mathcal{K}(\mathbb{C}), \qquad K \mapsto \overline{\phi(K)},$$

continuous for every $\phi \in C_0(\mathbb{C}^n)$. On $\mathcal{K}(\mathbb{C})$ the Hausdorff topology is considered. The ζ-topology is coarser than the Vietoris topology by Lemma 3.4.7.

Lemma 4.2.7. (C) *holds if and only if* Σ *is* ζ*-continuous.*

Proof. '⇐': The map

$$\mathcal{T} \ni t \mapsto \overline{\phi(\sigma_j(A_t))} \overset{\text{Lemma 3.3.7}}{=} \sigma(\phi(A_t)) \in \mathcal{K}(\mathbb{C})$$

is Hausdorff continuous for every C_0-function ϕ by assumption. Hence,

$$\mathcal{T} \ni t \mapsto \|\phi(A_t)\| = \sup |\sigma(\phi(A_t))| \in [0, \infty)$$

is continuous as argued before and (C) holds.

'⇒': For every $\phi \in C_0(\mathbb{C}^n)$ we have the continuous section $(\phi(A_t))_{t\in\mathcal{T}} \in \Gamma$. Then,

$$\mathcal{T} \ni t \mapsto \sigma(\phi(A_t)) = \overline{\phi(\sigma_j(A_t))} \in \mathcal{K}(\mathbb{C})$$

is Hausdorff continuous by Theorem 2.4.8. Hence, Σ is ζ-continuous. □

Corollary 4.2.8. *On $\mathcal{C}(\mathbb{R})$ the ζ-topology coincides with the γ-topology.*

Finally, we consider an approach where the continuous field of C^*-algebras is given independently of the operators A_t.

Definition 4.2.9 (Example 10.1.4 in [5])**.** Let $\mathcal{T}$ be a topological space. Let $\mathfrak{A}$ be a C^*-algebra. Let $\Gamma = C(\mathcal{T}, \mathfrak{A})$ be the algebra of continuous functions from $\mathcal{T}$ to $\mathfrak{A}$. Then, $((\mathfrak{A})_{t\in\mathcal{T}}, \Gamma)$ is a continuous field of C^*-algebras. It is called the *constant field* over $\mathcal{T}$ generated by $\mathfrak{A}$.

Lemma 4.2.10. *Let $\mathcal{T}$ be a topological space. Let $\mathfrak{A}$ be a C^*-algebra. Let $\mathfrak{C} := ((\mathfrak{A})_{t\in\mathcal{T}}, \Gamma)$ be the constant field generated by $\mathfrak{A}$. Let $a = (a_t)_{t\in\mathcal{T}} \subseteq \mathbb{L}(\mathfrak{A})$. Then, $a \,\eta\, \mathfrak{C}$ if and only if*

(i) for every $t \in \mathcal{T}$: $a_t \,\eta\, \mathfrak{A}$,

(ii) the map $\mathcal{T} \ni t \mapsto b(a_t) \in M(\mathfrak{A})$ is continuous in the strict topology.

Proof. '⇒': (i) holds by definition of $a \,\eta\, \mathfrak{C}$. For every $x \in \Gamma$: $b(a)x,\ b(a)^*x \in \Gamma$ by assumption. That is, for every $x \in \Gamma = C(\mathcal{T}, \mathfrak{A})$ the maps

$$\mathcal{T} \ni t \mapsto b(a_t)x_t \in \mathfrak{A}, \qquad \mathcal{T} \ni t \mapsto b(a_t)^*x_t \in \mathfrak{A}$$

are continuous. Let $(t_\lambda)_{\lambda\in\Lambda} \subseteq \mathcal{T}$ be a net converging to $t \in \mathcal{T}$. Let $x \in \mathfrak{A}$. Interpret x as the constant function $(t \mapsto x) \in \Gamma$. Then,

$$\lim_{\lambda\in\Lambda} \|b(a_{t_\lambda})x - b(a_t)x\| = \lim_{\lambda\in\Lambda} \|b(a_{t_\lambda})^*x - b(a_t)^*x\| = 0.$$

That is, the net $(b(a_{t_\lambda}))_{\lambda\in\Lambda}$ converges to $b(a_t)$ in the strict topology on $M(\mathfrak{A})$. Thus, $\mathcal{T} \ni t \mapsto b(a_t) \in M(\mathfrak{A})$ is continuous in the strict topology and (ii) holds.

'⇐': For every $t \in \mathcal{T}$: $a_t \,\eta\, \mathfrak{A}$ by (i). Let $x = (x_t)_{t\in\mathcal{T}} \in \Gamma$. Let $(t_\lambda)_{\lambda\in\Lambda} \subseteq \mathcal{T}$ be a net converging to $t \in \mathcal{T}$. Then,

$$\|b(a_{t_\lambda})x_{t_\lambda} - b(a_t)x_t\| \leq \underbrace{\|b(a_{t_\lambda})\|}_{\leq 1}\underbrace{\|x_{t_\lambda} - x_t\|}_{\xrightarrow{\lambda\in\Lambda} 0} + \underbrace{\|b(a_{t_\lambda})x_t - b(a_t)x_t\|}_{\xrightarrow{\lambda\in\Lambda} 0} \xrightarrow{\lambda\in\Lambda} 0$$

because x is continuous and $b(a)$ is continuous in the strict topology by (ii). Similarly, $\|b(a_{t_\lambda})^*x_{t_\lambda} - b(a_t)^*x_t\| \xrightarrow{\lambda\in\Lambda} 0$. Therefore, $b(a)x,\, b(a)^*x \in \Gamma$ and $a \,\eta\, \mathfrak{C}$. □

For the remainder of this section let $\mathcal{T}$ be a topological space and let $\mathcal{H}$ be a Hilbert space. Let $n \in \mathbb{N}$. For $t \in \mathcal{T}$ let $A_t = (a_{t,k})_{k=1}^n \subseteq \mathbb{L}(\mathcal{H})$ be such that $A = (A_t)_{t\in\mathcal{T}}$ is a normal strongly commuting field of families of unbounded operators on the constant field of Hilbert spaces $(\mathcal{H})_{t\in\mathcal{T}}$. For $k \in \{1,\dots,n\}$ we write $a_k = (a_{t,k})_{t\in\mathcal{T}}$. We denote $\Sigma : \mathcal{T} \to \mathcal{C}(\mathbb{C}^n),\ t \mapsto \sigma_j(A_t)$, as before.

Definition 4.2.11. Let $\mathfrak{B} \subseteq \mathbb{B}(\mathcal{H})$ be a C^*-subalgebra. Let $\mathfrak{C} = ((\mathfrak{B})_{t\in\mathcal{T}}, \Gamma)$ be the constant field generated by $\mathfrak{B}$. If $A\ \eta\ \mathfrak{C}$, then we say the field A is $\mathfrak{B}$-*continuous.*

Remark 4.2.12. We would like to relate the continuity of Σ to this notion. As seen in Example 4.1.13 every closed operator a on a Hilbert space $\mathcal{H}$ is affiliated with the compact operators $\mathbb{K}(\mathcal{H})$. We consider the C^*-algebras $\mathbb{K}(\mathcal{H})$ and $\mathbb{B}(\mathcal{H})$ as candidates for $\mathfrak{B}$ in the definition above.

Proposition 4.2.13. *The* $\mathbb{K}(\mathcal{H})$*-continuity of* A *does not imply the continuity of* Σ *with respect to any topology on* $\mathcal{C}(\mathbb{C}^n)$ *where limits are unique, in particular in no Hausdorff topology.*

Proof. Assume $\mathcal{H}$ to be separable. Consider an orthonormal Hilbert basis $(e_k)_{k\in\mathbb{N}}$ for $\mathcal{H}$. For $N \in \mathbb{N}$ define the orthogonal projections

$$p_N := \sum_{k=1}^{N} \langle e_k,\ \cdot\ \rangle\, e_k \in \mathbb{K}(\mathcal{H}) \subseteq \mathbb{B}(\mathcal{H}) = M(\mathbb{K}(\mathcal{H})).$$

Set $\mathcal{T} = \frac{1}{\mathbb{N}} \cup \{0\} \subseteq \mathbb{R}$. Consider the normal field of unbounded operators $a_{\frac{1}{N}} := p_N$ for $N \in \mathbb{N}$ and $a_0 := I$. For every $t \in \mathcal{T}$: $a_t\ \eta\ \mathbb{K}(\mathcal{H})$. The bounded transforms are given by $b(a_t) = a_t/\sqrt{2}$. This can be seen using the functional calculus. Let $x \in \mathbb{K}(\mathcal{H})$. Let $(t_\lambda)_{\lambda\in\Lambda} \subseteq \mathcal{T}$ be a net converging to $t \in \mathcal{T}$. Due to the topology of $\mathcal{T}$ there are two cases:
(i) The net $(t_\lambda)_{\lambda\in\Lambda}$ is eventually constant: There is a $\lambda_0 \in \Lambda\colon \forall\lambda \geq \lambda_0\colon t_\lambda = t$. Then, for $\lambda \geq \lambda_0$:

$$\|b(a_{t_\lambda})x - b(a_t)x\| = 0 \xrightarrow{\lambda\in\Lambda} 0.$$

(ii) $(t_\lambda)_{\lambda\in\Lambda}$ converges to $t = 0$. Then, for every $\lambda \in \Lambda$ there is an $N(\lambda) \in \mathbb{N}$ with $t_\lambda \leq 1/N(\lambda)$ and $N(\lambda) \xrightarrow{\lambda\in\Lambda} \infty$. Thus,

$$\|b(a_{t_\lambda})x - b(a_0)x\| \leq \frac{1}{\sqrt{2}}\|p_{N(\lambda)}x - x\| \xrightarrow{\lambda\in\Lambda} 0$$

because the sequence $(p_N)_{N\in\mathbb{N}}$ is an approximate unit for $\mathbb{K}(\mathcal{H})$.
In both cases $\|b(a_{t_\lambda})^* x_{t_\lambda} - b(a_t)^* x_t\| \xrightarrow{\lambda\in\Lambda} 0$ follows because the operators $b(a_t)$ are self-adjoint. Therefore, $(b(a_t))_{t\in\mathcal{T}}$ is continuous in the strict topology on $\mathbb{B}(\mathcal{H}) = M(\mathbb{K}(\mathcal{H}))$. Then, the field $A = (a_t)_{t\in\mathcal{T}}$ is affiliated with the constant field generated by $\mathbb{K}(\mathcal{H})$ by Lemma 4.2.10. So, A is $\mathbb{K}(\mathcal{H})$-continuous.

Here, the map Σ is given by

$$\mathcal{T} \mapsto \mathcal{C}(\mathbb{C}), \qquad \begin{cases} \frac{1}{N} \mapsto \sigma(p_N) = \{0,\, 1\},\ N \in \mathbb{N} \\ 0 \mapsto \sigma(I) = \{1\}. \end{cases}$$

Let τ be any topology on $\mathcal{C}(\mathbb{C})$. Assume Σ is continuous in this topology. Then,

$$\{0,\, 1\} = \sigma\left(a_{\frac{1}{N}}\right) \xrightarrow{N\to\infty} \sigma(a_0) = \{1\}$$

in τ. However, the assertion $\{0,\, 1\} \xrightarrow{N\to\infty} \{0,\, 1\}$ always holds. Hence, limits in τ cannot be unique. □

Remark 4.2.14. We conclude that $\mathbb{K}(\mathcal{H})$-continuity is too weak to be useful. In particular, just requiring the existence of any continuous field of C^*-algebras such that A is affiliated with it is not a useful notion for the continuity of the spectra.

Proposition 4.2.15. *The $\mathbb{B}(\mathcal{H})$-continuity of A implies the β-continuity of Σ.*

Proof. The C^*-algebra $\mathbb{B}(\mathcal{H})$ is unital. Hence, $M(\mathbb{B}(\mathcal{H})) = \mathbb{B}(\mathcal{H})$ by Lemma 4.1.4. Then, the strict topology on $\mathbb{B}(\mathcal{H})$ is the operator norm topology by Lemma 4.1.6. Therefore, for every $k \in \{1, \ldots, n\}$ the maps

$$\mathcal{B}_k \colon \mathcal{T} \to \mathbb{B}(\mathcal{H}), \qquad t \mapsto b(a_{t,k}) = b_k(A_t),$$

are continuous in the norm topology on $\mathbb{B}(\mathcal{H})$ by 4.2.10. Let $p \in \mathcal{P}_n$ be a polynomial in n variables and their complex conjugates. Then, the map

$$\mathcal{T} \ni t \mapsto \|p(b(A_t))\| = \|p(b_1(A_t), \ldots, b_n(A_t))\| \in [0,\, \infty)$$

is continuous because adjoining, scaling, products, sums and the norm map are continuous on $\mathbb{B}(\mathcal{H})$. Therefore, the spectrum of the bounded transforms of A_t is continuous in the Hausdorff distance by Theorem 2.5.1. This is equivalent to the β-continuity of Σ by Proposition 3.6.6. □

Remark 4.2.16. The proof shows that $\mathbb{B}(\mathcal{H})$-continuity implies the continuity of the bounded transforms in norm. We prove that this is too strong to be captured by the spectra.

Proposition 4.2.17. *In general there is no topology on $\mathcal{C}(\mathbb{C}^n)$ such that the continuity of Σ in this topology implies the $\mathbb{B}(\mathcal{H})$-continuity of A.*

Proof. Assume $\mathcal{H}$ to be separable. Let $(e_k)_{k\in\mathbb{N}_0} \subseteq \mathcal{H}$ be an orthonormal Hilbert basis. For $t \in \mathbb{R}$ denote the largest integer smaller than or equal to t by $\lfloor t \rfloor$. For $t \in [0, \infty) =: \mathcal{T}$ define the operators

$$a_t := \langle e_{\lfloor t \rfloor}, \cdot \rangle \, e_{\lfloor t \rfloor} \in \mathbb{B}(\mathcal{H}).$$

Then, $(a_t)_{t\in\mathcal{T}}$ is a normal bounded field of operators on the constant field of Hilbert spaces $(\mathcal{H})_{t\in\mathcal{T}}$. The operators a_t are orthogonal projections and not equal to the identity or the zero operator. Hence, $\sigma(a_t) = \{0, 1\}$ for all $t \in \mathcal{T}$. In particular the map Σ for this case is constant. Therefore, it is continuous independent of the topology on $\mathcal{C}(\mathbb{C})$. However, the map

$$\mathcal{T} \ni t \mapsto b(a_t) = \frac{1}{\sqrt{2}} a_t \in \mathbb{B}(\mathcal{H})$$

is not continuous in the operator norm topology on $\mathbb{B}(\mathcal{H})$:

$$\forall k \in \mathbb{N} \colon \left\| b\Big(a_{1-\frac{1}{k}}\Big) - b(a_1) \right\| = \frac{1}{\sqrt{2}} \| \langle e_0, \cdot \rangle \, e_0 - \langle e_1, \cdot \rangle \, e_1 \| = \frac{1}{\sqrt{2}} \neq 0.$$

Thus, $A = (a_t)_{t\in\mathcal{T}}$ cannot be $\mathbb{B}(\mathcal{H})$-continuous. □

Remark 4.2.18. We conclude that $\mathbb{B}(\mathcal{H})$-continuity is too strong to be useful.

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Nick von Selzam, Oktober 2021

Selbstständigkeitserklärung

Hiermit erkläre ich, dass ich die vorliegende Arbeit selbstständig und nur unter Verwendung der angegebenen Quellen und Hilfsmittel verfasst habe.

Göttingen, 31.10.2021

(Nick von Selzam)