Mathematisches Institut
Georg-August-Universität Göttingen
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Noncommutative geometry I
Summer semester 2024

Exercise sheet 4.

Name

Exercise group (tutor's name)

Exercise	1	2	3	4	Σ
Points					

Deadline: **Monday, 6.5.2024, 10:00**.

Please use this page as a cover sheet and enter your name and tutor in the appropriate fields. Please staple your solutions to this cover sheet.

Exercise 1. (This exercise describes the hereditary C*-subalgebras in the commutative case. In particular, it follows that any hereditary C*-subalgebra in a commutative C*-algebra is a (twosided) ideal.) Show that every hereditary subalgebra of $C_0(X)$ has the form

$$\mathcal{J}_E = \{f \in C_0(X) : f_{|E} = 0\}$$

for a closed subset E of X. (*Hint*: Use the Stone–Weierstrass Theorem.)

Exercise 2. (This result about approximate units in a commutative C*-algebra was already mentioned during the class. It may be useful for some students to work out the details of this.) Let X be a locally compact space. Show that a net $(e_\lambda)_{\lambda\in I}$ is an approximate unit in $C_0(X)$ if and only if

1. $0 \le e_\lambda(x) \le 1$ for all $x \in X$ and $\lambda \in I$;
2. $e_\lambda(x) \le e_\mu(x)$ for all $x \in X$ and $\lambda, \mu \in I$ with $\lambda \le \mu$;
3. $\lim_{\lambda\in I} e_\lambda(x) = 1$ for all $x \in X$ uniformly on compact subsets. Recall that a net of functions (f_λ) on a locally compact space X converges to a function f *uniformly on compact subsets* if for every compact subset $K \subseteq X$, $f_{\lambda_{|K}}$ converges uniformly to $f_{|K}$.

Exercise 3. (Next we describe a particularly nice approximate unit for the compact operators on a Hilbert space, consisting of projections. Such approximate units only exist in special cases, compare the commutative case, where you cannot expect this.) Let $\mathcal{H}$ be a Hilbert space and let $(e_i)_{i\in I}$ an orthonormal basis. For $S \subseteq I$, let p_S be the orthogonal projection onto $\operatorname{span}\{e_i : i \in S\}$. The set $\mathcal{F}(I)$ consisting of finite subsets of I ordered by inclusion is directed. Show that $(p_S)_{S\in\mathcal{F}(I)}$ is an approximate unit in $\mathbb{K}(\mathcal{H})$, the compact operators on a Hilbert space $\mathcal{H}$.

Exercise 4. (In general, a quotient seminorm is only a limit, there need not be a representative where the minimum is reached. This exercise shows that for quotient C*-algebras, the quotient norm is indeed the norm of a carefully chosen representative.) Show that if J is an ideal of a C*-algebra A and $a \in A$, then there is an element $j \in J$ such that $\|a - j\| = \operatorname{dist}(a, J)$. (Hint: You may let $a - j = af(a^*a)$ for a carefully chosen continuous function f on $[0, \infty[$.)