Mathematisches Institut
Georg-August-Universität Göttingen
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Exercise sheet 5.

Name

Exercise group (tutor's name)

Exercise	1	2	3	4	Σ
Points					

Deadline: **Monday, 13.5.2024, 10:00**.

Please use this page as a cover sheet and enter your name and tutor in the appropriate fields. Please staple your solutions to this cover sheet.

Exercise 1. (This exercise describes the hereditary C^*-subalgebra generated by a single positive element. This is often useful to produce C^*-hereditary subalgebras.) Let a be a positive element of a C^*-algebra A. Show that $\overline{aAa} \subseteq A$ is the hereditary C^*-subalgebra of A generated by a. (Hint: Use approximate units to show that a^2 lies in $\overline{aAa}$).

Exercise 2. (This exercise is about a counterexample, showing that the multiplication is not jointly continuous in the strong operator topology.) Show that multiplication is not jointly continuous in the strong operator topology. (Hint: let I consist of ordered pairs (M, x), where M is a finite-dimensional subspace and x is a unit vector orthogonal to M, ordered by the relation $(M, x) < (N, y)$ if M and x are both contained in N. Let e be a fixed unit vector. For two vectors $a, b \in \mathcal{H}$, define the rank-one operator ab^* by $ab^*(c) := a \cdot \langle b \mid c \rangle$. Set $A_{(M,x)} = (\dim M) e x^*$ and $B_{(M,x)} = (\dim M)^{-1} x e^*$.

Exercise 3. (This exercise is about a simple commutative example, where the statement of the Double Commutant Theorem is easy to check and where weak convergence has another nice interpretation.) Let μ be a probability measure – that is, $\mu(X) = 1$ – on a locally compact space X. Represent the C^*-algebra $A = L^\infty(X, \mu)$ of essentially bounded functions in $\mathbb{B}(L^2(X, \mu))$ by pointwise multiplication operators.

1. Show that $A' = A$ and deduce $A'' = A$.
2. Show that a net $(f_\lambda)_{\lambda \in I}$ in A converges weakly to f if and only if $\int_X f_\lambda g \, d\mu(x) \to \int_X f g \, d\mu(x)$ for all $g \in L^1(X, \mu)$.
3. Deduce that A is weakly closed. You may use that the dual space of $L^1(X, \mu)$ is $L^\infty(X, \mu)$.

Exercise 4. (This exercise is about the important case when the commutant of a C^*-subalgebra of $\mathbb{B}(\mathcal{H})$ is $\mathbb{C} \cdot 1$. We have already related this condition to irreducible representations.)

1. Let A be a C^*-subalgebra of $\mathbb{B}(\mathcal{H})$ containing the identity operator 1. Show that A is weakly dense in $\mathbb{B}(\mathcal{H})$ if and only if $A' = \mathbb{C} \cdot 1$.
2. Let S be the unilateral shift operator defined by
$$S\colon \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N}), \qquad \delta_n \mapsto \delta_{n+1},$$
for all $n \in \mathbb{N}$. Show that the commutant of $\{S, S^*\}$ is $\mathbb{C} \cdot 1$