Mathematisches Institut
Georg-August-Universität Göttingen
Prof. Dr. Ralf Meyer, Dr. Clément Cren.

Exercise sheet 1.

Name

Exercise group (tutor's name)

Exercise	1	2	3	4	Σ
Points					

Deadline: **Friday, 12.4.2024, 16:00**.

Please use this page as a cover sheet and enter your name and tutor in the appropriate fields. Please staple your solutions to this cover sheet.

Exercise 1. (This exercise practices working with two basic definitions (C^*-algebras, representations) and computing operator norms.)

1. Show that the vector space $A := L^\infty(\mathbb{R}, \mathrm{d}x)$ of essentially bounded functions with the essential supremum norm (with respect to Lebesgue measure) is a C^*-algebra under the pointwise multiplication and the involution defined by $f^*(x) := \overline{f(x)}$.
2. Let $\mathcal{H} := L^2(\mathbb{R}, \mathrm{d}x)$ be the Hilbert space of square-integrable functions. Define the multiplication operator $M_f \in \mathbb{B}(\mathcal{H})$ for $f \in A$ by $(M_f(g))(x) := f(x)g(x)$ for $g \in \mathcal{H}$ and $x \in \mathbb{R}$. Show that
$$A \to \mathbb{B}(\mathcal{H}), \qquad f \mapsto M_f,$$
is an isometric *-homomorphism.

Exercise 2. (This exercise practices working with two definition of a C^*-algebra by giving an equivalent variation.) Let A be a Banach algebra with a *-algebra structure that satisfies $\|x\|^2 \le \|x^*x\|$ for all $x \in A$. Show that $\|x^*\| = \|x\|$ and conclude that A is a C^*-algebra.

Exercise 3. (This exercise introduces an important result about general C^*-algebras in an easier special case. It refreshes your memory about a relevant concept from linear algebra.) Let $A \in \mathbb{M}_{n\times n}(\mathbb{C})$. Use the C^*-identity to show that $\|A\|^2 = \rho(A^*A)$, where ρ is the spectral radius of a matrix.

Exercise 4. (Several important classes of C^*-algebra elements such as projections, unitaries, (partial) isometries may be defined both geometrically or by algebraic conditions. The projections studied in this exercise are fundamental objects for K-theory, which will be the topic of a course in the next semester.)

1. Let $p \in \mathbb{B}(\mathcal{H})$ be a bounded operator on a Hilbert space $\mathcal{H}$. Show that p is a projection in the C^*-algebra $\mathbb{B}(\mathcal{H})$ – that is, $p^* = p$ and $p^2 = p$ – if and only if p is the orthogonal projection onto a closed subspace of $\mathcal{H}$.
2. If A is an arbitrary C^*-algebra, show that p is a projection if and only if $p^*p = p$.