Mathematisches Institut
Georg-August-Universität Göttingen
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Exercise sheet 2.

Name

Exercise group (tutor's name)

Exercise	1	2	3	Σ
Points				

Deadline: **Friday, 19.4.2024, 16:00**.

Please use this page as a cover sheet and enter your name and tutor in the appropriate fields. Please staple your solutions to this cover sheet.

Exercise 1. (This exercise practices once again showing that something is a C^*-algebra. The C^*-algebras of the form $\mathrm{C}_0(X, A)$ are important for many purposes. For instance, a homotopy between two *-homomorphisms $A \rightrightarrows B$ is defined as a *-homomorphism $A \to \mathrm{C}([0,1], B)$.) Let A be a C^*-algebra and let X be a locally compact Hausdorff space. Let

$$\mathrm{C}_0(X, A) := \{f : X \to A : f \text{ is continuous and vanishes at } \infty\}.$$

Show that $\mathrm{C}_0(X, A)$ with the norm $\|f\| = \sup\{\|f(x)\| : x \in X\}$ is a C^*-algebra.

Exercise 2. (Yet one more exercise showing that something is a C^*-algebra. If you are interested in category theory, you may want to check that A below is a product of (A_n) in the category of C^*-algebras. The ideal I does not have an obvious universal property, but it is actually the more important one, mainly because it occurs in a few important places in K-theory.) Let A_n for $n \in \mathbb{N}$ be C^*-algebras. Let A be the set of all elements $(a_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} A_n$ with $\|(a_n)\| := \sup\{\|a_n\|\} < \infty$. Give A this norm and the "pointwise" *-algebra structure using the *-algebra structures on the factors A_n. Show that A is a C^*-algebra.

Let $I \subseteq A$ be the subset of sequences $(a_n)_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} \|a_n\| = 0$. Show that I is a closed two-sided ideal in A.

Exercise 3. (This exercise shows that the canonical commutation relations $pq - qp = \mathrm{i}\hbar$ cannot be solved in a normed algebra. In particular, bounded operators cannot satisfy this relation.)

1. Let x, y be elements in a unital algebra A with unit 1 and let $\lambda \in \mathbb{C}$. Show that
$$(x(yx - \lambda 1)^{-1} y - I)(xy - \lambda 1) = \lambda 1.$$
2. Show that $\sigma(xy) \cup \{0\} = \sigma(yx) \cup \{0\}$.
3. Show that if A is a normed algebra, then $xy - yx$ cannot be a nonzero multiple of 1.