Exercise sheet 3.

Name

Exercise group (tutor's name)

Exercise	1	2	3	4	5	Σ
Points						

Deadline: **Monday, 29.4.2024, 10:00**.

Please use this page as a cover sheet and enter your name and tutor in the appropriate fields. Please staple your solutions to this cover sheet.

Exercise 1. (This exercise is an occasion to practice with the C^*-subalgebra generated by a single normal element.) Let $\mathcal{H}$ be a Hilbert space and let X, Y be normal operators on $\mathcal{H}$. Let $C^*_1(X)$ and $C^*_1(Y)$ be the unital C^*-subalgebras of $\mathbb{B}(\mathcal{H})$ generated by X and Y, respectively. Show that there is an isomorphism $\varphi\colon C^*_1(X) \to C^*_1(Y)$ with $\varphi(X) = Y$ if and only if X and Y have the same spectrum.

Exercise 2. (This exercise provides a very important result about C^*-algebra: the norm is already determined uniquely by its algebraic structure.) Let A be a *-algebra. Show that there is at most one norm for which A becomes a C^*-algebra. Use that $\|a\|^2$ must be the spectral radius of a^*a.

Exercise 3. (Positivity in C^*-algebras has some unintuitive properties. Many of them are about whether certain functions preserve operator inquealities. This exercise shows that the square function is not operator monotone. The next exercise gives a few more examples and non-examples of operator monotone functions.) Show that there are positive 2×2 matrices A and B such that $A \le B$ (that is, $B - A \ge 0$), but $A^2 \nleq B^2$. Prove that $A^2 \le B^2$ holds if A and B are *commuting* elements in a C^*-algebra with $0 \le A \le B$.

Exercise 4. A function f on $\mathbb{R}_+ = [0, \infty[$ is *operator monotone* if $0 \le A \le B$ implies $f(A) \le f(B)$.

1. Show that $f_s(t) = st(1 + st)^{-1}$ is operator monotone for $s > 0$.
2. Show that an integral of f_s by a positive measure is also operator monotone. Apply this to $\int_0^\infty f_s(t)s^{-(\beta+1)}\,\mathrm{d}s$ to show that $f(t) = t^\beta$ is operator monotone for $0 < \beta < 1$.
3. Show that $f(t) = \min\{t, 1\}$ is not operator monotone on the C^*-algebra $\mathbb{M}_{2\times 2}(\mathbb{C})$.