Georg-August-Universität Göttingen

master thesis

The Generalized Fixed Point Algebra for the Scaling Action on the Tangent Groupoid

Philip Miller

supervised by
Prof. Dr. Ralf Meyer
cosupervided by
Prof. Dr. Dorothea Bahns

March 23, 2017

Abstract

The (reduced) C^*-algebra of the tangent groupoid of $\mathbb{R}^n$ is a continuous field of C^*-algebras over the interval $[0, \infty)$, such that the fibre at zero is isomorphic to $C_0(\mathbb{R}^n \times \mathbb{R}^n)$ and all the other fibres are isomorphic to $\mathbb{K}(L^2(\mathbb{R}^n))$. This is called a deformation of the noncommutative C^*-algebra $\mathbb{K}(L^2(\mathbb{R}^n))$ to the commutative algebra $C_0(\mathbb{R}^n \times \mathbb{R}^n)$.
There is scaling action of $\mathbb{R}_+^*$ on the reduced C^*-algebra of the tangent groupoid. We prove that this scaling action is continuously square-integrable in the sense of Meyer [1] when restricted to an ideal. The generalised fixed point algebra for this action is an extension of $C_0(\mathbb{R}^n \times \mathbb{S}^{n-1})$ by $\mathbb{K}(L^2(\mathbb{R}^n))$. It is isomorphic to the so-called pseudodifferential extension as described by Higson and Roe.

Contents

Introduction

In the first chapter we present Meyers theory of square-integrable group actions on C^*-algebras[1]. For a G-C^*-algebra A with action $(\alpha_g)_{g\in G}$, we define the G-equivariant Hilbert A-module $L^2(G, A)$ and square-integrable functions $G \to A$. An element of A is called *square-integrable* if the function $g \mapsto \alpha_g(a)^* b$ is square-integrable for all $b \in B$. If the square-integrable elements are dense in A, then A is called square-integrable.
To define the generalised fixed point algebra one needs further requirements invoking the crossed product of the action. We represent the twisted convolution algebra $C_c(G, A)$ on $L^2(G, A)$ to obtain the reduced crossed product $C^*_r(G, A)$.
We define continuously square-integrable group actions. For these actios a $C^*_r(G, A)$-module is constructed and the generalised fixed point algebra is defined, such that this Hilbert module gets a bimodule over it and the crossed product.
The second chapter deals with Locally Compact Hausdorff Groupoids and their reduced C^*-algebras . We discuss the tangent bundle $T\mathbb{R}^n$ and the pair groupoid $P\mathbb{R}^n$ and prove that $C^*_r(T\mathbb{R}^n) \cong C_0(\mathbb{R}^n)$ and $C^*(P\mathbb{R}^n) \cong \mathbb{K}(L^2(\mathbb{R}^n))$. We introduce the tangent groupoid of $\mathbb{R}^n$ with its fibre epimorphisms.
In the third chapter We define the scaling action on the reduced C^*-algebra of the tangent groupoid and prove that it is continuously square integrable on an appropriate ideal. The generalised fixed point algebra is an extension of $C_0(\mathbb{R}^n \times \mathbb{S}^{n-1})$ by $\mathbb{K}(L^2(\mathbb{R}^n))$.
In the fourth chapter we consider the C^*-algebra $\mathcal{P}$ generated by pseudodifferential operators on Rn of the form $D_{gf} := M_g \circ \mathcal{F}^{-1} \circ M_f \circ \mathcal{F}$, where M_g is a multiplication operator by a C_0-function, $\mathcal{F}$ is the Fourir transform and M_f is a multiplication operator by a function on $\mathbb{R}^n$ that extends to the compactification of $\mathbb{R}^n$ to a closed n-dimensional ball. We prove that that $\mathcal{P}$ is isomorphic to the generalised fixed point algebra. We obtain a short exact sequence $\mathbb{K}(L^2(\mathbb{R}^n)) \subset \mathcal{P} \xrightarrow{\pi} C_0(\mathbb{R}^n \times \mathbb{S}^{n-1})$ with

$$\pi(D_{gf})(x, \xi) = g(x) \cdot \lim_{\lambda\to\infty} f(\lambda\xi).$$

Finally we prove that $\mathcal{P}$ is the C^*-algebra mentioned in Higson and Roe's Analytic K-homology [2, pages 46-48] generated by pseudodifferential operators.

1 Square-Integrable Group Actions and The Generalized Fixed Point Algebra

1.1 The Hilbert Module $L^2(G, A)$ and Square-Integrable Functions

We define and characterise the G-equivariant Hilbert module $L^2(G, A)$ for a locally compact group G acting continuously on a C^*-algebra A. This Hilbert A-module should generalise the Hilbert space $L^2(G)$ of square integrable functions $G \to \mathbb{C}$ with the action by left translation. We will define $L^2(G, A)$ as a completion of the space $C_c(G, A)$ of compactly supported functions $G \to A$ with respect to the A-valued inner product

$$\langle f_1, f_2\rangle = \int_G f_1(g)^* f_2(g) \ \mathrm{d}\mu(g).$$

We show that there is a natural isomorphism between $L^2(G, A)$ and $L^2(G) \otimes A$.

The Hilbert Module $L^2(X, A)$

Let us first review some facts about Hilbert modules and fix the notation.
Let $\mathcal{E}$ be a Hilbert A-module. We write $\mathbb{B}(\mathcal{E})$ for the C^*-algebra of adjointable operators on $\mathcal{E}$. For $\xi, \eta \in \mathcal{E}$, we define a so-called rank-one operator $\theta_{\xi\eta} \in \mathbb{B}(\mathcal{E})$ by $\theta_{\xi\eta}(\zeta) = \xi \cdot \langle \eta, \zeta \rangle$. The closed linear span of all rank-one operators is denoted by $\mathbb{K}(\mathcal{E})$. It is a closed two-sided ideal in $\mathbb{B}(\mathcal{E})$ and its elements are called compact operators on $\mathcal{E}$.
Let B be another C^*-algebra and $\mathcal{F}$ a Hilbert module over B. Given a $*$-homomorphism $f \colon B \to \mathbb{B}(\mathcal{E})$ we form the A-module $\mathcal{F} \otimes_B \mathcal{E}$. It is the completion of the algebraic B-balanced tensor product $\mathcal{F} \otimes_B^{\mathrm{alg}} \mathcal{E}$ with respect to the A-module structure defined by $(\eta \otimes \xi) \cdot a = \eta \otimes (\xi \cdot a)$ and the A-valued the inner product

$$\langle \eta_1 \otimes \xi_1, \eta_2 \otimes \xi_2 \rangle = \langle \xi_1, f\left(\langle \eta_1, \eta_2 \rangle\right) \xi_2 \rangle .$$

If $B = \mathbb{C}$ we use the canonical map $\mathbb{C} \to \mathbb{B}(\mathcal{E})$. In this case, we omit the subscript and just write $\mathcal{F} \otimes \mathcal{E}$.
Let X be a locally compact Hausdorff space and μ a locally finite, strictly positive Borel measure on X. Before we turn to the G-equivariant case, we define the Hilbert A-module $L^2(X, A)$ and show that it is isomorphic to $L^2(X) \otimes A$.
If $f \in C_c(X, A)$ and $a \in A$, then we define $(f \cdot a)(x) = f(x) \cdot a$ and obtain $f \cdot a \in C_c(X, A)$. Obviously, $C_c(X, A)$ is a right A- module with this multiplication. For $f_1, f_2 \in C_c(X, A)$ the map $x \mapsto f_1(x)^* f_2(x)$ is continuous and compactly supported. Hence the pairing

$$\langle f_1, f_2 \rangle = \int_X f_1(x)^* f_2(x) \;\, \mathrm{d}\mu(x) \in A$$

is well-defined by Proposition 5.7.

Proposition 1.1 ($C_c(X, A)$ is a pre-Hilbert A-module)**.**
The above multiplication and inner product turn $C_c(X, A)$ into a pre-Hilbert A-module.

Proof. For $f \in C_c(X, A)$ we have $f(x)^* f(x) \geq 0$ for all $x \in X$. Therefore, 5.13(iii) yields $\langle f, f \rangle \geq 0$.
If $\langle f, f \rangle = 0$ we have $f(x)^* f(x) = 0$ for all $x \in X$ by 5.13(iv). Hence $f = 0$ by the C^*-condition. If $f_1, f_2 \in C_c(X, A)$ and $a \in A$, then

$$\begin{aligned} \langle f_1, f_2 \rangle^* &= \left(\int_X f_1(x)^* f_2(x) \;\, \mathrm{d}\mu(x) \right)^* \overset{5.13(i)}{=} \int_X \left(f_1(x)^* f_2(x) \right)^* \;\, \mathrm{d}\mu(x) \\ &= \int_X f_2(x)^* f_1(x) \;\, \mathrm{d}\mu(x) = \langle f_2, f_1 \rangle \end{aligned}$$

and

$$\langle f_1, f_2 \cdot a \rangle = \int_X f_1(x)^* f_2(x) a \;\, \mathrm{d}\mu(x) \overset{5.13(v)}{=} \int_X f_1(x)^* f_2(x) \;\, \mathrm{d}\mu(x) \cdot a = \langle f_1, f_2 \rangle a. \qquad \square$$

Definition 1.2 (The Hilbert A-module $L^2(X,A)$)**.**
The Hilbert A-module $L^2(X,A)$ is the completion of the pre-Hilbert A-module $C_c(X,A)$.

We view $C_c(X,A)$ as a dense submodule of $L^2(X,A)$ and write $\|\cdot\|_2$ for the norm induced by the inner product. For $A=\mathbb{C}$, we obtain the Hilbert space $L^2(X)$ of square-integrable functions on X. Considering A as a Hilbert A-module with right multiplication and $\langle a_1,a_2\rangle = a_1^*a_2$, we obtain the Hilbert A-Module $L^2(X)\otimes A$, where the inner product simplifies to

$$\langle f_1\otimes a_1, f_2\otimes a_2\rangle = \langle a_1, \langle f_1,f_2\rangle a_2\rangle = \langle f_1,f_2\rangle a_1^*a_2.$$

Hence we have $\|f\otimes a\| = \|\langle f,f\rangle a^*a\|^{1/2} = \|f\|_2\cdot\|a\|$ by the C^*-condition. If $f\in C_c(X)$ and $a\in A$, then we write $(f.a)(x)=f(x)\cdot a$ for and get $f.a\in C_c(X,A)\subseteq L^2(X,A)$. The next proposition shows that the elements of the form $f.a$ span a dense subspace of $L^2(X,A)$.

Proposition 1.3. *The subspace*

$$M := \operatorname{span}\{f.a\colon f\in C_c(X), a\in A\}$$

is dense in $L^2(X,A)$.

Proof. Let $f\in C_c(X,A)$ and $\varepsilon>0$. Let $U\subseteq X$ be open with compact closure and $\operatorname{supp}(f)\subseteq U$. We have $0<\mu(U)\le\mu\left(\overline{U}\right)<\infty$. By Lemma 5.5, there is $h\in M$ such that

$$\operatorname{supp}(h)\subseteq U \quad\text{and}\quad \|f(x)-h(x)\| < \frac{\varepsilon}{\sqrt{\mu(\overline{U})}} \quad\text{for all } x\in X.$$

We estimate

$$\begin{aligned}\|f-h\|_2^2 &= \left\|\int_X (f(x)-h(x))^*(f(x)-h(x))\ \mathrm{d}\mu(x)\right\| \\ &\le \int_X \|f(x)-h(x)\|^2\ \mathrm{d}\mu(x) = \int_U \|f(x)-h(x)\|^2\ \mathrm{d}\mu(x) \\ &< \mu(\overline{U})\cdot\frac{\varepsilon^2}{\mu(\overline{U})} = \varepsilon^2.\end{aligned}$$

Hence $\|f-h\|_2<\varepsilon$. This shows $C_c(X,A)\subseteq\overline{M}$. So that $L^2(X,A)=\overline{C_c(X,A)}\subseteq\overline{M}$. Therefore, M is dense in $L^2(X,A)$. $\square$

Theorem 1.4. *There is a unique isomorphism of Hilbert A-modules*

$$\Phi\colon L^2(X)\otimes A \xrightarrow{\sim} L^2(X,A) \quad \textit{with } \Phi(f\otimes a)=f.a \quad \textit{for } f\in C_c(X) \textit{ and } a\in A.$$

Proof. Let $f\in L^2(X)$ and $a\in A$. There is a sequence $(h_n)_{n\in\mathbb{N}}\subseteq C_c(X,A)$ converging to f in the L^2-norm. We obtain

$$\|f\otimes a - h_n\otimes a\| = \|f-h_n\|_2\cdot\|a\| \xrightarrow{n\to\infty} 0.$$

Hence $\Phi(f \otimes a) = f.a$ for all $f \in C_c(X, A)$ determines Φ uniquely on all elementary tensors of $L^2(X) \otimes A$. Since these span a dense subspace of $L^2(X) \otimes A$, this shows, that Φ is unique, if it exits.
The bilinear map

$$C_c(X) \times A \to L^2(X, A) \quad \text{given by } (f, a) \mapsto f.a.$$

gives rise to a linear map

$$\phi_1 \colon C_c(X) \otimes^{\text{alg}} A \to L^2(X, A) \quad \text{with } f \otimes a \mapsto f.a.$$

If $b \in A$, then

$$\phi_1(f \otimes a) \cdot b = (f.a) \cdot b = f.(ab) = \phi_1(f \otimes ab).$$

Therefore, ϕ_1 is an A-module homomorphism. The range of ϕ_1 is dense by Proposition 1.3.
There is also an A-module homomorphism

$$\phi_2 \colon C_c(X) \otimes^{\text{alg}} A \to L^2(X) \otimes A \text{ given by } f \otimes a \mapsto f \otimes a.$$

The range of ϕ_2 is dense by the approximation argument above. If $f_1, f_2 \in C_c(X)$ and $a_1, a_2 \in A$, then

$$\begin{aligned}
\langle \phi_1(f_1 \otimes a_1), \phi_1(f_2 \otimes a_2) \rangle = \langle f_1.a_1, f_2.a_2 \rangle &= \int_X (f_1(x)a_1)^* \, (f_2(x)a_2) \; \mathrm{d}\mu(x) \\
&\overset{5.7}{=} \int_X \overline{f_1(x)} f_2(x) \; \mathrm{d}\mu(x) \cdot a_1^* a_2 \\
&= \langle f_1 \otimes a_1, f_2 \otimes a_2 \rangle \\
&= \langle \phi_2(f_1 \otimes a_1), \phi_2(f_2 \otimes a_2) \rangle.
\end{aligned}$$

Therefore, ϕ_1 and ϕ_2 induce the same inner product on the A-module $C_c(X) \otimes^{\text{alg}} A$. Both, ϕ_1 and ϕ_2, extend to unitaries Φ_1 and Φ_2 from the completion of $C_c(X) \otimes^{\text{alg}} A$ in this inner product to $L^2(X, A)$ and $L_2(X) \otimes A$ respectively. Therefore, $\Phi \colon = \Phi_1 \circ \Phi_2^{-1}$ is a unitary $L^2(X) \otimes A \to L^2(X, A)$. If $f \in C_c(X)$ and $a \in A$, then

$$\Phi(f \otimes a) = \Phi\left(\Phi_2(f \otimes a)\right) = \Phi_1(f \otimes a) = f.a. \qquad \square$$

Let $\mathcal{E}$ be a right Hilbert A-module and $I \lhd A$ an ideal with quotient map $\pi \colon A \to A/I$. By Cohen's factorisation theorem we have

$$\mathcal{E} \cdot I := \overline{\operatorname{span}}\{\xi \cdot i \colon \xi \in \mathcal{E}, i \in I\} = \{\xi \cdot i \colon \xi \in \mathcal{E}, i \in I\}.$$

$\mathcal{E} \cdot I$ is a submodule of $\mathcal{E}$. Using an approximate identity of I Cohen's factorisation theorem yields, that $\xi \in \mathcal{E} \cdot I$ if and only if $\langle \xi, \xi \rangle \in I$.
Let $q \colon \mathcal{E} \to \mathcal{E}/(\mathcal{E} \cdot I)$ be the quotient map. $\mathcal{E}/(\mathcal{E} \cdot I)$ becomes a right Hilbert A/I-module, when equipped with the well-defined multiplication $q(\xi) \cdot \pi(a) := q(\xi \cdot a)$ and inner product $\langle q(\xi), q(\eta) \rangle := \pi(\langle \xi, \eta \rangle)$. The bilinear map

$$\mathcal{E} \times (A/I) \to \mathcal{E}/(\mathcal{E} \cdot I) \quad \text{given by } (\xi, b) \mapsto q(\xi) \cdot b$$

induces an isomorphism $\phi\colon \mathcal{E} \otimes_A (A/I) \to \mathcal{E}/(\mathcal{E} \cdot I)$. Here A acts on A/I by $a \cdot \pi(b) := \pi(ab)$. Therefore we get a $*$-isomorphism

$$\mathbb{B}(\mathcal{E}/(\mathcal{E} \cdot I)) \to \mathbb{B}(\mathcal{E} \otimes_A A/I) \quad \text{by } T \mapsto \phi^{-1} \circ T \circ \phi.$$

Moreover we define

$$Q_1\colon \mathbb{B}(\mathcal{E}) \to \mathbb{B}(\mathcal{E}/(\mathcal{E} \cdot I)) \quad \text{by } Q_1(T)(q(\xi)) = q(T\xi) \quad \text{for all } T \in \mathcal{E}.$$

We have $T(\mathcal{E} \cdot I) = T(\mathcal{E}) \cdot I \subseteq \mathcal{E} \cdot I$ for all $T \in \mathbb{E}$. Therefore, Q_1 is a well defined $*$-homomorphism. Finally we define a $*$-homomorphism

$$Q_2\colon \mathbb{B}(\mathcal{E}) \to \mathbb{B}(\mathcal{E} \otimes (A/I)) \quad \text{by } T \mapsto T \otimes \mathrm{id}_{A/I}.$$

Then we obtain a commutative diagram

$$\begin{tikzcd}
 & \mathbb{B}(\mathcal{E}) \arrow[dl, "Q_1"'] \arrow[dr, "Q_2"] & \\
\mathbb{B}(\mathcal{E}/(\mathcal{E} \cdot I)) \arrow[rr] & & \mathbb{B}(\mathcal{E} \otimes_A (A/I))
\end{tikzcd}$$

We have

$$\ker(Q_1) = \ker(Q_2) = \{T \in \mathcal{E} \colon T(\mathcal{E}) \subseteq \mathcal{E} \cdot I\}.$$

Let $(u_j)_{j \in J}$ be an approximate identity of I and $T \in \ker(Q_1)$. Then

$$T(\xi) = \lim_j T(\xi) \cdot u_j = \lim_j T(\xi \cdot u_j) \quad \text{for all } \xi \in \mathcal{E}.$$

Therefore, $\|T\| = \|T|_{\mathcal{E} \cdot I}\|$.

Lemma 1.5 (Short Exact Sequences lift to $L^2(X, \cdot)$)**.**
Let $I \lhd A$ be an ideal. The inclusion map $C_c(X, I) \to C_c(X, A)$ extends to an isometry $i\colon L^2(X, I) \to L^2(X, A)$. The image of i is $L^2(X, A) \cdot I$.
The pointwise quotient map $C_c(X, A) \to C_c(X, A/I)$ extends to a surjective map $\pi\colon L^2(X, A) \to L^2(X, A/I)$.
There is an isomorphism $L^2(X, A/I) \cong L^2(X, A)/(L^2(X, A) \cdot I)$ of right Hilbert A/I-modules, such that the following diagram commutes

$$\begin{tikzcd}
 & L^2(X, A) \arrow[dl, "\pi"'] \arrow[dr] & \\
L^2(X, A/I) \arrow[rr, "\sim"] & & L^2(X, A)/(L^2(X, A) \cdot I)
\end{tikzcd}$$

Proof. The inclusion map $C_c(X, I) \to C_c(X, A)$ is isometric. Therefore, its extension to $L^2(X, I) \to L^2(X, A)$ is isometric. Let $f \in C_c(X, I)$. Then $\langle i(f), i(f) \rangle = \langle f, f \rangle \in I$. Therefore $i(f) \in L^2(X, A) \cdot I$.

Since $L^2(X,A)\cdot I$ is closed, this implies $i(L^2(X,I)) \subseteq L^2(X,A)\cdot I$.
If $f \in C_c(X,A)$ and $i \in I$, then $f \cdot i \in C_c(X,I)$, hence

$$C_c(X,A)\cdot I \subseteq i(C_c(X,I)) \subseteq i(L^2(X,I)).$$

Since $i(L^2(X,I))$ is closed, this implies that $i(L^2(X,I)) = L^2(X,A)\cdot I$.
The commutative diagram

$$\begin{tikzcd} & A \arrow[dl] \arrow[dr] & \\ A/I \arrow[rr, "\sim"] & & A \otimes_A (A/I) \end{tikzcd}$$

gives rise to a a commutative diagram

$$\begin{tikzcd} & L^2(X)\otimes A \arrow[dl] \arrow[dr] & \\ L^2(X)\otimes (A/I) \arrow[rr, "\sim"] & & L^2(X)\otimes A \otimes_A (A/I). \end{tikzcd}$$

By Theorem 1.4 this yields the diagram

$$\begin{tikzcd} & L^2(X,A) \arrow[dl] \arrow[d] \arrow[dr] & \\ L^2(X,A/I) \arrow[r, "\sim"] & L^2(X,A)\otimes_A (A/I) \arrow[r, "\sim"] & L^2(X,A)/(L^2(X,A)\cdot I). \end{tikzcd}$$

$\square$

Square-Integrable Functions

To define square-integrable group actions, we need a notion of square-integrable functions. We want to identify a square integrable function with an element of $L^2(X,A)$. As above, let X be a locally compact space and μ a locally finite and strictly positive Borel measure on X. This time, we also need to require that μ is inner regular on open sets. That is, for $U \subseteq X$ open, we have

$$\mu(U) = \sup\{\mu(K)\colon K \text{ is compact with } K \subseteq U\}.$$

We write $\mathcal{T}(X)$ for the set of all $h \in C_c(X)$ with $0 \le h \le 1$. For $f\colon X \to A$ continuous and $h \in \mathcal{T}(X)$, we obtain $h \cdot f \in C_c(X,A)$. If $h_1, h_2 \in \mathcal{T}(X)$ with $h_1 \le h_2$, then

$$0 \le h_2^2(x)f(x)^*f(x) - h_1^2(x)f(x)^*f(x) \quad \text{for all } x \in X.$$

Therefore,

$$\int_X h_1^2(x)f(x)^*f(x)\ \mathrm{d}\mu(x) \le \int_X h_2^2(x)f(x)^*f(x)\ \mathrm{d}\mu(x)$$

by 5.13(iii) and hence

$$\|h_1 \cdot f\|_2^2 = \left\| \int_X h_1^2(x) f(x)^* f(x) \ \mathrm{d}\mu(x) \right\| \leq \left\| \int_X h_2^2(x) f(x)^* f(x) \ \mathrm{d}\mu(x) \right\| = \|h_2 \cdot f\|_2^2 .$$

For $h \in \mathcal{T}(X)$ we define a linear map $C_c(X,A) \to C_c(X,A)$ by $f \mapsto h \cdot f$. Since $0 \leq f(x)^* f(x) - h^2(x) f(x)^* f(x)$ for all $x \in X$, we obtain $\|f \cdot h\|_2 \leq \|f\|_2$ by the same computation as above. Therefore, the map extends to a self-adjoint operator

$$M_h \colon L^2(X,A) \to L^2(X,A) \text{ with } \|M_h(f)\|_2 \leq \|f\|_2.$$

Lemma 1.6 (The Multiplication Operators M_h)**.**
Let $(\chi_i)_{i\in I} \subseteq \mathcal{T}(X)$ be a net with $\chi_i \longrightarrow 1$ uniformly on compact subsets.
For $f \in L^2(X,A)$, we have

$$\lim_i M_{\chi_i} f = f.$$

Proof. Let $f \in L_2(X,A)$ and $\varepsilon > 0$. There is $h \in C_c(X,A)$, such that $h \neq 0$ and $\|f-h\|_2 < \frac{\varepsilon}{3}$. There is $i_0 \in I$, such that $|1-\chi_i(x)| \leq \frac{\varepsilon}{3\|h\|_2}$ for all $x \in \operatorname{supp}(h)$ and $i \geq i_0$. We obtain

$$\begin{aligned}
\|h - h \cdot \chi_i\|_2 &= \left\| \int_X h(x)^* h(x) \left(1-\chi_i(x)\right)^2 \ \mathrm{d}\mu(x) \right\|^{1/2} \\
&\overset{5.13(iii)}{\leq} \frac{\varepsilon}{3\|h\|_2} \left\| \int_X h(x)^* h(x) \ \mathrm{d}\mu(x) \right\|^{1/2} = \frac{\varepsilon}{3}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|f - M_{\chi_i} f\|_2 &\leq \|f-h\|_2 + \|h - M_{\chi_i} h\|_2 + \|M_{\chi_i} h - M_{\chi_i} f\|_2 \\
&\leq 2\,\|f-h\|_2 + \|h - M_{\chi_i} h\|_2 < \varepsilon
\end{aligned}$$

for all $i \geq i_0$. This shows $\lim_i M_{\chi_i} f = f$. □

The set $\mathcal{T}(X)$ is a directed set with the pointwise order. $\max\{h_1,h_2\} \in \mathcal{T}(X)$ is an upper bound for $h_1, h_2 \in \mathcal{T}(X)$. For $K \subseteq X$ compact, there is $h \in \mathcal{T}(X)$ with $h(K) = \{1\}$. Hence the net $(h)_{h\in\mathcal{T}(X)}$ converges to 1 uniformly on compact subsets. This shows that there always exists a net as in Lemma 1.6.
The following corollary motivates the definition of a square-integrable function.

Corollary 1.7. *For $f \in L_2(X,A)$, we have*

$$\|f\|_2 = \sup\left\{\|M_h f\|_2 \colon h \in \mathcal{T}(X)\right\}.$$

Proof. If $h \in \mathcal{T}(X)$, then $\|M_h f\|_2 \leq \|f\|_2$. Therefore,

$$\sup\left\{\|M_h f\|_2 \colon h \in \mathcal{T}(X)\right\} \leq \|f\|_2.$$

Pick a net $(\chi_i)_{i\in I}$ in $\mathcal{T}(X)$ with $\chi_i \longrightarrow 1$ uniformly on compact subsets.
Then Lemma 1.6 implies $\lim_i \|M_{\chi_i} f\|_2 = \|f\|_2$, proving the assertion. □

Definition 1.8 (Square-Integrable Function)**.**
A continuous function $f\colon X \to A$ is called *square-integrable* if there is $M \in [0, \infty)$, such that

$$\|h \cdot f\|_2 \leq M \text{ for all } h \in \mathcal{T}(X).$$

We write $\mathcal{S}^2(X, A)$ for the set of all continuous and square-integrable functions.

Proposition 1.9 (The Normed Space $\mathcal{S}^2(X, A)$)**.**
$\mathcal{S}^2(X, A)$ is a vector space and

$$f \mapsto \|f\|_{\mathcal{S}^2} = \sup \{\|h \cdot f\|_2 \leq M \textit{ for all } h \in \mathcal{T}(X)\}$$

defines a norm on $\mathcal{S}^2(X, A)$.

Proof. Let $f_1, f_2 \in \mathcal{S}^2(X, A)$ and $\lambda_1, \lambda_2 \in \mathbb{C}$. By definition, there are $M_1, M_2 \in [0, \infty)$, such that

$$\|h \cdot f_1\|_2 \leq M_1 \text{ and } \|h \cdot f_2\|_2 \leq M_2 \text{ for all } h \in \mathcal{T}(X).$$

Put $M = |\lambda_1| M_1 + |\lambda_2| M_2$. Then $M \in [0, \infty)$ and

$$(h\lambda f_1 + \lambda f_2)\,\|_2 \leq |\lambda_1| \|h \cdot f_1\|_2 + |\lambda_2| \|h \cdot f_2\|_2 \leq M.$$

Therefore, $\lambda_1 f_1 + \lambda_2 f_2 \in \mathcal{S}^2(X, A)$, so $\mathcal{S}^2(X, A)$ is a subspace of the vector space of continuous functions $X \to A$.
It is easy to see that the map $f \mapsto \|h \cdot f\|_2$ for $h \in \mathcal{T}(X)$ is a seminorm on $\mathcal{S}^2(X, A)$. Therefore, $f \mapsto \|f\|_{\mathcal{S}^2}$ is a seminorm. It remains to check that it is positive definite. Let $f \in \mathcal{S}^2(X, A)$ with $\|f\|_{\mathcal{S}^2} = 0$. For $x \in X$ there is $h \in \mathcal{T}(X)$ with $h(x) = 1$. Since $\|h \cdot f\| = 0$, we obtain $f(x) = (h \cdot f)(x) = 0$. Hence $f = 0$. $\square$

Lemma 1.10. *Let $f\colon X \to A$ be continuous. If $\int_X \|f(x)\|^2 \, d\mu(x) < \infty$ then f is square-integrable.*

Proof. If $h \in \mathcal{T}(X)$, then

$$\begin{aligned}
\|f \cdot h\|_2 &= \left\| \int_X f(x)^* f(x) h^2(x) \ \mathrm{d}\mu(x) \right\|^{1/2} \\
&\overset{5.7}{\leq} \left(\int_X \|f(x)^* f(x)\| \cdot h^2(x) \ \mathrm{d}\mu(x) \right)^{1/2} \\
&\leq \left(\int_X \|f(x)\|^2 \ \mathrm{d}\mu(x) \right)^{1/2}.
\end{aligned}$$

Therefore, f is square-integrable. $\square$

The next lemma states the converse of Lemma 1.10 for $A = \mathbb{C}$. So in this case continuous and square-integrable functions are exactly continuous L^2-functions.

Lemma 1.11 (Square-Integrable Functions to $\mathbb{C}$)**.**
A continuous function $f\colon X \to \mathbb{C}$ is square-integrable in the sense of Definition 1.8 *if and only if $\int_X |f|^2 \, d\mu < \infty$.*

Proof. The reverse implication is Lemma 1.10, so we suppose $f\colon X \to \mathbb{C}$ continuous and square-integrable in the sense of Definition 1.8.
Let $M \in [0, \infty)$, such that

$$\|h \cdot f\|_2 \leq M \text{ for all } h \in \mathcal{T}(X).$$

For a Borel set $B \subseteq X$ put $\nu_f(B) = \int_B |f|^2 \, \mathrm{d}\mu$. Then ν_f is a Borel measure on X. Let $K \subset X$ be compact. Since there is $h \in \mathcal{T}(X)$ with $h(K) = \{1\}$, we obtain

$$\nu_f(K) = \int_K |f|^2 \, \mathrm{d}\mu \leq \int_X |f|^2 h^2 \, \mathrm{d}\mu = \|h \cdot f\|_2 \leq M.$$

Let $\delta > 0$ and put $U_\delta = \left\{x \in X\colon |f(x)|^2 > \delta\right\}$. Supposing $\mu(U_\delta) = \infty$, we would find $K \subseteq U_\delta$ compact with $\nu(K) > \frac{1}{\delta}$. Then we would get

$$\nu_f(K) = \int_K |f|^2 \, \mathrm{d}\mu \geq \int_K \delta \, \mathrm{d}\mu = \mu(K) \cdot \delta > 1,$$

contradicting $\nu_f(K) \leq 1$.
Therefore, $\mu(U_\delta) < \infty$ and there is $L_n \subseteq U_\delta$ compact with $\mu\left(U_\delta \setminus L_n\right) < \frac{1}{n}$. Then we obtain

$$\mu\left(U_\delta \setminus \bigcup_{n=1}^{\infty} L_n\right) \leq \mu\left(U_\delta \setminus L_n\right) < \frac{1}{n} \text{ for all } n \in \mathbb{N}.$$

Hence $\mu\left(U_\delta \setminus \bigcup_{n=1}^\infty L_n\right) = 0$. This implies $\nu_f\left(U_\delta \setminus \bigcup_{n=1}^\infty L_n\right) = 0$, and we obtain

$$\nu_f\left(U_\delta\right) = \nu_f\left(\bigcup_{n=1}^{\infty} L_n\right) = \lim_{n \to \mathbb{N}} \nu_f(L_1 \cup \ldots \cup L_n) \leq M,$$

since $L_1 \cup \ldots \cup L_n$ is compact for all $n \in \mathbb{N}$.
Put $U = \left\{x \in X\colon |f(x)|^2 > 0\right\}$. For $\delta = \frac{1}{n}$, we observe $U_{\frac{1}{n}} \subseteq U_{\frac{1}{n+1}}$ and $U = \bigcup_{n=1}^\infty U_{\frac{1}{n}}$. Therefore,

$$\nu_f(U) = \lim_{n \to \infty} \nu_f\left(U_{\frac{1}{n}}\right) \leq M.$$

Finally, we get

$$\int_X |f|^2 \, \mathrm{d}\mu = \int_U |f|^2 \, \mathrm{d}\mu = \nu_f(U) \leq M < \infty. \qquad \square$$

Example 1.12. The converse of Lemma 1.10 is false in general. Consider $\mathbb{N}$ with the discrete topology and counting measure μ. It is easy to see that μ is locally finite, strictly positive and inner regular on open sets.
Let V be a Banach space and $f \in C_c(\mathbb{N}, V)$, then

$$\phi\left(\sum_{n=1}^{\infty} f(n)\right) = \sum_{n=1}^{\infty} \phi\left(f(n)\right) = \int_{\mathbb{N}} \phi \circ f \, \mathrm{d}\mu \quad \text{for all } \phi \in V'.$$

Therefore, $\int_{\mathbb{N}} f \, \mathrm{d}\mu = \sum_{n=1}^\infty f(n)$ by Proposition 5.7.
Let $A = \mathbb{B}(\ell^2(\mathbb{N}))$ and $(e_n)_{n\in\mathbb{N}}$, the standard orthonormal basis of $\ell_2(\mathbb{N})$. Consider

the projections $(P_n)_{n\in\mathbb{N}}$ in A, where $P_n(x) = \langle x, e_n\rangle e_n$. Let $f\colon \mathbb{N} \to A$ be given by $n \mapsto \frac{1}{\sqrt{n}} P_n$. Then

$$\int_{\mathbb{N}} \|f(n)\|^2 \ \mathrm{d}\mu(n) = \sum_{n=1}^{\infty} \left\| \frac{1}{\sqrt{n}} P_n \right\|^2 = \sum_{n=1}^{\infty} \frac{1}{n} = \infty.$$

But given $h \in \mathcal{T}(\mathbb{N})$, we have

$$\sum_{n=1}^{\infty} \frac{1}{n} P_n h^2(k) \leq \sum_{n=1}^{\infty} P_n = \mathrm{id}_{\ell^2(\mathbb{N})}.$$

Hence

$$\|h \cdot f\| = \left\| \int_{\mathbb{N}} f(k)^* f(k) g^2(k) \ \mathrm{d}\mu(k) \right\|^{1/2} = \left\| \sum_{k=1}^{\infty} \frac{1}{k} P_k g(k) \right\|^{1/2} \leq 1.$$

Therefore, f is square-integrable.

Next we characterise square-integrability with nets. This characterisation is used as a definition in [3, page 22] and [1, page 175]. From now on, let $(\chi_i)_{i\in I} \subseteq \mathcal{T}(X)$ with $\chi_i \longrightarrow 1$ uniformly on compact subsets.

Lemma 1.13 (Square-Integrability with Nets)**.**
Let $f\colon X \to A$ be a continuous function. The following statements are equivalent:

(i) *f is square-integrable.*

(ii) *The net $(\chi_i \cdot f)_{i\in I}$ converges in $L^2(X, A)$.*

Proof.

(i)⇒(ii) Since $L^2(X, A)$ is complete, it suffices to show that $(\chi_i \cdot f)_{i\in I}$ is a Cauchy net. Let $\varepsilon > 0$.

Claim: There is a compact $K \subseteq X$ such that $\|h \cdot f\|_2 < \frac{\varepsilon}{2}$ for all $h \in \mathcal{T}(X)$ with $\mathrm{supp}(h) \subseteq X \setminus K$.

Proof of the claim: Since f is assumed to be square-integrable,

$$M = \sup \{\|h \cdot f\|_2 \colon h \in \mathcal{T}(X)\} < \infty.$$

There is $k \in \mathcal{T}(X)$ such that $\|k \cdot f\|_2^2 > M^2 - \frac{\varepsilon^2}{4}$. We define $K = \mathrm{supp}(k)$. Let $h \in \mathcal{T}(X)$ with $\mathrm{supp}(h) \subseteq X \setminus K$. Then $h + k \in \mathcal{T}(X)$ and

$$\langle h \cdot f, k \cdot f\rangle = \int_X h(x)k(x)f(x)^* f(x) \ \mathrm{d}\mu(x) = 0.$$

Therefore, $\|h \cdot f + k \cdot f\|_2^2 = \|h \cdot f\|_2^2 + \|k \cdot f\|_2^2$. Hence

$$\|h \cdot f\|_2 = \left(\|f \cdot (h + k)\|_2^2 - \|f \cdot k\|_2^2\right)^{1/2} < \left(M^2 - \left(M^2 - \frac{\varepsilon^2}{4}\right)\right)^{1/2} = \frac{\varepsilon}{2}.$$

Now let $V \subseteq X$ be open and $L \subseteq X$ compact with $K \subseteq V \subseteq L$. Choose $h_1 \in \mathcal{T}(X)$ such that $h_1(L) = \{1\}$ and put $h_2 = 1 - h_1$. There is i_0 such that $|1 - \chi_i(x)| < \frac{\varepsilon}{4M}$ for all $x \in \mathrm{supp}(h_1)$ and $i \geq i_0$. For $i, j \geq i_0$, we obtain

$$h_1(x) \cdot |\chi_i(x) - \chi_j(x)| \leq h_1(x) \cdot (|\chi_i(x) - 1| + |1 - \chi_j(x)|) \leq h_1(x) \cdot \frac{\varepsilon}{2M}.$$

As above this implies

$$\|h_1 (\chi_i - \chi_j) \cdot f\|_2 = \|h_1 |\chi_i - \chi_j| \cdot f\|_2 \leq \frac{\varepsilon}{2M} \cdot \|h_1 \cdot f\|_2 \leq \frac{\varepsilon}{2}.$$

If $x \in V$, then $h_1(x) = 1$. Therefore, $h_2(x) = 0$. Hence

$$\mathrm{supp}\,(h_2 |\chi_i - \chi_j|) \subseteq \mathrm{supp}(h_2) \subseteq X \setminus V \subseteq X \setminus K.$$

$|\chi_i - \chi_j| \in C_c(X)$ implies $h_2 |\chi_i - \chi_j| \in \mathcal{T}(X)$. The above claim implies

$$\|h_2 (\chi_i - \chi_j) \cdot f\|_2 = \|h_2 |\chi_i - \chi_j| \cdot f\|_2 < \frac{\varepsilon}{2}.$$

All in all, we get

$$\|(\chi_i - \chi_j) \cdot f\|_2 = \|(h_1 + h_2)(\chi_i - \chi_j) \cdot f\|_2 \leq \|h_1 (\chi_i - \chi_j) \cdot f\|_2 + \|h_2 (\chi_i - \chi_j) \cdot f\|_2 < \varepsilon.$$

Hence $(\chi_i \cdot f)_{i \in I}$ is a Cauchy net.

(ii)$\Rightarrow$(i) Assume $\lim_i (\chi_i \cdot f) = F \in L^2(X, A)$ and let $h \in \mathcal{T}(X)$. For $i \in I$, we have $\chi_i h \leq \chi_i$. Hence $\|\chi_i h \cdot f\|_2 \leq \|\chi_i \cdot f\|_2$ as above. Since $h \cdot f \in C_c(X, A)$ we obtain

$$\|h \cdot f\|_2 \overset{1.6}{=} \|\lim_i \chi_i \cdot (h \cdot f)\|_2 = \lim_i \|\chi_i h \cdot f\|_2 \leq \lim_i \|\chi_i \cdot f\|_2 = \|F\|_2.$$

Therefore, f is square-integrable. $\square$

The next lemma and its corollary prove that the limit of 1.13(ii) is independent of the chosen net $(\chi_i)_{i \in I}$.

Lemma 1.14. *Let $f \in \mathcal{S}^2(X, A)$ and $F = \lim_i (\chi_i \cdot f)$. Then*

$$M_h F = h \cdot f \text{ for all } h \in \mathcal{T}(X).$$

Proof. Let $h \in \mathcal{T}(X)$. Since M_h is bounded,

$$M_h F = M_h \left(\lim_i (\chi_i \cdot f) \right) = \lim_i M_h (\chi_i \cdot f) = \lim_i h (\chi_i \cdot f) = \lim_i \chi_i (h \cdot f) \overset{1.6}{=} h \cdot f. \quad \square$$

As a corollary of 1.14 we show that the limit in 1.13(ii) is independent of the chosen net $(\chi_i)_{i \in I}$:

Corollary 1.15. *Let $f \in \mathcal{S}^2(X, A)$ and $(\kappa_j)_{j \in J} \subseteq \mathcal{T}(X)$ be another net with $\kappa_j \longrightarrow 1$ uniformly on compact subsets. Then*

$$\lim_i (\chi_i f) = \lim_j (\kappa_i f).$$

Proof. Let $F_1 = \lim_i (\chi_i f)$ and $F_2 = \lim_j (\kappa_i f)$. If $h \in \mathcal{T}(X)$,then $M_h F_1 = h \cdot f = M_h F_2$ by Lemma 1.14. Therefore, by Corollary 1.7,

$$\|F_1 - F_2\|_2 = \sup \{\|M_h (F_1 - F_2)\|_2 \colon h \in \mathcal{T}(X)\} = 0.$$

Hence $F_1 = F_2$. $\square$

The next corollary allows us to view $\mathcal{S}^2(X, A)$ as a subspace of $L^2(X, A)$.

Corollary 1.16 (The Embedding of $\mathcal{S}^2(X, A)$ into $L^2(X, A)$)**.**
The linear map $\iota \colon \mathcal{S}^2(X, A) \to L^2(X, A)$ given by $f \mapsto \lim_i (\chi_i \cdot f)$ is isometric.

Proof. Let $f \in \mathcal{S}^2(X, A)$. Corollary 1.7 and Lemma 1.14 imply

$$\begin{aligned}\|\iota(f)\|_2 &= \sup \{\|M_h (\iota(f))\|_2 \colon h \in \mathcal{T}(X)\} \\ &= \sup \{\|h \cdot f\|_2 \colon h \in \mathcal{T}(X)\} = \|f\|_{\mathcal{S}^2}.\end{aligned}$$

$\square$

If $f \in \mathcal{S}^2(X, A)$ we want to compute inner products of $\iota(f)$ and elements of $C_c(G, A)$.

Lemma 1.17. *If $f_1 \in \mathcal{S}^2(X, A)$ and $f_2 \in C_c(X.A)$, then*

$$\langle \iota(f_1), f_2 \rangle = \int_X f_1(x)^* f_2(x) \ \ d\mu(x).$$

Proof. Since the statement is trivial for $f_2 = 0$, we assume $f_2 \neq 0$.
Let $\varepsilon > 0$. Since the net $(h \cdot f_1)_{h \in \mathcal{T}(X)}$ converges to $\iota(f_1)$, there is $h_0 \in \mathcal{T}(X)$, such that

$$\|\iota(f_1) - h \cdot f_1\|_2 < \frac{\varepsilon}{\|f_2\|_2} \quad \text{for all } h \in \mathcal{T}(X) \quad \text{with } h \geq h_0.$$

Choose $h_1 \in \mathcal{T}(X)$ with $h_1(\mathrm{supp}(f_2)) = \{1\}$. Put $h = \min\{h_0 + h_1, 1\}$. Then $h \in \mathcal{T}(X)$ with $h \geq h_0$. We estimate

$$\begin{aligned}\left\| \langle \iota(f_1), f_2 \rangle - \int_X f_1(x)^* f_2(x) \ \ \mathrm{d}\mu(x) \right\| &= \left\| \langle \iota(f_1), f_2 \rangle - \int_X h(x) f_1(x)^* f_2(x) \ \ \mathrm{d}\mu(x) \right\| \\ &= \|\langle \iota(f_1) - h \cdot f_1, f_2 \rangle\| \\ &\leq \|\iota(f_1) - h \cdot f_1\|_2 \cdot \|f_2\|_2 < \varepsilon.\end{aligned}$$

$\square$

Corollary 1.18. *Let $I \lhd A$ be an ideal of A.*
If $f \in \mathcal{S}^2(X, A)$ with $f(x) \in I$ for all $x \in X$, then $\iota(f) \in L^2(G, A) \cdot I$

Proof. Let $f_2 \in C_c(X, A)$. Then Lemma 1.17 yields

$$\langle \iota(f), f_2 \rangle = \int_X f(x)^* f_2(x) \ \ \mathrm{d}\mu(x)$$

Since I is an ideal, we gave $f(x)^* f_2(x) \in I$ for all $x \in I$. Therefore, we can view the integral as an I-valued integral. Since the inclusion $I \subseteq A$ is continuous, the integral viewed as an A-valued gives the same element. Hence $\langle \iota(f), f_2 \rangle \in I$ for all $f_2 \in C_c(X, A)$. Since $C_c(X, A)$ is dense in $L^2(X, A)$, we obtain $\langle \iota(f), f_2 \rangle \in I$ for all $f_2 \in L^2(X, A)$. In particular $\langle \iota(f), \iota(f) \rangle \in I$. So that $\iota(f) \in L^2(G, A) \cdot I$. $\square$

Another proof of the above Corollary is by approximating f with $C_c(X, I)$-functions. The following lemma allows us to identify pointwise limits with L^2-limits, if both exist.

Lemma 1.19 (Pointwise Convergence)**.**
Let $(f_n)_{n\in\mathbb{N}} \subseteq C_c(X, A)$ be uniform bounded a sequence. That is, there is a constant $C > 0$ with $\|f_n\|_\infty < C$ for all $n \in \mathbb{N}$. Assume $g \in C_b(X, A)$ and $f \in L^2(X, A)$, such that $\|f - f_n\|_2 \to 0$ for $n \to \infty$ and $f_n(x) \to g(x)$ for $n \to \infty$ for all $x \in X$.
Then $g \in \mathcal{S}^2(X, A)$ with $\iota(g) = f$.

Proof. Since $(f_n)_{n\in\mathbb{N}}$ is convergent in $\|\cdot\|_2$, there is $M > 0$ with $\|f_n\|_2 \leq M$. Let $h \in \mathcal{T}(X)$. Since $\|f_n\|_\infty < C$ for all $n \in \mathbb{N}$ Lemma 5.10 applies and we obtain

$$\lim_{n\to\infty}^{weak} \int_X (h \cdot f_n)(x)^* \cdot (h \cdot f_n)(x) \;\mathrm{d}\mu(x) = \int_X h(x)^2 \cdot g(x)^* \cdot g(x) \;\mathrm{d}\mu(x) = \|h \cdot g\|_2^2,$$

where

$$\left\| \int_X (h \cdot f_n)(x)^* \cdot (h \cdot f_n)(x) \;\mathrm{d}\mu(x) \right\| = \|h \cdot f_n\|_2^2 \leq \|f_n\|_2^2 \leq M^2.$$

The Hahn-Banach theorem implies $\|h \cdot g\|_2 \leq M$. Hence $g \in \mathcal{S}^2(X, A)$.
Let $k \in C_c(X, A)$. Then

$$\langle \iota(g), k\rangle = \int_X g(x)^* k(x) \;\mathrm{d}\mu(x) \overset{5.10}{=} \lim_{n\to\infty}^{weak} \int_G f_n(x)^* k(x) \;\mathrm{d}\mu(x) = \lim_{n\to\infty} \langle f_n, k\rangle = \langle f, h\rangle.$$

Since $C_c(X, A)$ is dense in $L^2(X, A)$ this implies $\iota(g) = f$. □

Remark 1.20. Let $A = \mathbb{B}(\ell^2(\mathbb{N}))$, $(P_n)_{n\in\mathbb{N}}$ as in Example 1.12, $X = [0, 1]$ and μ the Lebesgue measure on X. Let $(h_n)_{n\in\mathbb{N}}$ be a sequence of continuous functions $X \to [0, \infty)$, such that for all $x \in X$ the sequence $(h_n(x))$ is unbounded while $\|h_n\|_2 \xrightarrow{n\to\infty} 0$. Consider the functions $f_n = h_n.P_n \colon X \to A$ given by $f_n(x) = h_n(x)P_n$. For $N_1 < N_2$ we have

$$\begin{aligned}
\left\| \sum_{n=N_1}^{N_2} f_n \right\|_2 &= \left\| \left\langle \sum_{n=N_1}^{N_2} h_n.P_n, \sum_{k=N_1}^{N_2} h_k.P_k \right\rangle \right\|^{1/2} \\
&= \left\| \sum_{n=N_1}^{N_2} \sum_{k=N_1}^{N_2} \langle h_n, h_k\rangle P_k^* P_n \right\|^{1/2} \\
&= \left\| \sum_{n=N_1}^{N_2} \|h_n\|_2^2 P_n \right\|^{1/2} = \max_{N_1\leq n\leq N_2} \|h_n\|_2.
\end{aligned}$$

Hence $\sum_{n=1}^{\infty} f_n \in L_2(X, A)$.
But for $x \in X$, we obtain

$$\left\| \sum_{n=1}^{N} f_n(x) \right\| = \left\| \sum_{n=1}^{N} h_n(x) P_n \right\| = \max_{1\leq n\leq N} h_n(x) \xrightarrow{N\to\infty} \infty.$$

Hence not even a subsequence of $\left(\sum_{n=1}^{N} f_n\right)_{N\in\mathbb{N}}$ converges pointwise.
In this sense, there is no general possibility to view an element $f \in L_2(X, A)$ as a function from X to A.

The Hilbert G-A-Module $L^2(G, A)$

The Hilbert space $L^2(G)$ carries a natural action $(\lambda_g)_{g\in G}$ by left translation. We want to endow $L^2(G, A)$ with a G-action in a way that it is compatible with the G-action on A, so that the isomorphism of Theorem 1.4 gets G-equivariant.
If $f \in L^2(G)$ and $g, x \in G$, then $(\lambda_g(f))(x) = f\left(g^{-1}x\right)$. Hence for $f_1, f_2 \in L^2(G)$, we compute

$$\langle \lambda_g(f_1), \lambda_g(f_2)\rangle = \int_G \overline{f_1\left(g^{-1}x\right)} f_2\left(g^{-1}x\right)\ \mathrm{d}\mu(x) = \int_G \overline{f_1(x)} f_2(x)\ \mathrm{d}\mu(x) = \langle f_1, f_2\rangle,$$

using the translation invariance of the Haar measure. Since λ_g is invertible, this shows that λ_g is unitary.
The next lemma proves the strong continuity of $(\lambda_g)_{g\in G}$. Hence Example 5.15(ii) yields that $L^2(G)$ is Hilbert G-$\mathbb{C}$-module.

Lemma 1.21 (Strong Continuity of the Left Translation Action)**.**
The action $(\lambda_g)_{g\in G}$ on $L^2(G)$ is strongly continuous.

Proof. Let $0 \neq f \in C_c(G)$ and $\varepsilon > 0$. Let W be a compact neighbourhood of the identity element $1 \in G$. We put $K = W \operatorname{supp}(f)$. Then $\operatorname{supp}(f) \subseteq K$. Hence $\mu(K) > 0$. Since the multiplication map is continuous, K is compact. For $x \notin K$ and $h \in W$, we have $h^{-1}x \notin \operatorname{supp}(f)$, so that $|f(x) - f\left(h^{-1}x\right)| = 0$.
Let $\varepsilon > 0$. The essential step is to show that there is a neighbourhood U of $1 \in G$, such that for all $h \in U$ we have $\|f - \lambda_h(f)\|_2 < \varepsilon$.
Since f is continuous, every $x \in K$ has a neighbourhood U_x such that

$$|f(x) - f(y)| < \frac{\varepsilon}{2\sqrt{\mu(K)}} \text{ for every } y \in U_x.$$

The open set xU_x^{-1} is a neighbourhood of 1. Since the multiplication is continuous, there is an open neighbourhood W_x of 1 such that $g_1, g_2 \in W_x$ implies $g_1g_2 \in xU_x^{-1}$.
Since K is compact and $W_x^{-1}x$ is an open neighbourhood of $x \in K$, we find $x_1, \dots, x_n \in K$ such that $W_{x_1}^{-1}x_1, \dots W_{x_n}^{-1}x_n$ cover K.
Put $U = W \cap W_{x_1}^{-1}x_1 \cap \ldots \cap W_{x_n}^{-1}x_n$. Then U is a neighbourhood of 1.
Now let $x \in K$ and $h \in U$. There is $1 \leq i \leq n$, such that $x \in W_{x_i}^{-1}x_i$. We have $x_ix^{-1} \in W_{x_i}$ and $h \in W_{x_i}$. Hence $x_ix^{-1}h \in x_iU_{x_i}^{-1}$. Therefore, $h^{-1}x = \left(x^{-1}h\right)^{-1} \in U_{x_i}$. Since $W_{x_i} \subseteq x_iU_{x_i}^{-1}$, we have $x \in W_{x_i}^{-1}x_i \subseteq U_{x_i}$ and we get

$$|f\left(x\right) - f(h^{-1}x)| \leq |f(x) - f(x_i)| + |f(x_i) - f(h^{-1}x)| < \frac{\varepsilon}{\sqrt{\mu(K)}}.$$

All in all, we obtain

$$\|f-\lambda_h(f)\|_2 = \left(\int_K |f(x)-f(h^{-1}x)|^2 \ \mathrm{d}\mu(x)\right)^{1/2} < \left(\mu(K)\frac{\varepsilon^2}{\mu(K)}\right)^{1/2} = \varepsilon.$$

Finally, let $g_0 \in G$. Then g_0U is a neighbourhood of g_0 and for $g \in g_oU$, we have $g_0^{-1}g \in U$. Hence

$$\|\lambda_{g_0}(f)-\lambda_g(f)\|_2 = \|f-\lambda_{g_0^{-1}g}(f)\|_2 < \varepsilon.$$

This shows that the map $g \mapsto \lambda_g(f)$ is continuous. Since $C_c(G)$ is dense in $L^2(G)$ Lemma 5.16 yields the strong continuity of $(\lambda_g)_{g\in G}$. □

For $g \in G$ and $f \in C_c(G, A)$ we define

$$\gamma_g^c(f)\colon G \to A, x \mapsto \alpha_g\left(f\left(g^{-1}x\right)\right).$$

The map $\gamma_g^c(f)$ is continuous. Since $\operatorname{supp}\left(\gamma_g^c(f)\right) = g\operatorname{supp}(f)$ is compact, we obtain a linear map

$$\gamma_g\colon C_c(X,A) \to C_c(X,A), f \mapsto \gamma_g^c(f).$$

We have

$$\begin{aligned}
\|\gamma_g^c(f)\|_2 &= \left\|\int_G \left(\gamma_g^c(f)\right)(x)^* \left(\gamma_g^c(f)\right)(x) \ \mathrm{d}\mu(x)\right\|^{1/2} \\
&= \left\|\int_G \alpha\left(f\left(g^{-1}x\right)^* f\left(g^{-1}x\right)\right) \ \mathrm{d}\mu(x)\right\|^{1/2} \\
&\overset{5.8}{=} \left\|\int_G f\left(g^{-1}x\right)^* f\left(g^{-1}x\right) \ \mathrm{d}\mu(x)\right\|^{1/2} \\
&\overset{5.11}{=} \left\|\int_G f(x)^*f(x) \ \mathrm{d}\mu(x)\right\|^{1/2} = \|f\|_2
\end{aligned}$$

Therefore, γ_g^c extends to a linear isometry $\gamma_g\colon L^2(G,A) \to L^2(G,A)$.

Theorem 1.22. *The collection $(\gamma_g)_{g\in G}$ is a Hilbert module action on $L^2(G,A)$. The isomorphism $\Phi\colon L^2(G)\otimes A \xrightarrow{\sim} L^2(G,A)$ of Theorem 1.4 is G-equivariant and thus an isomorphism of Hilbert G-A-modules.*

Proof. We denote $(\delta_g)_{g\in G}$ for the G-action on $L^2(G)\otimes A$.
Let $f \in C_c(G)$ and $a \in A$. We have

$$\begin{aligned}
(\gamma_g(f.a))(x) &= \alpha_g\left((f.a)\left(g^{-1}x\right)\right) \\
&= f\left(g^{-1}x\right)\alpha_g(a) \\
&= (\lambda_g(f), \alpha_g(a))(x) \\
&= (\Phi(\lambda_g(f)\otimes\alpha_g(a)))(x) \\
&= (\Phi(\delta_g(f\otimes a)))(x) \\
&= \left(\Phi\circ\delta_g\circ\Phi^{-1}(f.a)\right)(x),
\end{aligned}$$

for all $x \in G$. By linearity, the continuous functions γ_g and $\Phi \circ \delta_g \circ \Phi^{-1}$ agree on the subspace of $L^2(G, A)$ spanned by elements of the form $f.a$ for $f \in C_c(G)$ and $a \in A$. This subspace is dense by Proposition 1.3. Therefore, $\gamma_g = \Phi \circ \delta_g \circ \Phi^{-1}$.
Because Φ is an isomorphism of Hilbert A-modules, it is clear that $(\gamma_g)_{g \in G}$ is a Hilbert module action on $L^2(G, A)$.
We have $\gamma_g \circ \Phi = \Phi \circ \delta_g$. Hence Φ is G-equivariant and an isomorphism of Hilbert G-A-modules. □

1.2 The Reduced Crossed Product

Let G be a unimodular[1] locally compact [2] group and A be a G-C^*-algebra with action $(\alpha_g)_{g \in G}$. We want to define the reduced product C^*-algebra $C^*_r(G, A)$ as a completion of the twisted convolution algebra $C_c(G, A)$.

The Completion to $C^*_r(G, A)$

We want to represent $C_c(G, A)$ on $L^2(G, A)$ as G-equivariant adjointable operators. We do this by defining representations of A and G on $L^2(G, A)$ separately and integrate to a representation of $C_c(G, A)$.
If $g \in G$ and $f \in C_c(G, A)$, then we define $(\delta_g(f))(x) = f(xg)$.
Since $\mathrm{supp}(\delta_g(f)) = \mathrm{supp}(f)g^{-1}$ is compact, we have $\delta_g(f) \in C_c(G, A)$. It is easy to see that $\delta_g \colon C_c(G, A) \to C_c(G, A)$ is linear. If $f_1, f_2 \in C_c(G, A)$, then

$$\langle \delta_g(f_1), \delta_g(f_2) \rangle = \int_G f_1(xg)^* f_2(xg) \ \mathrm{d}\mu(x) \overset{5.12}{=} \langle f_1, f_2 \rangle$$

Therefore $\|\delta_g(f)\|_2 = \|f\|_2$, hence δ_g extends uniquely to $\delta_g \colon L^2(G, A) \to L^2(G, A)$. If $g, h \in G$, then $\delta_g \circ \delta_h = \delta_{gh}$ and $\delta_1 = \mathrm{id}_{L^2(G,A)}$. Therefore, every δ_g is a unitary on $L^2(G, A)$. We have

$$(\delta_g(\gamma_h(f)))(x) = (\gamma_h(f))(xg) = \alpha_h(f(h^{-1}xg)) = \alpha_h((\delta_g(f))(h^{-1}x)) = (\gamma_h(\delta_g(f)))(x).$$

Therefore, $\delta \colon G \to \mathbb{B}^G(L^2(G, A))$, $g \mapsto \delta_g$ is an action on $L^2(G, A)$ by G-equivariant unitaries.
Now let $a \in A$. We define $(\pi_a(f))(x) = \alpha_x(a) \cdot f(x)$.
Since $(\alpha_g)_{g \in G}$ is continuous and $\mathrm{supp}(\pi_a(f)) = \mathrm{supp}(f)$ is compact, we obtain $\pi_a(f) \in C_c(G, A)$. Obviously $\pi_a \colon C_c(G, A) \to C_c(G, A)$ is linear. We estimate

$$\|\pi_a(f)\|_2 = \left\| \int_G f(x)^* \alpha_x(a^* a) f(x) \ \mathrm{d}\mu(x) \right\|^{1/2} \overset{5.13(iii)}{\leq} \|a^* a\|^{1/2} \cdot \|f\|_2 = \|a\| \cdot \|f\|_2.$$

Therefore, π_a extends to $\pi_a \colon L^2(G, A) \to L^2(G, A)$ with $\|\pi_a\| \leq \|a\|$.
If $f_1, f_2 \in C_c(G, A)$, then

$$\langle \pi_a(f_1), f_2 \rangle = \int_G f_1(x)^* \alpha_g(a^*) f_2(g) \ \mathrm{d}\mu(g) = \langle f_1, \pi_{a^*}(f_2) \rangle.$$

[1] We only assume G to be modular, to simplify the formulas. Everything we do works for general locally compact groups by adding modular functions in the correct places.(see [1])
[2] We always assume, that locally compact groups are Hausdorff.

This implies $\pi_a \in \mathbb{B}(L^2(G,A))$ with $\pi_a^* = \pi_{a^*}$. If $b \in A$, then

$$((\pi_a \circ \pi_b)(f))(x) = \alpha_g(a) \cdot \alpha_g(b) \cdot f(x) = \alpha_g(ab) \cdot f(x) = (\pi_{ab}(f))(x).$$

If $g \in G$, then

$$(\pi_a \circ \gamma_g)(f))(x) = \alpha_x(a)\alpha_g(f(g^{-1}x)) = \alpha_g\left(\alpha_{g^{-1}x}(a)f(g^{-1}x)\right) = ((\gamma_g \circ \pi_a)(f))(x).$$

Therefore, $\pi\colon A \to \mathbb{B}^G(L^2(G,A))$, $a \mapsto \pi_a$ is a $*$-homomorphism.
We compute

$$((\delta_g \circ \pi_a \circ \delta_g^*)(f))(x) = \alpha_{xg}(a)(\delta_{g^{-1}}(f))(xg) = \alpha_{xg}(a)f(x) = (\pi_{\alpha_g(a)}(f))(x).$$

Hence π and δ fulfil the *covariance condition* $\delta_g \circ \pi_a \circ \delta_g^* = \pi_{\alpha_g(a)}$.
If $f \in C_c(G,A)$, then we define $(\mathrm{inv}(f))(x) = f(x^{-1})$. Then $\mathrm{inv}(f) \in C_c(G,A)$. Hence $\mathrm{inv}\colon C_c(G,A) \to C_c(G,A)$ is well defined and linear. We have $\mathrm{inv}^2 = \mathrm{id}_{C_c(G,A)}$. If $f_1, f_2 \in C_c(G,A)$, then

$$\langle \mathrm{inv}(f_1), \mathrm{inv}(f_2)\rangle = \int_G f_1(x^{-1})^* f_2(x^{-1}) \ \mathrm{d}\mu(x) \overset{5.12}{=} \langle f_1, f_2\rangle.$$

Therefore inv extends to a unitary $L^2(G,A) \to L^2(G,A)$. To define the integrated representation $C_c(G,A) \to \mathbb{B}^G(L^2(G,A))$ we need the following lemma concerning continuity.

Lemma 1.23. *If $f \in C_c(G,A)$ and $h \in L^2(G,A)$, then the map $G \to L^2(G,A)$ given by $g \mapsto (\pi_{f(g)} \circ \delta_g)(h)$ is continuous.*

Proof. Let $\tilde{A} = A$ as a C^*-algebra and let G act trivially on $\tilde{A}$. The action $(\tilde{\gamma_g})_{g \in G}$ on $L^2(G,\tilde{A})$ is strongly continuous by 1.22.
If $k \in C_c(G,A)$, then

$$((\mathrm{inv} \circ \tilde{\gamma_g} \circ \mathrm{inv})(k))(x) = (\mathrm{inv}(k))(g^{-1}x^{-1}) = k(xg) = (\delta_g(k))(x)$$

Therefore, $\delta_g = \mathrm{inv} \circ \tilde{\gamma_g} \circ \mathrm{inv}$. Hence the map $g \mapsto \delta_g(h)$ is continuous.
Since $f \in C_c(G,A)$, there is $M > 0$, such that $\|f(g)\| \leq M$ for all $g \in G$. Let $g_0 \in G$ and $\varepsilon > 0$. Since the lemma is trivial for $h = 0$, we suppose $h \neq 0$. There is a neighbourhood U of g_0 in G, such that

$$\|f(g) - f(g_0)\| < \frac{\varepsilon}{2\|h\|_2} \text{ and } \|\delta_g(h) - \delta_{g_0}(h)\|_2 < \frac{\varepsilon}{2M} \text{ for all } x \in U$$

If $g \in U$, then

$$\begin{aligned}\|(\pi_{f(g)} \circ \delta_g)(h) - (\pi_{f(g_0)} \circ \delta_{g_0})(h)\|_2 &\leq \|\pi_{f(g)}(\delta_g(h) - \delta_{g_0}(h))\|_2 \\ &\quad + \|(\pi_{f(g)} - \pi_{f(g_0)})(\delta_{g_0}(h))\|_2 \\ &\leq M \cdot \|\delta_g(h) - \delta_{g_0}(h)\|_2 + \|h\|_2 \cdot \|f(g) - f(g_0)\| < \varepsilon\end{aligned}$$

This shows that $g \mapsto (\pi_{f(g)} \circ \delta_g)(h)$ is continuous. □

If $f \in C_c(G,A)$, we define $\rho_f : L^2(G,A) \to L^2(G,A)$ by $h \mapsto \int_G (\pi_{f(g)} \circ \delta_g)(h)\ \mathrm{d}\mu(g)$. The integrand is continuous by Lemma 1.23 and compactly supported. Hence the integral is well-defined by Proposition 5.7. If $h_1, h_2 \in L^2(G,A)$, then we have

$$\begin{aligned}
\langle \rho_f(h_1), h_2\rangle &= \langle \int_G (\pi_{f(g)} \circ \delta_g)(h_1)\ \mathrm{d}\mu(g), h_2\rangle \\
&\overset{5.8}{=} \int_G \langle (\pi_{f(g)} \circ \delta_g)(h_1), h_2\rangle\ \mathrm{d}\mu(g) \\
&= \int_G \langle h_1, (\delta_{g^{-1}} \circ \pi_{f(g)^*})(h_2)\rangle\ \mathrm{d}\mu(g) \\
&\overset{5.12}{=} \int_G \langle h_1, (\delta_g \circ \pi_{f(g^{-1})^*})(h_2)\rangle\ \mathrm{d}\mu(g) \\
&= \int_G \langle h_1, (\pi_{\alpha_g(f(g^{-1}))^*} \circ \delta_g)(h_2)\rangle\ \mathrm{d}\mu(g) \\
&= \langle h_1, \rho_{f^*}(h_2)\rangle.
\end{aligned}$$

Therefore, $\rho_f \in \mathbb{B}^G(L^2(G,A))$ with $\rho_f^* = \rho_{f^*}$. If $g \in G$, then

$$\rho_f(\gamma_g(h)) = \int_G (\pi_{f(x)} \circ \delta_x)(\gamma_g(h))\ \mathrm{d}\mu(x) = \int_G (\gamma_g \circ \pi_{f(x)} \circ \delta_x)(g)\ \mathrm{d}\mu(x) \overset{5.8}{=} \gamma_g(\rho_g(h)).$$

This shows that ρ_f is G-equivariant. Therefore, the map $\rho\colon C_c(G,A) \to \mathbb{B}^G(L^2(G,A))$ given by $f \mapsto \rho_f$ is well-defined. Obviously ρ is linear. If $f_1, f_2 \in C_c(G,A)$, then we obtain

$$\begin{aligned}
\pi_{(f_1 * f_2)(g)} \circ \delta_g &= \pi\left(\int_G f_1(x)\alpha_x(f_2(x^{-1}g))\ \mathrm{d}\mu(x)\right) \circ \delta_g \\
&\overset{5.8}{=} \int_G \pi_{f_1(x)} \circ \delta_x \circ \pi_{f_2(x^{-1}g)} \circ \delta_{x^{-1}g}\ \mathrm{d}\mu(x).
\end{aligned}$$

If $h \in L^2(G,A)$, this implies

$$\begin{aligned}
\rho_{f_1 * f_2}(h) &= \int_G (\pi_{(f_1 * f_2)(g)} \circ \delta_g)(h)\ \mathrm{d}\mu(g) \\
&\overset{5.8}{=} \int_G \left(\int_G (\pi_{f_1(x)} \circ \delta_x \circ \pi_{f_2(x^{-1}g)} \circ \delta_{x^{-1}g})(h)\ \mathrm{d}\mu(x).\right)\ \mathrm{d}\mu(g) \\
&\overset{5.9}{=} \int_G (\pi_{f_1(x)} \circ \delta_x)\left(\int_G (\pi_{f_2(x^{-1}g)} \circ \delta_{x^{-1}g})(h)\ \mathrm{d}\mu(g)\right)\ \mathrm{d}\mu(x) \\
&\overset{5.11}{=} \int_G (\pi_{f_1(x)} \circ \delta_x)(\rho_{f_2}(h))\ \mathrm{d}\mu(x) \\
&= (\rho_{f_1} \circ \rho_{f_2})(h).
\end{aligned}$$

Hence ρ is a $*$-homomorphism.
We have

$$\|\rho_f(h)\|_2 \leq \int_G \|(\pi_{f(g)} \circ \delta_g)(h)\|_2\ \mathrm{d}\mu(g) \leq \int_G \|\pi_{f(g)}\| \cdot \|h\|_2\ \mathrm{d}\mu(g) \leq \|f\|_1 \cdot \|h\|_2.$$

Therefore $\|\rho_f\| \leq \|f\|_1$.
The next lemma gives a more concrete formula for $\rho_f(h)$ with $h \in C_c(G, A)$.

Lemma 1.24. *If $f, h \in C_c(G, A)$, then*

$$(\rho_f(h))(g) = \int_G \alpha_g(f(x))h(gx) \ \ d\mu(x).$$

Therefore, $\rho_f(h) \in C_c(G, A)$.

Proof. If $x \in G$, then $(\pi_{f(x)} \circ \delta_x)(h) \in C_c(G, A)$ and

$$((\pi_{f(x)} \circ \delta_x)(h))(g) = \alpha_g(f(x)) \cdot (\delta_x(h))(g) = \alpha_g(f(x))h(gx).$$

The function $(x, g) \mapsto \alpha_g(f(x)) \cdot h(gx)$ is continuous by Lemma 1.23 and compactly supported. Therefore, $k\colon G \to A, g \mapsto \int_G \alpha_g(f(x)) \cdot h(gx) \ \mathrm{d}\mu(x)$ is continuous and compactly supported by Lemma 5.9. We compute

$$\begin{aligned}
\langle \rho_f(h), h_2 \rangle &\overset{5.8}{=} \int_G \langle (\pi_{f(x)} \circ \delta_x)(h), h_2 \rangle \ \ \mathrm{d}\mu(x) \\
&= \int_G \left(\int_G (\alpha_g(f(x)) \cdot h(gx))^* h_2(x) \ \ \mathrm{d}\mu(g) \right) \ \mathrm{d}\mu(x) \\
&\overset{5.9}{=} \int_G k(g)^* \circ h_2(g) \ \ \mathrm{d}\mu(g) = \langle k, h_2 \rangle
\end{aligned}$$

for all $h_2 \in C_c(G, A)$. This implies $\rho_f(h) = k$. $\square$

Corollary 1.25. *The $*$-homomorphism ρ is injective.*

Proof. Let $f \in C_c(G, A)$, such that $\rho_f = 0$. Put $h(x) = f(x)^*$ for all $x \in G$. Then $h \in C_c(G, A)$. Lemma 1.24 implies

$$0 = (\rho_f(h))(1) = \int_G f(g) \cdot h(g) \ \ \mathrm{d}\mu(g) = \int_G h(g)^* h(g) \ \ \mathrm{d}\mu(g) = \langle h, h \rangle.$$

Therefore, $h = 0$. Hence $f = 0$. $\square$

Definition 1.26 (The Reduced Crossed Product [1, page 173])**.**
The *reduced crossed product* $C_r^*(G, A)$ is the closure of $\rho(C_c(G, A))$ with respect to the operator norm on $\mathbb{B}^G(L^2(G, A))$.

Remark 1.27. Many authors define the reduced product slightly different. We want to show that the crossed product defined above is isomorphic to the usual one.
First we modify our representation ρ to simplify the formulas. If $a \in A$, $g \in G$, then we define

$$\pi'_a = \mathrm{inv} \circ \pi_a \circ \mathrm{inv} \quad \text{ and } \delta'_g = \mathrm{inv} \circ \delta_g \circ \mathrm{inv}.$$

For $h \in C_c(G, A)$ we get

$$(\pi'_a(h))(x) = \alpha_x^{-1}(a)h(x) \text{ and } (\delta'_g(h))(x) = h(g^{-1}x).$$

If $f \in C_c(G,A)$ and $h \in L^2(G,A)$, then integration yields

$$\rho'_f(h) := \int_G (\pi'_{f(g)} \circ \delta'_G)(h) \ \mathrm{d}\mu(g) = (\mathrm{inv} \circ \rho_f \circ \mathrm{inv})(h).$$

Since ρ is an injective $*$-homomorphism, so is ρ'. Since inv is unitary, we have

$$\|\rho'_f\| = \|\mathrm{inv} \circ \rho_f \circ \mathrm{inv}\| = \|\rho_f\|.$$

Next we present the usual definition of the reduced crossed product. Let $\phi\colon A \to \mathbb{B}(\mathcal{H})$ be a non-degenerate faithful representation. We get a representation $\Phi\colon A \to \mathbb{B}(L^2(G,\mathcal{H}))$ and a strongly continuous group homomorphism $\lambda\colon G \to \mathcal{U}(L^2(G,\mathcal{H}))$, where

$$(\Phi(a)h)(x) = \phi(\alpha_x^{-1}(a))h(x) \text{ and } (\lambda_g(h))(x) = h(g^{-1}x) \text{ for } h \in C_c(G,\mathcal{H}).$$

The covariant pair (Φ, λ) integrates to a $*$-homomorphism

$$\theta\colon C_c(G,A) \to \mathbb{B}(L^2(G,\mathcal{H})) \quad \text{,where } \theta_f(h) = \int_G (\Phi(f(g)) \circ \lambda_g)(h) \ \mathrm{d}\mu(g).$$

In this situation the crossed product is defined as the closure of $\theta(C_c(G,A))$ in the operator norm.
From the isomorphisms

$$L^2(G,A) \otimes_A \mathcal{H} \cong L^2(G) \otimes A \otimes_A \mathcal{H} \cong L^2(G) \otimes \mathcal{H} \cong L^2(G,H).$$

we obtain a isomorphism $L : L^2(G,A) \otimes_A \mathcal{H} \to L^2(G,\mathcal{H})$. If $f \in C_c(G,A)$ and $\xi \in \mathcal{H}$, then $(L(f \otimes \xi))(x) = \phi(f(x))\xi$. Using the faithfulness of ϕ one can check that $\Psi : \mathbb{B}(L^2(G,A)) \to \mathbb{B}(L^2(G,A) \otimes \mathcal{H})$, $\quad T \mapsto T \otimes \mathrm{id}_{\mathcal{H}}$ is an injective, hence isometric $*$-homomorphism.
Let $f \in C_c(G,A)$. It is not hard to see that $L \circ \Psi(\rho'_f) = \theta_f \circ L$. This implies

$$\|\rho_f\| = \|\rho'_f\| = \|L \circ \Psi(\rho'_f)\| = \|\theta_f\|.$$

Therefore the closure of $\theta(C_c(G,A))$ is isomorphic to the crossed product $C^*_r(G,A)$ defined in 1.26.

Lemma 1.28 ($C^*_r(G,A)$ has an approximate identity of C_c-functions)**.**
There is a net $(u_i)_{i\in I} \subseteq C_c(G,A)$ with $\|u_i\|_1 \leq 1$ and $u_i^* = u_i$ for all $i \in I$, such that

$$\|\psi - \psi \circ \rho_{u_i}\| \longrightarrow 0 \text{ and } \|\psi - \rho_{u_i} \circ \psi\| \longrightarrow 0 \quad \text{for all } \psi \in C^*_r(G,A).$$

Proof. Let $(u_i)_{i\in I}$ be a net as in Lemma 5.20.
For $f \in C_c(G,A)$, we obtain

$$\|\rho_f - \rho_f \circ \rho_{u_i}\| = \|\rho_{f - f * u_i}\| \leq \|f - f * u_i\|_1 \longrightarrow 0.$$

Likewise $\|\rho_f - \rho_{u_i} \circ \rho_f\| \longrightarrow 0$.
Since $\rho(C_c(G,A))$ is dense in $C^*_r(G,A)$ this implies the assertion for all $\psi \in C^*_r(G,A)$. $\square$

The following lemma allows us to identify continuous L^1-functions with elements of $C_r^*(G,A)$.

Lemma 1.29. *Let $f\colon G \to A$ be continuous, such that $\int_G \|f(x)\| \; d\mu(x) < \infty$. For $h \in C_c(G,A)$, we define*

$$(\rho_f h)(g) = \int_G \alpha_g(f(x))h(gx) \; d\mu(x).$$

Then $\rho_f h \in C_b(X,A)$. The image of the map $\rho_f\colon C_c(G,A) \to C_b(X,A)$ is contained in $\mathcal{S}^2(X,A)$ and $\iota \circ \rho_f$ extends to an operator $L^2(G,A) \to L^2(G,A)$.
We have $\iota \circ \rho_f \in C_r^*(G,A)$.

Proof. With similar arguments as in the proof of Lemma 1.11 we obtain a sequence $\omega_n\colon G \to [0,1]$ of continuous, compactly supported functions, such that

$$\int_G \omega_n(x)\|f(x)\| \; \mathrm{d}\mu(x) \longrightarrow \int_G \|f(x)\| \; \mathrm{d}\mu(x) \quad \text{for } n \to \infty.$$

If $h \in C_c(G,A)$, then Lemma 1.24 yields

$$\begin{aligned}\|(\rho_f h)(g) - (\rho_{\omega_n \cdot f} h)(g)\| &= \left\| \int_G (1-\omega_n(x)) \cdot \alpha_g(f(x)) \cdot h(gx) \; \mathrm{d}\mu(x) \right\| \\ &\leq \int_G (1-\omega_n(x)) \cdot \|f(x)\| \cdot \|h\|_\infty \; \mathrm{d}\mu(x) \xrightarrow{n\to\infty} 0.\end{aligned}$$

Hence $\|\rho_f(h) - \rho_{\omega_n \cdot f}(h)\|_\infty \longrightarrow 0$ for $n \to \infty$. Therefore, $\rho_f(h)$ is continuous.
We have $\|(\rho_f h)(g)\| \leq \int_G \|f(x)\| \; \mathrm{d}\mu(x) \cdot \|h\|_\infty$. Therefore, $\rho_f\colon C_c(G,A) \to C_b(G,A)$ is well defined. The sequence $(\rho_{\omega_n f})_{n\in\mathbb{N}} \subseteq C_r^*(G,A)$ is a Cauchy sequence. Let T be the limit. Then $T \in C_r^*(G,A)$. If $h \in C_c(G,A)$, then $\|\rho_{\omega_n f}(h) - T(h)\|_2 \xrightarrow{n\to\infty} 0$ and $(\rho_{\omega_n f} h)(x) \to (\rho_f(h))(x)$ for all $x \in G$. Hence Lemma 1.19 yields $\rho_f(h) \in \mathcal{S}^2(G,A)$ with $\iota(\rho_f(h)) = T(h)$. Therefore $\iota \circ \rho_f$ extends to $L^2(G,A)$ and we have $\iota \circ \rho_f = T \in C_r^*(G,A)$. ∎

G and A multiply $C_r^*(G,A)$

The next two lemmas shows that G and A multiply $C_r^*(G,A)$.

Lemma 1.30 (A multiplies $C_r^*(G,A)$)**.**
Let $a \in A$ and $\psi \in C_r^*(G,A)$. Then $\pi_a \circ \psi \in C_r^*(G,A)$.
If $(a_i)_{i\in I}$ is an approximate identity of A, then $\|\pi_{a_i} \circ \psi - \psi\| \longrightarrow 0$.

Proof. Let $f \in C_c(G,A)$. For $h \in L^2(G,A)$, we obtain

$$(\pi_a \circ \rho_f)(h) = \int_G (\pi_{a \cdot f(g)} \circ \delta_g)(h) \; \mathrm{d}\mu(g) = \rho_{a\cdot f}.$$

Hence $\pi_a \circ \rho_f \in C_r^*(G,A)$. Since $\rho(C_c(G,A))$ is dense in $C_r^*(G,A)$, this implies $\pi_a \circ \psi \in C_r^*(G,A)$ for all $\psi \in C_r^*(G,A)$.

Now let $(a_i)_{i\in I}$ is an approximate identity of A and $\varepsilon > 0$. Since $f(G) = f(\mathrm{supp}(f))$ is compact, a standard compactness argument shows that there is $i_0 \in I$, such that

$$\|f(g) - a_i f(g)\| < \frac{\varepsilon}{\mu(\mathrm{supp}(f))} \quad \text{for all } g \in G.$$

Therefore,

$$\|f - a_i \cdot f\|_1 = \int_G \|f(g) - u_i \cdot f(g)\| \ \mathrm{d}\mu(g) \le \varepsilon.$$

Hence

$$\|\rho_f - \pi_{a_i} \circ \rho_f\| = \|\rho_f - \rho_{a\cdot f}\| \le \|f - a_i \cdot f\|_1 \longrightarrow 0.$$

Using the density of $\rho(C_c(G,A))$ in $C_r^*(G,A)$ the assertion follows by an $\frac{\varepsilon}{3}$-argument. $\square$

Lemma 1.31 (G multiplies $C_r^*(G,A)$)**.**
Let $g \in G$ and $\psi \in C_r^*(G,A)$. Then $\delta_g \circ \psi \in C_r^*(G,A)$.
The function $G \to C_r^*(G,A)$ given by $g \mapsto \delta_g \circ \psi$ is continuous.

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Proof. Let $f \in C_c(G,A)$. For $h \in L^2(G,A)$, we have

$$\begin{aligned}
(\delta_g \circ \rho_f)(h) &= \delta_g \left(\int_G (\pi_{f(x)} \circ \delta_x)(h) \ \mathrm{d}\mu(x) \right) \\
&\overset{5.8}{=} \int_G (\delta_g \circ \pi_{f(x)} \circ \delta_g^* \circ \delta_{gx})(h) \ \mathrm{d}\mu(x) \\
&= \int_G (\pi_{\alpha_g(f(x))} \circ \delta_{gx})(h) \ \mathrm{d}\mu(x) \\
&\overset{5.11}{=} \int_G (\pi_{\alpha_g(f(g^{-1}x))} \circ \delta_x)(h) \ \mathrm{d}\mu(x) = \rho_{\gamma_g(f)}(h).
\end{aligned}$$

Hence $\delta_g \circ \rho_f \in C_r^*(G,A)$. Since $\rho(C_c(G,A))$ is dense in $C_r^*(G,A)$, this implies $\delta_g \circ \psi \in C_r^*(G,A)$ for all $\psi \in C_r^*(G,A)$.
With similar arguments as in the proof of Lemma 1.21 we see, that the function $G \to C_c(G,A)$ given by $g \mapsto \gamma_g(f)$ is continuous with respect to $\|\cdot\|_1$. Since ρ is continuous, this implies that the function $G \to C_r^*(G,A), g \mapsto \rho(\gamma_g(f)) = \delta_g \circ \rho_f$ is continuous. The continuity for an arbitrary element $\psi \in C_r^*(G,A)$ follows by an $\frac{\varepsilon}{3}$-argument. $\square$

Exactness of the Reduced Crossed Product

Let $I \lhd A$ be a G-invariant ideal of A. We view $C_c(G,I)$ as a subset of $C_c(G,A)$. Lemma 1.5 allows us to view $L^2(X,I) \cong L^2(X,A) \cdot I$ as a submodule of $L^2(G,A)$.
If $a \in I$ and $f \in C_c(G,A)$, then

$$(\pi_a(f))(x) = \alpha_x(a) \cdot f(x) \in I \quad \text{for all } x \in G.$$

Hence $\pi_a(f) \in L_2(G,A) \cdot I$. Since $C_c(G,A)$ is dense in $L^2(G,A)$ it follows

$$\pi_a(L^2(G,A)) \subseteq L^2(G,A) \cdot I.$$

We consider the representations

$$\rho^A\colon C_c(G,A)\to \mathbb{B}(L^2(G,A)) \text{ and } \rho^I\colon C_c(G,I)\to \mathbb{B}^G(L^2(G,I)).$$

Let $f\in C_c(G,I)$ and $h\in L^2(G,A)$ then $(\pi_{f(x)}\circ\delta_g)(h)\in L^2(G,A)\cdot I$ for all $x\in G$. Hence $\rho^A_f(h)\in L^2(G,A)\cdot I$. This shows $\rho^A_f(L^2(G,A))\subseteq L^2(G,A)\cdot I$. Therefore the considerations in the first chapter yield $\|\rho^A_f\| = \|\rho^A_f|_{L^2(G,A)\cdot I}\| = \|\rho^I_f\|$.
Therefore $C^*_r(G,I)$ is isomorphic to the closure of $\rho^A(C_c(G,I))\subseteq C^*_r(G,A)$. Thus we may identify $C^*_r(G,I)$ with the closure of $\rho^A(C_c(G,I))$.
The formula for the convolution shows, that $C_c(G,I)$ is a an ideal in $C_c(G.A)$. Therefore, $C^*_r(G,I)$ is a closed ideal in $C^*_r(G,A)$.
If $a\in I$ and $f\in C_c(G,A)$, then $\pi_a\circ\rho^A_f = \rho_{a\cdot f}\in\rho_A(C_c(G,I))\subseteq C^*_r(G,I)$. Hence $\pi_a\circ\psi\in C^*_r(G,I)$ for all $\psi\in C^*_r(G,A)$. Let $(u_j)_{j\in J}$ be an approximate unit of I and $\psi\in C^*_r(G,A)$. Using Lemma 1.30 and its proof we see, that $\psi\in C^*_r(G,I)$ if and only if $\|\psi-\pi_{u_j}\circ\psi\|\to 0$. Since $\mathcal{K} := \{T\in\mathbb{B}^G(L^2(G,A))\colon T(L^2(G,A))\subseteq L^2(G,A)\cdot I\}$ is closed in $\mathbb{B}^G(L^2(G,A))$, we obtain $C^*_r(G,I)\subseteq C^*_r(G,A)\cap\mathcal{K}$.
Now consider A/I and the quotient map $\pi\colon A\to A/I$. Since I is G-invariant, A/I is a G-C^*-algebra and π is G-equivariant. Let $q\colon C_c(G,A)\to C_c(G,A/I)$ be the pointwise quotient map. By Lemma 1.5 and the considerations above it, we obtain a commutative diagram of C^*-algebras

$$\begin{tikzcd}
\mathbb{B}^G(L^2(G,A)) \arrow[rr, "Q"] \arrow[dr, "Q_1"'] & & \mathbb{B}(L^2(G,A/I)) \\
& \mathbb{B}(L^2(G,A)/(L^2(G,A)\cdot I)) \arrow[ur, "\sim"'] &
\end{tikzcd}$$

Let $\pi^{A/I}$ be the action of A/I on $L^2(G,A/I)$ and $\rho^{A/I}$ the representation of $C_c(G,A/I)$ on $\mathbb{B}(L^2(G,A/I))$. Since

$$\pi^{A/I}_{q(a)}(q(f)) = \alpha_g(q(a))\cdot q(f(x)) = q(\pi^A_a(f)) = Q(\pi_a)(q(f))$$

the $*$-homomorphism Q is compatible with the representations ρ^A and $\rho^{A/I}$. That is we obtain a commutative diagram

$$\begin{tikzcd}
C_c(G,A) \arrow[rr, "q"] \arrow[d, "\rho^A"] & & C_c(G,A/I) \arrow[d, "\rho^{A/I}"] \\
\mathbb{B}^G(L^2(G,A)) \arrow[rr, "Q"] \arrow[dr, "Q_1"'] & & \mathbb{B}(L^2(G,A/I)) \\
& \mathbb{B}(L^2(G,A)/(L^2(G,A)\cdot I)) \arrow[ur, "\sim"'] &
\end{tikzcd}$$

From

$$Q(\rho^A(C_c(G,A)))\subseteq \rho^{A/I}(q(C_c(G,A)))\subseteq C^*_r(G,A/I),$$

we deduce $Q(C_r^*(G,A)) \subseteq C_r^*(G,A/I)$.
Let $f \in C_c(G)$ and $a \in A$, then $q(f.a) = f.(\pi(a))$, therefore $Q(C_r^*(G,A))$ contains all elements of the form $\rho^{A/I}_{f.\pi(a)}$. The elements of this form generate $C_r^*(G,A/I)$. Therefore, $Q(C_r^*(G,A)) = C_r^*(G,A/I)$. Hence $Q|_{C_r^*(G,A)}$ is a surjective $*$-homomorphism $C_r^*(G,A) \to C_r^*(G,A/I)$. $\mathcal{K}$ is the kernel of the map Q_1 of the diagram above. Therefore the kernel of $Q|_{C_r^*(G,A)}$ is $C_r^*(G,A) \cap \mathcal{K}$.
All in all we obtain a sequence

$$0 \to C_r^*(G,I) \to C_r^*(G,A) \to C_r^*(G,A/I) \to 0.$$

This sequence is in general not exact in the middle. If we fix the group G and obtain an exact sequence for all G-C^*-algebras A and G-invariant ideals, then the group G is called *exact*.
Our considerations show that the sequence is exact if and only if

$$\overline{\rho^A(C_c(G,I))} = \{\psi \in C_r^*(G,A) \colon \psi(L^2(G,A)) \subseteq L^2(G,A) \cdot I\}.$$

1.3 Square-Integrable Group Actions

Let G be a unimodular locally compact group and A be a G-C^*-algebra with action $(\alpha_g)_{g\in G}$. We view A as a Hilbert G-A-module.
The C^*-algebra A acts on itself by left multiplication. This action yields an embedding $A \hookrightarrow \mathbb{B}(A)$. $\mathcal{M}(A) = \mathbb{B}(A)$ is called the *multiplier algebra* of A. It carries the strict topology, where a net $(T_i)_{i\in I} \subseteq \mathcal{M}(A)$ converges to $T \in \mathcal{M}(A)$, if and only if

$$T_i(a) \longrightarrow T(a) \quad \text{and} \quad T_i^*(a) \longrightarrow T^*(x) \quad \text{for all } a \in A.$$

In this case, we write $T = \lim_i^s T_i$.
Let B be a C^*-algebra. A $*$-homomorphism $f \colon A \to B$ is called *nondegenerate* if $f(A) \cdot B = B$. A nondegenerate $*$-homomorphism $A \to B$ extends uniquely to a strictly continuous $*$-homomorphism $\tilde{f} \colon \mathcal{M}(A) \to \mathcal{M}(B)$. If $T \in \mathcal{M}(A)$ and $a \in A$, then $\tilde{f}(F)f(a) = f(Fa)$. If $g \in G$, then the $*$-automorphism α_g extends to $\mathcal{M}(A)$, by $T \mapsto \alpha_g \circ T \circ \alpha_{g^{-1}}$. We obtain an action of G on $\mathcal{M}(A)$ as in Section 5.3. The fixed points of this action are exactly the G-equivariant adjointable operators (or *multipliers*) $A \to A$. We write $\mathcal{M}^G(A) := \mathbb{B}^G(A)$.
Let $a, b \in A$. We define the *coefficient function* $c_{ab} \colon G \to A$ by $x \mapsto \alpha_x(a)^* b$. Moreover we define two linear maps

$$\Lambda_a \colon C_c(G,A) \to A \quad \text{by} \quad f \longmapsto \int_G \alpha_x(a) f(x) \, \mathrm{d}\mu(x).$$

and

$$\Gamma_a \colon A \to C_b(G,A) \quad \text{by} \quad b \longmapsto c_{ab}.$$

We collect some properties of Λ_a:

Proposition 1.32 (Properties of Λ_a).
Let $a \in A$.

(i) Λ_a *is* G*-equivariant.*

(ii) *If* $g \in G$*, then* $\Lambda_{\alpha_g(a)} = \Lambda_a \circ \delta_{g^{-1}}$*.*

(iii) *If* $b \in A$*, then* $\Lambda_{ab} = \Lambda_a \circ \pi_b$*.*

(iv) *If* $T \in \mathcal{M}^G(A)$*, then* $\Lambda_{T(a)} = T \circ \Lambda_a$

(v) *If* $\Lambda_a = 0$*, then* $a = 0$*.*

Proof. (i)

$$\begin{aligned}\Lambda_a(\gamma_g(f)) &= \int_G \alpha_x(a)\alpha_g(f(g^{-1}x)) \ \mathrm{d}\mu(x) \\ &\overset{5.8}{=} \alpha_g\left(\int_G \alpha_{g^{-1}x}(a)f(g^{-1}x) \ \mathrm{d}\mu(x)\right) \overset{5.11}{=} \alpha_g(\Lambda_a(f)).\end{aligned}$$

This shows, that Λ_a is G-equivariant.

(ii)

$$\begin{aligned}\Lambda_{\alpha_g(a)}(f) &= \int_G \alpha_{xg}(a)f(x) \ \mathrm{d}\mu(x) \\ &\overset{5.12}{=} \int_G \alpha_x(a)f(xg^{-1}) \ \mathrm{d}\mu(x) \\ &= \int_G \alpha_x(a)(\delta_{g^{-1}}(f))(x) \ \mathrm{d}\mu(x) \\ &= (\Lambda_a \circ \delta_{g^{-1}})(f).\end{aligned}$$

(iii)

$$\begin{aligned}\Lambda_{ab}(f) &= \int_G \alpha_x(ab)f(x) \ \mathrm{d}\mu\, x \\ &= \int_G \alpha_x(a)(\pi_b(f))(x) \ \mathrm{d}\mu\, x \\ &= (\Lambda_a \circ \pi_b)(f).\end{aligned}$$

(iv)

$$\begin{aligned}\Lambda_{T(a)}(f) &= \int_G \alpha_x(T(a))f(x) \ \mathrm{d}\mu(x) \\ &= T\left(\int_G \alpha_x(a)f(x) \ \mathrm{d}\mu(x)\right) \\ &= (T \circ \Lambda_a)(f).\end{aligned}$$

(v) Assume $\Lambda_a = 0$. Choose $h \in \mathcal{T}(G)$, with $h(1) = 1$. Define $f(x) = \alpha_x(a^*)h(x)$. Then $f \in C_c(G, A)$. We have

$$0 = \Lambda_a(f) = \int_G \alpha_x(aa^*)h(x) \ \mathrm{d}\mu(x).$$

Since $\alpha_x(aa^*)h(x) \geq 0$ for all $x \in G$, we obtain

$$aa^* = \alpha_1(aa^*)h(1) = 0$$

by Lemma 5.13(iv). Hence the C^*-condition yields $a = 0$.

□

Definition 1.33 (Square-Integrable Elements[3, page 222]).
An element $a \in A$ is called *square-integrable* if the function c_{ab} is square-integrable for all $b \in A$.

Theorem 1.34 (Characterisation of Square-Integrability).
An element $a \in A$ is square-integrable if and only if Λ_a extends to a G-equivariant adjointable operator $L^2(G, A) \to A$.

Proof.
"⇒" If $a \in A$ is square-integrable, then the image of Γ_a is contained in $\mathcal{S}^2(G, A)$. Using the canonical embedding $\iota\colon S^2(G, A) \to L^2(G, A)$, we obtain a linear map

$$\Gamma_a' := \iota \circ \Gamma_a \colon A \to L^2(G, A).$$

As a first step, we show that Γ_a' is bounded.
If $h \in \mathcal{T}(G)$, then we define $T_h\colon A \to L^2(G, A)$ by $b \mapsto h \cdot c_{ab}$. Then T_h is linear. Let $b \in A$. We estimate

$$\|T_h(b)\|_2 = \left\| \int_G h(x)^2 b^* \alpha_x(aa^*) b \ \mathrm{d}\mu(x) \right\|^{1/2} \leq \|a\| \cdot \|b\| \cdot \|h\|_2.$$

Since $h \in C_c(G)$, we have $\|h\|_2 < \infty$. Hence T_h is bounded.
Since c_{ab} is square integrable, there is $M_b > 0$, such that

$$\|T_h(b)\|_2 = \|h \cdot c_{ab}\|_2 \leq M_b \quad \text{for all } h \in \mathcal{T}(G).$$

The uniform boundedness principal implies, that there is a constant $C > 0$, such that $\|T_h\| \leq C$ for all $h \in \mathcal{T}(G)$. Using Corollary 1.16 we conclude

$$\|\Gamma_a'(b)\|_2 = \|c_{ab}\|_{S^2(G,A)} = \sup\{\|T_h(b)\|_2 \colon h \in \mathcal{T}(G)\} \leq C\|b\|.$$

Therefore Γ_a' is bounded.
Let $f \in C_c(G, A)$ and $b \in A$. We calculate

$$\Lambda_a(f)^* b = \int_G f(x)^* \alpha_x(a)^* \ \mathrm{d}\mu(x) b \overset{5.13(v)}{=} \int_G f(x)^* c_{ab}(x) \ \mathrm{d}\mu(x) \overset{1.17}{=} \langle f, \Gamma_a'(b)\rangle. \tag{1}$$

This implies

$$\|\Lambda_a(f)\|^2 = \|\Lambda_a(f)^*\Lambda_a(f)\| = \|\langle f, \Gamma'_a(\Lambda_a(f))\rangle\| \leq \|f\|_2\|\Gamma'_a\| \cdot \|\Lambda_a(f)\|.$$

Therefore, $\|\Lambda_a\| \leq \|\Gamma'_a\|$. Hence Λ_a extends to a G-equivariant bounded linear operator $L^2(G,A) \to A$. Since $C_c(G,A)$ is dense in $L^2(G,A)$. Equation (1) shows that Λ_a is adjointable with $\Lambda_a^* = \Gamma'_a$.

"$\Leftarrow$" Assume Λ_a extends to an adjointable operator $L^2(G,A) \to A$. Let $b \in A$. If $h \in \mathcal{T}(G)$ and $f \in C_c(G,A)$, then

$$\begin{aligned}
\langle f, M_h(\Lambda^*(b))\rangle &= \langle M_h(f), \Lambda_a^* b\rangle \\
&= \Lambda_a(M_h(f))^* \cdot b \\
&= \int_G f(x)^* \cdot h(x) \cdot c_{ab}(x) \ \mathrm{d}\mu(x) \\
&= \langle f, h \cdot c_{ab}\rangle.
\end{aligned}$$

Since $C_c(G,A)$ is dense in $L^2(G,A)$, this implies $h \cdot c_{ab} = M_h(\Lambda_a^*(b))$. Therefore,

$$\|h \cdot c_{ab}\|_2 = \|M_h(\Lambda_a^*(b))\|_2 \leq \|\Lambda_a^*(b)\|_2 \quad \text{for all } h \in \mathcal{T}(G).$$

Hence c_{ab} is square-integrable for all $b \in A$. That is, a is square-integrable. □

Remark 1.35. In the above proof, we showed that if $a \in A$ is square-integrable, then $\Lambda_a^* = \iota \circ \Gamma_a$.

The next computation motivates the definition of the generalised fixed point algebra.

Lemma 1.36. *Let $(\chi_i)_{i \in I} \subseteq \mathcal{T}(X)$ be a net with $\chi_i \longrightarrow 1$ uniformly on compact subsets. If $a, b \in A$ are square integrable elements, then*

$$\Lambda_a \circ \Lambda_b^* = \lim_i^s \int_G \chi_i(x)\alpha_x(ab^*) \ d\mu(x).$$

Proof. If $i \in I$, then we define

$$T_i = \int_G \chi_i(x)\alpha_x(ab^*) \ \mathrm{d}\mu(x) \in A.$$

We have

$$T_i^* \overset{5.13(i)}{=} \int_G \chi_i(x)\alpha_x(ba^*) \ \mathrm{d}\mu(x)$$

Let $d \in A$. Then

$$\Lambda_b^*(b) \overset{1.35}{=} (\iota \circ \Gamma_b)(d) = \iota(c_{bd}) = \lim_i(\chi_i \cdot c_{bd}).$$

Hence

$$
\begin{aligned}
(\Lambda_a \circ \Lambda_b^*)(d) &= \lim_i \Lambda_a(\chi_i \cdot c_{bd}) \\
&= \lim_i \left(\int_G \chi_i(g)\alpha_x(a) \cdot c_{bd}(x) \ \mathrm{d}\mu(x) \right) \\
&= \lim_i \left(\int_G \chi_i(g)\alpha_x(ab^*) \ \mathrm{d}\mu(x) \cdot d \right) \\
&= \lim_i (T_i \cdot d)
\end{aligned}
$$

By changing the roles of a and b we obtain

$$(\Lambda_a \circ \Lambda_b^*)^* = \Lambda_b \circ \Lambda_a^* = \lim_i (T_i^* \cdot d).$$

This shows that left multiplication with $(T_i)_{i\in I}$ converges strictly to $\Lambda_a \circ \Lambda_b^*$. □

Let A_{si} be the subset of square-integrable elements of A. Since $a \longmapsto c_{ab}$ is anti-linear, A_{si} is a linear subspace of A.

Proposition 1.37. *A_{si} is a G-invariant right ideal of A. We have $\mathcal{M}^G(A) \circ A_{si} \subseteq A_{si}$.*

Proof. Let $a \in A_{si}$ and $b \in A$. Then $\Lambda_{ab} = \Lambda_a \circ \pi_b$ by Proposition 1.32(iii). By Theorem 1.34 Λ_a extends to an adjointable operator $L^2(G, A) \to A$. Since $\pi_b \in \mathbb{B}^G(L^2(G, A))$, this implies, that Λ_{ab} extends to an adjointable operator $L^2(G, A) \to A$. Therefore, $ab \in A_{si}$ by Theorem 1.34. This proves, that A_{si} is a right ideal.
We use the same argument to prove that A_{si} is G-invariant and that $\mathcal{M}^G(A) \circ A_{si} \subseteq A_{si}$.. □

Definition 1.38 (Square-Integrable G-C^*-algebra)**.**
A is called *square-integrable* if A_{si} is dense in A.

For $a \in A_{si}$, we define $\|a\|_{si} = \|a\| + \|\Lambda_a\|$. It is easy to see, that $\|\cdot\|_{si}$ defines a norm on A_{si}.

Proposition 1.39. *A_{si} is complete with respect to $\|\cdot\|_{si}$.*
Let $a \in A_{si}$. We have the following estimations:

(i) *If $b \in A$, then $\|a \cdot b\|_{si} \leq \|a\|_{si} \cdot \|b\|$.*

(ii) *If $g \in G$, then $\|\alpha_g(a)\|_{si} = \|a\|_{si}$.*

(iii) *If $T \in \mathcal{M}^G(A)$, then $\|T(a)\|_{si} \leq \|T\| \cdot \|a\|_{si}$.*

Proof. Let $(a_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in A with respect to $\|\cdot\|_{si}$. Since $\|\cdot\| \leq \|\cdot\|_{si}$, $(a_n)_{n\in\mathbb{N}}$ is a norm Cauchy-sequence in A. Since A is complete, there is $a \in A$, such that $\|a_n - a\| \to 0$ for $n \to \infty$.
Likewise $\|\Lambda_{a_n} - \Lambda_{a_m}\| \leq \|a_n - a_m\|_{si}$. So that $(\Lambda_{a_n})_{n\in\mathbb{N}}$ is a Cauchy sequence in

$\mathbb{B}(L^2(G,A),A)$. Hence there is $\Lambda \in \mathbb{B}(L^2(G,A),A)$, such that $\Lambda_n \to \Lambda$ in the operator norm. For $f \in C_c(G,A)$, we have

$$\|\Lambda_{a_n}(f) - \Lambda_a(f)\| \leq \int_G \|a_n - a\| \cdot \|f(x)\| \ \mathrm{d}\mu(x) = \|a_n - a\| \cdot \|f\|_1 \xrightarrow{n\to\infty} 0.$$

Therefore, $\Lambda_a(f) = \lim_{n\to\infty} \Lambda_{a_n}(f) = \Lambda(f)$. Hence Λ_a extends to an adjointable operator $L^2(G,A) \to A$ with $\Lambda_a = \Lambda$. Theorem 1.34 implies $a \in A_{si}$. Since

$$\|\Lambda_{a_n} - \Lambda_a\| = \|\Lambda_{a_n} - \Lambda\| \to 0 \quad \text{for } n \to \infty,$$

we obtain $\|a_n - a\|_{si} \to 0$ for $n \to \infty$. This shows that A_{si} is complete. The estimations follow from Proposition 1.32 by elementary computations. □

Lemma 1.40. *Let $a \in A_{si}$. If $\Lambda_a^*\Lambda_a \in C_r^*(G,A)$, then the function $G \to A$, $g \longmapsto \alpha_g(a)$ is continuous with respect to $\|\cdot\|_{si}$.*

Proof. Let $g \in G$ and $\varepsilon > 0$. By Lemma 1.31 the function $g \longmapsto \delta_g \circ (\Lambda_a^*\Lambda_a)$ is continuous. Therefore, there is a neighbourhood U_1 of g in G, such that

$$\|\delta_g \circ (\Lambda_a^*\Lambda_a) - \delta_x \circ (\Lambda_a^*\Lambda_a)\| < \frac{1}{8}\varepsilon^2. \quad \text{for all } x \in U.$$

If $x \in U$, then

$$\begin{aligned}
\|\Lambda_{\alpha_g(a)} - \Lambda_{\alpha_x(a)}\|^2 &\overset{1.32(\mathrm{ii})}{=} \|\Lambda_a \circ (\delta_{g^{-1}} - \delta_{x^{-1}})\|^2 \\
&= \|(\delta_g - \delta_x)\Lambda_a^*\Lambda_a(\delta_{g^{-1}} - \delta_{x^{-1}})\| \\
&\leq 2\|(\delta_g - \delta_x)\Lambda_a^*\Lambda_a\| \leq \frac{1}{4}\varepsilon^2.
\end{aligned}$$

Since $G \to A$, $g \longmapsto \alpha_g(a)$ is continuous with respect to $\|\cdot\|$, there is a neighbourhood $U_2 \subseteq G$ of g, such that

$$\|\alpha_g(a) - \alpha_x(a)\| \leq \frac{\varepsilon}{2} \quad \text{for all } x \in U_2.$$

If $x \in U_1 \cap U_2$, then we obtain

$$\|\alpha_g(a) - \alpha_x(a)\|_{si} = \|\alpha_g(a) - \alpha_x(a)\| + \|\Lambda_{\alpha_g(a)} - \Lambda_{\alpha_x(a)}\| < \varepsilon.$$ □

The Right Module Structure over $C_c(G,A)$

In the following we turn A into a right module over the convolution algebra $C_c(G,A)$. Let $f \in C_c(G,A)$. We define $\breve{f}(x) = \alpha_x(f(x^{-1}))$. Then $\breve{f} \in C_c(G,A)$. If $a \in A$, then we define

$$a * f := \int_G \alpha_x(a \cdot f(x^{-1})) \ \mathrm{d}\mu(x) = \Lambda_a(\breve{f}).$$

Since $f \mapsto \check{f}$ and $a \mapsto \Lambda_a$ are linear the map $(a, f) \mapsto a * f$ is bilinear.
If $h \in C_c(G, A)$, then

$$\begin{aligned}
\Lambda_{a*f}(h) &= \int_G \alpha_g \left(\int_G \alpha_x(af(x^{-1})) \, d\mu(x) \right) \cdot h(g) \, d\mu(g) \\
&\overset{5.9}{=} \int_G \left(\int_G \alpha_g(\alpha_x(af(x^{-1}))) \cdot h(g) \, d\mu(g) \right) d\mu(x) \\
&= \int_G \Lambda_{\alpha_x(af(x^{-1}))}(h) \, d\mu(x) \\
&= \int_G (\Lambda_a \circ \pi_{f(x^{-1})} \circ \delta_{x^{-1}})(h) \, d\mu(x) \\
&\overset{5.12}{=} (\Lambda_a \circ \rho_f)(h).
\end{aligned}$$

Therefore, $\Lambda_{a*f} = \Lambda_a \circ \rho_f$. If $f_1, f_2 \in C_c(G, A)$, then this implies

$$\Lambda_{a*(f_1*f_2)} = \Lambda_a \circ \rho_{f_1*f_2} = \Lambda_a \circ \rho_{f_1} \circ \rho_{f_2} = \Lambda_{a*f_1} \circ \rho_{f_2} = \Lambda_{(a*f_1)*f_2}.$$

By Proposition 1.32(v) $a \mapsto \Lambda_a$ is injective. Hence $a * (f_1 * f_2) = (a * f_1) * f_2$. This shows that A is a right module over $C_c(G, A)$. If $F \in \mathcal{M}^G(A)$, then

$$F(a) * f = \Lambda_{F(a)}(\check{f}) = (F \circ \Lambda_a)(\check{f}) = F(a * f).$$

If $a \in A_{si}$, then $\Lambda_{a*f} = \Lambda \circ \rho_f$ extends to an adjointable operator $L^2(G, A) \to A$ therefore $a * f \in A_{si}$ by Theorem 1.34.
The following lemma collects three norm estimates for this module structure.

Lemma 1.41 (Norm Estimations for $C_c(G, A)$ module structure)**.**
Let $f \in C_c(G, A)$.

(i) *If $a \in A$, then $\|a * f\| \leq \|a\| \cdot \|f\|_1$.*

(ii) *If $a \in A_{si}$, then $\|a * f\|_{si} \leq \|a\|_{si} \cdot \|f\|_1$ and $\|a * f\|_{si} \leq \|\Lambda_a\| \cdot (\|\check{f}\|_2 + \|\rho_f\|)$.*

Proof.
From the definition of $a * f$, we obtain

$$\|a * f\| \leq \int_G \|\alpha_x(a \cdot f(x^{-1}))\| \, d\mu(x) \leq \|a\| \cdot \|f\|_1$$

If $a \in A_{si}$, then

$$\begin{aligned}
\|a * f\|_{si} &= \|a * f\| + \|\Lambda_{a*f}\| \\
&= \|a * f\| + \|\Lambda_a \circ \rho_f\| \\
&= \|a\| \cdot \|f\|_1 + \|\Lambda_a\| \cdot \|f\|_1 \\
&= \|a\|_{si} \cdot \|f\|_1.
\end{aligned}$$

Using $a * f = \Lambda_a(\check{f})$, we obtain

$$\|a * f\|_{si} = \|\Lambda_a(\check{f})\| + \|\Lambda_a \circ \rho_f\| \leq \|\Lambda_a\| \cdot (\|\check{f}\|_2 + \|\rho_f\|).$$

□

Lemma 1.42. *$A * C_c(G, A)$ is dense in A.*

Proof. Let $a \in A$. There is a $u \in A$, with $\|u\| \leq 1$ and $\|au - u\| < \frac{\varepsilon}{2}$.
Since $(\alpha_g)_{g \in G}$ is continuous, there is a compact neighbourhood U of 1 in G, such that $\|\alpha_x(a) - a\| \leq \frac{\varepsilon}{2}$ for all $x \in U$. Let $h\colon G \to [0, \infty)$ be a continuous function with $\int_G h \, \mathrm{d}\mu = 1$ and $\operatorname{supp}(h) \subseteq U$. Define $f(x) = h(x^{-1}) \cdot \alpha_x(u)$. Then $f \in C_c(G, A)$ with $\breve{f} = h.u$. We estimate

$$\begin{aligned}
\|a * f - a\| &= \|\Lambda_a(h.u) - a\| \\
&= \| \int_G h(x)(\alpha_x(a)u - a) \, \mathrm{d}\mu(x)\| \\
&\leq \int_G h(x) \cdot (\|(\alpha_x(a)u - au\| + \|au - a\|) \, \mathrm{d}\mu(x) \\
&< \varepsilon \cdot \int_G h \, \mathrm{d}\mu = \varepsilon.
\end{aligned}$$

□

Corollary 1.43. *Let $(u_i)_{i \in I}$ be an approximate identity as in Lemma 5.20. Then*

$$\|a - a * u_i\| \longrightarrow 0 \quad \text{for all } a \in A.$$

Proof. Let $a \in A$ and $\varepsilon > 0$. By the previous lemma, there is $b \in A$ and $f \in C_c(G, A)$, such that $\|a - b * f\| \leq \frac{\varepsilon}{3}$.
There is i_0, such that

$$\|f - f * u_i\| \leq \frac{\varepsilon}{3} \quad \text{for all } i \geq i_0.$$

If $i \geq i_0$, we estimate

$$\begin{aligned}
\|a - a * u_i\| &\leq \|a - b * f\| + \|b * f - b * (f * u_i)\| + \|(b * f) * u_i - a * u_i\| \\
&\overset{1.41(\mathrm{i})}{\leq} \frac{\varepsilon}{3} + \|b\| \cdot \|f - f * u_i\|_1 + \|b * f - a\| \|u_i\|_1 < \varepsilon.
\end{aligned}$$

□

Continuously Square-Integrable Subsets and the Generalized Fixed Point Algebra

We want to extend the module structure over $C_c(G, A)$ to a Hilbert module structure over the reduced crossed product $C^*_r(G, A)$. To get an inner product with values in $C^*_r(G, A)$, we define relatively continuous subsets.

Definition 1.44 (Relatively Continuous Subset and Complete Subspaces)**.**
Let $\mathcal{R} \subseteq A_{si}$ be a subset.

(i) $\mathcal{R}$ is called *relatively continuous* if

$$\Lambda^*_a \circ \Lambda_b \in C^*_r(G, A) \quad \text{for all } a, b \in \mathcal{R}.$$

(ii) $\mathcal{R}$ is called *complete* if $\mathcal{R}$ is a linear subspace of A_{si}, closed with respect to $\|\cdot\|_{si}$ and $\mathcal{R} * C_c(G, A) \subseteq \mathcal{R}$.

Definition 1.45 (Continuously Square-Integrable G-C^*-algebra).
A *continuously square-integrable G-C^*-algebra* $(A, \mathcal{R})$ is a G-C^*-algebra A together with a dense, complete and relatively continuous subspace $\mathcal{R}$.

Let $\mathcal{R}$ be a relatively continuous and complete subspace of A. We define $\mathcal{E}_{\mathcal{R}}$ as the closure of $\mathcal{E}^0_{\mathcal{R}} = \{\Lambda_a : a \in \mathcal{R}\} \subseteq \mathbb{B}^G(L^2(G, A), A)$ in the operator norm.

Proposition 1.46 ($\mathcal{E}_{\mathcal{R}}$ is a right Hilbert $C^*_r(G, A)$-module).
Let $\xi, \eta \in \mathcal{E}_{\mathcal{R}}$ and $\psi \in C^*_r(G, A)$. Then $\xi \circ \psi \in \mathcal{E}_{\mathcal{R}}$ and $\xi^* \circ \eta \in C^*_r(G, A)$.
$\mathcal{E}_{\mathcal{R}}$ becomes a right Hilbert $C^*_r(G, A)$-module, when equipped with the right module structure $\xi \cdot \psi := \xi \circ \psi$ and the $C^*_r(G, A)$-valued inner product $\langle \xi, \eta \rangle := \xi^* \circ \eta$.

Proof. There are sequences $(f_n)_{n\in\mathbb{N}} \subseteq C_c(G, A)$ and $(a_n)_{n\in\mathbb{N}} \subseteq \mathcal{R}$, such that $\rho_{f_n} \to \psi$ and $\Lambda_{a_n} \to \xi$ in the operator norms. Since $\mathcal{R}$ is complete $\Lambda_{a_n} \circ \rho_{f_n} = \Lambda_{a_n * f} \in \mathcal{E}^0_{\mathcal{R}}$ for all $n \in \mathbb{N}$. Therefore

$$\xi \circ \psi = \lim_{n\to\infty} \Lambda_{a_n} \circ \rho_{f_n} \in \mathcal{E}_{\mathcal{R}}.$$

To prove, that $\xi^* \circ \eta \in C^*_r(G, A)$, let $(b_n)_{n\in\mathbb{N}}$ be a sequence, such that $\Lambda_{b_n} \to \eta$. Since $\mathcal{R}$ is relatively continuous $\Lambda^*_{a_n} \circ \Lambda_{b_n} \in C^*_r(G, A)$ for all $n \in \mathbb{N}$. Therefore,

$$\xi^* \circ \eta = \lim_{n\to\infty} \Lambda^*_{a_n} \circ \Lambda_{b_n} \in C^*_r(G, A).$$

Hence the module structure is well-defined. The conditions $\langle \xi, \eta \cdot \psi \rangle = \langle \xi, \eta \rangle \circ \psi$ and $\langle \xi, \eta \rangle^* = \langle \eta, \xi \rangle$ are obviously satisfied.
Also $\langle \xi, \xi \rangle = \xi^* \circ \xi \geq 0$ for all $\xi \in \mathcal{E}_{\mathcal{R}}$. Since

$$\|\xi\| = \|\xi^* \circ \xi\|^{1/2} = \|\langle \xi, \xi \rangle\|^{1/2}.$$

the norm induced by the inner product equals the operator norm.
This shows, that $\mathcal{E}_{\mathcal{R}}$ is a right Hilbert $C^*_r(G, A)$-module. □

Definition 1.47 (The Generalised Fixed Point Algebra).
Let $\mathcal{R} \subseteq A_{si}$ be a relatively continuous and complete subspace of A. The *generalised fixed point algebra* $\mathrm{Fix}_{\mathcal{R}}$ is defined to be the norm-closed linear span of $\{\Lambda_a \circ \Lambda^*_b : a, b \in \mathcal{R}\}$ in $\mathcal{M}^G(A)$.

Proposition 1.48 ($\mathrm{Fix}_{\mathcal{R}}$ is a C^*-algebra).
$\mathrm{Fix}_{\mathcal{R}}$ *is a C^*-subalgebra of $\mathcal{M}^G(A)$. Therefore,* $\mathrm{Fix}_{\mathcal{R}}$ *is a C^*-algebra.*

Proof. Let $a_1, a_2, b_1, b_2 \in \mathcal{R}$. Since $\mathcal{R}$ is relatively continuous $\Lambda^*_{b_1} \circ \Lambda_{a_2} \in C^*_r(G, A)$. Therefore, there is a sequence $(f_n)_{n\in\mathbb{N}} \subseteq C_c(G, A)$, such that $\rho_{f_n} \to \Lambda^*_{b_1} \circ \Lambda_{a_2}$. We obtain

$$(\Lambda_{a_1} \circ \Lambda^*_{b_1}) \circ (\Lambda_{a_2} \circ \Lambda^*_{b_2}) = \lim_{n\to\infty} \Lambda_{a_1 * f_n} \circ \Lambda^*_{b_2} \in \mathrm{Fix}_{\mathcal{R}}.$$

Obviously $\mathrm{Fix}_{\mathcal{R}}{}^* = \mathrm{Fix}_{\mathcal{R}}$ and $\mathrm{Fix}_{\mathcal{R}}$ is closed by definition. Therefore $\mathrm{Fix}_{\mathcal{R}}$ is a C^*-subalgebra of $\mathcal{M}^G(A)$. □

Proposition 1.49 ($\mathcal{E}_\mathcal{R}$ is a Hilbert $\mathrm{Fix}_\mathcal{R}$-$C_r^*(G,A)$-bimodule).
Let $\xi,\eta \in \mathcal{E}_\mathcal{R}$ and $F \in \mathrm{Fix}_\mathcal{R}$. Then $F \circ \xi \in \mathcal{E}_\mathcal{R}$ and $\xi \circ \eta^* \in \mathrm{Fix}_\mathcal{R}$.
$\mathcal{E}_\mathcal{R}$ becomes a Hilbert $\mathrm{Fix}_\mathcal{R}$-$C_r^*(G,A)$-bimodule, when equipped with right Hilbert module structure from above and the left module structure defined by $F \cdot \xi := F \circ \xi$ and the $\mathrm{Fix}_\mathcal{R}$-valued inner product $\langle\langle \xi,\eta\rangle\rangle := \xi \circ \eta^*$.

Proof. Let $a,b,c \in \mathcal{R}$. Then

$$(\Lambda_a \circ \Lambda_b^*) \circ \Lambda_c = \Lambda_a \circ (\Lambda_b^* \circ \Lambda_c) \in \mathcal{E}_\mathcal{R} \circ C_r^*(G,A) \subseteq \mathcal{E}_\mathcal{R}\,.$$

Since ${\mathcal{E}_\mathcal{R}}^0$ is dense in $\mathcal{E}_\mathcal{R}$, this implies $(\Lambda_a \circ \Lambda_b^*) \cdot \xi \in \mathcal{E}_\mathcal{R}$ for all $\xi \in \mathcal{E}_\mathcal{R}$. Therefore $\mathrm{Fix}_\mathcal{R} \cdot \mathcal{E}_\mathcal{R} \subseteq \mathcal{E}_\mathcal{R}$.
Similarly $\Lambda_a \circ \Lambda_b^* \in \mathrm{Fix}_\mathcal{R}$ by definition and hence $\xi \circ \eta^* \in \mathrm{Fix}_\mathcal{R}$ for all $\xi,\eta \in \mathcal{E}_\mathcal{R}$.
Similar arguments as in the proof of Proposition 1.46 show, that $\mathcal{E}_\mathcal{R}$ is a left Hilbert $\mathrm{Fix}_\mathcal{R}$-module. Since the module structures are defined by composition of maps the conditions $(F \cdot \xi)\cdot \psi = F \cdot (\xi \cdot \psi)$ and $\langle\langle \xi,\eta\rangle\rangle \cdot \theta = \xi \cdot \langle \eta,\theta\rangle$ for $F \in \mathrm{Fix}_\mathcal{R}, \psi \in C_r^*(G,A)$ and $\xi,\eta,\theta \in \mathcal{E}_\mathcal{R}$ are obviously satisfied. □

Remark 1.50. The bimodule $\mathcal{E}_\mathcal{R}$ is full on the left by definition of $\mathrm{Fix}_\mathcal{R}$. Therefore $\mathrm{Fix}_\mathcal{R}$ is Morita-Rieffel equivalent to the ideal $\langle \mathcal{E}_\mathcal{R}, \mathcal{E}_\mathcal{R}\rangle$ of $C_r^*(G,A)$.

Lemma 1.51 ($\mathcal{E}_\mathcal{R}$ detects elements of $\mathcal{R}$).
Let $\mathcal{R}$ be a relatively continuous and complete subspace of A and $a \in A_{si}$.
If $\Lambda_a \in \mathcal{E}_\mathcal{R}$, then $a \in \mathcal{R}$.

Proof. Let $u \in C_c(G,A)$ and $\varepsilon > 0$. Assume $u \neq 0$. Define $C = \|\breve{u}\|_2 + \|\rho_u\|$. Since $\Lambda_a \in \mathcal{E}_\mathcal{R}$, there is $r \in \mathcal{R}$, such that $\|\Lambda_a - \Lambda_r\| < \frac{\varepsilon}{C}$. Therefore, Lemma 1.41(ii) yields $\|a * u - r * u\|_{si} < \varepsilon$. Since $\mathcal{R}$ is complete, we have $r * u \in \mathcal{R}$. Hence $a * u \in \mathcal{R}$, since $\mathcal{R}$ is $\|\cdot\|_{si}$- closed.
Let $(u_i)_{i\in I}$ be the approximate identity of 5.20. *Corollary* 1.43 yields $\|a - a * u_i\| \to 0$. By Lemma 1.28 $(\rho_{u_i})_{i\in I}$ is an approximate identity of $C_r^*(G,A)$. Since $\mathcal{E}_\mathcal{R}$ is a right Hilbert $C_r^*(G,A)$-module, this implies $\|\Lambda_a - \Lambda_a \circ \rho_{u_i}\| \to 0$.
Therefore,

$$\|a - a * u_i\|_{si} = \|a - a * u_i\| + \|\Lambda_a - \Lambda_a \circ \rho_{u_i}\| \longrightarrow 0.$$

Since $a * u_i \in \mathcal{R}$ for all $i \in I$ and $\mathcal{R}$ is $\|\cdot\|_{si}$ closed, this leads to $a \in \mathcal{R}$. □

Theorem 1.52 (Properties of a Complete, Relatively Continuous Subspace).
Let $\mathcal{R} \subseteq A$ be a complete, relatively continuous subspace.

(i) *$\mathcal{R}$ is G-invariant and the action of G on $\mathcal{R}$ is continuous with respect to $\|\cdot\|_{si}$.*

(ii) *$\mathcal{R} \cdot A = \mathcal{R}$.*

(iii) *If $a \in \mathcal{R}$ and $F \in \mathcal{F}_\mathcal{R}$, then $F(a) \in \mathcal{R}$.*

(iv) *If G is exact and $I \subseteq A$ is a G-invariant closed ideal, then $\mathcal{R} \cdot I = \mathcal{R} \cap I$*

Proof. (i) Let $r \in \mathcal{R}$ and $g \in G$. Proposition 1.37 yields $\alpha_g(r) \in A_{si}$.
Since $\Lambda_r \in \mathcal{E}_{\mathcal{R}}$ and $\mathcal{E}_{\mathcal{R}}$ is a right Hilbert $C_r^*(G,A)$-module, there is $\xi \in \mathcal{E}_{\mathcal{R}}$ and $\psi \in C_r^*(G,A)$, such that $\Lambda_r = \xi \cdot \psi$. By Lemma 1.31, we have

$$\psi \circ \delta_{g^{-1}} = (\delta_g \circ \psi^*)^* \in C_r^*(G,A).$$

Hence

$$\Lambda_{\alpha_g(r)} \overset{1.32(\mathrm{i})}{=} \Lambda_r \circ \delta_{g^{-1}} = \xi \circ \psi \circ \delta_{g^{-1}} = \mathcal{E}_{\mathcal{R}} \cdot C_r^*(G,A) \subseteq \mathcal{E}_{\mathcal{R}}\,.$$

Lemma 1.51 implies $\alpha_g(r) \in \mathcal{R}$. Since $\Lambda_r^* \circ \Lambda_r \in C_r^*(G,A)$ the map $G \to \mathcal{R}$ given by $g \mapsto \alpha_g(r)$ is continuous with respect to $\|\cdot\|_{si}$ by Lemma 1.40.

(ii) Let $r \in \mathcal{R}$ and $a \in A$. Proposition 1.37 yields $r \cdot a \in A_{si}$. As above, there is $\xi \in \mathcal{E}_{\mathcal{R}}$ and $\psi \in C_r^*(G,A)$ with $\Lambda_r = \xi \cdot \psi$. Hence Lemma 1.30 implies

$$\Lambda_{ra} \overset{1.32(\mathrm{iii})}{=} \Lambda_r \circ \pi_a = \xi \circ \psi \circ \pi_a \in \mathcal{E}_{\mathcal{R}} \cdot C_r^*(G,A) \subseteq \mathcal{E}_{\mathcal{R}}\,.$$

This shows $\mathcal{R} \cdot A \subseteq \mathcal{R}$.
Let $(a_i)_{i \in I}$ be an approximate identity of A. Then $\|r - r \cdot a_i\| \to 0$. Since $\Lambda_r^* \circ \Lambda_r \in C_r^*(G,A)$ Lemma 1.30 yields

$$\begin{aligned}\|\Lambda_r - \Lambda_{ra_i}\|^2 &= \|\Lambda_r - \Lambda_r \pi_{a_i}\|^2 \\ &= \|(\Lambda_r^* - \pi_{a_i}\Lambda_r^*)(\Lambda_r - \Lambda_r \pi_{a_i})\| \\ &\leq 2 \cdot \|\Lambda_r^*\Lambda_r - \pi_{a_i}\Lambda_r^*\Lambda_r\| \longrightarrow 0.\end{aligned}$$

Therefore,

$$\|r - r \cdot a_i\|_{si} = \|r - r \cdot a_i\| + \|\Lambda_r - \Lambda_{r \cdot a_i}\| \longrightarrow 0.$$

Cohen's factorisation theorem implies $r \in \mathcal{R} \cdot A$. Therefore $\mathcal{R} \subseteq \mathcal{R} \cdot A$.

(iii) Let $r \in \mathcal{R}$ and $F \in \mathrm{Fix}_{\mathcal{R}}$. Since $\mathrm{Fix}_{\mathcal{R}} \subseteq \mathcal{M}^G(A)$ we obtain $F(r) \in A_{si}$ from Proposition 1.37. Proposition 1.32(iv) yields

$$\Lambda_{F(r)} = F \circ \Lambda_r \in \mathrm{Fix}_{\mathcal{R}} \cdot \mathcal{E}_{\mathcal{R}} \subseteq \mathcal{E}_{\mathcal{R}}\,.$$

Therefore, $F(r) \in \mathcal{R}$ by Lemma 1.51.

(iv) Part (ii) yields $\mathcal{R} \cdot I \subseteq R \cdot A \subseteq R$. Also $\mathcal{R} \cdot I \subseteq I$, since I is an ideal of A. Hence $R \cdot I \subseteq R \cap I$.
To prove the other inclusion, we will use Cohen's Factorisation Theorem. Let $r \in R \cap I$. If $b \in A$, then $c_{rb}(g) = \alpha_x(r)^* b \in I$ for all $g \in G$. Since $r \in \mathcal{R} \subseteq A_{si}$, we have $c_{rb} \in \mathcal{S}^2(G,A)$. Corollary 1.18 yields $\iota(c_{rb}) \in L^2(G,A) \cdot I$. Hence Remark 1.35 implies

$$\Lambda_r^*(b) = \iota \circ \Gamma_r(b) = \iota(c_{rb}) \in L^2(G,A) \cdot I$$

Hence $(\Lambda_r^* \circ \Lambda_r)(L^2(G,A)) \subseteq L^2(G,A) \cdot I$.
Since G is assumed to be exact, this yields $\Lambda_r^* \circ \Lambda_r \in C_r^*(G,I)$ by the considerations in Section 1.2. Let $(u_j)_{j \in J}$ be an approximate identity of I. Then

$$\|\Lambda_r^* \circ \Lambda_r - \Lambda_r^* \circ \Lambda_r \circ \pi_{u_j}\| \longrightarrow 0.$$

The same computations as in Part (ii) show $\|r - r \cdot u_j\|_{si} \to 0$. Hence the Factorisation Theorem yields $r \in \mathcal{R} \cdot I$.

$\square$

The following lemma and its corollary give a more explicit criteria weather a given subset $\mathcal{R}$ is relatively continuous.

Lemma 1.53. *Let $\mathcal{R}$ be a dense subset of A, such that for all $a, b \in \mathcal{R}$ the image of the map $\Gamma_a \circ \Lambda_b \colon C_c(G,A) \to C_b(G,A)$ is contained in $S^2(G,A)$ and the map $\iota \circ \Gamma_a \circ \Lambda_b$ extends to a bounded operator $L^2(G,A) \to L^2(G,A)$ with $\iota \circ \Gamma_a \circ \Lambda_b \in C_r^*(G,A)$.
Then $\mathcal{R} \subseteq A_{si}$ and $\mathcal{R}$ is relatively continuous.*

Proof. Fix $a \in \mathcal{R}$.
If $b \in \mathcal{R}$ and $f_1, f_2 \in C_c(G,A)$, then

$$\begin{aligned}\langle(\iota \circ \Gamma_a \circ \Lambda_b)(f_1), f_2\rangle &\overset{1.17}{=} \int_G ((\Gamma_a \circ \Lambda_b)(f_1)(g)^* f_2(g) \ \mathrm{d}\mu(g) \\ &= \int_G (\alpha_g(a)^* \Lambda_b(f_1))^* f_2(g) \ \mathrm{d}\mu(g) \\ &= \Lambda_b(f_1)^* \circ \Lambda_a(f_2)\end{aligned}$$

If $f \in C_c(G,A)$, this implies This implies

$$\begin{aligned}\|\Lambda_a(f)\|^2 &= \|\Lambda_a(f)^* \Lambda_a(f)\| \\ &= \|\langle(\iota \circ \Gamma_a \circ \Lambda_b)(f), f\rangle\| \\ &\leq \|\iota \circ \Gamma_a \circ \Lambda_a\| \|f\|_2^2.\end{aligned}$$

Hence Λ_a extends to a bounded operator $L^2(G,A) \to A$. In view of Theorem 1.34 it remains to prove, that Λ_a is adjointable.
Let A_0 be the domain of Λ_a^*. That is $x \in A_0$ if and only if there is $f_x \in L^2(G,A)$, such that

$$x^* \Lambda_a(h) = \langle f_x, h\rangle \quad \text{for all } h \in L^2(G,A).$$

Since Λ_a is bounded it suffices to look at $h \in C_c(G,A)$. The computation at the beginning of this proof shows $b * k = \Lambda_b(\check{k}) \in A_0$ for all $b \in \mathcal{R}$ and $k \in C_c(G,A)$. Since $\mathcal{R}$ is dense 1.41(i) yields $A * C_c(G,A) \subset A_0$. By Lemma 1.42 $A * C_c(G,A)$ is dense in A. Hence A_0 is dense in A. Since Λ_a is bounded A_0 is closed. Therefore, $A_0 = A$ and the map $x \mapsto f_x$ serves as an adjoint for Λ_a.
Theorem 1.34 yields $\mathcal{R} \subseteq A_{si}$. By Remark 1.35 we obtain

$$\Lambda_a^* \circ \Lambda_b = \iota \circ \Gamma_a \circ \Lambda_b \in C_r^*(G,A).$$

Therefore $\mathcal{R}$ is relatively continuous. $\square$

If $a, b \in A$ we define a function $f_{ab} \in C_b(G,A)$ by $f_{ab}(g) = a^* \alpha_g(b)$.

Corollary 1.54. *Let $\mathcal{R}$ be a dense subset of A, such that*

$$\int_G \|f_{ab}(g)\| \ d\mu(g) < \infty \quad \textit{for all } a, b \in \mathcal{R}.$$

Then $\mathcal{R} \subseteq A_{si}$ and $\mathcal{R}$ is relatively continuous.

Proof. Let $a, b \in \mathcal{R}$ and $k \in C_c(G, A)$ We compute

$$\begin{aligned}((\Gamma_a \circ \Lambda_b)(k))(g) &= \alpha_g(a)^* \Lambda_b(k) \\ &= \alpha(a)^* \cdot \int_G \alpha_x(b) \cdot f(x) \ \mathrm{d}\mu(x) \\ &\overset{5.11}{=} \int_G \alpha_g(a^* \alpha_x(f)) \cdot k(gx) \ \mathrm{d}\mu(x) \\ &\int_G \alpha_g(f_{ab}(x)) \cdot k(gx) \ \mathrm{d}\mu(x) = (\rho_{f_{ab}})h)(g).\end{aligned}$$

By Lemma 1.29 the image of $\Gamma_a \circ \Lambda_b = \rho_{f_{ab}}$ is contained in $\mathcal{S}^2(G, A)$ and $\iota \circ \Gamma_a \circ \Lambda_b = \iota \circ \rho_{f_{ab}}$ extends to an operator $L^2(G, A) \to L^2(G, A)$ with $\iota \circ \Gamma_a \circ \Lambda_b = \iota \circ \rho_{f_{ab}} \in C_r^*(G, A)$. Therefore, $\mathcal{R} \subseteq A_{si}$ and $\mathcal{R}$ is relatively continuous by Lemma 1.53. □

The last of this section is customized for our application to the scaling action of the tangent groupoid.

Lemma 1.55. *Let $\mathcal{R}_0 \subseteq A_{si}$ be a dense, relatively continuous, G-invariant subspace, such that $\mathcal{R}_0 \cdot \mathcal{R}_0 \subseteq \mathcal{R}_0$.*
Then the closure $\mathcal{R}$ of $\mathcal{R}_0$ with respect to $\|\cdot\|_{si}$ is dense, complete and relatively continuous. Therefore $(A, \mathcal{R})$ is a continuously square-integrable G-C^*-algebra.

Proof. Clearly $\mathcal{R} \subset A_{si}$ is a closed linear subspace of the Banach space A_{si}. Since $\|\Lambda_a\| \leq \|a\|_{si}$ for all $a \in A_{si}$ we get $\Lambda_a^* \circ \Lambda_b \in C_r^*(G, A)$ for all $a, b \in \mathcal{R}$ by approximation with elements of $\mathcal{R}_0$. Therefore, $\mathcal{R}$ is relatively continuous.
It remains to prove $\mathcal{R} * C_c(G, A) \subseteq \mathcal{R}$.

Claim: $\mathcal{R} \cdot A \subseteq \mathcal{R}$.

Proof of the claim: Let $r \in \mathcal{R}$ and $a \in A$. There is a sequence $(r_n)_{n \in \mathbb{N}} \subseteq \mathcal{R}_0$ such that $\|r - r_n\|_{si} \to 0$. Since $\mathcal{R}_0$ is dense, there is a sequence $(a_n)_{n \in \mathbb{N}} \subseteq \mathcal{R}_0$, such that $\|a - a_n\| \to 0$. since every convergent sequence is bounded and by Lemma 1.41(ii) we obtain

$$\|ra - r_n a_n\|_{si} \leq \|r\|_{si} \cdot \|a - a_n\| + \|r - r_n\|_{si} \cdot \|a_n\| \xrightarrow{n \to \infty} 0$$

As $r_n a_n \in \mathcal{R}_0 \cdot \mathcal{R}_0 \subseteq \mathcal{R}$ for all $n \in \mathbb{N}$, we get $ra \in \mathcal{R}$. This proves the Claim.
Since $\mathcal{R}_0$ is G-invariant $\mathcal{R}$ is G-invariant by Proposition 1.39(ii).
If $r \in \mathcal{R}$ and $f \in C_c(G, A)$, then

$$r * f = \Lambda_r(\check{f}) = \int_G \alpha_x(r) \cdot \check{f}(x) \ \mathrm{d}\mu(x).$$

By the above claim and since $\mathcal{R}$ is G-invariant $\alpha_x(r) \cdot \check{f}(x) \in \mathcal{R}$ for all $x \in G$. By Lemma 1.40 the map $G \to \mathcal{R}$ given by $x \mapsto \alpha_x(r)$ is continuous with respect to $\| \cdot \|_{si}$. The estimation 1.39(i) shows that the multiplication map $\mathcal{R} \times A \to \mathcal{R}$ is continuous with respect to $\| \cdot \|_{si}$ on $\mathcal{R}$. Hence the map $x \mapsto \alpha_x(r) \cdot \check{f}(x)$ is continuous with respect to $\| \cdot \|_{si}$. Hence the integral above makes sense as an integral with values in $(\mathcal{R}, \| \cdot \|_{si})$. Since the inclusion $(\mathcal{R}, \| \cdot \|_{si}) \to (A, \| \cdot \|)$ is continuous, we obtain $r * f \in \mathcal{R}$.

□

2 The Tangent Groupoid of $\mathbb{R}^n$

2.1 Locally Compact Hausdorff Groupoids and their reduced C^*-Algebras

Notation and Definitions

Definition 2.1 (Locally Compact Hausdorff Groupoid)**.**
A *locally compact Hausdorff groupoid* is a groupoid $\mathcal{G}$ with object set $\mathcal{G}^{(0)}$ together with locally compact Hausdorff topologies on $\mathcal{G}$ and on $\mathcal{G}^{(0)}$, such that the structure maps

$$\begin{aligned} r &\colon \mathcal{G} \to \mathcal{G}^{(0)}, \\ s &\colon \mathcal{G} \to \mathcal{G}^{(0)}, \\ inv &\colon \mathcal{G} \to \mathcal{G}, \\ mult &\colon \mathcal{G}^{(2)} \to \mathcal{G}, \end{aligned}$$

are continuous. Here the set $\mathcal{G}^{(2)} \subseteq \mathcal{G} \times \mathcal{G}$ of composable arrows carries the induced topology from $\mathcal{G} \times \mathcal{G}$.

For $x \in \mathcal{G}^{(0)}$, we define

$$\mathcal{G}^x = \{\gamma \in \mathcal{G} \colon r(\gamma) = x\} \quad \text{and } \mathcal{G}_x = \{\gamma \in \mathcal{G} \colon s(\gamma) = x\}.$$

To construct a C^*-algebra associated to a locally compact Hausdorff groupoid, we need our groupoid to provide a Haar system.

Definition 2.2 (Left Haar System)**.**
Let $\mathcal{G}$ be a locally compact Hausdorff groupoid. A family $(\mu^x)_{x \in \mathcal{G}^{(0)}}$ of positive Radon measures μ^x on $\mathcal{G}^x$ is called a right Haar system $\mathcal{G}$ if satisfies the following conditions for all $f \in C_c(\mathcal{G})$.

(i) The function $\mathcal{G}^{(0)} \to \mathbb{C}$, $x \mapsto \int_{\mathcal{G}^x} f \ \mathrm{d}\mu^x$ is continuous.

(ii) If $\gamma \in \mathcal{G}$, then

$$\int_{\mathcal{G}^{r(\gamma)}} f(\gamma_2) \ \mathrm{d}\mu^{r(\gamma)}(\gamma_2) = \int_{\mathcal{G}^{s(\gamma)}} f(\gamma\gamma_2) \ \mathrm{d}\mu^{s(\gamma)}(\gamma_2).$$

The definition for a right Haar system is analogously. Every left Haar system $(\mu^x)_{x\in\mathcal{G}^{(0)}}$ gives rise to a right Haar system $(\mu_x)_{x\in\mathcal{G}^{(0)}}$ defined by

$$\int_{\mathcal{G}_x} f(\gamma)\ \mathrm{d}\mu_x(\gamma) = \int_{\mathcal{G}^x} f(\gamma^{-1})\ \mathrm{d}\mu^x(\gamma) \quad \text{for } f \in C_c(\mathcal{G}).$$

The Reduced C^*- Algebra of a Locally Compact Hausdorff Groupoid

For $f_1, f_2 \in C_c(\mathcal{G})$ the convolution is defined by

$$\begin{aligned}(f_1 * f_2)(\gamma) &= \int_{\mathcal{G}^{s(\gamma)}} f_1(\gamma\gamma_2)\cdot f_2(\gamma_2^{-1})\ \mathrm{d}\mu^{s(\gamma)}(\gamma_2)\\ &= \int_{\mathcal{G}^{r(\gamma)}} f_1(\gamma_2)\cdot f_2(\gamma_2^{-1}\gamma)\ \mathrm{d}\mu^{r(\gamma)}(\gamma_2)\end{aligned}$$

If $f \in C_c(\mathcal{G})$ the involution is defined by $f^*(\gamma) = \overline{f(\gamma^{-1})}$. With this operators $C_c(\mathcal{G})$ becomes a $*$-algebra.
We define a norm $\|f\|_I$ as the maximum of

$$\|f\|_{I,r} := \sup_{x\in\mathcal{G}^{(0)}} \int_{\mathcal{G}^x} |f(\gamma)|\ \mathrm{d}\mu^x(\gamma) \quad \text{and } \|f\|_{I,s} = \sup_{x\in\mathcal{G}^{(0)}} \int_{\mathcal{G}_x} |f(\gamma)|\ \mathrm{d}\mu_x(\gamma)$$

Then $\|\cdot\|_I$ is a submultiplicative norm on $C_c(\mathcal{G})$. We have $\|f\|_{I,r} = \|f^*\|_{I,s}$. Therefore, $\|f^*\|_I = \|f\|_I$.
Let $x \in \mathcal{G}^{(0)}$. There is a representation

$$\lambda_x \colon C_c(\mathcal{G}) \to \mathbb{B}(L^2(\mathcal{G}_x), \mu_x))$$

defined by the formula

$$(\lambda_x(f)h)(\gamma) = \int_{\mathcal{G}^x} f(\gamma\gamma_2)h(\gamma_2^{-1})\ \mathrm{d}\mu^x(\gamma_2).$$

We call these representations the *regular representations* of $\mathcal{G}$.
The *reduced C^*-algebra* $C_r^*(\mathcal{G})$ is defined as the completion of $C_c(\mathcal{G})$ in the norm

$$\|f\|_r = \sup_{x\in\mathcal{G}^{(0)}} \|\lambda_x(f)\|.$$

We have $\|f\|_r \leq \|f\|_I$ for all $f \in C_c(\mathcal{G})$.

2.2 The Tangent Bundle of $\mathbb{R}^n$

Let $T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$ be the tangent bundle of $\mathbb{R}^n$. We view $T\mathbb{R}^n$ as a locally compact Hausdorff groupoid with object set $\mathbb{R}^n$ and arrow set $\mathbb{R}^n \times \mathbb{R}^n$. The structure maps are given by

$$\begin{aligned}s(y,x) = r(y,x) &= y,\\ (y,x_1)(y,x_2) &= (y, x_1+x_2),\\ (y,x)^{-1} &= (y,-x).\end{aligned}$$

If $y \in \mathbb{R}^n$, then

$$(T\mathbb{R}^n)^y = (T\mathbb{R}^n)_y = \{(y, x) \colon x \in \mathbb{R}^n\}.$$

The Lebesgue measure on $\mathbb{R}^n$ gives a left Haar-system $(\mu^y)_{y \in \mathbb{R}^n}$ for $T\mathbb{R}^n$ with

$$\int_{(T\mathbb{R}^n)^y} f(y, x) \ \mathrm{d}\mu^y(y, x) = \int_{\mathbb{R}^n} f(y, x) \ \mathrm{d}x \quad \text{for all } f \in C_c(T\mathbb{R}^n).$$

Let $\mathcal{F} \colon C_c(\mathbb{R}^n) \to C_0(\mathbb{R}^n)$ be the Fourier-transformation. That is

$$(\mathcal{F}(f))(\xi) = \int_{\mathbb{R}^n} f(x) \cdot e^{-2\pi i \langle x, \xi \rangle} \mathrm{d}x \quad \text{for all } \xi \in \mathbb{R}^n.$$

If $f \in C_c(T\mathbb{R}^n)$ and $y \in \mathbb{R}^n$, then we define $f_y \in C_c(\mathbb{R}^n)$ by $f_y(x) = f(y, x)$ for all $x \in \mathbb{R}^n$. The function $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ given by $(y, \xi) \mapsto \mathcal{F}(f_y)(\xi)$ is continuous and vanishes at ∞. Therefore, we obtain a linear map $\mathcal{F}_1 \colon C_c(T\mathbb{R}^n) \to C_0(\mathbb{R}^n \times \mathbb{R}^n)$ with

$$(\mathcal{F}_1(f))(y, \xi) = \mathcal{F}(f_y)(\xi) = \int_{\mathbb{R}^n} f(y, x) \cdot e^{-2\pi i \langle x, \xi \rangle} \ \mathrm{d}x.$$

Theorem 2.3. *The map $\mathcal{F}_1$ lifts to a $*$-isomorphism of C^*-algebras*

$$\mathcal{F}_1 \colon C_r^*(T\mathbb{R}^n) \to C_0(\mathbb{R}^n \times \mathbb{R}^n).$$

Proof. First we prove that $\mathcal{F}_1$ is a $*$-homomorphism on the C_c-level. If $f_1, f_2 \in C_c(T\mathbb{R}^n)$, then

$$\begin{aligned}(f_1 * f_2)_y(x) = (f_1 * f_2)(y, x) &= \int_{\mathbb{R}^n} f_1(y, z) \cdot f_2((y, -z) \cdot (y, x)) \ \mathrm{d}z \\ &= \int_{\mathbb{R}^n} f_{1y}(z) \cdot f_{2y}(x - z) \ \mathrm{d}z = (f_{1y} * f_{2y})(x).\end{aligned}$$

Since $\mathcal{F}$ is a $*$-homomorphism, this implies

$$\begin{aligned}\mathcal{F}_1(f_1 * f_2)(y, \xi) &= \mathcal{F}((f_1 * f_2)_y)(\xi) \\ &= \mathcal{F}(f_{1y} * f_{2y})(\xi) \\ &= \mathcal{F}(f_1^y)(\xi) \cdot \mathcal{F}(f_2^y)(\xi) \\ &= (\mathcal{F}_1(f_1) \cdot \mathcal{F}_1(f_2))(y, \xi).\end{aligned}$$

If $f \in C_c(T\mathbb{R}^n)$, then

$$(f_y)^*(x) = \overline{f_y(-x)} = \overline{f((y, x)^{-1})} = f^*(y, x) = (f^*)_y(x).$$

Therefore,

$$\begin{aligned}(\mathcal{F}_1(f)^*)(y, \xi) &= \overline{(\mathcal{F}_1(f))(y, \xi)} = \overline{(\mathcal{F}(f_y))(\xi)} = (\mathcal{F}(f_y))^*(\xi) \\ &= (\mathcal{F}(f_y)^*)(\xi) = \mathcal{F}((f^*)_y)(\xi) = \mathcal{F}_1(f^*)(y, \xi).\end{aligned}$$

This proves, that $\mathcal{F}_1$ is a $*$-homomorphism.
Let $\alpha\colon C_c(\mathbb{R}^n) \to \mathbb{B}(L^2(\mathbb{R}^n))$ be given by $\alpha(f)(h) = f * h$ for $h \in C_c(R^n)$. The Pancharel theorem yields $\|\alpha(f)\| = \|\mathcal{F}(f)\|_\infty$. Hence $\mathcal{F}$ extends to an isomorphism $C_r^*(\mathbb{R}^n) \to C_0(\mathbb{R}^n)$.
Let $y \in \mathbb{R}^n = (T\mathbb{R}^n)^{(0)}$. The regular representation α_y of $C_c(T\mathbb{R}^n)$ on $L^2(T\mathbb{R}^n_y) \cong L^2(\mathbb{R}^n)$ identifies with

$$(\alpha_y(f))h)(x) = \int_{\mathbb{R}^n} f(y, x-z) \cdot h(z) \, \mathrm{d}z = (f_y * h)(x) = (\alpha(f_y)h)(x).$$

Therefore,

$$|\mathcal{F}_1(f)(y,\xi)| = |\mathcal{F}(f_y)(\xi)| \leq \|\mathcal{F}(f_y)\|_\infty = \|\alpha(f_y)\| \leq \sup_{y\in\mathbb{R}^n} \|\alpha_y(f)\| = \|f\|_r. \tag{2}$$

This implies $\|\psi(f)\|_\infty \leq \|f\|_r$. Choosing appropriate $y, \xi \in \mathbb{R}^n$, we see that (2) implies $\|\mathcal{F}_1(f)\|_\infty = \|f\|_r$. Hence $\mathcal{F}_1$ extends to a isometric $*$-homomorphism

$$\mathcal{F}_1\colon C_r^*(T\mathbb{R}^n) \to C_0(\mathbb{R}^n \times \mathbb{R}^n).$$

Let $h, k \in C_0(\mathbb{R}^n)$. We define $f \in C_0(\mathbb{R}^n \times \mathbb{R}^n)$ by $f(y,\xi) = h(y) \cdot k(\xi)$. There are sequences $(g_n)_{n\in\mathbb{N}}, (h_n)_{n\in\mathbb{N}}, \subseteq C_c(\mathbb{R}^n)$, such that

$$\|h_n - h\|_\infty \xrightarrow{n\to\infty} 0 \quad \text{and } \|\mathcal{F}(g_n) - k\|_\infty \xrightarrow{n\to\infty} 0.$$

We define $f_n \in C_c(T\mathbb{R}^n)$ by $f_n(y,x) = h_n(y) \cdot g_n(x)$. Then $\mathcal{F}_1(f_n)(y,\xi) = h_n(y) \cdot \mathcal{F}(g_n)(\xi)$. Therefore, $\|\mathcal{F}_1(f_n) - f\|_\infty \xrightarrow{n\to\infty} 0$.
Since the range of $\mathcal{F}_1$ is closed, this shows that f is in the range of $\mathcal{F}_1$.
By the Stone-Weierstrass theorem elements of the form $h(y) \cdot k(\xi)$ generate $C_0(\mathbb{R}^n \times \mathbb{R}^n)$. Hence $\mathcal{F}_1$ is surjective.
Therefore, $\mathcal{F}_1$ is a $*$-isomorphism of C^*-algebras. □

2.3 The Pair Groupoid of $\mathbb{R}^n$

The locally compact space $P\mathbb{R}^n := \mathbb{R}^n \times \mathbb{R}^n$ becomes a is a locally compact Hausdorff groupoid with object set $\mathbb{R}^n$ and arrow set $\mathbb{R}^n \times \mathbb{R}^n$ together with the structure maps

$$\begin{aligned} s(x,y) &= y, \\ r(x,y) &= x, \\ (x,y)(y,z) &= (x,z) \\ (x,y)^{-1} &= (y,x). \end{aligned}$$

$P\mathbb{R}^n$ is called the *pair groupoid* of $\mathbb{R}^n$. If $x \in \mathbb{R}^n$, then

$$(P\mathbb{R}^n)^x = \{(x,y) \colon y \in \mathbb{R}^n\} \quad \text{and } (P\mathbb{R}^n)_x = \{(y,x) \colon y \in \mathbb{R}^n\}.$$

As above the Lebesgue measure gives a left Haar system $(\mu^x)_{x\in\mathbb{R}^n}$ for $P\mathbb{R}^n$ with

$$\int_{(P\mathbb{R}^n)^x} f(x,y) \, \mathrm{d}\mu^x(x,y) = \int_{\mathbb{R}^n} f(x,y) \, \mathrm{d}y \quad \text{for all } f \in C_c(P\mathbb{R}^n).$$

For $f \in C_c(P\mathbb{R}^n)$ and $h \in C_c(\mathbb{R}^n)$, we define $(K_f h)(x) = \int_{\mathbb{R}^n} f(x,y) \cdot h(y) \, \mathrm{d}y$.

Theorem 2.4 ($C_c(P\mathbb{R}^n)$-functions as Integral Kernels)**.**
The linear map $K_f : C_c(\mathbb{R}^n) \to C_c(\mathbb{R}^n)$ extends to a bounded operator on $L_2(\mathbb{R}^n)$. The map $C_c(P\mathbb{R}^n) \to \mathbb{B}(L^2(\mathbb{R}^n))$ given by $f \mapsto K_f$ extends to a $*$-isomorphism

$$K\colon C_r^*(P\mathbb{R}^n) \to \mathbb{K}(L^2(\mathbb{R}^n)).$$

Proof. Let $y \in \mathbb{R}^n$ and $h \in C_c(\mathbb{R}^n)$. The regular representation α_y of $C_c(P\mathbb{R}^n)$ on $L^2(P\mathbb{R}^n_y) \cong L^2(\mathbb{R}^n)$ identifies with

$$\begin{aligned}(\alpha_y(f)h)(x) &= \int_{\mathbb{R}^n} f((x,y)\cdot(y,z))\cdot h(z)\ \mathrm{d}z\\ &= \int_{\mathbb{R}^n} f(x,z)\cdot h(z)\ \mathrm{d}z\\ &= (K_f h)(x).\end{aligned}$$

Hence $\alpha_y(f) = K_f$. The theory of regular representations of groupoids implies, that K_f extends to a bounded operator on $L^2(\mathbb{R}^n)$ and that $C_c(P\mathbb{R}^n) \to \mathbb{B}(L^2(\mathbb{R}^n))$, $f \mapsto K_f$ is a $*$-homomorphism.
If $f \in C_c(R\mathbb{R}^n)$ is of the form $f(x,y) = f_1(x)\cdot f_2(y)$ for $f_1, f_2 \in C_c(\mathbb{R}^n)$, then K_f is a rank-one operator. Using the Stone-Weierstraß theorem, we see that the linear span of elements of the form $f_1(x)\cdot f_2(y)$ is dense in $C_c(P\mathbb{R}^n)$ in the I-norm. Hence $C_r^*(P\mathbb{R}^n)$ is generated by elements of this form. Therefore, $K(C_r^*(P\mathbb{R}^n)) \subseteq \mathbb{K}(L^2(\mathbb{R}^n))$. Since $\mathbb{K}(L^2(\mathbb{R}^n))$ is generated by rank-one operators and every rank-one operator is in the image of K, we obtain $K(C_r^*(P\mathbb{R}^n)) \supseteq \mathbb{K}(L^2(\mathbb{R}^n))$. This shows, that K is a $*$-isomorphism. $\square$

2.4 The Transformation Groupoid of a Group Action

Let G be a locally compact group with left Haar measure μ and X a locally compact space together with a continuous action

$$G \times X \to X, (g,x) \mapsto gx.$$

The transformation groupoid $X \rtimes G$ has arrow set $X \times G$ and object set X. The structure maps are given by

$$\begin{aligned}s(x,g) &= x,\\ r(x,g) &= gx,\\ (hx,g)(x,h) &= (x,gh)\\ (x,g)^{-1} &= (gx,g^{-1}).\end{aligned}$$

If $x \in X$, then

$$(X \rtimes G)^x = \{(g^{-1}x,g)\colon g \in G\} \quad \text{and } (X \rtimes G)_x = \{(x,g)\colon g \in G\}.$$

We define a measure μ^x on $(X \rtimes G)^x$ by

$$\int_{(X \rtimes G)^x} f(g^{-1}x, g)\ \mathrm{d}\mu^x(g^{-1}x, g) = \int_G f(g^{-1}x, g)\ \mathrm{d}\mu(g) \quad \text{for all } f \in C_c(X \rtimes G)$$

The family $(\mu^x)_{x \in X}$ provides a left Haar system for $X \rtimes G$.

Example 2.5. The tangent bundle $T\mathbb{R}^n$ is $\mathbb{R}^n \rtimes \mathbb{R}^n$, where the group $\mathbb{R}^n$ acts trivially on the space $\mathbb{R}^n$.

The action of G on X induces a continuous action on the C^*-algebra $C_0(X)$ by

$$(g \cdot f)(x) = f(g^{-1}x) \quad \text{for all } f \in C_0(X).$$

We define a map

$$\Phi \colon C_c(G \ltimes X) \to C_c(G, C_0(X)) \quad \text{by } (\Phi(f)(g))(x) = f(g^{-1}x, g).$$

If $f \in C_c(X \rtimes G)$, then

$$\begin{aligned}
(\Phi(f^*)(g))(x) &= f^*(g^{-1}x, g) \\
&= \overline{f(x, g^{-1})} \\
&= \overline{(\Phi(f)(g^{-1}))(g^{-1}x)} \\
&= (g \cdot (\Phi(f)(g^{-1})))^*(x) \\
&= (\Phi(f)^*(g))(x)
\end{aligned}$$

If $f_1, f_2 \in C_c(G \ltimes X)$, then

$$\begin{aligned}
(\Phi(f_1 * f_2)(g))(x) &= (f_1 * f_2)(g^{-1}x, g) \\
&= \int_G f_1(h^{-1}x, h) \cdot f_2(g^{-1}x, h^{-1}g)\ \mathrm{d}\mu(h) \\
&= \left(\int_G \Phi(f_1)(h) \cdot (h \cdot (\Phi(f_2)(h^{-1}g)))\ \mathrm{d}\mu(h) \right)(x) \\
&= ((\Phi(f_1) * \Phi(f_2))(g))(x)
\end{aligned}$$

This shows that Φ is a $*$-homomorphism. By merging the regular representations of $X \rtimes G$ together to a representation on a Hilbert bundle and comparing to the representation of $C_c(G, C_0(X))$ one proves, that Φ extends to an isomorphism of C^*-algebras

$$C_r^*(G \ltimes X) \to C_r^*(G.C_0(X)).$$

2.5 The Tangent Groupoid of $\mathbb{R}^n$

Before we define the tangent groupoid of $\mathbb{R}^n$ we give the definition of a continuous bundle of C^*-algebras.

Definition 2.6 (Continuous Bundle of C^*-algebras)**.**
A *continuous bundle of* $C*$*-algebras* is a triple $(X, (\pi_t\colon A \to A_t)_{t\in X}, A)$, where A is a C^*-algebra, and for each $t \in X$, A_t is a C^*-algebra (called the *fibre* at $t \in X$) and $\pi_t\colon A \to A_x$ a surjective $*$-homomorphism, such that the following conditions are satisfied.

(i) $\|a\| = \sup_{t\in X} \|\pi_t(a)\|$.

(ii) For $f \in C_0(X)$ and $a \in A$, there is an element $f \cdot a \in A$, such that

$$\pi_t(f \cdot a) = f(t) \cdot \pi_t(a) \quad \text{for all } t \in X$$

.

(iii) The function $X \to [0, \infty)$ given by $t \mapsto \|\pi_t(a)\|$ belongs to $C_0(X)$ for all $a \in A$.

The group $\mathbb{R}^n$ acts continuously on $\mathbb{R}^n \times [0, \infty))$ by $x \cdot (y, t) = (y + tx, t)$. We obtain a continuous action of $\mathbb{R}^n$ on $C_0(\mathbb{R}^n \times [0, \infty)$ given by

$$x \cdot f(y, t) = f(y - tx, t) \quad \text{for } f \in C_0(\mathbb{R}^n \times [0, \infty)).$$

Let $t \in [0, \infty)$. We define $\mathbb{R}^n_t = \mathbb{R}^n \times \{t\} \subset \mathbb{R}^n \times [0, \infty)$. Then $\mathbb{R}^n_t$ is an $\mathbb{R}^n$-invariant closed subset. Thus the restriction $s_t\colon C_0(\mathbb{R}^n \times [0, \infty)) \to C_0(\mathbb{R}^n_t)$ to $\mathbb{R}^n_t$ is a surjective $\mathbb{R}^n$-equivariant $*$-homomorphism.
As in the consideration of 1.2 we obtain a surjective $*$-homomorphism

$$\tilde{s}_t\colon C^*_r(\mathbb{R}^n, C_0(\mathbb{R}^n \times [0, \infty))) \to C^*_r(\mathbb{R}^n, C_0(\mathbb{R}^n_t)) \quad \text{for } t \in \mathbb{R}.$$

The triple $([0, \infty), (s_t)_{t\in\mathbb{R}}, C_0(\mathbb{R}^n \times [0, \infty)))$ is a continuous bundle of C^*- algebras. The group $\mathbb{R}^n$ is exact. Therefore, [4, Theorem 4.2] implies, that

$$([0, \infty), (\tilde{s})_{t\in\mathbb{R}}, C^*_r(\mathbb{R}^n, C_0(\mathbb{R}^n \times [0, \infty)))$$

is again a continuous field of C^*-algebras.

Definition 2.7 (The Tangent Groupoid of $\mathbb{R}^n$)**.**
The transformation groupoid $\mathcal{G}\mathbb{R}^n\colon = (\mathbb{R}^n \times [0, \infty)) \rtimes \mathbb{R}^n$ is called the *tangent groupoid* of $\mathbb{R}^n$. It has the set of objects $\mathbb{R}^n \times [0, \infty)$ and the set of arrows $\mathbb{R}^n \times [0, \infty) \times \mathbb{R}^n$. The structure maps of $\mathcal{G}\mathbb{R}^n$ are given by

$$\begin{aligned} s(y, t, x) &= (y, t), \\ r(y, t, x) &= (y + tx, t), \\ (y + tx, t, z) \cdot (y, t, x) &= (y, t, z + x), \\ (y, t, x)^{-1} &= (y + tx, t, -x). \end{aligned}$$

Let $t \in [0, \infty)$. We define the subgroupoid

$$\mathcal{G}\mathbb{R}^n_t := \mathbb{R}^n_t \rtimes \mathbb{R}^n = \{(y, t, x)\colon y, x \in \mathbb{R}^n\} \subset \mathcal{G}\mathbb{R}^n.$$

We use the isomorphisms

$$C_r^*(\mathcal{G}\mathbb{R}^n) \cong C_r^*(\mathbb{R}^n, C_0(\mathbb{R}^n \times [0,\infty))) \quad \text{and} \quad C_r^*(\mathcal{G}\mathbb{R}^n_t) \cong (\mathbb{R}^n, C_0(\mathbb{R}^n_t))$$

to define a surjective $*$-homomorphism $\tau_t \colon C_r^*(\mathcal{G}\mathbb{R}^n) \to C_r^*(\mathcal{G}\mathbb{R}^n_t)$, such that the following diagram commutes

$$\begin{tikzcd}
C_r^*(\mathcal{G}\mathbb{R}^n) \arrow[r, dashed, "\tau_t"] \arrow[d, "\sim"] & C_r^*(\mathcal{G}\mathbb{R}^n_t) \arrow[d, "\sim"] \\
C_r^*(\mathbb{R}^n, C_0(\mathbb{R}^n \times [0,\infty))) \arrow[r, "\tilde{s}_t"] & C_r^*(\mathbb{R}^n, C_0(\mathbb{R}^n_t))
\end{tikzcd}$$

Since the vertical arrows are isomorphisms $([0,\infty), (\tau_t)_{t\in\mathbb{R}}, C_r^*(\mathcal{G}\mathbb{R}^n))$ is a continuous field of C^*-algebras.
If $f \in C_c(\mathcal{G}\mathbb{R}^n)$, then $\tau_t(f)$ is just the restriction of f to $\mathcal{G}\mathbb{R}^n_t$.
The map $T\mathbb{R}^n \to \mathcal{G}\mathbb{R}^n_0$ given by $(y,x) \mapsto (y,0,x)$ is a groupoid isomorphism compatible with the Haar-measures. By Theorem 2.3 we get isomorphisms

$$C_r^*(\mathcal{G}\mathbb{R}^n_0) \longrightarrow C_r^*(T\mathbb{R}^n) \xrightarrow{\mathcal{F}_1} C_0(\mathbb{R}^n \times \mathbb{R}^n)$$

of C^*-algebras.
We define $\pi_0 \colon C_r^*(\mathcal{G}\mathbb{R}^n) \to C_0(\mathbb{R}^n \times \mathbb{R}^n)$ as the isomorphism $C_r^*(\mathcal{G}\mathbb{R}^n_0) \to C_0(\mathbb{R}^n \times \mathbb{R}^n)$ composed with τ_0. Therefore, π_0 is a surjective $*$-homomorphism. For $f \in C_c(\mathcal{G}\mathbb{R}^n)$ we have

$$(\pi_0(f))(y,\xi) = \int_{\mathbb{R}^n} f(y,0,x) \cdot e^{-2\pi i \langle x,\xi\rangle} \,\mathrm{d}x.$$

Let $t > 0$. The map $P\mathbb{R}^n \to \mathcal{G}\mathbb{R}^n_t$ given by $(x,y) \mapsto (ty, t, x-y)$ is a groupoid isomorphism compatible with the Haar measures. By Theorem 2.4 we get isomorphisms

$$C_r^*(\mathcal{G}\mathbb{R}^n_t) \longrightarrow C_r^*(P\mathbb{R}^n) \xrightarrow{K} \mathbb{K}(L^2(\mathbb{R}^n))$$

of C^*-algebras.
We define $\pi_t \colon C_r^*(\mathcal{G}\mathbb{R}^n) \to \mathbb{K}(L^2(\mathbb{R}^n))$ as the isomorphism $C_r^*(\mathcal{G}\mathbb{R}^n_t) \to \mathbb{K}(L^2(\mathbb{R}^n))$ composed with τ_t. Therefore, π_t is a surjective $*$-homomorphism. For $f \in C_c(\mathcal{G}\mathbb{R}^n)$ we obtain

$$(\pi_t(f)h)(x) = \int_{\mathbb{R}^n} f(ty, t, x-y) \cdot h(y) \,\mathrm{d}y \quad \text{for all } h \in C_c(\mathbb{R}^n).$$

We changed the fibres of $([0,\infty), (\tau_t)_{t\in\mathbb{R}}, C_r^*(\mathcal{G}\mathbb{R}^n))$ in an isomorphic way, that is compatible with the τ_t. Therefore, $([0,\infty), (\pi_t)_{t\in\mathbb{R}}, C_r^*(\mathcal{G}\mathbb{R}^n))$ is still a continuous bundle.
Before we introduce the scaling action on the tangent groupoid, we will prove a lemma about continuous bundles of C^*-algebras, that will be useful later.

Lemma 2.8 (The Kernels of the Fibre Epimorphisms are Essential)**.**
Let $(X, (\pi_t \colon A \to A_t)_{t\in X}, A)$ be a continuous bundle of C^*-algebras. If $t \in X$ is not isolated, then $\ker(\pi_t)$ is an essential ideal of A.

Proof. Let $a \in A$, such that $a \cdot k = 0$ for all $k \in \ker(\pi_t)$. Let $s \in X$ with $s \neq t$. There is $f \in C_0(X)$, such that $f(t) = 0$ and $f(s) = 1$. We have $\pi_t(f \cdot a^*) = f(t)\pi_t(a^*) = 0$. Hence $f \cdot a^* \in \ker(\pi_t)$, so that $a \cdot (f \cdot a^*) = 0$. Hence

$$0 = \pi_s(a \cdot (f \cdot a^*)) = \pi_s(a) \cdot f(s)\pi_s(a^*) = \pi_s(a) \cdot \pi_s(a^*).$$

Using the C^*-condition in A_s, we obtain $\pi_s(a) = 0$.
Since t is not isolated and the function $s \mapsto \|\pi_s(a)\|$ is continuous, this implies $\pi_t(a) = 0$. Therefore $a \in \ker(\pi_t)$. Using the C^*-condition in A, we achieve $a = 0$. □

Since $[0, \infty)$ has no isolated points $\ker(\pi_t)$ is essential in $C_r^*(\mathcal{G}\mathbb{R}^n)$ for all $t \in [0.\infty)$.

3 The Scaling Action on the Tangent Groupoid and its Generalized Fixed Point Algebra

3.1 The Scaling Action

The multiplicative group $\mathbb{R}_+^*$ of positive real numbers is a locally compact group with Haar measure $A \mapsto \int_A \frac{\mathrm{d}\lambda}{\lambda}$ for all Borel sets $A \subseteq \mathbb{R}_+^*$. If $\lambda \in \mathbb{R}_+^*$ and $f \in C_c(\mathcal{G}\mathbb{R}^n)$, then we define

$$(\sigma_\lambda(f))(y, t, x) = \lambda^n \cdot f(y, \lambda^{-1}t, \lambda x).$$

We obtain $\sigma_\lambda(f) \in C_c(\mathcal{G}\mathbb{R}^n)$. It is easy to check, that $(\sigma_\lambda)_{\lambda \in \mathbb{R}_+^*}$ is a linear action of $\mathbb{R}_+^*$ on $C_c(\mathcal{G}\mathbb{R}^n)$. If $f_1, f_2 \in C_c(\mathcal{G}\mathbb{R}^n)$, then

$$\begin{aligned}
(\sigma_\lambda(f_1 * f_2))(y, t, x) &= \lambda^n \cdot (f_1 * f_2)(y, \lambda^{-1}t, \lambda x) \\
&= \lambda^n \int_{\mathbb{R}^n} f_1(y - \lambda^{-1}tz, \lambda^{-1}t, \lambda x + z) \cdot f_2(y, \lambda^{-1}t, -z)\ \mathrm{d}z \\
&= \int_{\mathbb{R}^n} \lambda^n f_1(y - tz, \lambda^{-1}t, \lambda x - \lambda z) \cdot \lambda^n f_2(y, \lambda^{-1}t, -\lambda z)\ \mathrm{d}z \\
&= \int_{\mathbb{R}^n} \sigma_\lambda(f_1)(y - tz, t, x + z) \cdot \sigma_\lambda(f_2)(y, t, -z)\ \mathrm{d}z \\
&= (\sigma_\lambda(f_1) * \sigma_\lambda(f_2))(x, y, t).
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\sigma_\lambda(f^*)(y, t, x) &= \lambda^n f^*(y, \lambda^{-1}t, \lambda x) \\
&= \overline{\lambda^n f(y + tx, \lambda^{-1}t, -\lambda x)} \\
&= \overline{(\sigma_\lambda f)(y + tx, t, -x} \\
&= (\sigma_\lambda(f))^*(y, t, x).
\end{aligned}$$

The function $\mathbb{R}_+^* \to C_c(\mathcal{G}\mathbb{R}^n)$, $\lambda \mapsto \sigma_\lambda(f)$ is continuous with respect to the I-norm on $C_c(\mathcal{G}\mathbb{R}^n)$. Since the I-norm dominates the reduced norm the action $(\sigma_\lambda)_{\lambda \in \mathbb{R}_+^*}$ extends uniquely to a continuous action on $C_r^*(\mathcal{G}\mathbb{R}^n)$. We denote this extended action again by

$(\sigma_\lambda)_{\lambda\in\mathbb{R}_+^*}$. At this point we also prove that σ_λ is isometric with respect to the I-norm, because we need this later. Let $f \in C_c(\mathcal{G}\mathbb{R}^n)$. For $\|\cdot\|_{I,s}$ we compute

$$\begin{aligned}
\|\sigma_\lambda(f)\|_{I,s} &= \sup_{(y,t)} \int_{\mathbb{R}^n} |(\sigma_\lambda(f))(y,t,x)| \, \mathrm{d}x \\
&= \sup_{(y,t)} \int_{\mathbb{R}^n} \lambda^n |f(y,\lambda^{-1}t,\lambda x)| \, \mathrm{d}x \\
&= \sup_{(y,t)} \int_{\mathbb{R}^n} |f(y,\lambda^{-1}t,x)| \, \mathrm{d}x \\
&= \|f\|_{I,s}.
\end{aligned}$$

This implies

$$\|\sigma_\lambda(f)\|_{I,r} = \|\sigma_\lambda(f^*)\|_{I,s} = \|f^*\|_{I,s} = \|f\|_{I,r}.$$

Hence $\|\sigma_\lambda(f)\|_I = \|f\|_I$.

The Scaling Action and the Fibre Epimorphisms

We define a continuous action of $\mathbb{R}_+^*$ on $\mathbb{R}^n \times \mathbb{R}^n$ by $\sigma_\lambda(y,\xi) = (y,\lambda\xi)$. This action induces a continuous action of $\mathbb{R}_+^*$ on $C_0(\mathbb{R}^n \times \mathbb{R}^n)$ by $(\sigma_\lambda(f))(y,\xi) = f(y,\lambda^{-1}\xi)$.

Proposition 3.1 (π_0 is $\mathbb{R}_+^*$-equivariant)**.**
The $*$-homomorphism $\pi_0\colon C_r^*(\mathcal{G}\mathbb{R}^n) \to C_0(\mathbb{R}^n \times \mathbb{R}^n)$ is $\mathbb{R}_+^*$-equivariant.

Proof. Let $f \in C_c(\mathcal{G}\mathbb{R}^n)$ and $\lambda \in \mathbb{R}_+^*$. We compute

$$\begin{aligned}
(\pi_0(\sigma_\lambda(f)))(y,\xi) &= \int_{\mathbb{R}^n} (\sigma_\lambda(f))(y,0,x) \cdot e^{-2\pi i\langle x,\xi\rangle} \, \mathrm{d}x \\
&= \int_{\mathbb{R}^n} \lambda^n f(y,0,\lambda x) \cdot e^{-2\pi i\langle x,\xi\rangle} \, \mathrm{d}z \\
&= \int_{\mathbb{R}^n} f(y,x) \cdot e^{-2\pi i\langle x,\lambda^{-1}\xi\rangle} \, \mathrm{d}z \\
&= \pi_0(f)(y,\lambda^{-1}\xi) = (\sigma_\lambda(\pi_0(f)))(y,\xi).
\end{aligned}$$

Since $C_c(\mathcal{G}\mathbb{R}^n)$ is dense in $C_r^*(\mathcal{G}\mathbb{R}^n)$ this implies that π_0 is $\mathbb{R}_+^*$-equivariant. □

Let $\lambda \in \mathbb{R}_+^*$ and $h \in C_c(\mathbb{R}^n)$. We define $(U_\lambda h)(x) = \lambda^{-n/2} h(\lambda^{-1}x)$. Then $U_\lambda h \in C_c(\mathbb{R}^n)$. We have

$$\langle U_\lambda h_1, U_\lambda h_2\rangle = \int_{\mathbb{R}^n} \lambda^{-n} \cdot \overline{h_1(\lambda^{-1}x)} \cdot h_2(\lambda^{-1}x) \, \mathrm{d}x = \langle h_1, h_2\rangle.$$

The family $(U_\lambda)_{\lambda\in\mathbb{R}_+^*}$ fulfils $U_{\lambda_1} \circ U_{\lambda_2} = U_{\lambda_1\lambda_2}$. Therefore U_λ extends to a unitary operator $L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$. Hence $(U_\lambda)_{\lambda\in\mathbb{R}_+^*}$ defines a unitary action of $\mathbb{R}_+^*$ on $L^2(\mathbb{R}^n)$. Let $U \in \mathbb{B}(L^2(\mathbb{R}^n))$ be a unitary. If $T \in \mathbb{K}(L^2(\mathbb{R}^n))$, we define $\mathrm{Ad}_U(T) = U^* \circ T \circ U$. Then $\mathrm{Ad}_U\colon \mathbb{B}(L^2(\mathbb{R}^n)) \to \mathbb{B}(L^2(\mathbb{R}^n))$ is a $*$-automorphism.

Proposition 3.2. *Let $t > 0$. If $\lambda \in \mathbb{R}^*_+$, then*

$$\pi_t(\sigma_\lambda f) = U^*_\lambda \circ \pi_{t/\lambda}(f) \circ U_\lambda \quad \textit{for all } f \in C^*_r(\mathcal{G}\mathbb{R}^n).$$

Hence $\pi_t \circ \sigma_\lambda = \mathrm{Ad}_{U_\lambda} \circ \pi_{t/\lambda}$.

Proof. Let $f \in C_c\mathcal{G}\mathbb{R}^n$. We compute

$$\begin{aligned}
((\pi_t(\sigma_\lambda f))h)(x) &= \int_{\mathbb{R}^n} (\sigma_\lambda f)(ty, t, x-y) \cdot h(y) \, \mathrm{d}y \\
&= \int_{\mathbb{R}^n} \lambda^n \cdot f(ty, \lambda^{-1}t, \lambda x - \lambda y) \cdot h(y) \, \mathrm{d}y \\
&= \int_{\mathbb{R}^n} \lambda^{n/2} \cdot f(\lambda^{-1}ty, \lambda^{-1}t, \lambda x - y) \cdot (U_\lambda h)(y) \, \mathrm{d}y \\
&= \lambda^{n/2} \cdot ((\pi_{t/\lambda}(f) \circ U_\lambda)h)(\lambda \cdot x) \\
&= ((U^*_\lambda \circ \pi_{t/\lambda}(f) \circ U_\lambda)h)(x)
\end{aligned}$$

Since $C_c(\mathcal{G}\mathbb{R}^n)$ is dense in $C^*_r(\mathcal{G}\mathbb{R}^n)$ this implies the assertion for all $f \in C^*_r(\mathcal{G}\mathbb{R}^n)$. □

The $\mathbb{R}^*_+$-invariant Ideal $J \lhd C^*_r(\mathcal{G}\mathbb{R}^n)$

Since $\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$ is open in $\mathbb{R}^n \times \mathbb{R}^n$, we may view

$$C_0(\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})) = \{f \in C_0(\mathbb{R}^n \times \mathbb{R}^n) \colon f(y,0) = 0 \text{ for all } y \in \mathbb{R}^n\} \subseteq C_0(\mathbb{R}^n \times \mathbb{R}^n).$$

Then $C_0(\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}))$ is a closed $\mathbb{R}^*_+$-invariant ideal in $C_0(\mathbb{R}^n \times \mathbb{R}^n)$. Therefore $J := \pi_0^{-1}(C_0(\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}))$ is a closed $\mathbb{R}^*_+$-invariant ideal in $C^*_r(\mathcal{G}\mathbb{R}^n)$. In particular J is a $\mathbb{R}^*_+$-C^*-algebra. Our aim is to prove that J is continuously square-integrable.
Let $f \in C_c(\mathcal{G}\mathbb{R}^n)$. If $y \in \mathbb{R}^n$, then

$$\pi_0(f)(y,0) = \int_{\mathcal{R}^n} f(y,0,x) \, \mathrm{d}x.$$

Hence $f \in J$ if and only if

$$\int_{\mathbb{R}^n} f(y,0,x) \, \mathrm{d}x = 0 \quad \text{for all } y \in \mathbb{R}^n.$$

Let $\mathcal{R}_0$ be the set of smooth functions $f \colon \mathcal{G}\mathbb{R}^n \to \mathbb{C}$ with compact support and

$$\int_{\mathbb{R}^n} f(y,0,x) \, \mathrm{d}x = 0 \quad \text{for all } y \in \mathbb{R}^n.$$

Then $\mathcal{R}_0$ is a $\mathbb{R}^*_+$-invariant linear subspace of J. The Leibniz integral rule yields $\mathcal{R}_0 * \mathcal{R}_0 \subseteq \mathcal{R}_0$. Moreover, we have $\mathcal{R}_0^* = \mathcal{R}_0$.

Proposition 3.3 ($\mathcal{R}_0$ is dense)**.**
$\mathcal{R}_0$ is a dense subspace of J.

Proof. Let $f \in J$ and $\varepsilon > 0$. Then $f \in C_r^*(\mathcal{G}\mathbb{R}^n)$. There is $f_1 \in C_c(\mathcal{G}\mathbb{R}^n)$ with $\|f - f_1\|_r < \frac{\varepsilon}{4}$. There is $R > 0$, such that $f_1(y,t,x) = 0$, if $\|y\| \geq R$ or $t \geq R$ or $\|x\| \geq R$. We write vol for the Lebesgue measure on $\mathbb{R}^n$. By the Stone-Weierstrass theorem, there is a smooth function $h \in C_0(\mathbb{R}^n \times \mathbb{R}^n \times [0,\infty))$, such that

$$\|f_1 - h\|_\infty < \frac{\varepsilon}{4 \cdot \mathrm{vol}(\{\|x\| < R+1\})}.$$

Let $\theta \colon \mathbb{R}^n \times \mathbb{R}^n \times [0,\infty) \to [0,1]$ be smooth, such that

$$\begin{aligned} \theta(x,y,t) &= 1 \quad \text{if } \|y\|, t, \|x\| \leq R \text{ and} \\ \theta(x,y,t) &= 0 \quad \text{if } \|y\| \geq R+1 \text{ or } t \geq R+1 \text{ or } \|x\| \geq R+1. \end{aligned}$$

Then $f_2\colon = \theta \cdot h$ is smooth and compactly supported.
Let $(y,t) \in \mathbb{R}^n \times [0,\infty)$. We estimate

$$\int_{\mathbb{R}^n} |f_1(y,t,x) - f_2(y,t,x)| \ \mathrm{d}x \leq \frac{\varepsilon}{4 \cdot \mathrm{vol}(\{\|x\| < R+1\})} \cdot \int_{\|x\| \leq R+1} 1 \ \mathrm{d}x = \frac{\varepsilon}{4}.$$

Therefore, $\|f_1 - f_2\|_{I,s} \leq \frac{\varepsilon}{4}$. Likewise

$$\int_{\mathbb{R}^n} |f_1(y-tx,t,x) - f_2(y-tx,t,x)| \ \mathrm{d}x \leq \frac{\varepsilon}{4}.$$

Hence $\|f_1 - f_2\|_{I,r} \leq \frac{\varepsilon}{4}$. This implies

$$\|f - f_2\|_r \leq \|f - f_1\|_r + \|f_1 - f_2\|_r < \frac{\varepsilon}{4} + \|f_1 - f_2\|_I \leq \frac{\varepsilon}{2}.$$

We define $g(y) = (\pi_0(f_2))(y,0) = \int_{\mathbb{R}^n} f_2(y,0,x) \ \mathrm{d}x$. Then g is smooth and compactly supported. Since $f \in J$, we have

$$|g(y)| = |((\pi_0(f))(y,0) - (\pi_0(f_2))(y,0)| \leq \|\pi_0(f) - \pi_0(f_2)\|_\infty \leq \|f - f_2\|_r < \frac{\varepsilon}{2}.$$

Let $h \in C_c(\mathbb{R}^n)$ be smooth, such that $\int_{\mathbb{R}^n} h(x) \ \mathrm{d}x = 1$ and $h \geq 0$. Let $\omega \in C_c([0,\infty))$ be smooth, such that $\omega(0) = 1$ and $\|\omega\|_\infty = 1$. We define $k(x,y,t) = g(y) \cdot \omega(t) \cdot h(x)$. Then k is smooth and $k \in C_c(\mathcal{G}\mathbb{R}^n)$.
Therefore, $f_3\colon = f_2 - k \in C_c(\mathcal{G}\mathbb{R}^n)$ is smooth. If $(y,t) \in \mathbb{R}^n \times [0,\infty)$, then

$$\int_{\mathbb{R}^n} |k(y,t,x)| \ \mathrm{d}x = |g(y)| \cdot |\omega(t)| \cdot \int_{\mathbb{R}^n} h(x) \ \mathrm{d}x \leq \frac{\varepsilon}{2}$$

and

$$\int_{\mathbb{R}^n} |k(y-tx,t,x)| \ \mathrm{d}x = |\omega(t)| \cdot \int_{\mathbb{R}^n} |g(y+tx)| \cdot h(x) \ \mathrm{d}x \leq \frac{\varepsilon}{2} \int_{\mathbb{R}^n} h(x) \ \mathrm{d}x = \frac{\varepsilon}{2}.$$

Hence $\|k\|_I \leq \frac{\varepsilon}{2}$.
If $y \in \mathbb{R}^n$, then

$$\int_{\mathbb{R}^n} f_3(y,0,x) \ \mathrm{d}x = \int_{\mathbb{R}^n} f_2(y,0,x) \ \mathrm{d}x - g(y) \cdot \omega(0) \cdot \int_{\mathbb{R}^n} \xi(x) \ \mathrm{d}x = 0.$$

Therefore, $f_3 \in \mathcal{R}_0$. Finally we estimate

$$\|f - f_3\|_r \leq \|f - f_2\|_r + \|f_2 - f_3\|_r < \frac{\varepsilon}{2} + \|k\|_I \leq \varepsilon.$$

This proves that $\mathcal{R}_0$ is dense in J. $\square$

The following lemma is the main ingredient to prove, that J is a continuously square-integrable $\mathbb{R}_+^*$-C^*-algebra.

Lemma 3.4 (Main Estimation)**.**
If $f_1, f_2 \in \mathcal{R}$, then

$$\int_{\mathbb{R}_+^*} \|f_1^* * \sigma_\lambda(f_2)\|_I \, \frac{d\lambda}{\lambda} < \infty.$$

Proof. If $(y, t, x) \in \mathcal{G}\mathbb{R}^n$, then

$$\begin{aligned}
(f_1^* * \sigma_\lambda(f_2))(y, t, x) &= \int_{\mathbb{R}^n} f_1^*(y - tz, t, x + z) \cdot (\sigma_\lambda(f_2))(y, t, -z) \, \mathrm{d}z \\
&= \lambda^n \int_{\mathbb{R}^n} \overline{f_1(y + tx, t, -x - z)} \cdot f_2(y, \lambda^{-1}t, -\lambda z) \, \mathrm{d}z \\
&= \lambda^n \int_{\mathbb{R}^n} \overline{f_1(y + tx, t, z - x)} \cdot f_2(y, \lambda^{-1}t, \lambda z) \, \mathrm{d}z
\end{aligned}$$

Since f_1 and f_2 are compactly supported and continuous, there is $R > 0$, such that $f_1(y, t, x) = f_2(y, t, x) = 0$ whenever $\|y\| \geq R$ or $t \geq R$ or $\|x\| \geq R$. There is $M > 0$, such that $|f_1(y, t, x)| \leq M$ and $f_2(y, t, x) \leq M$ for all $(y, t, x) \in \mathcal{G}\mathbb{R}^n$.
If $t \geq R$, then $(f_1^* * \sigma_\lambda(f_2))(y, t, x) = 0$ for all $\lambda \in \mathbb{R}_+^*$ and $y, x \in \mathbb{R}^n$.
Fix $\lambda \geq 1$. If $\|x\| \geq 2R$, then $\|z - x\| \geq \|x\| - \|z\| \geq 2R - \frac{R}{\lambda} \geq R$ for all $z \in \mathbb{R}^n$ with $\|\lambda z\| < R$. Hence $(f_1^* * \sigma_\lambda(f_2))(y, t, x) = 0$ for all $y \in \mathbb{R}^n$ and $t \in [0, \infty)$.
Using the Mean value theorem for real and imaginary part separately we obtain $r_2 \colon \mathbb{R}^n \times [0, \infty) \times \mathbb{R}^n \to \mathbb{C}$ and $C_2 > 0$ such that

$$f_2(y, t, x) = f_2(y, 0, x) + r_2(y, t, x) \quad \text{with } |r_2(y, t, x)| \leq C_2 \cdot t$$

for all $(y, t, x) \in \mathcal{G}\mathbb{R}^n$. If $\|x\| \geq R$, then $r_2(y, t, x) = 0$.
By the Mean value theorem in multiple variables there are $r_1 \colon \mathbb{R}^n \times [0, \infty) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ and $C_1 > 0$, such that

$$f_1(y, t, x + h) = f_1(y, t, x) + r_1(y, t, x, h) \quad \text{with } |r_1(y, t, x, h)| \leq C_1 \|h\|$$

for all $(y, t, x, h) \in \mathbb{R}^n \times [0, \infty) \times \mathbb{R}^n \times \mathbb{R}^n$.
Since $f_2 \in \mathcal{R}_0$ we have

$$\lambda^n \int_{\mathbb{R}^n} \overline{f_1(y + tx, t, -x)} \cdot f_2(y, 0, \lambda z,) \, \mathrm{d}z = \overline{f_1(y + tx, t, -x)} \cdot \int_{\mathbb{R}^n} f_2(y, 0, z) \, \mathrm{d}z = 0.$$

Therefore,

$$\begin{aligned}
\left(f_1^* * \sigma_\lambda(f_2)\right)(y,t,x) &= \lambda^n \int_{\mathbb{R}^n} \overline{f_1(y+tx,t,z-x)} \cdot f_2(y,0,\lambda z) \ \mathrm{d}z \\
&\quad + \lambda^n \int_{\mathbb{R}^n} \overline{f_1(y+tx,t,z-x)} \cdot r_2(y,\lambda^{-1}t,\lambda z) \ \mathrm{d}z \\
&= \lambda^n \int_{\|z\|\le \frac{R}{\lambda}} \overline{r_1(y,t,-x,z)} \cdot f_2(y,0,\lambda z) \ \mathrm{d}z \\
&\quad + \lambda^n \int_{\|z\|\le \frac{R}{\lambda}} \overline{f_1(y+tx,t,z-x)} \cdot r_2(y,\lambda^{-1}t,\lambda z) \ \mathrm{d}z
\end{aligned}$$

Let $(y,t,x) \in \mathcal{G}\mathbb{R}^n$. We estimate

$$\begin{aligned}
\left|\left(f_1^* * \sigma_\lambda(f_2)\right)(y,t,x)\right| &= \left|\left(f_1^* * \sigma_\lambda(f_2)\right)(y,t,x)\right| \cdot \chi_{\{x\in\mathbb{R}^n :\ \|x\|\le 2R\}}(x) \\
&\le \lambda^n M \left(\int_{\|z\|\le \frac{R}{\lambda}} C_1 \cdot \|z\| + C_2 \lambda^{-1} t \ \mathrm{d}z \right) \cdot \chi_{\{x\in\mathbb{R}^n :\ \|x\|\le 2R\}}(x) \\
&\le \lambda^{n-1} M R (C_1 + C_2) \cdot \mathrm{vol}\left(\{\|z\| \le \lambda^{-1} R\}\right) \cdot \chi_{\{x\in\mathbb{R}^n :\ \|x\|\le 2R\}}(x) \\
&\le C_3 \lambda^{-1} \cdot \chi_{\{x\in\mathbb{R}^n :\ \|x\|\le 2R\}}(x),
\end{aligned}$$

where $C_3 = MR(C_1+C_2) \cdot \mathrm{vol}\left(\{\|z\| \le R\}\right)$.
Hence

$$\begin{aligned}
\|f_1^* * \sigma_\lambda(f_2)\|_{I,s} &= \sup_{(y,t)} \left(\int_{\mathbb{R}^n} \left|(f_1^* * \sigma_\lambda(f_2))(y,t,x)\right| \ \mathrm{d}x \right) \\
&\le C_3 \lambda^{-1} \cdot \mathrm{vol}\left(\{\|x\| \le 2R\}\right)
\end{aligned}$$

and likewise

$$\begin{aligned}
\|f_1^* * \sigma_\lambda(f_2)\|_{I,r} &= \sup_{(y,t)} \left(\int_{\mathbb{R}^n} \left|(f_1^* * \sigma_\lambda(f_2))(y-tx,t,x)\right| \ \mathrm{d}x \right) \\
&\le C_3 \lambda^{-1} \cdot \mathrm{vol}\left(\{\|x\| \le 2R\}\right)
\end{aligned}$$

Therefore,

$$\int_{[1,\infty)} \|f_1^* * \sigma_\lambda(f_2)\|_I \ \frac{\mathrm{d}\lambda}{\lambda} \le C_3 \cdot \mathrm{vol}\left(\{\|x\| \le 2R\}\right) \cdot \int_{[1,\infty)} \lambda^{-2} \ \mathrm{d}\lambda < \infty.$$

We analyse $\lambda \in (0,1)$ using the following trick:

$$\begin{aligned}
\int_{(0,1)} \|f_1^* * \sigma_\lambda(f_2)\|_I \ \frac{\mathrm{d}\lambda}{\lambda} &= \int_{(1,\infty)} \|f_1^* * \sigma_{\lambda^{-1}}(f_2)\|_I \ \frac{\mathrm{d}\lambda}{\lambda} \\
&= \int_{(1,\infty)} \|\sigma_\lambda(f_1^*) * f_2\|_I \ \frac{\mathrm{d}\lambda}{\lambda} \\
&= \int_{(1,\infty)} \|f_2^* * \sigma_\lambda(f_1)\|_I \ \frac{\mathrm{d}\lambda}{\lambda} < \infty
\end{aligned}$$

Finally,

$$\int_{\mathbb{R}_+^*} \|f_1^* * \sigma_\lambda(f_2)\|_I \ \frac{\mathrm{d}\lambda}{\lambda} = \int_{(0,1)} \|f_1^* * \sigma_\lambda(f_2)\|_I \ \frac{\mathrm{d}\lambda}{\lambda} + \int_{[1,\infty)} \|f_1^* * \sigma_\lambda(f_2)\|_I \ \frac{\mathrm{d}\lambda}{\lambda} < \infty.$$

□

Theorem 3.5 (J is a Continuously Square-Integrable $\mathbb{R}_+^*$-C^*-Algebra)**.**
The $\|\cdot\|_{si}$-closure $\mathcal{R}$ of $\mathcal{R}_0 \subset J_{si}$ is a dense, complete and relatively continuous subspace of J. Therefore, $(J, \mathcal{R})$ is a continuously square-integrable $\mathbb{R}_+^*$-C^*-algebra.

Proof. By Lemma 3.4 we obtain

$$\int_{\mathbb{R}_+^*} \|f_1^* * \sigma_\lambda(f_2)\| \ \frac{\mathrm{d}\lambda}{\lambda} \leq \int_{\mathbb{R}_+^*} \|f_1^* * \sigma_\lambda(f_2)\|_I \ \frac{\mathrm{d}\lambda}{\lambda} < \infty$$

for all $f_1, f_2 \in \mathcal{R}_0$.
By Proposition 3.3 $\mathcal{R}_0$ is dense in J. Therefore, Corollary 1.54 yields $\mathcal{R}_0 \subseteq J_{si}$ and $\mathcal{R}_0$ is relatively continuous. By Lemma 1.55 $\mathcal{R}$ is a dense, complete and relatively continuous subspace of J. □

3.2 The Generalized Fixed Point Algebra

Since $(J, \mathcal{R})$ is continuously square-integrable we obtain the generalized fixed point algebra $\mathrm{Fix}_\mathcal{R}$ and the Hilbert $\mathrm{Fix}_\mathcal{R}$-$C_r^*(\mathbb{R}_+^*, J)$ bimodule $\mathcal{E}_\mathcal{R}$ as in Section 1.3. The following technical lemma provides functions of the form $f_1 * f_2^*$ with separated variables. We write $C_c^\infty(\mathbb{R}^n)$ and $C_c^\infty([0,\infty))$ for the vector spaces of smooth, compactly supported $\mathbb{C}$-valued functions on $[0,\infty)$ and $\mathbb{R}^n$ respectively.

Lemma 3.6 (Functions of a Spezial Form)**.**
Let $\omega \in C_c^\infty([0,\infty))$ and $k_1, k_2, h \in C_c^\infty(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} k_1(x)\ dx = \int_{\mathbb{R}^n} k_2(x)\ dx = 0$. Then there are $f_1, f_2 \in \mathcal{R}_0$, such that

$$(f_1 * f_2^*)(y,t,x) = h(y+tx) \cdot \omega(t) \cdot (k_1 * k_2^*)(x).$$

Proof. Define $f_1(y,t,x) = h(y+tx) \cdot \omega(t) \cdot k_1(x)$. Then f_1 is smooth, compactly supported and

$$\int_{\mathbb{R}^n} f_1(y,0,x)\ \mathrm{d}x = h(y) \cdot \omega(t) \cdot \int_{\mathbb{R}^n} k_1(x)\ \mathrm{d}x = 0.$$

Therefore, $f_1 \in \mathcal{R}_0$.
There is $R > 0$, such that

$$\begin{aligned} \omega(t) &= 0 \quad \text{for} \quad t > R \text{ and} \\ h(y) &= 0 \quad \text{for} \quad \|y\| > R \text{ and} \\ (k_1 * k_2^*)(x) &= 0 \quad \text{for} \quad \|x\| > R. \end{aligned}$$

Let $h_2 \in C_c^\infty(\mathbb{R}^n)$, such that $h_2(y) = 1$ for all $y \in \mathbb{R}^n$ with $\|y\| \leq R + R^2$.
Let $\omega_2 \in C_c^\infty([0,\infty))$, such that $\omega_2|_{\mathrm{supp}(\omega)} = 1$. Define $f_2(y,t,x) = h_2(y+tx)\cdot\omega_2(t)\cdot k_2(x)$. Then $f_2 \in \mathcal{R}_0$ as above. We compute

$$\begin{aligned}(f_1 * f_2^*)(y,t,x) &= \int_{\mathbb{R}^n} f_1(y-tz,t,x+z)\cdot\overline{f_2(y-tz,t,z)}\,\mathrm{d}z\\ &= h(y+tx)\cdot\overline{h_2(y)}\cdot\omega(t)\cdot\overline{\omega_2(t)}\cdot\int_{\mathbb{R}^n} k_1(x+z)\cdot\overline{k_2(z)}\,\mathrm{d}z\\ &= h(y)\cdot\omega(t)\cdot\int_{\mathbb{R}^n} k_1(x+z)\cdot k_2^*(-z)\,\mathrm{d}z\\ &= h(y)\cdot\omega(t)\cdot(k_1 * k_2^*)(x).\end{aligned}$$

The second last equation holds, because $\omega(t)\cdot\overline{\omega_2(t)} = \omega(t)$ for all $t \in [0,\infty)$. If $\|y+tx\| \leq R$, $t \leq R$ and $\|x\| \leq R$, then $\|y\| \leq R + R^2$, hence $\overline{h_2(y+tx)} = 1$. ☐

The Epimorphism $\pi_0^r\colon \mathrm{Fix}_{\mathcal{R}} \to C_0(\mathbb{R}^n \times \mathbb{S}^{n-1})$

The $\mathbb{R}_+^*$-invariant surjective $*$-homomorphism $\pi_0\colon J \to C_0(\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}))$ extends uniquely to a strictly continuous $*$-homomorphism

$$\mathcal{M}(J) \to \mathcal{M}(C_0(\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}))) = C_b(\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})).$$

This $*$-homomorphism restricts to a $*$-homomorphism $\pi_0\colon \mathrm{Fix}_{\mathcal{R}} \to C_b(\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}))$.[3]
Let $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n \colon \|x\| = 1\}$ be the unit sphere. By restriction to the closed subset $\mathbb{R}^n \times \mathbb{S}^{n-1} \subset \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$, we obtain a $*-$homomorphism

$$\mathrm{res}\colon C_b(\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})) \to C_b(\mathbb{R}^n \times \mathbb{S}^{n-1}) \quad \text{given by } f \mapsto f|_{\mathbb{R}^n \times \mathbb{S}^{n-1}}.$$

We define $\pi_0^r := \mathrm{res} \circ \pi_0 \colon \mathrm{Fix}_{\mathcal{R}} \to C_b(\mathbb{R}^n \times \mathbb{S}^{n-1})$.
Let $(\chi_k)_{k\in\mathbb{N}} \subseteq C_c(\mathbb{R}_+^*)$ be a sequence, such that $0 \leq \chi_k \leq 1$ and $\chi_k \leq \chi_{k+1}$ for all $k \in \mathbb{N}$ and $\chi_k \longrightarrow 1$ uniformly on compact subsets.

Theorem 3.7 (The $*$-Homomorphism π_0^r).
If $F \in \mathrm{Fix}_{\mathcal{R}}$, then $\pi_0^r \in C_0(\mathbb{R}^n \times \mathbb{S}^{n-1})$. The $*$-homomorphism $\pi_0^r\colon \mathrm{Fix}_{\mathcal{R}} \to C_0(\mathbb{R}^n \times \mathbb{S}^{n-1})$ is surjective.

Proof. Let $f_1, f_2 \in \mathcal{R}_0$. By the definition of π_0 there is $R > 0$, such that $R > 0$, such that $\pi_0(f_1)(y,\xi) = 0$ for all $(y,\xi) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$ with $\|y\| \geq R$.
In view of Lemma 1.36, we define

$$T_k = \int_{\mathbb{R}_+^*} \chi_k(\lambda)\sigma_\lambda(f_1 * f_2^*)\,\frac{\mathrm{d}\lambda}{\lambda} \in J.$$

[3]This is a slight abuse of notion. It will be clear from context if we talk about

$$\pi_0\colon J \to C_0(\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})) \quad \text{or} \quad \pi_0\colon \mathrm{Fix}_{\mathcal{R}} \to C_b(\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})).$$

Since the evaluation homomorphisms are continuous, we can compute

$$\begin{aligned}\pi_0(T_k)(y,\xi) &= \int_{\mathbb{R}_+^*} \chi_k(\lambda)\cdot \pi_0(f_1 * f_2^*)(y,\lambda^{-1}\xi)\ \frac{\mathrm{d}\lambda}{\lambda}\\ &= \int_{\mathbb{R}_+^*} \chi_k(\lambda)\cdot \pi_0(f_1)(y,\lambda^{-1}\xi)\cdot \overline{\pi_0(f_2)(y,\lambda^{-1}\xi)}\ \frac{\mathrm{d}\lambda}{\lambda}\end{aligned}$$

If $(y,\xi)\in\mathbb{R}^n\times(\mathbb{R}^n\setminus\{0\})$ with $\|y\|\geq R$, then $\pi_0(T_k)(y,\xi)=0$ for all $k\in\mathbb{N}$. By Lemma 1.36 the sequence $(T_k)_{k\in N}$ converges strictly to $\Lambda_{f_1}\circ\Lambda_{f_2}^*$. Since the extension of π_0 to the multiplier algebras is strictly continuous, we obtain

$$\pi_0(\Lambda_{f_1}\circ\Lambda_{f_2}^*) = \overset{s}{\lim_{k\to\infty}}\pi_0(T_k).$$

Since the strict topology on $C_b(\mathbb{R}^n\times(\mathbb{R}^n\setminus\{0\}))$ is the topology of uniform convergence on compact subsets, this implies

$$\pi_0^r(\Lambda_{f_1}\circ\Lambda_{f_2}^*)(y,\xi) = \lim_{k\to\infty}\pi_0(T_k)(y,\xi)=0$$

for all $(y,\xi)\in\mathbb{R}^n\times\mathbb{S}^{n-1}$ with $\|y\|\geq R$. Therefore

$$\pi_0^r(\Lambda_{f_1}\circ\Lambda_{f_2}^*)\in C_c(\mathbb{R}^n\times\mathbb{S}^{n-1})\subset C_0(\mathbb{R}^n\times\mathbb{S}^{n-1}).$$

Since $\mathrm{Fix}_{\mathcal{R}}$ is spanned by elements of the form $\Lambda_{f_1}\circ\Lambda_{f_2}^*$ this shows $\pi_0^r(F)\in C_0(\mathbb{R}^n\times\mathbb{S}^{n-1})$ for all $F\in\mathrm{Fix}_{\mathcal{R}}$.
To prove that π_0^r is surjective, we use the Stone-Weierstraß theorem.
Let $A:=\pi_0^r(\mathrm{Fix}_{\mathcal{R}})\subseteq C_0(\mathbb{R}^n\times\mathbb{S}^{n-1})$. Let $h_1\in C_c(\mathbb{R}^n)$. Then $h(y,\xi):=h_1(y)$ defines an element of $C_0(\mathbb{R}^n\times\mathbb{S}^{n-1})$. We want to show $h\in A$.
Let $g\in C_c^\infty([0,\infty))$, such that $g\neq 0$ and $\int_0^\infty g(r)r^{n-1}\ \mathrm{d}r=0$. Put $f(x)=g(\|x\|)$. Then $f\in C_c^\infty(\mathbb{R}^n)$ and

$$\int_{\mathbb{R}^n} f(x)\ \mathrm{d}x = \int_0^\infty g(r)r^{n-1}\ \mathrm{d}r\cdot\mathrm{area}(\mathbb{S}^{n-1})=0.$$

If $R\in\mathrm{SO}_n$, then

$$\begin{aligned}(\mathcal{F}f)(R\xi) &= \int_{\mathbb{R}^n} f(x)e^{-2\pi i\langle x,R\xi\rangle}\ \mathrm{d}x\\ &= \int_{\mathbb{R}^n} f(Rx)e^{-2\pi i\langle Rx,R\xi\rangle}\ \mathrm{d}x\\ &= (\mathcal{F}f)(\xi).\end{aligned}$$

Therefore $(\mathcal{F}f)(\xi)=\tilde{g}(\|\xi\|)$ for $\tilde{g}(\lambda)=(\mathcal{F}f)(\lambda\xi_0)$ independent of $\xi_0\in\mathbb{S}^{n-1}$. The function $\tilde{g}:[0,\infty)$ is smooth. Since $\tilde{g}(0)=(\mathcal{F}f)(0)=\int_{\mathbb{R}^n}f(x)\ \mathrm{d}x=0$ and $\tilde{g}$ has rapid decay $\lambda\to\infty$, the integral $\int_{\mathbb{R}_+^*}|\tilde{g}(\lambda)|^2\ \frac{\mathrm{d}\lambda}{\lambda}$ is finite. The integral is not zero, since $\mathcal{F}f$ is

not zero. By a normalisation of g, we assume $\int_{\mathbb{R}_+^*} |\tilde{g}(\lambda)|^2 \frac{\mathrm{d}\lambda}{\lambda} = 1$.
Now fix $\omega \in C_c(\mathbb{R})$ with $\omega(0) = 1$. By Lemma 3.6 there are $f_1, f_2 \in \mathcal{R}_0$, such that

$$(f_1 * f_2^*)(y,t,x) = h(y+tx) \cdot \omega(t) \cdot (f * f^*)(x).$$

We compute

$$\pi_0(f_1 * f_2^*)(\xi, y) = \omega(0) \cdot h(y) \cdot |\mathcal{F}(f)(\xi)|^2.$$

If $(y, \xi_0) \in \mathbb{R}^n \times \mathbb{S}^{n-1}$, then the Monotone Convergence Theorem yields

$$\begin{aligned}
\pi_0^r(\Lambda_{f_1} \circ \Lambda_{f_2}^*)(y, \xi_0) &= \lim_{k\to\infty} \int_{\mathbb{R}_+^*} \chi_k(\lambda) \cdot \pi_0(f_1 * f_2^*)(y, \lambda^{-1}\xi_0) \, \frac{\mathrm{d}\lambda}{\lambda} \\
&= h_1(y) \cdot \lim_{k\to\infty} \int_{\mathbb{R}_+^*} \chi_k(\lambda) \cdot |(\mathcal{F}f)(\lambda^{-1}\xi_0)|^2 \, \frac{\mathrm{d}\lambda}{\lambda} \\
&= h_1(y) \cdot \int_{\mathbb{R}_+^*} |\tilde{g}(\lambda^{-1})|^2 \, \frac{\mathrm{d}\lambda}{\lambda} \\
&= h_1(y) = h(y, \xi_0)
\end{aligned}$$

Hence $h \in A$. This shows that for every $(y, \xi) \in \mathbb{R}^n \times \mathbb{S}^{n-1}$, there is $h \in A$, such that $h(y, \xi) \neq 0$. It remains the show that A separates points.
Let $(y_1, \xi_1), (y_2, \xi_2) \in \mathbb{R}^n \times \mathbb{S}^{n-1}$ with $(y_1, \xi_1) \neq (y_2, \xi_2)$. If $y_1 \neq y_2$, then there is $h \in C_c^\infty(\mathbb{R}^n)$, with $h(y_1) \neq h(y_2)$. Therefore A separates (y_1, ξ_1) and (y_2, ξ_2) by the above computations.
If $y_1 = y_2$, so that $\xi_1 \neq \xi_2$, then there is $1 \leq j \leq n$, such that ξ_1 and ξ_2 differ in the j'th component. Take $h_1 \in C_c(\mathbb{R}^n)$ with $h(y_1) = h(y_2) = 1$. Let $k(x) = (\partial_{x_j} f)(x)$, then $k \in C_c^\infty(\mathbb{R}^n)$ with $(\mathcal{F}k)(\xi) = 2\pi i \cdot \xi_j \cdot (\mathcal{F}f)(\xi)$. Hence $\int_{\mathbb{R}^n} k(x) \, \mathrm{d}x = (\mathcal{F}f)(0) = 0$.
Take f and ω as above. Again by Lemma 3.6 there are $g_1, g_2 \in \mathcal{R}_0$, such that

$$(g_1 * g_2^*)(y,t,x) = h(y+tx) \cdot \omega(t) \cdot (k * f^*)(x).$$

We compute

$$\begin{aligned}
\pi_0(g_1 * g_2^*)(\xi, y_1) &= h(y_1) \cdot \omega(0) \cdot (\mathcal{F}k)(\xi) \cdot (\mathcal{F}f)(\xi) \\
&= 2\pi i \cdot \xi_j |(\mathcal{F}f)(\xi)|.
\end{aligned}$$

If $\xi_0 \in \mathbb{S}^{n-1}$, then the Monotone Convergence Theorem yields

$$\begin{aligned}
\pi_0^r(\Lambda_{g_1} \circ \Lambda_{g_2}^*)(y_1, \xi_0) &= \lim_{k\to\infty} \int_{\mathbb{R}_+^*} \chi(\lambda) \pi_0(f_1 * f_2^*)(y_1, \lambda^{-1}\xi_0) \, \frac{\mathrm{d}\lambda}{\lambda} \\
&= 2\pi i \cdot \xi_{0j} \cdot \int_{\mathbb{R}_+^*} \lambda^{-1} |(\mathcal{F}f)(\lambda^{-1}\xi_0)|^2 \, \frac{\mathrm{d}\lambda}{\lambda} \\
&= 2\pi i \cdot \xi_{0j} \cdot \int_{\mathbb{R}_+^*} |\tilde{g}(\lambda)|^2 \, \mathrm{d}\lambda
\end{aligned}$$

Since $\tilde{g} \neq 0$, we have $\int_{\mathbb{R}_+^*} |\tilde{g}(\lambda)|^2 \, \mathrm{d}\lambda \neq 0$. Therefore,

$$\pi_0^r(\Lambda_{g_1} \cdot \Lambda_{g_2}^*)(y_1, \xi_1) \neq \pi_0^r(\Lambda_{g_1} \cdot \Lambda_{g_2}^*)(y_2, \xi_2).$$

This proves that A separates the points of $\mathbb{R}^n \times \mathbb{S}^{n-1}$.
A is the image of the $*$-homomorphism π_0^r. Therefore A is closed and $A^* = A$. The Stone-Weierstraß theorem implies that A is dense in $C_0(\mathbb{R}^n \times \mathbb{S}^{n-1})$.
Therefore, $A = C_0(\mathbb{R}^n \times \mathbb{S}^{n-1})$, so that π_0^r is surjective. □

π_1 as a Faithful Representation of $\mathrm{Fix}_{\mathcal{R}}$ on $L^2(\mathbb{R}^n)$

Let $t > 0$. The restriction of the $*$-homomorphism $\pi_t \colon C_r^*(\mathcal{G}\mathbb{R}^n) \to \mathbb{K}(L^2(\mathbb{R}^n))$ is still surjective. To see this, let $K \in \mathbb{K}(L^2(\mathbb{R}^n))$. There is $f \in C_r^*(\mathcal{G}\mathbb{R}^n)$, such that $\pi_t(f) = K$. Let $\omega \in C_0(\mathbb{R})$, such that $\omega(0) = 0$ and $\omega(t) = 1$. Since $(\mathbb{R}, (\pi_t)_{t \in \mathbb{R}}, C_r^*(\mathcal{G}\mathbb{R}^n))$ is a continuous bundle, there is $f \cdot \omega \in C_r^*(\mathcal{G}\mathbb{R}^n)$, such that $\pi_s(f \cdot \omega) = \omega(t) \cdot \pi_s(f)$ for all $s \in \mathbb{R}$. In particular $\pi_t(\omega \cdot f) = \pi_t(f) = K$ and $\pi_0(\omega \cdot f) = 0$. Therefore even the restriction of π_t to $\ker(\pi_0) \subseteq J$ is surjective.
As is the case $t = 0$ the $*$-homomorphism π_t extends uniquely to a strictly continuous $*$-homomorphism

$$\mathcal{M}(J) \to \mathcal{M}(\mathbb{K}(L^2(\mathbb{R}^n))) = \mathbb{B}(L^2(\mathbb{R}^n))$$

This $*$-homomorphism restricts to a $*$-homomorphism $\pi_t \colon \mathrm{Fix}_{\mathcal{R}} \to \mathbb{B}(L^2(\mathbb{R}^n))$. By Proposition 3.2 we have $\pi_t \circ \sigma_\lambda = \mathrm{Ad}_{U_\lambda} \circ \pi_{t/\lambda}$. The same formula still holds for the extensions to $\mathrm{Fix}_{\mathcal{R}}$. Since $\mathrm{Fix}_{\mathcal{R}} \subseteq \mathcal{M}^G(A)$, we obtain

$$\pi_t(F) = \pi_t(\sigma_t(F)) = U_t^* \circ \pi_1(F) \circ U_t \quad \text{for all } T \in \mathrm{Fix}_{\mathcal{R}}.$$

Therefore, the representations $\pi_t \colon \mathrm{Fix}_{\mathcal{R}} \to \mathbb{B}(L^2(\mathbb{R}^n))$ are unitary equivalent.

Theorem 3.8 (π_1 is a Faithful Representation of $\mathrm{Fix}_{\mathcal{R}}$ in $L^2(\mathbb{R}^n)$)**.**
The $*$-homomorphism $\pi_1 \colon \mathrm{Fix}_{\mathcal{R}} \to \mathbb{B}(L^2(\mathbb{R}^n))$ is injective.

Proof. Let $F \in \mathrm{Fix}_{\mathcal{R}}$ with $\pi_1(F) = 0$. Then $\pi_t(F) = U_t^* \circ \pi_1(F) \circ U_t = 0$ for all $t > 0$. If $g \in \ker(\pi_0)$, then $\pi_t(F(g)) = \pi_t(F) \cdot \pi_t(g) = 0$ for all $t \in [0, \infty)$. Hence $\|F(g)\|_r = \sup_{t \in [0,\infty)} \|\pi_t(F(g))\| = 0$. Now let $f \in J$, then $F(f) * g = F(f * g) = 0$ for all $g \in \ker(\pi_0)$. Lemma 2.8 yields $F(f) = 0$. Therefore $F = 0$. □

3.3 Pseudodifferential Operator Extension

Our aim is to prove, that π_1 restricts to a $*$-isomorphism $\ker(\pi_0^r) \cong \mathbb{K}(L^2(\mathbb{R}^n))$. Since π_1 is injective, it suffices to prove $\mathbb{K}(L^2(\mathbb{R}^n)) = \pi_1(\ker(\pi_0^r))$. We use the following lemma to prove $\mathbb{K}(L^2(\mathbb{R}^n)) \subseteq \pi_1(\ker(\pi_0^r))$.

Lemma 3.9. *Let $\omega \colon [0, \infty) \to [0, \infty)$ be smooth and compactly supported with $\mathrm{supp}(\omega) \subset (0, \infty)$. If $h_1, h_2 \in C_c^\infty(\mathbb{R}^n)$, then there are $f_1, f_2 \in \mathcal{R}_0$ with*

$$(f_1 * f_2^*)(y, t, x) = h_1(y + tx) \cdot \overline{h_2(y)} \cdot \omega(t) \quad \text{for all } (y, t, x) \in \mathcal{G}\mathbb{R}^n.$$

Proof. There is $R > 0$ and $\varepsilon > 0$, such that

$$\begin{aligned} h_1(x) = h_2(x) = 0 &\quad \text{for } \|x\| > R \\ \omega(t) = 0 &\quad \text{for } t < \varepsilon \text{ or } t > R. \end{aligned}$$

There is $\omega_1\colon [0,\infty) \to [0,\infty)$ smooth and compactly supported with $\operatorname{supp}(\omega_1) \subset (0,\infty)$, such that $\omega_1(t) = 1$ for all $\varepsilon \le t \le R$. Then $\omega(t) = \omega(t) \cdot \omega_1(t)$ for all $t \in [0,\infty)$.
Let $k_1 \in C_c^\infty(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} k_1(x)\ \mathrm{d}x = 1$ and $k_1(x) = 0$ for $\|x\| > 1$. There is $k_2 \in C_c(\mathbb{R}^n)$, such that $k_2(x) = 1$ for $\|x\| \le \frac{2R}{\varepsilon} + 1$. We define $f_1(y,t,x) = h_1(y+tx) \cdot \omega_1(t) \cdot k_1(x)$ and $f_2(y,t,x) = h_2(y+tx) \cdot \omega(t) \cdot k_2(x)$. Since $\omega_1(0) = \omega(0) = 0$ we obtain $f_1, f_2 \in \mathcal{R}_0$.
We compute

$$\begin{aligned} (f_1 * f_2^*)(y,t,x) &= \int_{\mathbb{R}^n} f_1(y-tz,t,x+z) \cdot \overline{f_2(y-tz,t,z)}\ \mathrm{d}z \\ &= h_1(y+tx) \cdot \overline{h_2(y)} \cdot \omega(t) \cdot \int_{\mathbb{R}^n} k_1(x+z) \cdot \overline{k_2(z)}\ \mathrm{d}z \\ &= h_1(y+tx) \cdot \overline{h_2(y)} \cdot \omega(t) \cdot \int_{\mathbb{R}^n} k_1(z)\ \mathrm{d}z \\ &= h_1(y+tx) \cdot \overline{h_2(y)} \cdot \omega(t). \end{aligned}$$

□

Proposition 3.10.
$\mathbb{K}(L^2(\mathbb{R}^n)) \subseteq \pi_1(\ker(\pi_0^r))$.

Proof. For $h_1, h_2 \in L^2(\mathbb{R}^n)$, we write θ_{h_1,h_2} for the rank-one operator given by

$$\theta_{h_1,h_2}(g) = \langle h_2, g\rangle h_1 \quad \text{for } g \in L^2(\mathbb{R}^n).$$

$C_c^\infty(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$ and $\|\theta_{h_1,h_2}\| = \|h_1\|_2 \cdot \|h_2\|_2$. Therefore $\mathbb{K}(L^2(\mathbb{R}^n))$ is generated by the set of θ_{h_1,h_2} for $h_1, h_2 \in C_c^\infty(\mathbb{R}^n)$.
Let $h_1, h_2 \in C_c^\infty(\mathbb{R}^n)$. There is $\omega\colon [0,\infty) \to [0,\infty)$ compactly supported and smooth, such that $\operatorname{supp}(\omega) \subset (0,\infty)$ and $\int_{\mathbb{R}_+^*} \lambda^n \omega(\lambda^{-1})\ \frac{\mathrm{d}\lambda}{\lambda} = 1$.
By the previous lemma, there is $f_1, f_2 \in \mathcal{R}_0$, such that

$$(f_1 * f_2^*)(y,t,x) = h_1(y+tx) \cdot \overline{h_2(y)} \cdot \omega(t) \quad \text{for all } (y,t,x) \in \mathcal{G}\mathbb{R}^n.$$

If $\lambda \in \mathbb{R}_+^*$ and $g \in C_c(\mathbb{R}^n)$, then we compute

$$\begin{aligned} (\pi_1(\sigma_\lambda(f_1 * f_2^*))g)(x) &= \int_{\mathbb{R}^n} \lambda_n (f_1 * f_2^*)(y, \lambda^{-1}, \lambda(x-y)) \cdot h(y)\ \mathrm{d}y \\ &= \lambda_n \omega(\lambda^{-1}) \cdot h_1(x) \cdot \int_{\mathbb{R}^n} \overline{h_2(y)} \cdot g(y)\ \mathrm{d}y \\ &= ((\lambda_n \omega(\lambda^{-1}) \cdot \theta_{h_1,h_2})g)(x) \end{aligned}$$

Using Lemma 1.36 and the Monotone convergence theorem, we archive

$$\begin{aligned}
\pi_1(\Lambda_{f_1} \circ \Lambda_{f_2}^*) &= \pi_1 \left(\overset{s}{\lim_{k\to\infty}} \int_{\mathbb{R}_+^*} \chi_k(\lambda) \cdot \sigma_\lambda(f_1 * f_2^*) \, \frac{\mathrm{d}\lambda}{\lambda} \right) \\
&= \overset{s}{\lim_{k\to\infty}} \int_{\mathbb{R}_+^*} \chi_k(\lambda) \cdot \pi_1(\sigma_\lambda(f_1 * f_2^*)) \, \frac{\mathrm{d}\lambda}{\lambda} \\
&= \left(\lim_{k\to\infty} \int_{\mathbb{R}_+^*} \chi_k(\lambda) \cdot \lambda^n \omega(\lambda^{-1}) \, \frac{\mathrm{d}\lambda}{\lambda} \right) \theta_{h_1,h_2} \\
&= \left(\int_{\mathbb{R}_+^*} \lambda^n \omega(\lambda^{-1}) \, \frac{\mathrm{d}\lambda}{\lambda} \right) \theta_{h_1,h_2} = \theta_{h_1,h_2}.
\end{aligned}$$

Since $\omega(0) = 0$, we have $\pi_0(f_1 * f_2^*) = 0$. Therefore,

$$\pi_0(\Lambda_{f_1} \circ \Lambda_{f_2}) = \overset{s}{\lim_{k\to\infty}} \int_{\mathbb{R}_+^*} \chi_k(\lambda) \cdot \sigma_\lambda(\pi_0(f_1 * f_2^*)) \, \frac{\mathrm{d}\lambda}{\lambda} = 0.$$

Hence $\pi_0^r(\Lambda_{f_1} \circ \Lambda_{f_2}) = 0$. This shows $\theta_{h_1,h_2} \in \pi_1(\ker(\pi_0^r))$.
Since $\pi_1(\ker(\pi_0^r))$ is a C^*-algebra we conclude $\mathbb{K}(L^2(\mathbb{R}^n)) \subseteq \pi_1(\ker(\pi_0^r))$. □

Corollary 3.11 (π_1 is nondegenerate)**.**
The representation $\pi_1\colon \mathrm{Fix}_{\mathcal{R}} \to \mathbb{B}(L^2(\mathbb{R}^n))$ *is nondegenerate. That is*

$$\pi_1(\mathrm{Fix}_{\mathcal{R}})L^2(\mathbb{R}^n) = L^2(\mathbb{R}^n).$$

Proof. $L^2(\mathbb{R}^n) = \mathbb{K}(L^2(\mathbb{R}^n))L^2(\mathbb{R}^n) \subseteq \pi_1(\ker(\pi_0^r))L^2(\mathbb{R}^n) \subseteq \pi_1(\mathrm{Fix}_{\mathcal{R}})L^2(\mathbb{R}^n) \subseteq L^2(\mathbb{R}^n)$. □

The following lemmas and their corollaries are the key ingredients for the proof of the remaining inclusion $\pi_1(\ker(\pi_0^r)) \subseteq \mathbb{K}(L^2(\mathbb{R}^n))$.

Lemma 3.12. *Let* $f \in C_c(\mathcal{G}\mathbb{R}^n)$ *be smooth, such that* $\pi_0(f) = 0$. *The sequence* $(T_k)_{k\in\mathbb{N}} \subseteq \mathbb{K}(L^2(\mathbb{R}^n))$ *, with*

$$T_n = \int_{\mathbb{R}_+^*} \chi_k(\lambda) \cdot \pi_1(\sigma_\lambda(f)) \, \frac{d\lambda}{\lambda}$$

converges in the operator-norm. The limit is a compact operator.

Proof. Since π_0 is defined as an isomorphism composed with τ_0 we obtain $\tau_0(f) = 0$. As $\tau_0(f)$ is the restriction to $\mathcal{G}\mathbb{R}_0^n$, this implies $f(y, 0, x) = 0$ for all $y, x \in \mathbb{R}^n$.
We define

$$g(y,t,x) = \begin{cases} \dfrac{f(y,t,y)}{t} & \text{if } t > 0 \\ (\partial_t f)(y,0,x) & \text{if } t = 0 \end{cases}.$$

Using the Mean Value Theorem for the real and imaginary part of f separately, we obtain $g \in C_c(\mathcal{G}\mathbb{R}^n)$. We have $f(y,t,x) = t \cdot g(y,t,x)$ for all $(y,t,x) \in G$. Let $t > 0$ and $h \in C_c(\mathbb{R}^n)$. Then

$$\begin{aligned}(\pi_t(f)h)(x) &= \int_{\mathbb{R}^n} f(ty,t,x-y) \cdot h(y)\ \mathrm{d}y \\ &= t \cdot \int_{\mathbb{R}^n} g(ty,t,x-y) \cdot h(y)\ \mathrm{d}y \\ &= ((t \cdot \pi_t(g))h)(x)\end{aligned}$$

Hence $\pi_t(f) = t \cdot \pi_t(g)$. Therefore,

$$\|\pi_t(f)\| \leq t \cdot \|\pi_t(g)\| \leq t \cdot \|g\|_r \quad \text{for all } t > 0.$$

Since f is compactly supported, there is $R > 0$, such that $\pi_t(f) = 0$ for all $t > R$. Let $k, m \in \mathbb{N}$ with $k < m$. We estimate

$$\begin{aligned}\|T_m - T_k\| &= \left\| \int_{\mathbb{R}_+^*} (\chi_m(\lambda) - \chi_k(\lambda)) \cdot \pi_1(\sigma_\lambda(f))\ \frac{\mathrm{d}\lambda}{\lambda} \right\| \\ &\overset{3.2}{\leq} \int_{\mathbb{R}_+^*} (\chi_m(\lambda) - \chi_k(\lambda)) \cdot \|U_\lambda^* \circ \pi_{\lambda^{-1}}(f) \circ U_\lambda\|\ \frac{\mathrm{d}\lambda}{\lambda} \\ &\leq \int_{[\frac{1}{R},\infty)} (1 - \chi_k(\lambda)) \|\pi_{\lambda^{-1}}(f)\|\ \frac{\mathrm{d}\lambda}{\lambda} \\ &\leq \|g\| \cdot \int_{[\frac{1}{R},\infty)} (1 - \chi_k(\lambda)) \cdot \frac{1}{\lambda^2}\ \mathrm{d}\lambda.\end{aligned}$$

The Dominated Convergence Theorem yields

$$\int_{[\frac{1}{R},\infty)} (1 - \chi_k(\lambda)) \cdot \frac{1}{\lambda^2}\ \mathrm{d}\lambda \longrightarrow 0 \quad \text{for } k \to \infty.$$

Hence $(T_k)_{k\in\mathbb{N}} \subseteq \mathbb{K}(L^2(\mathbb{R}^n))$ is a Cauchy-sequence in the operator-norm. Therefore, $(T_k)_{k\in\mathbb{N}} \subseteq \mathbb{K}(L^2(\mathbb{R}^n))$ converges. Since $\pi_1(\sigma_\lambda(f)) \in \mathbb{K}(L^2(\mathbb{R}^n))$ for all $\lambda \in \mathbb{R}_+^*$ and $\mathbb{K}(L^2(\mathbb{R}^n))$ is closed in $\mathbb{B}(L^2(\mathbb{R}^n))$, we obtain $T_k \in \mathbb{K}(L^2(\mathbb{R}^n))$ for all $k \in \mathbb{N}$. This implies $\lim_k T_k \in \mathbb{K}(L^2(\mathbb{R}^n))$. ☐

Corollary 3.13. *If $f \in \mathcal{R}_0$, with $\pi_0(f) = 0$, then $\pi_1(\Lambda_f \circ \Lambda_g^*) \in \mathbb{K}(L^2(\mathbb{R}^n))$ for all $g \in \mathcal{R}$.*

Proof. Let first $g \in \mathcal{R}_0$. Then $\pi_0(f * g^*) = \pi_0(f) * \pi_0(g)^* = 0$. Hence Lemma 3.12 shows

$$\begin{aligned}\pi_1(\langle\langle \Lambda_f, \Lambda_g \rangle\rangle) &= \pi_1(\Lambda_f \circ \Lambda_g^*) \\ &\overset{1.36}{=} \overset{s}{\lim_{k\to\infty}} \int_{\mathbb{R}_+^*} \chi_k(\lambda) \cdot \pi_1(\sigma_\lambda(f * g^*))\ \frac{\mathrm{d}\lambda}{\lambda} \\ &= \lim_{k\to\infty} \int_{\mathbb{R}_+^*} \chi_k(\lambda) \cdot \pi_1(\sigma_\lambda(f * g^*))\ \frac{\mathrm{d}\lambda}{\lambda} \in \mathbb{K}(L^2(\mathbb{R}^n)).\end{aligned}$$

If $g \in \mathcal{R}$ arbitrary, then there is a sequence $(g_k)_{k\in\mathbb{N}} \subseteq \mathcal{R}_0$ with

$$\|\Lambda_g - \Lambda_{g_k}\| \leq \|g - g_k\|_{si} \longrightarrow 0 \quad \text{for } k \to \infty.$$

Therefore

$$\pi_1(\Lambda_f \circ \Lambda_g^*) = \lim_{k\to\infty} \pi_1(\Lambda_f \circ \Lambda_{g_k}^*) \in \mathbb{K}(L^2(\mathbb{R}^n)). \qquad \square$$

To generalise the statement of the previous corollary to $f \in \mathcal{R}$ arbitrary, we need another lemma.

Lemma 3.14. *$\mathcal{R}_0 \cap \ker(\pi_0)$ is $\|\cdot\|_{si}$-dense in $\mathcal{R} \cap \ker(\pi_0)$.*

Proof. First we prove, that $\mathcal{R}_0 \cap \ker(\pi_0)$ is $\|\cdot\|_r$-dense in $\ker(\pi_0)$.
Let $f \in C_r^*(\mathcal{G}\mathbb{R}^n)$ with $\pi_0(f) = 0$. Since the function $t \mapsto \|\pi_t(f)\|$ is continuous, there is a $\delta > 0$, such that $\|\pi_t(f)\| < \frac{\varepsilon}{3}$ for all $0 \leq t < \delta$. By the argument in the beginning of the proof of Proposition 3.3 there is a smooth $f_1 \in C_c(\mathcal{G}\mathbb{R}^n)$, with $\|f - f_1\|_r < \frac{\varepsilon}{3}$.
Let $\omega \in C_c^\infty([0,\infty))$, such that $0 \leq \omega \leq 1$, $\omega(0) = 1$ and $\omega(t) = 0$ for $t \geq \delta$. Let $k(y,t,x) = f_1(y,t,x) \cdot \omega(t)$. Then k is compactly supported and smooth. If $t \in [0,\infty)$, then

$$\|\pi_t(k)\| = \omega(t) \cdot \|\pi_t(f_1)\| = \omega(t) \cdot (\|\pi_t(f_1 - f)\| + \|\pi_t(f)\|) \leq \frac{2\varepsilon}{3}.$$

Therefore

$$\|k\|_r = \sup_{t\in[0,\infty)} \|\pi_t(k)\| \leq \frac{2\varepsilon}{3}.$$

We define $f_2 := f_1 - k$. Then f_2 is smooth and compactly supported. We have $f_2(y,0,x) = 0$ for all $y, x \in \mathbb{R}^n$. Therefore $f_2 \in \mathcal{R}_0 \cap \ker(\pi_0)$. We estimate

$$\|f - f_2\|_r \leq \|f - f_1\|_r + \|k\|_r < \varepsilon.$$

This shows, that $\mathcal{R}_0 \cap \ker(\pi_0)$ is $\|\cdot\|_r$-dense in $\ker(\pi_0)$.
Now let $f \in \mathcal{R} \cap \ker(\pi_0)$ and $\varepsilon > 0$. The group $\mathbb{R}_+^*$ is exact. Theorem 1.52(iv) yields $\mathcal{R} \cap \ker(\pi_0) = \mathcal{R} * \ker(\pi_0)$. Hence there is $r \in \mathcal{R}$ and $k \in \ker(\pi_0)$, such that $f = r * k$. There is $k_1 \in \mathcal{R}_0 \cap \ker(\pi_0)$, such that $\|k - k_1\|_r \leq \frac{\varepsilon}{2\|r\|_{si}}$. Since $\mathcal{R}$ is the $\|\cdot\|_{si}$ closure of $\mathcal{R}_0$ there is $r_1 \in \mathcal{R}_0$, such that $\|r - r_1\|_{si} \leq \frac{\varepsilon}{2\|k_1\|_r}$. Then $r_1 * k_1 \in \mathcal{R}_0 \cap \ker(\pi_0)$. Using Proposition 1.39(i) we obtain

$$\begin{aligned}\|f - r_1 * k_1\|_{si} &= \|r * k - r_1 * k_1\|_{si} \\ &\leq \|r\|_{si} \cdot \|k - k_1\|_r + \|r - r_1\|_{si} \cdot \|k_1\|_r < \varepsilon\end{aligned}$$

This shows that $\mathcal{R}_0 \cap \ker(\pi_0)$ is $\|\cdot\|_{si}$-dense in $\mathcal{R} \cap \ker(\pi_0)$. $\square$

Corollary 3.15. *Let $f \in \mathcal{R}$. If $\pi_0(f) = 0$, then $\pi_1(\Lambda_f \circ \Lambda_g^*) \in \mathbb{K}(L^2(\mathbb{R}^n))$ for all $g \in \mathcal{R}$.*

Proof. By the previous lemma there is a sequence $(f_k)_{k\in\mathbb{N}} \subseteq \mathcal{R}_0 \cap \ker(\pi_0)$, such that

$$\|\Lambda_f - \Lambda_{f_k}\| \leq \|f - f_k\|_{si} \longrightarrow 0 \quad \text{for } k \to \infty.$$

Corollary 3.13 yields $\pi_1\left(\Lambda_{f_k} \circ \Lambda_g^*\right) \in \mathbb{K}(L^2(\mathbb{R}^n))$ for all $k \in \mathbb{N}$. Therefore,

$$\pi_1(\Lambda_f \circ \Lambda_g^*) = \lim_{k\to\infty} \pi_1(\Lambda_{f_k} \circ \Lambda_g^*) \in \mathbb{K}(L^2(\mathbb{R}^n)). \qquad \square$$

Theorem 3.16 (ker(π_0) is isomorph to $\mathbb{K}(L^2(\mathbb{R}^n))$)**.**
The $*$-homomorphism $\pi_1\colon \mathrm{Fix}_{\mathcal{R}} \to \mathbb{B}(L^2(\mathbb{R}^n))$ restricts to a $*$-isomorphism

$$\pi_1|_{\ker(\pi_0^r)}\colon \ker(\pi_0^r) \to \mathbb{K}(L^2(\mathbb{R}^n)).$$

Proof. By Theorem 3.8 $\pi_1\colon \mathrm{Fix}_{\mathcal{R}} \to \mathbb{B}(L^2(\mathbb{R}^n))$ is injective. Therefore, $\pi_1|_{\ker(\pi_0^r)}$ is injective. Let $F \in \mathrm{Fix}_{\mathcal{R}}$ with $\pi_0^r(F) = 0$. We have to show $\pi_1(F) \in K$.
Let $(y,\xi) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$. Then

$$\begin{aligned}\pi_0(F)(y,\xi) &= (\sigma_{\|\xi\|^{-1}}(\pi_0(F))\left(y, \frac{\xi}{\|\xi\|}\right)\\ &= (\pi_0(\sigma_{\|\xi\|^{-1}}F))\left(y, \frac{\xi}{\|\xi\|}\right)\\ &= \pi_0^r(F)\left(y, \frac{\xi}{\|\xi\|}\right) = 0\end{aligned}$$

Let $f, g \in \mathcal{R}$. Then $F(f) \in \mathcal{R}$ by Theorem 1.52(iii). Furthermore,

$$\pi_0(F(f)) = \pi_0(F)(\pi_0(f)) = 0.$$

Corollary 3.15 yields

$$\pi_1(F) \circ \pi_1(\Lambda_f \circ \Lambda_g^*) = \pi_1(F \circ (\Lambda_f \circ \Lambda_g^*)) = \pi_1(\Lambda_{F(f)} \circ \Lambda_g^*) \in \mathbb{K}(L^2(\mathbb{R}^n)).$$

Since $\mathrm{Fix}_{\mathcal{R}}$ is the closed linear span of $\{\Lambda_a \circ \Lambda_b^*\colon a, b \in \mathcal{R}\}$, we conclude

$$\pi_1(X) \circ \pi_1(X)^* \in \mathbb{K}(L^2(\mathbb{R}^n)).$$

Let $q\colon \mathbb{B}(L^2(\mathbb{R}^n)) \to \mathbb{B}(L^2(\mathbb{R}^n))/\mathbb{K}(L^2(\mathbb{R}^n))$ be the quotient map onto the Calkin algebra. Then

$$\|q(\pi_1(X))\|^2 = \|q(\pi_1(X))q(\pi_1(X))^*\| = \|q(\pi_1(X) \circ \pi_1(X)^*)\| = 0.$$

Therefore $\pi_1(X) \in \ker(q) = \mathbb{K}(L^2(\mathbb{R}^n))$.
Hence $\pi_1|_{\ker(\pi_0^r)}\colon \ker(\pi_0^r) \to \mathbb{K}(L^2(\mathbb{R}^n))$ is well-defined. By Proposition 3.10 we have $\mathbb{K}(L^2(\mathbb{R}^n)) \subseteq \pi_1(\ker(\pi_0^r))$. Therefore, $\pi_1|_{\ker(\pi_0^r)}$ maps onto $\mathbb{K}(L^2(\mathbb{R}^n))$. ☐

Corollary 3.17 (Pseudodifferential Operator Extension)**.**
The map $\mathrm{sym} := \pi_0^r \circ \pi_1^{-1}\colon \pi_1(\mathrm{Fix}_{\mathcal{R}}) \to C_0(\mathbb{R}^n \times \mathbb{S}^{n-1})$ yields a short exact sequence

$$0 \to \mathbb{K}(L^2(\mathbb{R}^n)) \longrightarrow \pi_1(\mathrm{Fix}_{\mathcal{R}}) \xrightarrow{\mathrm{sym}} C_0(\mathbb{R}^n \times \mathbb{S}^{n-1}) \to 0.$$

Proof. Consider the diagram

$$\begin{tikzcd} 0 \arrow[r] & \ker(\pi_0^r) \arrow[r] \arrow[d, "\pi_1|_{\ker(\pi_0^r)}"] & \mathrm{Fix}_{\mathcal{R}} \arrow[r, "\pi_0^r"] \arrow[d, "\pi_1"] & C_0(\mathbb{R}^n \times \mathbb{S}^{n-1}) \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\ 0 \arrow[r] & \mathbb{K}(L^2(\mathbb{R}^n)) \arrow[r] & \pi_1(\mathrm{Fix}_{\mathcal{R}}) \arrow[r, "\mathrm{sym}"] & C_0(\mathbb{R}^n \times \mathbb{S}^{n-1}) \arrow[r] & 0. \end{tikzcd}$$

The upper sequence is exact and all vertical arrows are isomorphisms. Therefore, the lower sequence is exact. ☐

4 Connection to Pseudo-Differential Operators

Let $H \subseteq C_b(\mathbb{R}^n)$ be the set of bounded continuous complex-valued function on $\mathbb{R}^n$ satisfying the condition $h(\lambda\xi) = h(\xi)$ for all $\|\xi\| \geq 1$ and $\lambda \geq 1$.
Let $B = \{\xi \in \mathbb{R}^n \colon \|x\| < 1\}$ be the open unit ball. Then $\phi\colon B \to \mathbb{R}^n$, $x \mapsto \frac{x}{\|x\|} \cdot \tan\left(\frac{\pi}{2}\|x\|\right)$ is a homeomorphism. For $f \in C_b(\mathbb{R}^n)$, $\lambda > 0$ and $\xi \in \mathbb{S}^{n-1}$ we define $f^\lambda(\xi) = f(\lambda\xi)$ and obtain $f^\lambda \in C(\mathbb{S}^{n-1})$.
We define $f^\infty(\xi) = \lim_{\lambda\to\infty} f(\lambda\xi)$ if the limit exists for all $\xi \in \mathbb{S}^{n-1}$. The function $f^\infty\colon \mathbb{S}^{n-1} \to \mathbb{C}$ need not to be continuous even if the limit exists for all $\xi \in \mathbb{S}^{n-1}$. The following proposition describes functions $f \in C_b(\mathbb{R}^n)$, where f^∞ exists and gives a continuous function on $\mathbb{S}^{n-1}$.

Proposition 4.1. *For $f \in C_b(\mathbb{R}^n)$ the following statements are equivalent:*

(i) *$f = h + k$ for $h \in H$ and $k \in C_0(\mathbb{R}^n)$.*

(ii) *$f^\infty \in C(\mathbb{S}^{n-1})$ and the net $(f^\lambda)_{\lambda>0}$ converges uniformly to f^∞.*

(iii) *$f \circ \phi$ extends to a continuous function $\overline{B} \to \mathbb{C}$.*

Proof.

(i)⇒ (ii): If $\xi \in \mathbb{S}^{n-1}$, then

$$\lim_{\lambda\to\infty} f(\lambda\xi) = \lim_{\lambda\to\infty} h(\lambda\xi) + k(\lambda\xi) = h\left(\frac{\xi}{\|\xi\|}\right).$$

Let $\varepsilon > 0$, There is $R > 0$, such that $|k(\xi)| < \varepsilon$ for $\|\xi\| > R$. For $\xi \in \mathbb{S}^{n-1}$ and $\lambda > R$, we obtain

$$|f^\infty(\xi) - f^\lambda(\xi)| = |h(\xi) - h(\lambda\xi) + k(\lambda\xi)| < \varepsilon.$$

Hence $\|f^\infty - f^\lambda\|_\infty \leq \varepsilon$. Therefore, $f^\infty \in C(\mathbb{S}^{n-1})$ with $f^\lambda \longrightarrow f^\infty 0$ uniformly.

(ii)⇒ (iii): Let

$$u(\xi) = \begin{cases} (f \circ \phi)(\xi) & \text{if } \|\xi\| < 1 \\ f^\infty(\xi) & \text{if } \|\xi\| = 1 \end{cases}.$$

Then $u\colon \overline{B} \to \mathbb{C}$. Obviously u is continuous in all $\xi \in B$ and extends $f \circ \phi$. Let $\varepsilon > 0$ and $\xi_0 \in \mathbb{S}^{n-1}$. There is a δ_1, such that

$$|f^\infty(\xi_0) - f^\infty(\xi_1)| < \frac{\varepsilon}{2} \quad \text{for all } \xi_1 \in \mathbb{S}^{n-1} \text{ with } \|\xi_0 - \xi_1\| < \delta_1.$$

There is $\lambda_0 > 0$, such that $\|f^\infty - f^\lambda\|_\infty < \frac{\varepsilon}{2}$ for all $\lambda > \lambda_0$.
We define $\delta = \frac{1}{2}\min\{\delta_1, 1 - \frac{2}{\pi}\tan^{-1}(\lambda_0)\}$. Let $\xi \in B$ with $\|\xi_0 - \xi\| < \delta$. We estimate

$$\left\|\xi_0 - \frac{\xi}{\|\xi\|}\right\| \leq \|\xi_0 - \xi\| + \left\|\xi - \frac{\xi}{\|\xi\|}\right\| < \delta + (1 - \|\xi\|) \leq 2\delta \leq \delta_1.$$

We have

$$(f\circ\phi)(\xi) = f\left(\frac{\xi}{\|\xi\|}\cdot\tan\left(\frac{\pi}{2}\|\xi\|\right)\right) = f^{\tan\left(\frac{\pi}{2}\|\xi\|\right)}\left(\frac{\xi}{\|\xi\|}\right).$$

Since $\tan\left(\frac{\pi}{2}\|\xi\|\right)\geq\lambda_0$, we obtain

$$|f^\infty(\xi_0)-(f\circ\phi)(\xi)|\leq\left|f^\infty(\xi_0)-f^\infty\left(\frac{\xi}{\|\xi\|}\right)\right|+\left|f^\infty\left(\frac{\xi}{\|\xi\|}\right)\right|\leq\varepsilon.$$

Hence $|u(\xi_0)-u(\xi)|\leq\varepsilon$ for all $\xi\in\overline{B}$ with $\|\xi_0-\xi\|<\delta$.

(iii)$\Rightarrow$ (i): Let u be the extension of $f\circ\phi$ to $\overline{B}$. Let $\omega\colon\overline{B}\to[0,1]$ be continuous, such that $\omega(0)$ and $\omega(\xi)=1$ for $\|\xi\|\geq\frac{1}{2}$. Define $h_B(\xi)=\omega(\xi)\cdot u\left(\frac{\xi}{\|\xi\|}\right)$. Then $h_b\in C(\overline{B})$. Put $k_B=u-h_B$. Then $k_B|_{\mathbb{S}^{n-1}}=0$. Hence $k_B\in C_0(B)$. Define $k=k_B\circ\phi^{-1}$ and $h=h_B\circ\phi^{-1}$. Then $k\in C_0(\mathbb{R}^n)$. If $\|\xi\|\geq 1$ and $\lambda\geq 1$, then

$$\begin{aligned}h(\lambda\xi)&=(h_B\circ\phi^{-1})(\lambda\xi)\\&=\omega(\phi^{-1}(\lambda\xi))u\left(\frac{\xi}{\|\xi\|}\right)\\&=\omega(\phi^{-1}(\xi))u\left(\frac{\xi}{\|\xi\|}\right)=h(\xi).\end{aligned}$$

Finally $h+k=(h_B+k_B)\circ\phi^{-1}=f$. □

Let $S\subset C_b(\mathbb{R}^n)$ be the set of functions satisfying one and therefore all conditions of the previous proposition. We have a map $s\colon S\to C(\mathbb{S}^{n-1})$, $f\mapsto f^\infty$.

Proposition 4.2. *S is a C^*-subalgebra of $C_b(\mathbb{R}^n)$. The sequence*

$$0\to C_0(\mathbb{R}^n)\longrightarrow S\longrightarrow C(\mathbb{S}^{n-1})\to 0$$

is exact.

Proof. By Proposition 4.1 the map $\alpha\colon C(\overline{B})\to S$ given by $\alpha(u)=u|_B\circ\phi^{-1}$ is a well defined, surjective $*$-homomorphism. Since $B\subset\overline{B}$ is dense α is injective. If $u\in C(\overline{B})$, then

$$\begin{aligned}s(\alpha(u))(\xi)&=\lim_{\lambda\to\infty}(\alpha(u))(\lambda\xi)\\&=\lim_{\lambda\to\infty}u|_B(\phi^{-1}(\lambda\xi))=u(\xi)\end{aligned}$$

for all $\xi\in\mathbb{S}^{n-1}$. Let $\phi^*\colon C_0(\mathbb{R}^n)\to C_0(B)$, $f\mapsto f\circ\phi$. we obtain a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & C_0(B) \arrow[r] & C(\overline{B}) \arrow[r] \arrow[d, "\alpha"] & C(\mathbb{S}^{n-1}) \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
0 \arrow[r] & C_0(\mathbb{R}^n) \arrow[r] \arrow[u, "\phi^*"] & S \arrow[r, "s"] & C(\mathbb{S}^{n-1}) \arrow[r] & 0.
\end{tikzcd}$$

The upper sequence is exact and all vertical arrows are isomorphisms. Hence the upper sequence is exact. □

Since $S \cong C(\overline{B})$ the spectrum of S is homeomorph to $\overline{B}$. Thus S corresponds to the compactification of $\mathbb{R}^n$ to a closed ball.
If $f \in C_b(\mathbb{R}^n)$ we define $M_f \colon L^2(\mathbb{R}^n) \to L_2(\mathbb{R}^n)$ by $(M_f g)(x) = f(x) \cdot g(x)$. Then $M \colon C_b(\mathbb{R}^n) \to \mathbb{B}(L^2(\mathbb{R}^n))$ is a $*$-homomorphism. M is the unique strictly continuous extension of $M \colon C_0(\mathbb{R}^n) \to \mathbb{B}(L^2(\mathbb{R}^n))$ to $\mathcal{M}(C_0(\mathbb{R}^n)) = C_b(\mathbb{R}^n)$. The strict topology on $C_b(\mathbb{R}^n)$ is the topology of uniform convergence on compact subsets.
For $g \in C_0(\mathbb{R}^n)$ and $f \in S$, we define $D_{gf} = M_g \circ \mathcal{F}^{-1} \circ M_f \circ \mathcal{F} \in \mathbb{B}(L^2(\mathbb{R}^n))$. Then $\|D_{gf}\| \leq \|g\|_\infty \cdot \|f\|_\infty$. Let $\mathcal{P}$ be the C^*-algebra generated by $\{D_{gf} \colon g \in C_0(\mathbb{R}^n), f \in S\}$.
Our aim is to prove $\mathcal{R} = \pi_1(\mathrm{Fix}_{\mathcal{R}})$.
If $h \in C_c(\mathbb{R}^n)$, then

$$\begin{aligned}(D_{gf}h)(x) &= g(x) \cdot ((\mathcal{F}^{-1} \circ M_f \circ \mathcal{F})h)(x) \\ &= \int_{\mathbb{R}^n} g(x) \cdot f(\xi) \cdot (\mathcal{F}h)(\xi) \cdot e^{2\pi\langle x,\xi\rangle} \,\mathrm{d}\xi.\end{aligned}$$

Therefore, D_{gf} is an order zero pseudo-differential operator.

Proposition 4.3. *If $g, f \in C_0(\mathbb{R}^n)$, then $D_{gf} \in \mathbb{K}(L^2(\mathbb{R}^n))$.*

Proof. Let first $g, f \in C_c(\mathbb{R}^n)$. Then the computation above shows

$$((M_g \circ \mathcal{F}^{-1} \circ M_f)h)(x) = \int_{\mathbb{R}^n} g(x) \cdot f(\xi) \cdot e^{2\pi\langle x,\xi\rangle} \cdot h(\xi) \,\mathrm{d}\xi.$$

Hence $M_g \circ \mathcal{F}^{-1} \circ M_f$ is an operator with compactly supported integral kernel. Theorem 2.4 shows $M_g \circ \mathcal{F}^{-1} \circ M_f \in \mathbb{K}(L^2(\mathbb{R}^n))$. Since $\mathbb{K}(L^2(\mathbb{R}^n))$ is an ideal in $\mathbb{B}(L^2(\mathbb{R}^n))$, we obtain $D_{gf} \in \mathbb{K}(L^2(\mathbb{R}^n))$
The assertion follows for $f, g \in C_0(\mathbb{R}^n)$, since $C_c(\mathbb{R}^n)$ is dense in $C_0(\mathbb{R}^n)$. □

The following lemma is helpful to prove the inclusion $\mathcal{R} \subseteq \pi_1(\mathrm{Fix}_{\mathcal{R}})$.

Lemma 4.4. *Let $A \subseteq \mathbb{B}(L^2(\mathbb{R}^n))$ be a C^*-algebra with a $*$-homomorphism*

$$\pi \colon A \to C_0(\mathbb{R}^n \times \mathbb{S}^{n-1}) \quad \textit{such that} \quad \mathbb{K}(L^2(\mathbb{R}^n)) \subseteq \ker(\pi).$$

Let $\mathcal{D}_1 \subseteq C_0(\mathbb{R}^n)$ be a dense set and $\mathcal{D}_2 \subseteq C(\mathbb{S}^{n-1})$ a set, such that the linear span of $\mathcal{D}_2$ is dense in $C(\mathbb{S}^{n-1})$.
Assume that for every $\tilde{f} \in \mathcal{D}_2$, there is $f \in S$ with $f^\infty = \tilde{f}$, such that $D_{gf} \in A$ and

$$\pi(D_{gf})(y,\xi) = g(y) \cdot f^\infty(\xi) \quad \textit{for all } g \in \mathcal{D}_1.$$

Then $\mathcal{P} \subseteq A$ and

$$\pi(D_{gf})(y,\xi) = g(y) \cdot f^\infty(\xi) \quad \textit{for all} \quad g \in C_0(\mathbb{R}^n) \textit{ and } f \in S.$$

Proof. Let $f_0 \in S$ with $f_0^\infty \in \mathrm{span}(\mathcal{D}_2)$. Then there are $m \in \mathbb{N}$, $\mu_1, \ldots, \mu_m \in \mathbb{C}$ and $\tilde{f}_1, \ldots, \tilde{f}_m \in \mathcal{D}_2$, such that $f_0^\infty = \sum_{i=1}^m \mu_i \tilde{f}_1$. By assumption there are $f_i \in S$, such that $f_i^\infty = \tilde{f}_i$ with

$$D_{gf_i} \in A \quad \text{and} \quad \pi(D_{gf_i})(y,\xi) = g(y) \cdot f_i^\infty(\xi) \text{ for all } g \in \mathcal{D}_1.$$

We define $f = \sum_{i=1}^m \mu_i f_i$. Then $f \in S$ with $f^\infty = f_0^\infty$. We have $D_{gf} = \sum_{i=1}^m \mu_i D_{gf_i} \in A$ and

$$\pi(D_{gf})(y,\xi) = \sum_{i=1}^m \pi(D_{gf_i})(y,\xi) = \sum_{k=1}^m g(y) \cdot f_i^\infty(\xi) = g(y) \cdot f^\infty(\xi) \text{ for all } g \in \mathcal{D}_1.$$

Since $(f_0 - f)^\infty = 0$ we have $f_0 - f \in C_0(\mathbb{R}^n)$ by Proposition 4.2. The previous proposition yields

$$D_{g(f_0-f)} \in \mathbb{K}(L^2(\mathbb{R}^n)) \subseteq A \quad \text{with} \quad \pi(D_{g(f_0-f)}) = 0 \text{ for all } g \in \mathcal{D}_1.$$

Hence $D_{gf_0} = D_{gf} + D_{g(f_0-f)} \in A$ with

$$\pi(D_{gf_0})(y,\xi) = g(y) \cdot f^\infty(\xi) = g(y) \cdot f_0^\infty(\xi) \text{ for all } g \in \mathcal{D}_1.$$

If $g \in C_0(\mathbb{R}^n)$, then there is a sequence $(g_k)_{k\in\mathbb{N}} \subseteq \mathcal{D}_1$ with $\|g - g_k\|_\infty \to 0$ for $k \to \infty$. We obtain $D_{gf_0} = \lim_{k\to\infty} D_{g_k f_0} \in A$ with

$$\pi(D_{gf_0})(y,\xi) = \lim_{k\to\infty} \pi(D_{g_k f_0})(y,\xi) = \lim_{k\to\infty} g_k(y) \cdot f_0^\infty(\xi) = g(y) \cdot f_0^\infty(\xi).$$

Now let $g \in C_0(\mathbb{R}^n)$ and $f \in S$ arbitrary. The $*$-homomorphism $s\colon S \to C(\mathbb{S}^{n-1})$ is surjective and continuous. Hence it is open by the Open Mapping Theorem. Therefore, the density of $\operatorname{span}(\mathcal{D}_2) \subseteq C(\mathbb{S}^{n-1})$ implies that $s^{-1}(\mathcal{D}_2) = \{f \in S\colon f^\infty \in \operatorname{span}(\mathcal{D}_2)\}$ is dense in S. Hence there is a sequence $(f_k)_{k\in\mathbb{N}} \subseteq s^{-1}(\mathcal{D}_2)$, such that $\|f - f_k\| \to 0$ for $k \to \infty$. We obtain $D_{gf} = \lim_{k\to\infty} D_{gf_k} \in A$ with

$$\pi(D_{gf})(y,\xi) = \lim_{k\to\infty} \pi(D_{gf_k})(y,\xi) = \lim_{k\to\infty} g(y) \cdot f_k^\infty(\xi) = g(y) \cdot f^\infty(\xi).$$

Since all the generators D_{gf} of $\mathcal{P}$ are in A. We conclude $\mathcal{P} \subseteq A$. $\square$

Proposition 4.5. $\mathbb{K}(L^2(\mathbb{R}^n)) \subseteq \mathcal{P}$.

Proof. Let $k \in L^2(\mathbb{R}^n \times \mathbb{R}^n)$. If $h \in L^2(\mathbb{R}^n)$, we define $(T_k h)(x) = \int_{\mathbb{R}^n} k(x,y)h(y)\,\mathrm{d}y$. Using the Cauchy–Schwarz inequality we obtain $\|T_k\| \leq \|k\|_2 \cdot \|h\|_2$. Hence T_k extends to a bounded operator $L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ with $\|T_k\| \leq \|k\|_2$. We obtain a bounded operator $T\colon L^2(\mathbb{R}^n \times \mathbb{R}^n) \to \mathbb{B}(L^2(\mathbb{R}^n))$, $k \mapsto T_k$. For $g, f \in C_c(\mathbb{R}^n)$, we define $(g \otimes f)(x,\xi) = g(x) \cdot f(\xi)$. Then $g \otimes f \in L^2(\mathbb{R}^n \times \mathbb{R}^n)$. Using Lemma 5.5 we see, that the linear span of $D = \{g \otimes f\colon g, f \in C_c(\mathbb{R}^n)\}$ is dense in $L^2(\mathbb{R}^n \times \mathbb{R}^n)$. $T_{g\otimes f}$ is a rank-one operator and every rank one operator is of this form. Hence $T(L^2(\mathbb{R}^n \times \mathbb{R}^n)) \subseteq \mathbb{K}(L^2(\mathbb{R}^n))$ is dense.[4] We define $\Psi\colon L^2(\mathbb{R}^n \times \mathbb{R}^n) \to L^2(\mathbb{R}^n \times \mathbb{R}^n)$ by $(\Psi(k))(x,\xi) = k(x,\xi) \cdot e^{2\pi i\langle x,\xi\rangle}$. Then Ψ is a unitary operator. If $g, f \in C_c(\mathbb{R}^n)$, then

$$\begin{aligned}(T_{\Psi(g\otimes f)}h)(x) &= \int_{\mathbb{R}^n} g(x) \cdot f(\xi) e^{2\pi i\langle x,\xi\rangle} h(\xi)\,\mathrm{d}\xi \\ &= ((M_g \cdot \mathcal{F}^{-1} \circ M_f)h)(x).\end{aligned}$$

[4] $T(L^2(\mathbb{R}^n \times \mathbb{R}^n))$ is the set of Hilbert-Schmidt operators on $L^2(\mathbb{R}^n)$.

Hence $\mathcal{F} \cdot T_{\Psi(g\otimes f)} = D_{gf} \in \mathcal{P}$. Since Ψ is unitary $\Psi(\mathrm{span}(D))$ is dense in $L^2(\mathbb{R}^n \times \mathbb{R}^n)$. Therefore, $(T\circ\Psi)(\mathrm{span}(D))$ is dense in $\mathbb{K}(L^2(\mathbb{R}^n))$. Since $\mathcal{P}$ is closed, we have $\mathcal{F}\circ S \in \mathcal{P}$ for all $S \in (T\circ\Psi)(\mathrm{span}(D))$. Let $K \in \mathbb{K}(L^2(\mathbb{R}^n))$ and $\varepsilon > 0$. Then $\mathcal{F}^{-1}\circ K \in \mathbb{K}(L^2(\mathbb{R}^n))$. So there is $S \in (T\circ\Psi)(\mathrm{span}(D))$ such that

$$\|K - \mathcal{F}\circ S\| = \|\mathcal{F}^{-1}\circ K - S\| < \varepsilon.$$

Since $\mathcal{F}\cdot S \in \mathcal{P}$ this implies $K \in \overline{\mathcal{P}} = \mathcal{P}$. $\square$

From now on fix $\omega \in C_c^\infty([0,\infty))$ with $0 \le \omega \le 1$, $\omega|_{[0,1]} = 1$ and $\omega|_{[2,\infty)} = 0$.

Lemma 4.6. *Let $f \in C_c^\infty(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} f(x)\ dx = 0$. For every $\xi \in \mathbb{R}^n$ the integral*

$$L_f(\xi) := \int_{\mathbb{R}^n} \omega(\lambda^{-1}) \cdot (\mathcal{F}f)(\lambda^{-1}\xi)\ \frac{d\lambda}{\lambda}$$

exists. We have $L_f \in S$ with

$$L_f^\infty(\xi_0) = \int_{\mathbb{R}^n} (\mathcal{F}f)(\lambda^{-1}\xi_0)\ \frac{d\lambda}{\lambda} \qquad \text{for } \xi_0 \in \mathbb{S}^{n-1}.$$

The sequence $(L_{f,k})_{k\in\mathbb{N}} \subseteq C_0(\mathbb{R}^n)$ given by

$$L_{f,k}(\xi) = \int_{\mathbb{R}^n} \chi_k(\lambda)\cdot\omega(\lambda^{-1}) \cdot (\mathcal{F}f)(\lambda^{-1}\xi)\ \frac{d\lambda}{\lambda}$$

converges uniformly on compact subsets to L_f.

Proof. Let $R > 0$. $\mathcal{F}f$ is smooth with $(\mathcal{F}f)(0) = \int_{\mathbb{R}^n} f(x)\ \mathrm{d}x = 0$. The Mean Value Theorem yields a constant $C > 0$, such that $|(\mathcal{F}f)(\xi)| \le C\|\xi\|$ for $\|\xi\| \le 2R$. If $\|\xi\| \le R$, then

$$|\omega(\lambda^{-1}) \cdot (\mathcal{F}f)(\lambda^{-1}\xi)| \le \chi_{(\frac{1}{2},\infty)}(\lambda) \cdot C \cdot \|\lambda^{-1}\xi\| \le CR\lambda^{-1} \cdot \chi_{(\frac{1}{2},\infty)}(\lambda).$$

Since $CR\lambda^{-1} \cdot \chi_{(\frac{1}{2},\infty)}(\lambda) \in L^1(\mathbb{R}_+^*,\ \frac{\mathrm{d}\lambda}{\lambda})$ this shows the existence of the integral. The Dominated Convergence Theorem yields

$$|L_f(\xi) - L_{f,k}(\xi)| \le \int_{(\frac{1}{2},\infty)} (1-\chi_k(\lambda)) \cdot CR \cdot \frac{1}{\lambda^2}\ \mathrm{d}\lambda \longrightarrow 0 \quad \text{for } k \to \infty.$$

Hence $L_{f,k} \xrightarrow{k\to\infty} L_f$ uniformly on $\{\xi \in \mathbb{R}^n \colon \|\xi\| \le R\}$. Therefore, L_f is continuous. Next we prove that L_f is bounded. As above there is $C > 0$, such that $|(\mathcal{F}f)(\xi)| \le C\|\xi\|$ for $\|\xi\| \le 1$. From

$$\left(\mathcal{F}\left(\sum_{j=1}^n \partial_{x_j}^2 f\right)\right)(\xi) = (2\pi i)^{2n} \cdot \|\xi\|^2 (\mathcal{F}f)(\xi)$$

we get $|(\mathcal{F}f)(\xi)| \le D\|\xi\|^{-2}$ for a constant $D > 0$. For $\xi_0 \in \mathbb{S}^{n-1}$ we estimate

$$|(\mathcal{F}f)(\lambda^{-1}\xi_0)| \le D \cdot \chi_{(0,1]}\lambda^2 + C\cdot\chi_{(1,\infty)}\lambda^{-1}.$$

We define $\theta(\lambda) = D \cdot \chi_{(0,1]}\lambda^2 + C \cdot \chi_{(1,\infty)}\lambda^{-1}$. Then $\theta \in L^1(\mathbb{R}_+^*, \frac{\mathrm{d}\lambda}{\lambda})$. We estimate

$$\begin{aligned}|L_f(\xi)| &\leq \int_{\mathbb{R}_+^*} |(\mathcal{F}f)(\lambda^{-1}\xi)| \frac{\mathrm{d}\lambda}{\lambda} \\ &= \int_{\mathbb{R}_+^*} \left|(\mathcal{F}f)\left(\lambda^{-1}\frac{\xi}{\|\xi\|}\right)\right| \frac{\mathrm{d}\lambda}{\lambda} \\ &= \int_{\mathbb{R}_+^*} \theta(\lambda) \frac{\mathrm{d}\lambda}{\lambda}.\end{aligned}$$

Therefore, L is bounded.
Let $\varepsilon > 0$. Since $\theta \in L^1(\mathbb{R}_+^*, \frac{\mathrm{d}\lambda}{\lambda})$, there is $\delta > 0$, such that $\int_{(0,\delta)} \theta(\lambda) \frac{\mathrm{d}\lambda}{\lambda} < \varepsilon$. Let $s > \frac{1}{\delta}$ and $\xi_0 \in \mathbb{S}^{n-1}$. Then

$$\begin{aligned}\left|\int_{\mathbb{R}_+^*} (\mathcal{F}f)(\lambda^{-1}\xi_0) \frac{\mathrm{d}\lambda}{\lambda} - L_f^s(\xi_0)\right| &= \left|\int_{\mathbb{R}_+^*} (\mathcal{F}f)(\lambda^{-1}\xi_0) - \int_{\mathbb{R}_+^*} \omega(\lambda^{-1})(\mathcal{F}f)(\lambda^{-1}s\xi_0) \frac{\mathrm{d}\lambda}{\lambda}\right| \\ &\leq \int_{\mathbb{R}_+^*} (1 - \omega((s\lambda)^{-1}))\|(\mathcal{F}f)(\lambda^{-1}\xi_0) \frac{\mathrm{d}\lambda}{\lambda} \\ &\leq \int_{(0,\frac{1}{s})} \theta(\lambda) \frac{\mathrm{d}\lambda}{\lambda} \leq \int_{(0,\delta)} \frac{\mathrm{d}\lambda}{\lambda} < \varepsilon.\end{aligned}$$

Hence the net $(L_f^s)_{s>0}$ converges uniformly to $\int_{\mathbb{R}_+^*} (\mathcal{F}f)(\lambda^{-1}\xi_0) \frac{\mathrm{d}\lambda}{\lambda}$ on $\mathbb{S}^{n-1}$. Hence $L \in S$ with $L^\infty(\xi_0) = \int_{\mathbb{R}_+^*} (\mathcal{F}f)(\lambda^{-1}\xi_0) \frac{\mathrm{d}\lambda}{\lambda}$ by Proposition 4.1. □

If $k, g \in C_c(\mathbb{R}^n)$, we define $\mathrm{Con}_k(g) = k * g$. We obtain

$$(\mathcal{F} \circ \mathrm{Con}_k)(g) = \mathcal{F}(k * g) = \mathcal{F}(k) \cdot \mathcal{F}(g) = (M_{\mathcal{F}(k)} \circ \mathcal{F})(g).$$

Hence $\mathrm{Con}_k = \mathcal{F}^{-1} \circ M_{\mathcal{F}(k)} \circ \mathcal{F} \in \mathbb{B}(L^2(\mathbb{R}^n))$.

Theorem 4.7. $\pi_1(\mathrm{Fix}_{\mathcal{R}}) = \mathcal{P}$. *If $g \in C_0(\mathbb{R}^n)$ and $f \in S$, then*

$$\mathrm{sym}(D_{gf})(y, \xi) = g(y) \cdot f^\infty(\xi).$$

Proof. Let $g, k_1, k_2 \in C_c^\infty(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} k_1(x)\,\mathrm{d}x = \int_{\mathbb{R}^n} k_2(x)\,\mathrm{d}x = 0$. We define $f = k_1 * k_2^*$ and $f_\lambda(x) = \omega(\lambda^{-1}) \cdot \lambda^n \cdot f(\lambda x)$. By Lemma 3.6 there are $f_1, f_2 \in \mathcal{R}_0$, such that $(f_1 * f_2^*)(y, t, x) = g(y + tx) \cdot \omega(t) \cdot f(x)$. Let $\lambda \in \mathbb{R}_+^*$. We compute

$$\begin{aligned}(\pi_1(\sigma_\lambda(f_1 * f_2^*))h)(x) &= \int_{\mathbb{R}^n} \lambda^n \cdot (f_1 * f_2^*)(y, \lambda^{-1}, \lambda(x - y)) \cdot h(y)\,\mathrm{d}y \\ &= \int_{\mathbb{R}^n} \lambda^n \cdot g(x) \cdot \omega(\lambda^{-1}) \cdot f(\lambda(x - y)) \cdot h(y)\,\mathrm{d}y \\ &= g(x) \cdot \int_{\mathbb{R}^n} f_\lambda(x - y) \cdot h(y)\,\mathrm{d}y \\ &= ((M_g \circ \mathrm{Con}_{f_\lambda})h)(x)\end{aligned}$$

Hence $\pi_1(\sigma_\lambda(f_1 * f_2^*)) = M_g \circ \mathcal{F}^{-1} \circ M_{\mathcal{F}(f_\lambda)} \circ \mathcal{F}$. If $\xi \in \mathbb{R}^n$, then

$$(\mathcal{F}f_\lambda)(\xi) = \int_{\mathbb{R}^n} \lambda^n \omega(\lambda^{-1}) \cdot f_\lambda(x) \cdot e^{2\pi i \langle x, \xi \rangle} \, \mathrm{d}x = \omega(\lambda^{-1}) \cdot (\mathcal{F}f)(\lambda^{-1}\xi).$$

Let L_f and $L_{f,k}$ as in Lemma 4.6. We have

$$\begin{aligned}
\pi_1(\Lambda_{f_1} \circ \Lambda_{f_2}^*) &\overset{1.36}{=} \overset{s}{\lim_{k\to\infty}} \int_{\mathbb{R}_+^*} \chi_k(\lambda) \cdot \pi_1(\sigma_\lambda(f_1 * f_2^*)) \, \frac{\mathrm{d}\lambda}{\lambda} \\
&= \overset{s}{\lim_{k\to\infty}} \int_{\mathbb{R}_+^*} \chi_k(\lambda)(M_g \circ \mathcal{F}^{-1} \circ M_{\mathcal{F}(f_\lambda)}\mathcal{F}) \, \frac{\mathrm{d}\lambda}{\lambda} \\
&= M_g \circ \mathcal{F}^{-1} \circ \left(\overset{s}{\lim_{k\to\infty}} M_{L_k} \right) \circ \mathcal{F} \\
&= M_g \circ \mathcal{F}^{-1} \circ M_L \circ \mathcal{F} = D_{gL},
\end{aligned}$$

because $L_{f,k} \to L_f$ uniformly on compact subsets. Let $(y, \xi_0) \in \mathbb{R}^n \times \mathbb{S}^{n-1}$. We compute

$$\begin{aligned}
\mathrm{sym}(D_{gL})(y, \xi) &= (\pi_0^r \circ \pi_1^{-1})(D_{gL})(y, \xi_0) \\
&= \pi_0^r(\Lambda_{f_1} \circ \Lambda_{f_2}^*)(y, \xi_0) \\
&= \left(\overset{s}{\lim_{k\to\infty}} \int_{\mathbb{R}_+^*} \chi_k(\lambda) \sigma_\lambda(\pi_0(f_1 * f_2^*)) \, \frac{\mathrm{d}\lambda}{\lambda} \right)(y, \xi_0) \\
&= \lim_{k\to\infty} \int_{\mathbb{R}_+^*} \chi_k(\lambda) \pi_0(f_1 * f_2^*)(y, \xi_0) \, \frac{\mathrm{d}\lambda}{\lambda} \\
&= g(y) \cdot \lim_{k\to\infty} \int_{\mathbb{R}_+^*} \chi_k(\lambda)(\mathcal{F}f)(\lambda^{-1}\xi_0) \, \frac{\mathrm{d}\lambda}{\lambda} \\
&= g(y) \cdot \int_{\mathbb{R}_+^*} (\mathcal{F}f)(\lambda^{-1}\xi_0) \, \frac{\mathrm{d}\lambda}{\lambda} \\
&\overset{4.6}{=} g(y) \cdot L_f^\infty(\xi_0).
\end{aligned}$$

In Lemma 4.6 we saw that $|(\mathcal{F}f)(\lambda^{-1}\xi)|$ is dominated by a $L^1(\mathbb{R}_+^*, \frac{\mathrm{d}\lambda}{\lambda})$-function. Hence second last equality holds by the Dominated Convergence Theorem.

Let $E = \pi_1^{-1}(\mathcal{P}) \subseteq \mathrm{Fix}_\mathcal{R}$. Then E is a C^*-subalgebra of $\mathrm{Fix}_\mathcal{R}$. The computation above shows, that $\Lambda_{f_1} \circ \Lambda_{f_2}^* \in E$, for all $f_1, f_2 \in \mathcal{R}_0$ that are obtained from the construction in Lemma 3.6. For the Stone-Weierstraß argument in the proof of Theorem 3.7 we only used functions obtained by the construction in Lemma 3.6. Therefore, the same arguments as in the proof of Lemma 3.6 show $\pi_0^r(E) = C_0(\mathbb{R}^n \times \mathbb{S}^{n-1})$.

Let $F \in \ker(\pi_0^r)$. Then $\pi_1(F) \in \mathbb{K}(L^2(\mathbb{R}^n))$ by Corollary 3.17. Hence $\pi_1(F) \in \mathcal{P}$ by Proposition 4.5. That is $F \in E$. Therefore, $\ker(\pi_0^r) \subseteq E$.

Let $F_0 \in \mathrm{Fix}_\mathcal{R}$. Then there is $F \in E$ with $\pi_0^r(F_0) = \pi_0^r(F)$. Hence $F_0 = (F_0 - F) + F \in E$. Therefore, $\mathrm{Fix}_\mathcal{R} = E$, so that $\pi_1(\mathrm{Fix}_\mathcal{R}) \subseteq \mathcal{P}$.

To prove the other inclusion, we use Lemma 4.4. By the computations above, we have $D_{gL_{k_1 * k_2^*}} \in \pi_1(\mathrm{Fix}_\mathcal{R})$ and $\mathrm{sym}(D_{gL_{k_1 * k_2}})(y, \xi_0) = g(y) \cdot L^\infty_{k_1 * k_2^*}(\xi_0)$ for all $g \in C_c^\infty(\mathbb{R}^n)$ and

$L_{k_1 * k_2}$ with $k_1, k_2 \in C_c^\infty(\mathbb{R}^n)$ integrating to 0. Let $\mathcal{D}_1 = C_c^\infty(\mathbb{R}^n)$ and $\mathcal{D}_2 \subset C(\mathbb{S}^{n-1})$ be the set of functions $q \in C(\mathbb{S}^{n-1})$ of the form $q(\xi) = \xi_1^{m_1} \cdot \ldots \cdot \xi_n^{m_n}$ for $m_1, \ldots, m_n \in \mathbb{N}$. Then $\mathcal{D}_1$ is dense in $C_0(\mathbb{R}^n)$ and the linear span of $\mathcal{D}_2$ is dense in $C(\mathbb{S}^{n-1})$ by the Stone-Weierstraß Theorem. By Corollary 3.17 we have $\mathbb{K}(L^2(\mathbb{R}^n)) = \ker(\mathrm{sym})$.
Let $q \in \mathcal{D}_2$ with $q(\xi) = \xi_1^{m_1} \cdot \ldots \cdot \xi_n^{m_n}$. Let $k_2 \in C_c^\infty(\mathbb{R}^n)$, such that $\int_{\mathbb{R}^n} k_2(x)\ \mathrm{d}x = 0$ and $(\mathcal{F}k_2) = \tilde{g}(\|\xi\|)$ with $\tilde{g} \in C_c^\infty([0,\infty))$ not equal to 0. We constructed such a function in Theorem 3.7. We define $m = \sum_{j=1}^n m_j$ and

$$C = (2\pi i)^m \cdot \int_{\mathbb{R}_+^*} \lambda^m |\tilde{g}(\lambda)|^2 \ \frac{\mathrm{d}\lambda}{\lambda}.$$

Then $C \neq 0$. Let $k_1 = \frac{1}{C} \cdot \partial_{x_1}^{m_1} \ldots \partial_{x_n}^{m_n} k_2$. Then $k_1 \in C_c^\infty(\mathbb{R}^n)$ with

$$(\mathcal{F}k_1)(\xi) = \frac{1}{C}(2\pi i)^m q(\xi)(\mathcal{F}k_2)(\xi).$$

Since $q(0) = 0$ we have $\int_{\mathbb{R}^n} k_1(x)\ \mathrm{d}x = 0$. Let $\xi_0 \in \mathbb{S}^{n-1}$. We compute

$$\begin{aligned} L^\infty_{k_1 * k_2^*}(\xi_0) &= \int_{\mathbb{R}_+^*} \mathcal{F}(k_1 * k_2^*)(\lambda^{-1}\xi_0)\ \frac{\mathrm{d}\lambda}{\lambda} \\ &= \int_{\mathbb{R}_+^*} (\mathcal{F}k_1)(\lambda^{-1}\xi_0) \cdot \overline{(\mathcal{F}k_2)(\lambda^{-1}\xi_0)}\ \frac{\mathrm{d}\lambda}{\lambda} \\ &= \frac{1}{C}(2\pi i)^m \cdot q(\xi_0) \cdot \int_{\mathbb{R}_+^*} \lambda^{-m} |(\mathcal{F}k_2)(\lambda^{-1}\xi_0)\ \frac{\mathrm{d}\lambda}{\lambda} = q(\xi_0). \end{aligned}$$

This shows that for every $q \in \mathcal{D}_2$ there is $L \in S$, such that $L^\infty = q$ and $D_{gL} \in \pi_1(\mathrm{Fix}_\mathcal{R})$ and $\mathrm{sym}(D_{gL})(y, \xi_0) = g(y) \cdot L^\infty(\xi_0)$ for all $g \in \mathcal{D}_1$. Hence Lemma 4.4 applies and yields $\mathcal{P} \subseteq \pi_1(\mathrm{Fix}_\mathcal{R})$, with $\mathrm{sym}(D_{gf})(y, \xi_0) = g(y) \cdot f^\infty(\xi_0)$ for all $g \in C_0(\mathbb{R}^n \times \mathbb{S}^{n-1})$ and $f \in S$. ☐

Corollary 4.8. *The sequence*

$$0 \to \mathbb{K}(L^2(\mathbb{R}^n)) \longrightarrow \mathcal{P} \xrightarrow{\mathrm{sym}} C_0(\mathbb{R}^n \times \mathbb{S}^{n-1}) \to 0$$

is exact.

Proof. Since $\mathcal{P} = \pi_1(\mathrm{Fix}_\mathcal{R})$, this is the same the statement as in Corollary 3.17. ☐

4.1 Comparison to the Pseudodifferential Operator Extension described by Higson and Roe

In Higson and Roe's Analytic K-homology there is a brief discussion on a C^*-algebra generated by pseudodifferential operators on a smooth manifold.[2, pages 46-48] In the case of $\mathbb{R}^n$ the construction is as follows:
Consider complex-valued function u on $T^*\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$ with the following properties:

- $u(x, t\xi) = u(x, \xi)$ for $t \geq 1$ and $\|\xi\| \geq 1$.

- $u(x,\xi)$ vanishes for x outside a compact subset of $\mathbb{R}^n$.

Define the linear map $T_u\colon C_c^\infty(\mathbb{R}^n) \to C_c^\infty(\mathbb{R}^n)$ by the formula

$$(T_u f)(x) = \int_{\mathbb{R}^n} u(x,\xi)(\mathcal{F}f)(\xi)e^{2\pi i\langle x,\xi\rangle}\,\mathrm{d}\xi.$$

The first condition on u implies that u gives rise to a function on the cosphere bundle $S^*\mathbb{R}^n = \mathbb{R}^n \times \mathbb{S}^{n-1}$. This function is called the *symbol* of T_u. T_u extends to $L^2(\mathbb{R}^n)$, so define the C^*-algebra $\mathfrak{P}(\mathbb{R}^n)$ generated by the T_u. Higson and Roe claim that the map sending T_u to its symbol extends to a surjective $*$-homomorphism from $\mathfrak{P}(\mathbb{R}^n)$ to $C_0(\mathbb{R}^n \times \mathbb{S}^{n-1})$ whose kernel is precisely $\mathbb{K}(L^2(\mathbb{R}^n))$.
Using functions of the form $u(x,\xi) = g(x)\cdot f(\xi)$ where $g \in C_c^\infty(\mathbb{R}^n)$ and $f \in H$ smooth we obtain $T_u = D_{gf} \in \mathfrak{P}(\mathbb{R}^n)$ and the symbol of D_u is

$$\mathrm{sym}(D_u)(x,\xi) = g(x)\cdot f^\infty(\xi) = g(x)\cdot f(\xi).$$

Lemma 4.4 yields $\mathcal{P} \subseteq \mathfrak{P}(\mathbb{R}^n)$. Both $\mathcal{P}$ and $\mathfrak{P}(\mathbb{R}^n)$ are extensions of $C_0(\mathbb{R}^n \times \mathbb{S}^{n-1})$ by $\mathbb{K}(L^2(\mathbb{R}^n))$ with compatible $*$-homomorphisms onto $C_0(\mathbb{R}^n \times \mathbb{S}^{n-1})$. Hence $\mathcal{P} = \mathfrak{P}(\mathbb{R}^n)$.
Therefore, the generalised fixed point algebra $\mathrm{Fix}_{\mathcal{R}}$ of the scaling action on $J \lhd C_r^*(\mathcal{G}\mathbb{R}^n)$ is isomorphic to $\pi_1(\mathrm{Fix}_{\mathcal{R}}) = \mathcal{P} = \mathfrak{P}(\mathbb{R}^n)$.
By Remark 1.50 the C^*-algebra $\mathfrak{P}(\mathbb{R}^n)$ is Morita-Rieffel equivalent to an ideal of $C_r^*(\mathbb{R}_+^*, J)$. The existence of such a Morita-Rieffel equivalence is a special case of Debords and Skandalis' results in [5].

5 Appendix

5.1 Positive Linear Functionals

To define an inner product on $C_c(G,A)$ and in many other places in this work we need to integrate functions with values in C^*-algebras. We define and describe the integral using positive linear functionals and deduce the needed properties from the standard $\mathbb{C}$-valued integral.
For this we first need two lemmas about positive linear functionals. A positive linear functional ϕ on A is a linear map $\phi\colon A \to \mathbb{C}$, such that $\phi(a) \geq 0$ for all positive $a \in A$. We denote the set of all positive linear functionals on A by $\mathcal{P}(A)$.

Lemma 5.1 (Selfadjointness)**.**
Let ϕ be a positive linear functional, then

$$\phi(a^*) = \overline{\phi(a)} \text{ for all } a \in A \text{ and } \phi(a) \in \mathbb{R} \text{ if } a \text{ is self-adjoint.}$$

Proof. First let $a \in A$ be self-adjoint. Then we have $a = a^+ - a^-$ for $a^+, a^- \in A$ positive. Therefore, $\phi(a) = \phi(a^+) - \phi(a^-) \in \mathbb{R}$ as a difference of two non-negative numbers.
Now take an arbitrary $a \in A$ and write $a = a_1 + ia_2$, where a_1, a_2 are the real and imaginary part of a. Since a_1, a_2 are self-adjoint, we obtain

$$\phi(a^*) = \phi(a_1 + ia_2) = \phi(a_1) - i\phi(a_2) = \overline{\phi(a_1) + i\phi(a_2)} = \overline{\phi(a)}. \qquad \square$$

Lemma 5.2 (Boundedness [6, Lemma I.9.5]).
Positive linear functionals are bounded.

Proof. Assume that ϕ is a positive linear functional and that there is a sequence $(a_n)_{n\in N} \subseteq A$ of positive elements with $\|a_n\| \leq 1$, such that $\phi(a_n) > 2^n$. Since the positive cone is closed, we have $\sum_{n=1}^{\infty} 2^{-n} a_n \geq \sum_{n=1}^{N} 2^{-n} a_k$ so

$$f\left(\sum_{n=1}^{\infty} 2^{-n} a_n\right) \geq f\left(\sum_{n=1}^{N} 2^{-n} a_n\right) \geq N$$

for all $N \in \mathbb{N}$, which is absurd. Therefore, there is $M > 0$, such that $\phi(a) \leq M$ for all $a \in A$ positive with $\|a\| \leq 1$.
If $a = a^* \in A$ with $\|a\| \leq 1$, then we write $a = a^+ - a^-$ for $a^+, a^- \in A$ positive with $\|a^+\|, \|a^-\| \leq \|a\| \leq 1$ and obtain $|\phi(a)| \leq \phi(a^+) + \phi(a^-) \leq 2M$.
Finally, for $a \in A$ arbitrary, write $a = a_1 + ia_2$ for a_1, a_2 self-adjoint. Then $\|a_1\|, \|a_2\| \leq \|a\|$, so that $|\phi(a)| \leq |\phi(a_1)| + |\phi(a_2)| \leq 4M$. This proves that ϕ is bounded on the unit ball of A. □

The next lemma says that there are enough positive linear functionals to detect the elements of A and their positivity or self-adjointness.
Its proof uses that for every self-adjoint element $a \in A$ there is a positive linear functional ϕ, such that $|\phi(a)| = \|a\|$ and $\|\phi\| = 1$. This fact is proved by a Hahn-Banach type argument and is used in the proof of the Gelfand–Naimark Theorem, which says that every C^*-algebra has a faithful representation (see for example [6, Lemma I.9.10] or [7, Chapter 4.3]).

Lemma 5.3 (Separation and Positivity via positive linear functionals).
For $a, b \in A$ we have

(i) *$\phi(a) = \phi(b)$ for all $\phi \in \mathcal{P}(A)$ iff $a = b$;*

(ii) *$\phi(a) \geq 0$ for all $\phi \in \mathcal{P}(A)$ iff $a \geq 0$.*

Proof. (i) First assume $a \in A$ such that $\phi(a) = 0$ for every $\phi \in \mathcal{P}(A)$. As mentioned above, for a self-adjoint, we can pick $\phi \in \mathcal{P}(A)$ with $\|a\| = \phi(a) = 0$. Hence $a = 0$. For $a \in A$ arbitrary, we decompose $a = a_1 + ia_2$ for a_1, a_2 self-adjoint. From

$$0 = \phi(a) = \phi(a_1) + i\phi(a_2),$$

we deduce $\phi(a_1) = \phi(a_2) = 0$, since those are real numbers by Lemma 5.1. From the self-adjoint case, we get $a_1 = a_2 = 0$. Hence $a = 0$.
Now assume $a, b \in A$, such that $\phi(a) = \phi(b)$ for all $\phi \in \mathcal{P}(A)$. Then $\phi(b - a) = 0$ for all $\phi \in \mathcal{P}(A)$ by linearity. Therefore, $a - b = 0$. Hence $a = b$. The other implication is trivial.

(ii) [8, Remark 2.6] Let $\pi\colon A \to \mathbb{B}(\mathcal{H})$ be a faithful representation on a Hilbert space $\mathcal{H}$. To $\xi \in \mathcal{H}$ we associate the positive linear functional $\phi_\xi(a) = \langle \pi(a)\xi, \xi\rangle$.
For $a \in A$ with $\phi(a) \geq 0$ for all $\phi \in \mathcal{P}(A)$ we obtain $0 \leq \phi_\xi(a) = \langle \pi(a)\xi, \xi\rangle$ for all $\xi \in \mathcal{H}$. Hence $\pi(a)$ is a positive operator on $\mathcal{H}$. Those are exactly the positive elements of $\mathbb{B}(\mathcal{H})$. We have $\pi(a^*) = \pi(a)^* = \pi(a)$, so a is self-adjoint by injectivity of π. By spectral permanence we have $\sigma_A(a) = \sigma_{\mathbb{B}(\mathcal{H})}(\pi(a)) \subseteq [0, \infty)$. Therefore, a is positive. The other implication is true by definition of positive linear functionals. $\square$

Remark 5.4. In Lemma 5.3 we used the existence of $\phi \in \mathcal{P}(A)$ with $|\phi(a)| = \|a\|$ and $\|\phi\| = 1$ when a is self-adjoint. The norm of a general element is not detected by its values on positive linear functionals in that way. For example, consider the element $a = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) \in \mathbb{M}_2(\mathbb{C})$. We have $\|a\| = 1$, but for $\phi \in \mathcal{P}(\mathbb{M}_2(\mathbb{C}))$ with $\|\phi\| = 1$ we obtain

$$|\phi(a)| = \frac{1}{2}\left|\phi\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) + i\phi\left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right)\right| = \frac{1}{2}\sqrt{\left|\phi\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\right|^2 + \left|\phi\left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right)\right|^2} \leq \frac{1}{2}\sqrt{2} < 1.$$

5.2 C^*-Valued Integration

Let X be a locally compact Hausdorff space and μ a locally finite and strictly positive Borel measure on X. Locally finite means that μ is finite on compact subsets of X. Equivalently, every point $x \in X$ has a neighbourhood of finite measure. Strictly positive means, that $\mu(U) > 0$ for every nonempty open set $U \subseteq X$.
We will first work in the more general setting of Banach space-valued functions. Let V be a Banach space. We denote the vector space of compactly supported continuous functions $X \to V$ by $C_c(X, V)$ and we write $C_c(X)$ for $C_c(X, \mathbb{C})$.
For $f \in C_c(X, V)$, the function $X \to \mathbb{R}_{\geq 0}$ given by $x \mapsto \|f(x)\|$ is compactly supported and continuous. Therefore, the Lebesgue integral $\|f\|_1\colon = \int_X \|f(x)\| \, \mathrm{d}\mu(x)$ exists. It is easy to verify that $\|\cdot\|_1$ defines a norm on $C_c(X, V)$.
For $f \in C_c(X)$ and $a \in V$ we obtain $f.a \in C_c(X, V)$ with $(f.a)(x) = f(x)a$ for $x \in X$. Let

$$M = \operatorname{span}\{f.a\colon f \in C_c(X), a \in V\} \subseteq C_c(X, V).$$

To define the integral, we are going to use the following lemma to approximate compactly supported functions by elements of M.

Lemma 5.5. *Let $f \in C_c(X, V)$. For every $\varepsilon > 0$ and $U \subseteq X$ open with* $\operatorname{supp}(f) \subseteq U$, *there is $h \in M$ with* $\operatorname{supp}(h) \subseteq U$ *and $\|f(x) - h(x)\| < \varepsilon$ for all $x \in X$.*

Proof. Every $x \in \operatorname{supp}(f)$ has an open neighbourhood $U_x \subseteq U$ with compact closure and $\|f(x) - f(y)\| < \varepsilon$ for all $y \in U_x$. By the compactness of $\operatorname{supp}(f)$, we find $x_1, \ldots, x_n \in \operatorname{supp}(f)$ such that $U_{x_1}, \ldots U_{x_n}$ cover $\operatorname{supp}(f)$. Put $a_i = f(x_i)$ and let $h_1, \ldots, h_n\colon X \to [0, 1]$ be a continuous partition of unity for $\operatorname{supp}(f)$ subordinate to the U_{x_i}'s. That is, $0 \leq \sum_{i=1}^n h_i(x) \leq 1$ for all $x \in X$, $\sum_{i=1}^n h_i(x) = 1$ for $x \in \operatorname{supp}(f)$ and $\operatorname{supp}(h_i) \subseteq U_i$. Since $\operatorname{supp}(h_i) \subseteq \overline{U_{x_i}}$ is compact, we have $h_i \in C_c(X)$. Therefore,

$h = \sum_{i=1}^n h_i.a_i \in M$ and $\operatorname{supp}(h) \subseteq \operatorname{supp}(h_1) \cup \ldots \cup \operatorname{supp}(h_n) \subseteq U$. We estimate

$$\begin{aligned}\|f(x) - h(x)\| &= \left\| \left(\sum_{i=1}^n h_i(x) \right) f(x) - \sum_{i=1}^n h_i(x) a_i \right\| \\ &\leq \sum_{i=1}^n \|h_i(x) f(x) - h_i(x) a_i\| \\ &= \sum_{i=1}^n h_i(x) \cdot \|f(x) - f(x_i)\| \\ &< \sum_{i=1}^n h_i(x) \cdot \varepsilon = \varepsilon. \qquad \square\end{aligned}$$

Corollary 5.6. *The subspace M is dense in $C_c(X, V)$ with respect to $\|\cdot\|_1$.*

Proof. Let $f \in C_c(X, A)$ and $\varepsilon > 0$. Since X locally compactness, there is an open $U \subseteq X$ with compact closure and $\operatorname{supp}(f) \subseteq U$. Because μ is locally finite and strictly positive, we have $0 < \mu(U) \leq \mu\left(\overline{U}\right) < \infty$.
By Lemma 5.5, there is $h \in M$ with $\operatorname{supp}(h) \subseteq U$ and

$$\|f(x) - h(x)\| < \frac{\varepsilon}{\mu(U)} \qquad \text{for all } x \in U.$$

Then

$$\|f - h\|_1 = \int_X \|f(x) - h(x)\| \, \mathrm{d}\mu(x) = \int_U \|f(x) - h(x)\| \, \mathrm{d}\mu(x) < \mu(U) \cdot \frac{\varepsilon}{\mu(U)} = \varepsilon. \ \square$$

The next proposition defines and characterises the integral. We write V' for the topological dual space of V.

Proposition 5.7 (Banach space-valued integral)**.**
For every $f \in C_c(X, V)$, there is a unique $I(f) \in V$, such that

$$\phi\left(I(f)\right) = \int_X \phi \circ f \ \ d\mu \ \text{ for all } \phi \in V'. \tag{3}$$

The map $I\colon C_c(X, V) \to V$ given by $f \mapsto I(f)$ is linear and bounded with $\|I(f)\| \leq \|f\|_1$. For $f \in C_c(X)$ and $a \in V$, we obtain $I(f.a) = \int_X f \ \ d\mu \cdot a$.

Proof. Let $f \in C_c(X, V)$. To prove that (3) determines $I(f)$ uniquely if it exists, we assume $a, b \in V$, such that $\phi\left(a\right) = \int_X \phi \circ f \ \mathrm{d}\mu = \phi(b)$ for all $\phi \in V'$. Then $a = b$, since V' separates points of V.
Assume that we have $f_1, f_2 \in C_c(X, V)$ with $I(f_1), I(f_2) \in V$, such that (3) holds for both. For $\lambda_1, \lambda_2 \in \mathbb{C}$, we obtain

$$\phi\left(\lambda_1 I(f_1) + \lambda_2 I(f_2)\right) \overset{(3)}{=} \lambda_1 \int_X \phi \circ f_1 \ \mathrm{d}\mu + \lambda_2 \int_X \phi \circ f_2 \ \mathrm{d}\mu = \int_X \phi \circ \left(\lambda_1 f_1 + \lambda_2 f_2\right) \ \mathrm{d}\mu\,.$$

This shows the existence of $I(\lambda_1 f_1 + \lambda_2 f_2) = \lambda_1 I(f_1) + \lambda_2 I(f_2)$.
By the Hahn-Banach Theorem, there is $\phi \in V'$ with $|\phi(I(f))| = \|I(f)\|$ and $\|\phi\| = 1$.
We obtain

$$\begin{aligned}\|I(f)\| = |\phi(I(f)))| &\overset{(3)}{=} \left|\int_X \phi(f(x))\ \mathrm{d}\mu(x)\right| \\ &\leq \int_X |\phi(f(x))|\ \mathrm{d}\mu(x) \\ &\leq \int_X \|f(x)\|\ \mathrm{d}\mu(x) = \|f\|_1.\end{aligned}$$

For $f \in C_c(X)$, $a \in V$ and $\phi \in V'$,

$$\begin{aligned}\phi\left(\left(\int_X f\ \mathrm{d}\mu\right) a\right) &= \int_X f\ \mathrm{d}\mu \cdot \phi(a) = \int_X f(x)\phi(a)\ \mathrm{d}\mu(x) = \int_X \phi(f(x)a)\ \mathrm{d}\mu(x) \\ &= \int_X \phi \circ (f.a)\ \mathrm{d}\mu,\end{aligned}$$

showing that $I(f.a)$ exists with $I(f.a) = \int_X f\ \mathrm{d}\mu \cdot a$.
This implies the existence of $I(h)$ for all $h \in M$ and that I is linear and bounded on M.
Since M is dense in $C_c(X,V)$ (Corollary 5.6) and V is complete, the bounded map I extends to a linear and bounded map $I : C_c(X,V) \to V$.
Now let $f \in C_c(X,V)$ arbitrary. We find a sequence $(h_n)_{n\in\mathbb{N}} \subseteq M$ converging to f. For $\phi \in V'$, we obtain

$$\begin{aligned}\left|\int_X \phi \circ f\ \mathrm{d}\mu - \int_X \phi \circ h_n\ \mathrm{d}\mu\right| &\leq \int_X |\phi(f(x) - h_n(x))|\ \mathrm{d}\mu(x) \\ &\leq \|\phi\| \int_X \|f(x) - h_n(x)\|\ \mathrm{d}\mu(x) \\ &= \|\phi\| \cdot \|f - h_n\|_1 n \to \infty 0.\end{aligned}$$

Hence

$$\phi(I(f)) = \phi\left(\lim_{n\to\infty} I(h_n)\right) = \lim_{n\to\infty} \phi(I(h_n)) = \lim_{n\to\infty} \int_X \phi \circ h_n\ \mathrm{d}\mu = \int_X \phi \circ f\ \mathrm{d}\mu$$

This yields the existence of $I(f)$. □

From now on, we write $\int_X f\ \mathrm{d}\mu$ or $\int_X f(x)\ \mathrm{d}\mu(x)$ for the element $I(f)$ from Proposition 5.7. Next we collect some properties of the integral that we need later.

Lemma 5.8 (Bounded linear maps and integration)**.**
Let W be another Banach space together with a bounded linear map $T : V \to W$.
For $f \in C_c(X,V)$ we obtain $T \circ f \in C_c(X,W)$ and

$$\int_X T \circ f\ d\mu = T\left(\int_X f\ d\mu\right).$$

Proof. If $f(x) = 0$, then $(T \circ f)(x) = 0$. Hence $\operatorname{supp}(T \circ f) \subseteq \operatorname{supp}(f)$ is compact.
Let $\psi \in W'$, then $\psi \circ T \in V'$ and

$$\psi\left(T\left(\int_X f \, \mathrm{d}\mu\right)\right) = (\psi \circ T)\left(\int_X f \, \mathrm{d}\mu\right) \overset{(3)}{=} \int_X (\psi \circ T) \circ f \, \mathrm{d}\mu = \int_X \psi \circ (T \circ f) \, \mathrm{d}\mu \,.$$

By Proposition 5.7 this proves the claim. □

Lemma 5.9 (Iterated Integration)**.**
Let Y be another locally compact Hausdorff space with a locally finite Borel measure ν. Let $f \in C_c(X \times Y, V)$. The functions

$$F\colon Y \to V;\; y \mapsto \int_X f(x,y) \; d\mu(x) \qquad \textit{and} \qquad G\colon X \to V;\; x \mapsto \int_Y f(x,y) \; d\nu(y)$$

are continuous with compact support. $\int_Y F \; d\nu = \int_X G \; d\mu$, that is,

$$\int_Y \left(\int_X f(x,y) \; d\mu(x)\right) \; d\nu(y) = \int_X \left(\int_Y f(x,y) \; d\nu(y)\right) \; d\mu(x).$$

Proof. First we show that F is well-defined and continuous.
Let $y_0 \in Y$. There are $K_1 \subseteq X$ and $K_2 \subseteq Y$, such that $\operatorname{supp}(f) \subseteq K_1 \times K_2$. The support of the function $x \mapsto f(x, y_0)$ is a subset of K_1, therefore, $F(y_0) = \int_X f(x, y_0) \; \mathrm{d}\mu(X) \in V$ by Proposition 5.7. Hence F is well-defined.
Let $\varepsilon > 0$. For all $x \in K_1$, there is a neighbourhood $U_x \subseteq X$ of x and a neighbourhood V_x of y_0, such that

$$\|f(x', y') - f(x, y_0)\| < \frac{\varepsilon}{2((K_1) + 1)} \qquad \text{for all } x' \in U_x \text{ and } y' \in V_x.$$

Since K_1 is compact, there are $x_1, \ldots, x_n \in K_1$, such that $U_{x_1}, \ldots, U_{x_n}$ cover K_1. Put $V = V_{x_1} \cap \ldots \cap V_{x_n}$ and let $y \in V$. For $x \in K_1$, we obtain an $i = 1, \ldots, n$, such that $x \in U_{x_i}$. We estimate

$$\|f(x,y) - f(x,y_0)\| \le \|f(x,y) - f(x_i,y_0)\| + \|f(x_i,y_0) - f(x,y_0)\| < \frac{\varepsilon}{\mu(K_1) + 1}.$$

This leads to

$$\begin{aligned}\|F(y) - F(y_0)\| &= \|\int_X f(x,y) \; \mathrm{d}\mu(x) - \int_X f(x,y_0) \; \mathrm{d}\mu(x)\| \\ &\le \int_{K_1} \|f(x,y) - f(x,y_0)\| \; \mathrm{d}\mu(x) < \varepsilon,\end{aligned}$$

proving that F is continuous. If $y \notin K_2$, then $f(x,y) = 0$ for all $x \in X$. Hence $F(y) = 0$. This shows $\operatorname{supp}(F) \subseteq K_2$. Therefore, $F \in C_c(Y, V)$.
Using the same arguments we get $G \in C_c(X, V)$.
Since $\mu|_{K_1}$ and $\nu|_{K_2}$ are finite, there is a unique product measure $\mu|_{K_1} \otimes \nu|_{K_2}$ on $K_1 \times K_2$.

Since $f \in C_c(X \times Y, V)$, there is $M > 0$, such that $\|f(x,y)\| \leq M$ for all $x \in X$ and $y \in Y$. Let $\phi \in V'$, we have

$$\int_{K_1 \times K_2} |(\phi \circ f)(x,y)| \, \mathrm{d}\,(\mu|_{K_1} \otimes \nu|_{K_2}) \leq \|\phi\| M \mu(K_1)\nu(K_2) < \infty.$$

Therefore, we can apply Fubini's theorem to get

$$\begin{aligned}
\phi\left(\int_Y F \ \mathrm{d}\nu\right) &= \int_Y \phi \circ F \ \mathrm{d}\nu \\
&= \int_{K_2} \left(\int_{K_1} \phi \circ f \ \mathrm{d}\mu\right) \ \mathrm{d}\nu \\
&= \int_{K_1} \left(\int_{K_2} \phi \circ f \ \mathrm{d}\nu\right) \ \mathrm{d}\mu \\
&= \int_X \phi \circ G \ \mathrm{d}\mu \, .
\end{aligned}$$

Hence $\int_Y F \ \mathrm{d}\nu = \int_X G \ \mathrm{d}\mu$, by Proposition 5.7. $\square$

Lemma 5.10 (Dominated Convergence Theorem).
Let $(f_n)_{n\in\mathbb{N}} \subseteq C_c(X,V)$ be a sequence and $f \in C_c(G,V)$ with $f_n(x) \longrightarrow f(x)$ for $n \to \infty$ for all $x \in X$. If there is $g \in L^1(X)$, with $\|f_n(x)\| \leq g(x)$ for all $x \in X$, then $\int_X f_n \ d\mu$ converges weakly to $\int_X f \ d\mu$.

Proof. Let $\phi \in V'$. Then $\phi \circ f_n$ converges pointwise to $\phi \circ f$ and

$$|(\phi \circ f_n)(x)| \leq \|\phi\| \|f_n(x)\| \leq \|phi\| \cdot g(x).$$

Hence Lebesgue's Dominated Convergence Theorem applies and we obtain

$$\lim_{n\to\infty} \phi\left(\int_X f_n \ \mathrm{d}\mu\right) \stackrel{3}{=} \lim_{n\to\infty} \int_X \phi \circ f_n \ \mathrm{d}\mu \stackrel{3}{=} \int_X \phi \circ f \ \mathrm{d}\mu = \phi\left(\int_X f \ \mathrm{d}\mu\right).$$

$\square$

Since a Haar measure μ on a locally compact group G is locally finite and strictly positive, we can use our integral for C_c-functions $G \to V$. The next lemma states the translation invariance of the integral.

Lemma 5.11 (Translation Invariance).
Let $g \in G$ and $f \in C_c(X,V)$. Then $\lambda_g(f) \in C_c(X,V)$, where $(\lambda_g(f))(x) = f(g^{-1}x)$ for $x \in X$, and

$$\int_X f \ d\mu = \int_X \lambda_g(f) \ d\mu \ \ \textit{or} \ \int_X f(x) \ d\mu(x) = \int_X f(g^{-1}x) \ d\mu(x) \ \textit{in the other notation.}$$

Proof. Since $\nu_g \colon G \to G$, $x \mapsto g^{-1}x$ is continuous, so is $\lambda_g(f) = f \circ \nu_g$. We have $g\,\mathrm{supp}(f) = \mathrm{supp}(\lambda_g(f))$. Therefore, $\lambda_g(f) \in C_c(G, V)$.
We assume the Haar measure μ to be left invariant, that is, $\mu(gA) = \mu(A)$ for a Borel set $A \subseteq G$. We have

$$((\nu_g)_* \mu)(A) = \mu\left((\nu_g)^{-1}(A)\right) = \mu(gA) = \mu(A)$$

for all Borel sets $A \subseteq G$, showing $(\nu_g)_*\mu = \mu$. For $\phi \in V'$ we obtain

$$\int_X \phi \circ (\lambda_g(f)) \ \mathrm{d}\mu = \int_X \phi \circ f \circ \nu_g \ \mathrm{d}\mu = \int_X \phi \circ f \ \mathrm{d}((\nu_g)_*\mu) \stackrel{(3)}{=} \phi\left(\int_X f \ \mathrm{d}\mu\right).$$

This proves the claim by Proposition 5.7. □

Remark 5.12. If G is unimodular, such that μ is also right invariant. We obtain

$$\int_X f(x) \ \mathrm{d}\mu(x) = \int_X f(xg^{-1}) \ \mathrm{d}\mu(x)$$

with the same arguments as in lemma 5.11. In this case we have $\mu(A) = \mu(A^{-1})$ for all Borel sets $A \subseteq G$, and a similar argument as above shows

$$\int_G f(g^{-1}) \ \mathrm{d}\mu(g) = \int_G f(g) \ \mathrm{d}\mu(g).$$

Next we collect properties of the integral with values in a C^*- algebra A.

Lemma 5.13 (Properties of the C^*-valued Integral)**.**
Let $f \in C_c(X, A)$.

(i) $\left(\int_X f(x) \ d\mu(x)\right)^* = \int_X f(x)^* \ d\mu(x)$.

(ii) *If $f(x)$ is self-adjoint for all $x \in X$, then $\int_X f \ d\mu$ is self-adjoint.*

(iii) *If $f(x)$ is positive for all $x \in X$, then $\int_X f \ d\mu$ is positive.*

(iv) *If $f(x)$ is positive for all $x \in X$ and $\int_X f \ d\mu = 0$, then $f = 0$.*

(v) *If $a \in A$, then $f \cdot a \in C_c(X, A)$, where $(f \cdot a)(x) = f(x)a$, and*

$$\int_X f \cdot a \ d\mu = \int_X f \ d\mu \cdot a.$$

(vi) *If $a \in A$,then $a \cdot f \in C_c(X, A)$, where $(a \cdot f)(x) = af(x)$ and*

$$\int_X a \cdot f \ d\mu = a \cdot \int_X f \ d\mu\,.$$

Proof. (i) Since the involution on A is isometric, (i) follows immediately from Lemma 5.8.

(ii) Provided $f(x)$ is self-adjoint for all $x \in X$, we use (i) to obtain

$$\left(\int_X f(x)\ \mathrm{d}\mu(x)\right)^* \overset{\text{(i)}}{=} \int_X f(x)^*\ \mathrm{d}\mu(x) = \int_X f(x)\ \mathrm{d}\mu(x).$$

(iii) Positive linear functionals are continuous by Lemma 5.2. Therefore, we can use them in equation (3).
Let $f(x)$ be positive for all $x \in X$ and $\phi \in \mathcal{P}(A)$. Then

$$\phi\left(\int_X f\ \mathrm{d}\mu\right) \overset{(3)}{=} \int_X \phi \circ f\ \mathrm{d}\mu \geq 0,$$

because $\phi\,(f(x)) \geq 0$ for all $x \in X$. Therefore, 5.3(ii) implies $\int_X f\ \mathrm{d}\mu \geq 0$.

(iv) If $f(x)$ is positive for all $x \in X$ and $\int_X f\ \mathrm{d}\mu = 0$, then

$$0 = \phi\left(\int_X f\ \mathrm{d}\mu\right) \overset{(3)}{=} \int_X \phi \circ f\ \mathrm{d}\mu \text{ for all } \phi \in \mathcal{P}(A).$$

Since $\phi \circ f \geq 0$ and μ is strictly positive, this implies $\phi \circ f = 0$. Hence for $x \in X$, we obtain $\phi\,(f(x)) = 0$ for all $\phi \in \mathcal{P}$. Using 5.3(i), we conclude that $f(x) = 0$.

(v) The left multiplication map $m_a \colon A \to A$ given by $b \mapsto ab$ is bounded. Therefore, Lemma 5.8 applies and yields

$$\int_X f \cdot a\ \mathrm{d}\mu = m_a\left(\int_X f\ \mathrm{d}\mu\right) = \int_X m_a \circ f\ \mathrm{d}\mu = \int_X f\ \mathrm{d}\mu \cdot a.$$

(vi) One could argue as in (v) or use (i) to get

$$\begin{aligned}\int_X (a \cdot f)(x)\ \mathrm{d}\mu(x) &= \left(\int_X (a \cdot f)(x)^*\ \mathrm{d}\mu(x)\right)^* \\ &\overset{\text{(v)}}{=} \left(\int_X f(x)^*\ \mathrm{d}\mu(x) a^*\right)^* = a \cdot \int_X f(x)\ \mathrm{d}\mu(x).\end{aligned}$$

□

5.3 Equivariant Hilbert Modules

Let G be a locally compact group with Haar measure μ.
The Hilbert space $L^2(G)$ carries a natural action by left translation of G. We want to define an action of G on $L^2(G, A)$ that is compatible with the Hilbert module structure and with a given action of G on A.
An action $(\alpha_g)_{g \in G}$ of G on A by $*$-automorphisms is called continuous if for all $a \in A$ the map $G \to A$ given by $g \mapsto \alpha_g(a)$ is norm continuous.
Let A be a C^*-algebra with a continuous G-action or briefly, a G-C^*-algebra.

Definition 5.14 (Equivariant Hilbert Module)**.** A linear action $(\gamma_g)_{g\in G}$ on a Hilbert A-module $\mathcal{E}$ is called *Hilbert module action* if it satisfies the following conditions.

(i) The action is strongly continous, that is, for every $\xi \in \mathcal{E}$ the map $G \to \mathcal{E}$ given by $g \mapsto \gamma_g(\xi)$ is norm continuous.

(ii) $\gamma_g(\xi \cdot a) = \gamma_g(\xi) \cdot \alpha_g(a)$ for $\xi \in \mathcal{E}, a \in A$ and $g \in G$.

(iii) $\langle \gamma_g(\xi), \gamma_g(\eta)\rangle = \alpha_g(\langle \xi, \eta\rangle)$ for all $\xi, \eta \in \mathcal{E}$ and $g \in G$.

A Hilbert A-module $\mathcal{E}$ equipped with a Hilbert module action $(\gamma_g)_{g\in G}$ is called a *G-equivariant Hilbert A-module* or briefly, a *Hilbert G-A-module.*
We write $\mathbb{B}^G(\mathcal{E})$ for the set of of G-equivariant adjointable maps $\mathcal{E} \to \mathcal{E}$.

It is easy to see, that $\mathbb{B}^G(\mathcal{E})$ is a C^*-subalgebra of $\mathbb{B}(\mathcal{E})$.

Example 5.15. (i) If we consider A as a Hilbert A-module the action $(\alpha_g)_{g\in G}$ turns it into a Hilbert G-A-module. The first condition is just the continuity, and the second condition is the multiplicativity of the C^*-algebra action $(\alpha_g)_{g\in G}$.
The third condition is satisfied, since for $a, b \in A$, we have

$$\langle \alpha_g(a), \alpha_g(b)\rangle = \alpha_g(a)^*\alpha_g(b) = \alpha_g(a^*b) = \alpha_g\left(\langle a, b\rangle\right).$$

(ii) For $A = \mathbb{C}$ with the trivial G-action, Hilbert module actions are just strongly continuous actions by unitaries on Hilbert spaces.

For a Hilbert module action $(\gamma_g)_{g\in G}$, it follows that

$$\|\gamma_g(\xi)\| = \|\langle \gamma_g(\xi), \gamma_g(\xi)\rangle\|^{1/2} \overset{\text{(iii)}}{=} \|\alpha_g\left(\langle \xi, \xi\rangle\right)\|^{1/2} = \|\langle \xi, \xi\rangle\|^{1/2} = \|\xi\|.$$

Therefore, every γ_g is an isometry.
To construct a tensor product of Hilbert G-A-modules we need to extend actions on dense submodules.

Lemma 5.16 (Extension of G-actions)**.** *Let $\mathcal{D} \subseteq \mathcal{E}$ be a dense submodule of a Hilbert A-module $\mathcal{E}$, such that there is a linear action $(\gamma_g)_{g\in G}$ on $\mathcal{D}$ satisfying* (i)*,* (ii) *and* (iii) *from Definition* 5.14*. Then $(\gamma_g)_{g\in G}$ extends uniquely to a Hilbert module action on $\mathcal{E}$, so that $\mathcal{E}$ becomes a Hilbert G-A-module.*

Proof. Let $g \in G$. As above, we see that γ_g is an isometry on $\mathcal{D}$. Since $\mathcal{E}$ is complete, the map $\gamma_g : \mathcal{D} \to \mathcal{D} \subseteq \mathcal{E}$ extends to a linear and isometric map $\widetilde{\gamma_g} : \mathcal{E} \to \mathcal{E}$. Since this extension is the only isometric one, this shows the uniqueness part.
We still have $\widetilde{\gamma_g} \circ \widetilde{\gamma_g} = \widetilde{\gamma_{gh}}$, since these maps agree on the dense subset $\mathcal{D}$. Therefore, $(\widetilde{\gamma_g})_{g\in G}$ is a linear action on $\mathcal{E}$. It remains to verify (i), (ii) and (iii) from definition 5.14.

(i) Let $\xi \in \mathcal{E}$. Choose $\xi_{\mathcal{D}} \in \mathcal{D}$ with $\|\xi - \xi_{\mathcal{D}}\| < \frac{\varepsilon}{3}$.
Let $g_0 \in G$. Since $g \mapsto \gamma_g(\xi_{\mathcal{D}})$ is continuous, we find a neighbourhood U of g_0 such that
$$\|\widetilde{\gamma_{g_0}}(\xi_{\mathcal{D}}) - \widetilde{\gamma_g}(\xi_{\mathcal{D}})\| = \|\gamma_{g_0}(\xi_{\mathcal{D}}) - \gamma_g(\xi_{\mathcal{D}})\| < \frac{\varepsilon}{3} \text{ for all } g \in U.$$
For $g \in U$, we obtain
$$\begin{aligned}\|\widetilde{\gamma_{g_0}}(\xi) - \widetilde{\gamma_g}(\xi)\| &\leq \|\widetilde{\gamma_{g_0}}(\xi) - \widetilde{\gamma_{g_0}}(\xi_{\mathcal{D}})\| + \|\widetilde{\gamma_{g_0}}(\xi_{\mathcal{D}}) - \widetilde{\gamma_g}(\xi_{\mathcal{D}})\| + \|\widetilde{\gamma_g}(\xi_{\mathcal{D}}) - \widetilde{\gamma_g}(\xi)\| \\ &< \|\xi - \xi_{\mathcal{D}}\| + \frac{\varepsilon}{3} + \|\xi_{\mathcal{D}} - \xi\| < \varepsilon.\end{aligned}$$
This shows that $g \mapsto \widetilde{\gamma_g}(\xi)$ is continuous.

(ii) Let $\xi \in \mathcal{E}$ and $a \in A$. Choose $(\xi_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}$ with $\xi_n \longrightarrow \xi$. Then $\xi_n \cdot a \in \mathcal{D}$ for all $n \in \mathbb{N}$ and $\xi_n \cdot a \longrightarrow \xi \cdot a$. Thus
$$\widetilde{\gamma_g}(\xi \cdot a) = \lim_{n\in\mathbb{N}} \gamma_g(\xi_n \cdot a) = \lim_{n\in\mathbb{N}} \gamma_g(\xi)\alpha_g(a) = \widetilde{\gamma_g}(\xi) \cdot \alpha_g(a).$$

(iii) Let $\xi, \eta \in \mathcal{E}$. There are $(\xi_n)_{n\in\mathbb{N}}, (\eta_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}$ with $\xi_n \longrightarrow \xi$ and $\eta_n \longrightarrow \eta$. Then
$$\langle \widetilde{\gamma_g}(\xi), \widetilde{\gamma_g}(\eta)\rangle = \lim_{n\to\infty} \langle \gamma_g(\xi_n), \gamma_g(\eta)\rangle = \lim_{n\to\infty} \alpha_g\left(\langle \xi_n, \eta_n\rangle\right) = \alpha\left(\langle \xi, \eta\rangle\right). \qquad \square$$

Let $\mathcal{E}$ be a Hibert G-A-module. The G-action $(\gamma_g)_{g\in G}$ on $\mathcal{E}$ induces an action on $\mathbb{B}(\mathcal{E})$ by $*$-automorphisms in the following way.
For $T \in \mathbb{B}(\mathcal{E})$, we obtain
$$\begin{aligned}\langle \left(\gamma_g \circ T \circ \gamma_{g^{-1}}\right)(\xi), \eta\rangle &= \alpha_g\left(\langle (T \circ \gamma_{g^{-1}})(\xi), \gamma_{g^{-1}}(\eta)\rangle\right) \\ &= \alpha_g\left(\langle \gamma_{g^{-1}}(\xi), (T^* \circ \gamma_{g^{-1}})(\eta)\rangle\right) \\ &= \langle \xi, \left(\gamma_g \circ T^* \circ \gamma_{g^{-1}}\right)(\eta)\rangle\end{aligned}$$
This shows that $\gamma_g \circ T \circ \gamma_{g^{-1}}$ is adjointable with adjoint $\gamma_g \circ T^* \circ \gamma_{g^{-1}}$.
Hence we have a well defined map $\rho_g \colon \mathbb{B}(\mathcal{E}) \to \mathbb{B}(\mathcal{E})$ given by $T \mapsto \gamma_g \circ T \circ \gamma_{g^{-1}}$.
We have
$$\rho_g \circ \rho_h(T) = \rho_g(\gamma_h \circ T \circ \gamma_{h^{-1}}) = \gamma_g \circ (\gamma_h \circ T \circ \gamma_{h^{-1}}) \circ \gamma_{g^{-1}} = \gamma_{gh} \circ T \circ \gamma_{(gh)^{-1}} = \phi_{gh}(T).$$
Furthermore, $\gamma_1 = \mathrm{id}_{\mathcal{E}}$ implies $\rho_1 = \mathrm{id}_{\mathbb{B}(\mathcal{E})}$, hence every ρ_g is invertible.
Obviously, ρ_g is multiplicative, and above we proved $\rho_g(T^*) = \rho_g(T)^*$. Therefore, $(\rho_g)_{g\in G}$ is an action by $*$-automorphisms. We always endow $\mathbb{B}(\mathcal{E})$ with this canonical action.
Now let B be another G-C^*-algebra with action $(\beta_g)_{g\in G}$ and $\mathcal{F}$ a Hilbert G-B-module with action $(\delta_g)_{g\in G}$ together with a G-equivariant $*$-homomorphism $\rho \colon B \to \mathbb{B}(\mathcal{E})$.

Lemma 5.17 (Induced Action on Tensor Products)**.**
The Hilbert A- module $\mathcal{F} \otimes_B \mathcal{E}$ is a Hilbert G-A module with the unique action $(\pi_g)_{g\in G}$ satisfying $\pi_g(\xi \otimes \eta) = \delta_g(\xi) \otimes \gamma_g(\eta)$.

Proof. For $g \in G$, we have a bilinear map

$$\mathcal{F} \times \mathcal{E} \to \mathcal{F} \otimes_B^{\text{alg}} \mathcal{E}, (\xi, \eta) \mapsto \delta_g(\xi) \otimes \gamma_g(\eta).$$

Since

$$\begin{aligned} \delta_g(\xi \cdot b) \otimes \gamma_g(\eta) &\overset{\text{(ii)}}{=} \delta_g(\xi) \cdot \beta_g(b) \otimes \gamma_g(\eta) \\ &= \delta_g(\xi) \otimes \phi\left(\beta_g(b)\right) \gamma_g(\eta) \\ &= \delta_g(\xi) \otimes \rho_g\left(\phi(b)\right) \gamma_g(\eta) \\ &= \delta_g(\xi) \otimes \gamma_g\left(\phi(b)\eta\right), \end{aligned}$$

it induces a linear map $\pi_g \colon \mathcal{F} \otimes_B^{\text{alg}} \mathcal{E} \to \mathcal{F} \otimes_B^{\text{alg}} \mathcal{E}$. Now

$$\pi_{gh}(\xi \otimes \eta) = \delta_{gh}(\xi) \otimes \gamma_{gh}(\eta) = \delta_g\left(\delta_h(\xi)\right) \otimes \gamma_g\left(\gamma_g(\eta)\right) = \pi_g \circ \pi_h(\xi \otimes \eta).$$

Linearity yields $\pi_{gh} = \pi_g \circ \pi_h$. And since $\pi_1 = \mathrm{id}_{\mathcal{F} \otimes_{\mathcal{B}}^{\text{alg}} \mathcal{E}}$, every π_g is invertible. Hence $(\pi_g)_{g \in G}$ is a linear action of G on $\mathcal{F} \otimes_B^{\text{alg}} \mathcal{E}$.
Now we verify properties (i), (ii) and (iii) from Definition 5.14.

(i) Since $\|\xi \otimes \eta\| \leq \|\xi\| \cdot \|\eta\|$, the canonical map $\mathcal{F} \times \mathcal{E} \to \mathcal{F} \otimes_B^{\text{alg}} \mathcal{E}$ is continuous with respect to the product topology on $\mathcal{F} \times \mathcal{E}$.
Now let $g \in G$. The map $g \mapsto (\delta_g(\xi), \gamma_g(\eta))$ is continuous, since both entries are continuous by the strong continuity of $(\gamma_g)_{g \in g}$ and $(\delta_g)_{g \in g}$. Thus the map $g \mapsto \pi_g(\xi \otimes \eta) = \delta_g(\xi) \otimes \gamma_g(\eta)$ is continuous. An arbitrary element $\zeta \in \mathcal{F} \otimes_B^{\text{alg}}$ is a linear combination of elementary tensors. Therefore, $g \mapsto \pi_g(\zeta)$ is continuous as a sum of continuous functions.

(ii) Let $a \in A$. We have

$$\begin{aligned} \pi_g\left((\xi \otimes \eta) \cdot a\right) &= \pi_g\left(\xi \otimes (\eta \cdot a)\right) \\ &= \delta_g(\xi) \otimes \gamma_g(\eta \cdot a) \\ &= \delta_g(\xi) \otimes \left(\gamma_g(\eta) \cdot \alpha_g(a)\right) \\ &= \pi_g(\xi \otimes \eta) \cdot \alpha_g(a). \end{aligned}$$

By linearity, (ii) holds for every $\zeta \in \mathcal{F} \otimes_B^{\text{alg}} \mathcal{E}$.

(iii) Let $\xi_1, \xi_2 \in \mathcal{F}$ and $\eta_1, \eta_2 \in \mathcal{E}$. Then

$$\begin{aligned} \langle \pi_g(\xi_1 \otimes \eta_1), \pi_g(\xi_2 \otimes \eta_2) \rangle &= \langle \delta_g(\xi_1) \otimes \delta_g(\eta_1), \delta_g(\xi_2) \otimes \gamma_g(\eta_2) \rangle \\ &= \langle \gamma_g(\eta_1)\, \phi\left(\langle \delta_g(\xi_1), \delta_g(\xi_2) \rangle\right) \gamma_g(\eta_2) \rangle \\ &= \langle \gamma_g(\eta_1), \rho_g\left(\phi(\langle \xi_1, \xi_2 \rangle)\right) \gamma_g(\eta_2) \rangle \\ &= \langle \gamma_g(\eta_1), \gamma_g\left(\phi(\langle \xi_1, \xi_2 \rangle) \eta_2\right) \rangle \\ &= \alpha_g\left(\langle \xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2 \rangle\right). \end{aligned}$$

Linearity yields (iii) for arbitrary elements of $\mathcal{F} \otimes_B^{\text{alg}} \mathcal{E}$.

By Lemma 5.16, the action $(\pi_g)_{g \in G}$ on $\mathcal{F} \otimes_B^{\text{alg}} \mathcal{E}$ extends uniquely to an action on $\mathcal{F} \otimes_B \mathcal{E}$, such that $\mathcal{F} \otimes_B \mathcal{E}$ becomes a Hilbert G-A-module. $\square$

5.4 The Twisted Convolution Algebra $C_c(G, A)$

To define a the $*$-algebra structure on $C_c(G, A)$ we need the following lemma.

Lemma 5.18. *If $m\colon G \times G \to A$ is a continuous function, then the function $f\colon G \times G \to A$ given by $(g, x) \mapsto \alpha_x(m(g, x))$ is continuous.*

Proof. Let $(x_0, g_0) \in G \times G$ and $\varepsilon > 0$. Since $(\alpha_g)_{g\in G}$ is continuous, there is a neighbourhood U_1 if $x_0 \in G$, such that

$$\|\alpha_x(m(g_0, x_0)) - \alpha_{x_0}(m(g_0, x_0))\| < \frac{\varepsilon}{2} \text{ for all } x \in U_1.$$

By continuity of m, there are neighbourhoods U_2 of x_0 and V of g_0 in G, such that

$$\|m(g, x) - m(g_0, x_0)\| < \frac{\varepsilon}{2} \text{ for all } x \in U_2 \text{ and } y \in V$$

For $(g, x) \in (U_1 \cap U_2) \times V$ we obtain

$$\begin{aligned}\|f(g, x) - f(g_0, x_0)\| \leq{}& \|\alpha_x(m(g, x)) - \alpha_x(m(g_0, x_0))\| \\ &+ \|\alpha_{x_0}(m(g_0, x_0)) - \alpha_x(m(g_0, x_0))\| < \varepsilon.\end{aligned}$$

□

Let $f_1, f_2 \in C_c(G, A)$. The function

$$h\colon G \times G \to A \text{ given by } (g, x) \mapsto f_1(x)\alpha_x(f_2(x^{-1}g))$$

is continuous by Lemma 5.18. If $x \in \operatorname{supp}(f_1)$ and $x^{-1}g \in \operatorname{supp}(f_2)$, then $g \in x \operatorname{supp}(f_2) \subseteq \operatorname{supp}(f_1)\operatorname{supp}(f_2)$. Hence

$$\operatorname{supp}(h) \subseteq \operatorname{supp}(f_1) \times (\operatorname{supp}(f_1)\operatorname{supp}(f_2))$$

is compact. Therefore, $h \in C_c(G \times G, A)$. We define

$$(f_1 * f_2)(g) = \int_G h(g, x)\ \mathrm{d}\mu(x) = \int_G f_1(x) \cdot \alpha_x(f_2(x^{-1}g))\ \mathrm{d}\mu(x).$$

Lemma 5.9 yields $f_1 * f_2 \in C_c(G, A)$.
If $f \in C_c(G, A)$ we define $f^*(g) = \alpha_g(f(g^{-1}))^*$. Since $\operatorname{supp}(f^*) = \operatorname{supp}(f)$ is compact, we obtain $f^* \in C_c(G, A)$.

Proposition 5.19. *The above structures turn $C_c(G, A)$ into a $*$-algebra. The norm $\|\cdot\|_1$ is submultiplicative and $\|f^*\|_1 = \|f\|_1$ for all $f \in C_c(G, A)$.*

Proof. It is easy to see, that $(f_1, f_2) \mapsto f_1 * f_2$ is bilinear. For $f, f_1, f_2, f_3 \in C_c(G, A)$ we obtain

$$\begin{aligned}((f_1 * f_2) * f_3)(g) &= \int_G \left(\int_G f_1(y)\alpha_y(f_2(y^{-1}x))\ \mathrm{d}\mu(y)\right) \cdot \alpha_x(f_3(x^{-1}g))\ \mathrm{d}\mu(x) \\ &\overset{5.9}{=} \int_G f_1(y) \left(\int_G \alpha_y(f_2(y^{-1}x))\alpha_x(f_3(x^{-1}g))\ \mathrm{d}\mu(x)\right)\ \mathrm{d}\mu(y) \\ &\overset{5.8}{=} \int_G f_1(y)\alpha_y \left(\int_G f_2(y^{-1}x)\alpha_{y^{-1}x}(f_3(x^{-1}g))\ \mathrm{d}\mu(x)\right)\ \mathrm{d}\mu(y) \\ &\overset{5.11}{=} \int_G f_1(y)\alpha_y\left((f_2 * f_3)(y^{-1}g)\right)\ \mathrm{d}\mu(y) = (f_1 * (f_2 * f_3))(g).\end{aligned}$$

Hence $*$ is associative. It is easy to see $(f_1+f_2)^* = f_1^* - f_2^*$ and $(\lambda f)^* = \overline{\lambda} f^*$ for $\lambda \in \mathbb{C}$. Furthermore we have

$$(f^*)^*(g) = \alpha_g(f^*(g^{-1}))^* = \alpha_g \circ \alpha_{g^{-1}}(f((g^{-1})^{-1})) = f(g)$$

and

$$\begin{aligned}
(f_1 * f_2)^*(g) &\overset{5.13(i)}{=} \alpha_g\left(\int_G \alpha_x(f_2(x^{-1}g^{-1}))^* \cdot f_1(x)^* \ \mathrm{d}\mu(x)\right) \\
&\overset{5.8}{=} \int_G \alpha_{gx}(f_2(x^{-1}g^{-1}))^* \cdot \alpha_g(f_1(x))^* \ \mathrm{d}\mu(x) \\
&\overset{5.11}{=} \int_G \alpha_x(f_2(x^{-1}))^* \cdot (\alpha_x \circ \alpha_{x^{-1}g})(f_1(g^{-1}x))^* \ \mathrm{d}\mu(x) \\
&= \int_G f_2^*(x) \cdot \alpha_x(f_1^*(x^{-1}g)) \ \mathrm{d}\mu(x) = (f_2^* * f_1^*)(g).
\end{aligned}$$

Hence $C_c(G,A)$ is a $*$-algebra.
Using Fubini's theorem, we obtain

$$\begin{aligned}
\|f_1 * f_2\|_1 &\overset{5.7}{\leq} \int_G \left(\int_G \|f_1(x)\| \cdot \|\alpha_x(f_2(x^{-1}g))\| \ \mathrm{d}\mu(x)\right) \ \mathrm{d}\mu(x) \\
&= \int_G \|f_1(x)\| \cdot \left(\int_G \|f_2(x^{-1}g)\| \ \mathrm{d}\mu(g)\right) \ \mathrm{d}\mu(x) \\
&\overset{5.11}{=} \int_G \|f_1(x)\| \ \mathrm{d}\mu(x) \cdot \int_G \|f_2(g)\| \ \mathrm{d}\mu(g) = \|f_1\|_1 \cdot \|f_2\|_1.
\end{aligned}$$

Finally, Remark 5.12 implies

$$\|f^*\|_1 = \int_G \|\alpha_g(f(g^{-1}))^*\| \ \mathrm{d}\mu(g) = \int_G \|f(g^{-1})\| \ \mathrm{d}\mu(g) = \|f\|_1. \qquad \square$$

The following lemma shows the existence of approximate identities in $C_c(G,A)$.

Lemma 5.20 (Approximate Identities)**.**
There is a net $(u_i)_{i\in I} \subseteq C_c(G,A)$ with $\|u_i\| \leq 1$ and $u_i^* = u_i$ for all $i \in I$, such that

$$\|f - f * u_i\|_1 \longrightarrow 0 \text{ and } \|f - u_i * f\|_1 \longrightarrow 0 \quad \text{for all } f \in C_c(G,A).$$

Proof. Let $(a_j)_{j\in J}$ be an approximate identity of A. Let $\mathcal{K}$ be the set of compact neighbourhoods of the identity in G. For every $U \in \mathcal{K}$, there is a continuous function $\psi_U \colon G \to [0,\infty)$, such that $\mathrm{supp}(\psi_U) \subseteq U$ and $\int_G \psi_U \ \mathrm{d}\mu = 1$. We define $I = J \times \mathcal{K}$. Then I is directed by

$$(j_1, U_1) \leq (j_2, U_2) \quad \text{if and only if } j_1 \leq j_2 \text{and} U_2 \subseteq U_1.$$

For $(j,U) \in I$, we define $u_{(j,U)} = \psi_U . a_j$. Then $(u_{(j,U)})_{(j,U)\in I}$ is a net in $C_c(G,A)$ with $\|u_{(j,U)}\| \leq 1$.

Let $f \in C_c(G)$ and $a \in A$, with $f \neq 0$ and $a \neq 0$. The proof of Lemma 1.21 shows, that there is $U_1 \in \mathcal{K}$ and $K \subseteq G$ compact, such that $U_1 \operatorname{supp}(f) \subseteq K$ and

$$|f(x) - f(g^{-1}x)| < \frac{\varepsilon}{3\|a\| \cdot \mu(K)} \quad \text{for all } x \in G \text{ and } g \in U_0.$$

There is $U_2 \in \mathcal{K}$, such that

$$\|a - \alpha_g(a)\| \leq \frac{\varepsilon}{3\|f\|1} \quad \text{for all } g \in U_2.$$

Define $U_0 = U_1 \cap U_2$.
There is $j_0 \in J$, such that $\|a - a_j a\| \leq \frac{\varepsilon}{3 \cdot \|f\|_1}$. For $(j, U) > (j_0, U_0)$, we estimate

$$\begin{aligned}
\|f(x)a_j a - (u_(U,j) * f.a)(x)\| &\leq \left\| \int_G \psi_U(g) f(x) a_j (a - \alpha_g(a)) \ \mathrm{d}\mu(g) \right\| \\
&+ \left\| \int_G \psi_U(g) a_j \alpha_g(a) (f(x) - f(g^{-1}x)) \ \mathrm{d}\mu(g) \right\| \\
&\leq \frac{\varepsilon}{3\|f\|_1} \|f(x)\| + \mathbb{1}_K(x) \cdot \frac{\varepsilon}{3\mu(K)}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\|f.a - u_{(j,u)} * f.a\|_1 &\leq \|f.a - f.(a_j a)\|_1 + \|f.(a_j a) - u_(U,j) * f.a\|_1 \\
&\leq \|f\|_1 \|a - a_i a\| + \frac{2}{3}\varepsilon < \varepsilon
\end{aligned}$$

This implies $\|f - u_i * f\|_1 \to 0$ for all $f \in M$. By Corollary 5.6 M is dense in $C_c(G, A)$.
Therefore we obtain $\|f - u_i * f\|_1 \to 0$ for all $f \in C_c(G, A)$ by an $\frac{\varepsilon}{3}$-argument.
Using $\|f\|_1 = \|f^*\|_1$, we obtain $\|f - f * u_i\|_1 \longrightarrow 0$ for all $f \in C_c(G, A)$.
We can arrange $u_i^* = u_i$ by taking $\frac{1}{2}(u_i + u_i^*)$ instead of u_i. □

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Selbstständigkeitserklärung

Hiermit erkläre ich, dass ich die vorliegende Arbeit selbstständig und nur unter Verwendung der angegebenen Quellen und Hilfsmittel verfasst habe.

Göttingen, den